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A Note on a Problem by Welsh in First-Passage Percolation

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Consider first-passage percolation on the square lattice. Welsh, who together with Hammersley introduced the subject in 1963, has formulated a problem about mean first-passage times, which, although seemingly simple, has not been proved in any non-trivial case. In this paper we give a general proof of Welsh's problem.

1. Introduction

First-passage percolation was introduced by Hammersley and Welsh [2] in 1963. For a survey of the area, see Smythe and Wierman [3].

Associate to the links, e, of the square lattice, S, non-negative, i.i.d. random variables, $t = t(e)$, with distribution G.

A path π in S (from n_0 to n_l) is an alternating sequence of nodes and links

$$(n_0, e_1, n_1, \ldots, e_l, n_l),$$

such that e_i is a link between n_{i-1} and n_i, $i = 1, \ldots, l$. The path is self-avoiding if no node appears more than once in the path.

The passage time for the path π is defined as

$$T(\pi) = \sum_{e \in \pi} t(e).$$

For $x, y \in \mathbb{Z}^2$, define the *first-passage time* between x and y as

$$T(x, y) = \inf\{T(\pi) \colon \pi \text{ is a path from } x \text{ to } y\}.$$

Remark 1. $T(x, y) = \inf\{T(\pi) \colon \pi \text{ is a self-avoiding path from } x \text{ to } y\}$.

Lemma 1.1. *For all* $x \neq y \in \mathbb{Z}^2$, $\mathrm{E}(T(x, y)) < \infty$ *iff* $\mathrm{E}(\min(t_1, \ldots, t_4)) < \infty$, *where* $t_1, \ldots, t_4$ *are i.i.d. with distribution* G.

Proof. See [3]. □

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Let $G \in \mathscr{G}$ if $\mathrm{E}(\min(t_1, \dots, t_4)) < \infty$, when $t_1, \dots, t_4$ are i.i.d. with distribution G. Denote the origin of $\mathbb{Z}^2$ by O.

Very little is known about the mean first-passage times, although there are several conjectures about them, most of them very natural. The difficulty of the area is indicated by the fact that one such conjecture about cylinder first-passage times, which was generally believed to be true, was proved wrong: see [1].

In this paper, we will solve a problem, formulated by Welsh, related to another conjecture.

2. Welsh's problem

Welsh [4] formulated the following natural conjecture.

Conjecture 2.1. $\mathrm{E}(T(O,(n,0))) \leqslant \mathrm{E}(T(O,(n,y)))$ for all y.

As Conjecture 2.1 has not been proved, Welsh [4] posed the following simpler problem, with $T_0 = T(O,(1,0))$ and $T_1 = T(O,(1,1))$.

Welsh's Problem. *Prove that* $\mathrm{E}(T_0) \leqslant \mathrm{E}(T_1)$.

The inequality trivially holds when G has support on an interval (a, b), where $b < 2a$, as then $T_0 < b < 2a < T_1$, but the problem has not been solved in *any* non-trivial case.

Theorem 2.1. $\mathrm{E}(T_0) \leqslant \mathrm{E}(T_1)$ *for all* $G \in \mathscr{G}$.

Proof. With notation as in Figure 1, we want to prove that

$$\mathrm{E}(T(\mathrm{A},\mathrm{B})) \leqslant \mathrm{E}(T(\mathrm{A},\mathrm{C})).$$

Note that

$$T(O,(1,0)) \stackrel{d}{=} T(\mathrm{A},\mathrm{B}) \stackrel{d}{=} T(\mathrm{C},\mathrm{D}) \,(\stackrel{d}{=} T(\mathrm{A},\mathrm{D}) \stackrel{d}{=} T(\mathrm{B},\mathrm{C}))$$

and

$$T(O,(1,1)) \stackrel{d}{=} T(\mathrm{A},\mathrm{C}) \stackrel{d}{=} T(\mathrm{B},\mathrm{D}).$$

Let π_{AC} be an arbitrary path from A to C, and π_{BD} be an arbitrary path from B to D. These must contain at least one common node, P. (To get a unique node, let P denote the one closest to A on π_{AC}.)

Introduce the following notation. If two paths have a common endpoint, they can be joined to form a new path. This operation is denoted by $\oplus$, *e.g.* $\pi_{\mathrm{AC}} = \pi_{\mathrm{AP}} \oplus \pi_{\mathrm{PC}}$, with $T(\pi_{\mathrm{AC}}) = T(\pi_{\mathrm{AP}}) + T(\pi_{\mathrm{PC}})$. Given a path π_{PQ} from P to Q, by $\bar{\pi}_{\mathrm{PQ}}$ we refer to the path π_{PQ} in reverse order, *i.e.* from Q to P. Obviously, $T(\pi_{\mathrm{PQ}}) = T(\bar{\pi}_{\mathrm{PQ}})$.

Partition

$$\pi_{\mathrm{AC}} = \pi_{\mathrm{AP}} \oplus \pi_{\mathrm{PC}}$$

and

$$\pi_{\mathrm{BD}} = \pi_{\mathrm{BP}} \oplus \pi_{\mathrm{PD}}$$

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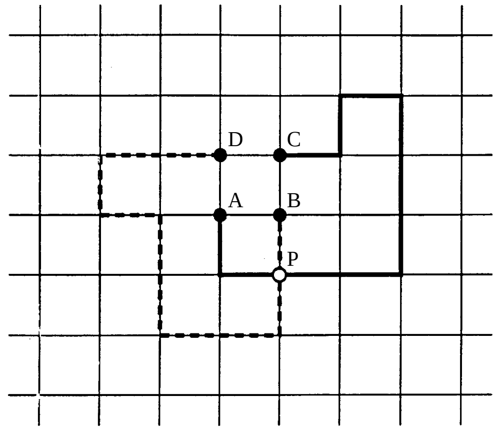

Figure 1

and create the new paths

$$\tilde{\pi}_{AB} = \pi_{AP} \oplus \pi_{PB} = \pi_{AP} \oplus \bar{\pi}_{BP}$$

and

$$\tilde{\pi}_{CD} = \pi_{CP} \oplus \pi_{PD} = \pi_{CP} \oplus \bar{\pi}_{DP}.$$

Then,

$$T(\tilde{\pi}_{AB}) = T(\pi_{AP}) + T(\pi_{PB})$$

and

$$T(\tilde{\pi}_{CD}) = T(\pi_{CP}) + T(\pi_{PD}),$$

so that

$$\begin{aligned} T(\tilde{\pi}_{AB}) + T(\tilde{\pi}_{CD}) &= T(\pi_{AP}) + T(\pi_{PB}) + T(\pi_{CP}) + T(\pi_{PD}) \\ &= T(\pi_{AC}) + T(\pi_{BD}). \end{aligned}$$

Now,

$$\begin{aligned} T(A,B) + T(C,D) &= \inf T(\pi_{AB}) + \inf T(\pi_{CD}) \\ &\leqslant T(\tilde{\pi}_{AB}) + T(\tilde{\pi}_{CD}) \\ &= T(\pi_{AC}) + T(\pi_{BD}). \end{aligned}$$

Since π_{AC} and π_{BD} are arbitrary, we have proved

$$T(A,B) + T(C,D) \leqslant T(A,C) + T(B,D), \tag{2.1}$$

which implies

$$2 \cdot E(T_0) \leqslant 2 \cdot E(T_1). \tag{2.2}$$

□

Remark 2. Note that (2.1) holds without *any* assumptions on the t-variables! For (2.1) to imply (2.2), we only need conditions that guarantee

$$T(A,B) \stackrel{d}{=} T(C,D) \quad \text{and} \quad T(A,C) \stackrel{d}{=} T(B,D).$$

We may, for instance, have different distributions on $t(e)$ for horizontal and vertical edges, and the t-variables do not have to be independent.

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For all non-trivial distributions the inequality of Theorem 2.1 is strict.

Theorem 2.2. *For all* $G \in \mathscr{G}$: $\mathrm{E}(T_0) < \mathrm{E}(T_1)$ *iff* $G(0) < 1$.

Proof. (i) If $G(0) = 1$, then $\mathrm{P}(T_0 = T_1 = 0) = 1$, so that $\mathrm{E}(T_0) = \mathrm{E}(T_1) = 0$.
(ii) Suppose $G(0) < 1$. We know from (2.1) that $T(\mathrm{A},\mathrm{B}) + T(\mathrm{C},\mathrm{D}) \leqslant T(\mathrm{A},\mathrm{C}) + T(\mathrm{B},\mathrm{D})$. The theorem follows if we can show that there is a positive probability that the inequality is strict.

Let $e_1, e_2, \ldots, e_{12}$ be the edges from the points A, B, C, and D as in Figure 2, and let $t_i = t(e_i)$, $i = 1, \ldots, 12$.

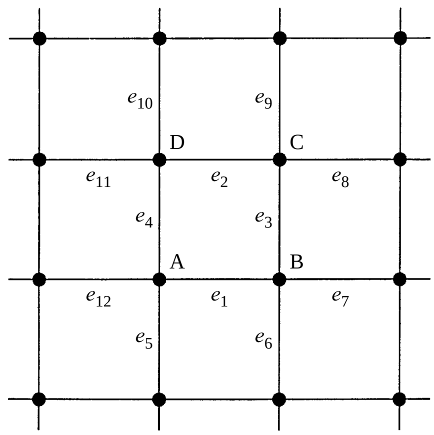

Figure 2

Then, $\mathrm{P}(T(\mathrm{A},\mathrm{B}) + T(\mathrm{C},\mathrm{D}) < T(\mathrm{A},\mathrm{C}) + T(\mathrm{B},\mathrm{D})) \geqslant \mathrm{P}(t_i < t_j,\ i = 1, 2 \text{ and } j = 3, \ldots, 12) > 0$ unless G is a one-point distribution, *i.e.* $\mathrm{P}(t = a) = 1$ for some a, in which case $a > 0$, as $G(0) < 1$, and $T_1 = 2a > a = T_0$. □

Note that we have not proven Conjecture 2.1 for $n = 1$ as the argument above cannot be used for $y > 1$. The argument also fails for $n > 1$, because if we, for example, try to prove that $\mathrm{E}(T(O,(2,0)) \leqslant \mathrm{E}(T(O,(2,2))$ by letting $\mathrm{A} = O$, $\mathrm{B} = (2,0)$, $\mathrm{C} = (2,2)$, and $\mathrm{D} = (0,2)$, then π_{AC} and π_{BD} do not necessarily have a common node P.

For the same reason, the argument fails in three dimensions.

However, if G is an exponential distribution, we can use the lack of memory property of the distribution to arrive at a different proof of Theorem 2.1 in this case. We have used the same technique to show that, in fact, Conjecture 2.1 holds for $n = 1$ in the exponential case.

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