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On the control of a single flexible arm robot via Youla-Kucera parameterization

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SUMMARY
In this paper, based on the Youla-Kucera (Y-K) parameterization, the control of a flexible beam acting as a flexible robotic manipulator is investigated. The method of Youla parameterization is the simple solution and proper method for describing the collection of all controllers that stabilize the closed-loop system. This collection comprises function of the Youla parameter which can be any proper transfer function that is stable. The main challenge in this approach is to obtain a Youla parameter with infinite dimension. This parameter is approximated by a subspace with finite dimensions, which makes the problem tractable. It is required to be generated from a finite number of bases within that space and the considered system can be approximated by an expansion of the orthonormal bases such as FIR, Laguerre, Kautz and generalized bases. To calculate the coefficients for each basis, it is necessary to define the problem in the form of an optimization problem that is solved by optimization techniques. The Linear Quadratic Regulator (LQR) optimization tool is employed in order to optimize the controller gains. The main aim in controller design is to merge the closed-loop system and the second order system with the desirable time response characteristic. The results of the Youla stabilizing controller for a planar flexible manipulator with lumped tip mass indicate that the proposed method is very efficient and robust for the time-continuous instances.

KEYWORDS: Flexible manipulator; Youla-Kucera parameterization; Orthonormal bases; Optimization problem; Linear quadratic regulator.

1. Introduction

The Youla parameterization is probably the most well-known controller one which parameterizes all stabilizing controllers of a system over an infinite dimensional space.[1] The problem of designing a stabilizing controller for a linear time invariant (LTI) plant can be cast into an optimization problem using a closed-loop approach. A design approach for linear continuous time controllers and a strategy to identify plants in closed loop based on Y-K parameterization are presented in this paper. A survey of the literature related to dynamic analyses and control of flexible robotic manipulators is carried out in the succeeding one.[2] An effort is made to critically examine the methods used in these analyses, their advantages and shortcomings and possible extension of these methods to be applied to a general class of problems. The review of the recent literature shows that almost all cases applied linearized models of the link flexibility which reduced the complexity of the model based controller. The experiment to control the end-effector of a flexible manipulator by measuring the tip position is initiated by Cannon and Schmitz,[3] and used as a basis for applying torque to the joint of the beam. However, they considered solely a linearized model and also the arm can sweep only in the XY plane so that it is not affected by the gravity. Since then many new control strategies are developed to control the flexible link vibration. A robust control scheme for the single-link manipulator is used in ref. [4]. The optimal

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control scheme is applied in the presented works in refs. [5, 6 and 7]. In the next one,[8] the singular perturbation technique is used and developed sliding mode control to attenuate the vibration in ref. [9]. The sensor-based feedback controls are carried out in ref. [10]. Some papers as,[11,12] and[13] bear combined feedback linearization with input shaping technique to control the vibration of a single-link flexible manipulator. An infinite dimensional distributed base controller for the regulation of the angular displacement of a one-link flexible robot arm is studied by several researchers.[14,15] In refs. [16, 17] a description regarding PD controller and piezoelectric actuators to control the vibration of the single-link flexible manipulator is presented. The nonlinear feedback, PID state feed-forward and Lyapunov-based stabilization procedure is used to control a flexible industrial manipulator.[18] Paper[19] presents an adaptive impedance control strategy for flexible manipulators by using an end-effector trajectory control approach. A parameterization of a class of stabilizing vibrational controllers for single-input single-output (SISO) systems is presented in ref. [20], and uses this parameterization to design a controller that satisfies unit step response specifications for the original system. In paper[21] a stability and performance preserving controller order reduction method for LTI continuous-time SISO systems is developed. In this method, the error between the complementary sensitivity functions of the nominal closed-loop system and closed-loop system using the reduced-order controller is converted to a frequency-weighted error between the Youla parameters of the full-order and reduced-order controllers and then the H^{∞} norm of this error, subject to a set of linear matrix inequality constraints, is minimized. The paper[22] offers a specific synthesis technique for designing H^{∞} controllers with an observer structure on which the Youla parameterization and a heuristic choice of the poles of the Youla parameter can be based. A hybrid frequency domain neurocontrol algorithm is discussed in ref. [23]. Neural network techniques are used to adjust the Youla parameter in an appropriately parameterized control system. The resulting neurocontroller is stable, robust, and reconfigurable. The learning algorithm is capable of choosing the order of the Youla parameter. In ref. [24], a parameterized output feedback dynamic sliding mode controller is proposed and the internal stability, boundary input boundary output (BIBO) stability, and external disturbance rejection problem of a generalized plant is studied. The proposed controller is described by a solution of Bezout equation and is parameterized by a Youla's free parameter on RH^{∞}. It is shown that the sliding mode is achieved in finite time by the mentioned controller, and thereafter, the ideal sliding mode controller stabilizes the generalized plant in the sense that internal stability is assured. An adaptive LQ controller design procedure is presented in ref. [25]. The design method is based on the idea of Y-K parameterization of the controller and the plant model. The algorithm is applied to a Continuous Stirred Tank Reactor (CSTR). The advantage of this method resides in the fact that in the case of CSTR only one parameter needs to be identified in order to update the controller. Adaptive feed forward broadband vibration (or noise) compensation is currently used when an image of the disturbance is available. The 26$^{\text{th}}$ paper deals with a new Y-K parameterization of the compensator.[26] The central compensator assures the stability of the system and its performances are enhanced in real time by the adaptation of the Q-parameters. A method for estimation of parameters or uncertainties in closed-loop systems is described in ref. [27]. The method is based on an application of the dual YJBK (after Youla, Jabr, Bongiorno and Kucera) parameterization of all systems stabilized by a given controller. The dual YJBK transfer function is a measure for the variation in the system seen through the feedback controller. In ref. [28] an adaptive algorithm based on the dual Y-K parameterization is introduced enabling simple closed-loop identification and adaptation of a class of symmetric MIMO systems. The methodology exploits the algebraic approach to control system design. Necessary conditions for usage of the developed method are discussed and the related results are presented for the case of coupled drives control. The 29$^{\text{th}}$ is concerned with a convex optimization approach for optimal synthesis in systems in which the overall control scheme is required to have certain structure.[29] These classes can be associated with several practical applications in integrated flight propulsion systems, platoons of vehicles, networked control, production lines, chemical processes, etc. The common thread in all of these classes is that by taking an input-output point of view, all stabilizing controllers can be characterized in terms of convex constraints concerning the Y–K parameter. The design of a motion control system for a powered reciprocating gait orthosis is considered in ref. [30]. Models for the orthosis are obtained using least squares identification. The control system design is based on pole-placement techniques and a restricted Youla parameterization of the controller. A straightforward basis for continuous time purposes is introduced in ref. [31]. The present paper assumes an extension to our earlier work.[32] The main contribution of this work aims at developing an efficient control approach for a single flexible

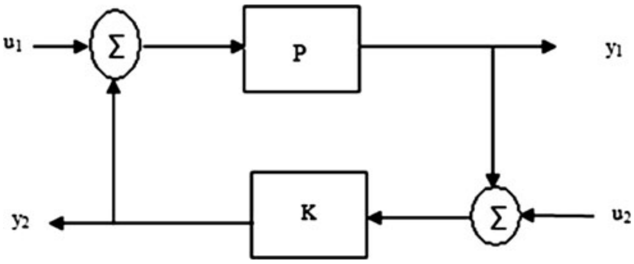

Fig. 1. The feedback system (P, K).

link robot together with tip mass free from linearization in formulation of the dynamics equations. To deal with the infinite dimensional set of Y-K parameters for time continuous applications, a finite dimensional basis is introduced in this paper. The current paper treats Youla-Kucera parameterization, design of the stabilize controller, and detector for the plant in section 2, and then finite dimensional parameterization including orthonormal bases and transfer from time-discrete into time-continuous in section 3, stabilizing controller implementation in section 4, and design of Youla stabilizing controller for a flexible arm with lumped tip mass and simulation results are presented in section 5. Finally, the relevant conclusions are considered.

2. Youla-Kucera Parameterization

This paper considers the robust stabilization of a plant model P by the use of a possibly feedback controller K in the scheme of Fig. 1. With the assumption that transfer function of the system P is not stable, its Coprime factors can be determined such that;[33]

$$P = NM^{-1}, \quad (N, M) \in RH^{\infty} \tag{1}$$

Where N and M are rational stable transfer functions. The transfer functions N and M are called Coprime factorization of P over RH^{∞} if and only if the two rational stable transfer functions X and Y exist such that Bezout equality is satisfied:

$$NX + MY = I \quad N, M, X, Y \in RH^{\infty} \tag{2}$$

Theorem (1): Assume that $P(s) = C(sI - A)^{-1}B + D$ and the pair (A, C) is detectable and the pair (A, B) is controllable. Thus, the real matrix F with dimension $1 \times n$ can be obtained such that the matrix $A - BF$ becomes stable. The matrices $N(s)$, $M(s)$ are given as follows;[33]

$$M(s) = \left[\begin{array}{c|c} A - BF & B \\ \hline -F & 1 \end{array}\right], \quad N(s) = \left[\begin{array}{c|c} A - BF & B \\ \hline C - DF & D \end{array}\right] \tag{3}$$

To determine the transfer functions X and Y, a detector must be designed so that the matrix $(A - HC)$ becomes stable and with the use of the matrix $H_{(n\times 1)}$, these can be written as

$$X(s) = \left[\begin{array}{c|c} A - HC & H \\ \hline F & 0 \end{array}\right], \quad Y(s) = \left[\begin{array}{c|c} A - HC & B - HD \\ \hline F & 1 \end{array}\right] \tag{4}$$

Theorem (2): all controllers that make the closed-loop internally stable are given by ref. [34]:

$$\left\{\frac{X + MQ}{Y - NQ}, \quad Q \in RH^{\infty}\right\} \tag{5}$$

It is important to note that when the transfer function P is stable, the above theorem turns to:

$$\left\{\frac{Q}{1 - PQ}, \quad Q \in RH^{\infty}\right\} \tag{6}$$

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The parameter Q in the above equations is known as Youla parameter.

2.1. *Design of the stabilizable controller for the plant*

The ultimate objective of the control process is to design Youla stabilize controller. However, in this section, the aim is to express the optimum conditions to determine the gain matrix F so that the matrix $A - BF$ becomes stable. For this purpose, the poles placement approach is used. To stabilize the matrix $A - BF$, it is necessary that all eigenvalues of the system are located in the left-hand side of the imaginary axis. With respect to the objective of the control process, which is to obtain the optimum design of Youla controller, it is necessary to calculate the gain matrix F for certain reasons. Recall that the gain matrix F is used to determine the transfer functions M and N. This can provide a proper tool to determine the gain matrix F. Since the eigenvalues of $A - BF$ are poles of the transfer functions M and N and the zeros of N and M are also zeros and poles of the transfer function P, then the gain matrix F can be designed so that the rank of the transfer functions N and M can be reduced to eliminate zero and pole condition. For generating such condition, it is necessary that the eigenvalues of $A - BF$ are taken equal to zeros of N and M that are located in the left-hand side of imaginary axis.

2.2. *Design of detector for the plant*

To calculate the transfer functions X and Y, the problem of designing detector to find the gain matrix H must be solved such that the matrix $A - HC$ becomes stable. The state equation of this detector is given by:

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u + Hy \tag{7}$$

Where u and y are input and output of the system, respectively. Matrices H, $\hat{B}$, $\hat{A}$ must be chosen such that the detector error, i.e. difference between the actual state variables and the estimated one become minimum,

$$\begin{aligned} &e = x - \hat{x} \\ &\dot{e} = \dot{x} - \dot{\hat{x}} = (Ax + Bu) - \left(\hat{A}\hat{x} + \hat{B}u + Hy\right) = Ax + Bu - \hat{A}\,(x - e) - \hat{B}u - HCx \end{aligned} \tag{8}$$

and therefore,

$$\dot{e} = \hat{A}e + (A - \hat{A} - HC)x + (B - \hat{B})u \tag{9}$$

Now, in order e approaches zero, the coefficients of x and u in the above relations should be zero. Thus,

$$\hat{A} = A - HC, \quad \hat{B} = B \tag{10}$$

To determine the matrix $\hat{A}$, the gain matrix H should be designed so that the eigenvalues of the matrix $\hat{A}$ become stable. Since the eigenvalues of the matrix $\hat{A}$ are determined by solving the equation $|\lambda I_n - \hat{A}| = 0$ and the open-loop system poles are also obtained from solving the equation $|sI_n - A| = 0$, where $|\cdots|$ represents determinate of the matrix, then the characteristic polynomial equation for the closed-loop system ($c.l.c.p$) is given by:

$$\begin{aligned} c.l.c.p\,\left|\lambda I_n - \hat{A}\right| &= 0 \\ \Rightarrow |\lambda I_n - A + HC| &= 0 \\ |\lambda I_n - A|\left|I_n + (\lambda I_n - A)^{-1}HC\right| &= 0 \\ a\,(\lambda)\left|I_n + (\lambda I_n - A)^{-1}HC\right| &= 0 \end{aligned} \tag{11}$$

Where $a(\lambda)$ represents the characteristic equation for the open-loop system with the eigenvalue λ. Considering the matrix $n \times n$ of $\phi(\lambda) = (\lambda I_n - A)^{-1}$ and using relation $|I_n + \phi(\lambda)HC| =$

$|I_m + C\phi(\lambda)H|$, one can obtain

$$c.l.c.p\ a(\lambda)\,|I_m + C\phi(\lambda)\,H| = 0 \tag{12}$$

Thus, the gain matrix H should be selected so as $|I_m + C\phi(\lambda)H| = 0$ is valid for any eigenvalues $\lambda_i\ i = 1, 2, \cdots, n$. This is equivalent to choosing the gain matrix H so that the matrix $I_m + C\phi(\lambda)H$ becomes singular. In order the matrix $I_m + C\phi(\lambda)H$ to become singular, the values of one of the rows of the matrix $C\phi(\lambda)H$ should be equal to the same values with negative sign of the corresponding row of the identity matrix I_m as follows:

$$row_k\,(C\phi\,(\lambda_i)\,H) = -row_k\,(I_m) \tag{13}$$

Where $row_k\,()$ shows the k^{th} row of the matrix. Note that by equating one row of the matrix $C\phi(\lambda_i)H$ and I_m with one row of the matrix $I_m + C\phi(\lambda_i)H$ becomes zero; therefore, the determinant of the matrix $I_m + C\phi(\lambda)H$ equates with zero. Hence, in order that $|I_m + C\phi(\lambda_i)H| = 0$ equates with zero for n eigenvalues it is require that

$$row_k\,(C\phi\,(\lambda_i)\,H) = -row_k\,(I_m) \quad \text{or} \quad \frac{d}{d\lambda}\left(row_k\left(C\phi\,(\lambda)\,|_{\lambda=\lambda_i}\ H\right)\right) = 0^T \tag{14}$$

Where 0^T is the row matrix of $1 \times m$ dimension with zero elements. The equation $\frac{d}{d\lambda}(row_k(C\phi(\lambda)|_{\lambda=\lambda_i} H)) = 0^T$ is used to determine the repeated eigenvalues and for the unrepeated eigenvalues, the left-hand side of the above equation is used. With the above relation and the definition of a nonsingular matrix $n \times n$, it can be considered:

$$G_c = \begin{bmatrix} row_j\,(C\phi\,(\lambda_1)) \\ \vdots \\ row_k\,(C\phi\,(\lambda_n)) \end{bmatrix} \tag{15}$$

and by considering the matrix J_c with the dimension $n \times 1$ where its rows are equal to rows of the matrices I_m or 0^T, we have:

$$G_c H = -J_c, \quad H = -G_c^{-1} J_c \tag{16}$$

Following the above process to calculate the gain matrix H, it is necessary to determine the eigenvalues of the matrix $A - HC$ with certain considerations. For this reason, the problem of reduced order of stabilize controller of Youla is considered as the criterion for choosing the poles of detector. Since the stabilize controllers are determined from $\{\frac{X+MQ}{Y-NQ}, Q \in RH^\infty\}$, the order of controller depends on the transfer functions X and Y. Thus to reduce the order of the stabilize controller of Youla, the poles of transfer functions X and Y (which are the eigenvalues of $A - HC$) can be taken equal to the poles of the transfer functions N and M.

3. Finite Dimensional Parameterization

The method of Youla parameterization is the simple solution and proper method for describing the collection of all controllers that stabilize the closed-system cycle of the shown block diagram. This collection is function of the parameter Q that can be any proper transfer function and is stable with dimension of $m \times l$. Here, m equals to the number of inputs and l equals to the number of system outputs. This method of solution is very suitable because instead of finding the matrix of the transfer function $K(s)$, it is only required to determine the parameter $Q(s)$. Hence to find the Youla parameter, it is needed to design a control problem for certain purposes. Note that the stable and proper condition (subspace of RH^∞) of Youla parameter is the only constraint for choosing $Q(s)$. Hence, if in the general case that the transfer function matrix of $Q(s)$ is calculated using the aforementioned conditions; in fact, the designed controller stabilize the closed-loop system. In order that $Q(s)$ becomes subspace of RH^∞, it is required to be generated from a finite number of bases from that space. Thus by choosing some bases for the space RH^{∞}, it is only sufficient to determine coefficients of these bases for Q.

3.1. Orthonormal bases

This section describes the method for calculating orthonormal bases in time discrete linear systems. From the stand point that the considered system can be approximated by expansion of orthonormal bases, it is assumed that one of the system major poles is located in ζ_0 such that $|\zeta_0| < 1$ and one of the bases can be taken as ref. [35]:

$$B_0(z) = \frac{Az^d}{z - \zeta_0} \quad d = 0, 1 \tag{17}$$

By choosing d, the considered base leads to proper or strictly proper base. To determine A, the condition in which the base is unit should be taken into account such as:

$$\langle B_0(z), B_0(z)\rangle = \frac{A^2}{2\pi j} \oint \frac{dz}{(z - \zeta_0)\left(1 - \zeta_0^* z\right)} = \frac{A^2}{1 - |\zeta_0|^2} \quad d = 0 \tag{18}$$

In order that the norm B_0 becomes unity, the coefficient $A = \sqrt{1 - |\zeta_0|^2}$ is adopted. Therefore for the first base, we have:

$$B_0(z) = \frac{\sqrt{1 - |\zeta_0|^2}}{z - \zeta_0} \quad d = 0 \tag{19}$$

Now, the second pole of the main system is assumed to be located in the point ζ_1 with the condition $|\zeta_1| < 1$. For the second base of the collection of orthonormal bases, we have:

$$B_1(z) = \frac{A'\left(1 - \zeta_0^* z\right)}{(z - \zeta_0)(z - \zeta_1)} \quad d = 0, 1 \tag{20}$$

Perhaps at the first glance, the chosen base seems unusual; however, by evaluating the orthogonal condition of inner product, it can be recognized that for equating $\langle B_0, B_1\rangle = 0$, it is essential that B_1^* (the complex conjugate of $B_1(z)$) holds zeros equal to the number of poles of $B_0(z)$. Hence by equating zero of B_1^* and pole of $B_0(z)$, it is concluded that the function $B_1^*(z)\ B_0(z)$ is analytic in D (There are no singular points in this space). According to Cushy integral theorem, the solution is $\langle B_0, B_1\rangle = 0$.

$$\langle B_0(z), B_1(z)\rangle = \frac{\sqrt{1 - |\zeta_0|^2}}{2\pi j} \oint_T \frac{A'(z - \zeta_0)\, dz}{(z - \zeta_0)\left(1 - \zeta_0^* z\right)\left(1 - \zeta_1^* z\right)} = 0 \tag{21}$$

Again, to determine the coefficient A', we must use the condition of unity of the used base such that:

$$\langle B_1(z), B_1(z)\rangle = \frac{A'^2}{2\pi j} \oint_T \frac{(\zeta_0 - z)\left(\zeta_0^* z - 1\right) dz}{(z - \zeta_0)(z - \zeta_1)\left(1 - \zeta_0^* z\right)\left(1 - \zeta_1^* z\right)} = \frac{A'^2}{1 - |\zeta_1|^2} \tag{22}$$

In order that norm of $B_1(z)$ becomes unity, it is required that:

$$A' = \sqrt{1 - |\zeta_1|^2} \tag{23}$$

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Therefore, by choosing the main system poles in $[\zeta_0\ \zeta_1\ \zeta_2 \cdots \zeta_n]$ and following the above-discussed process, the collection of orthonormal bases can be determined by:

$$B_n(z) = \frac{\sqrt{1-|\zeta_n|^2}}{z-\zeta_n}\prod_{k=0}^{n-1}\frac{1-\zeta_k^* z}{z-\zeta_k} \tag{24}$$

The above bases are called the generalized orthogonal bases and are the most general shape of these bases.

- If the above bases of poles are located at $\zeta_i = 0\ i = 1, 2, \cdots, n$ then the collection of orthogonal unit bases FIR are obtained in general form as:

$$B_i(z) = \left(\frac{1}{z}\right)^i \quad i = 0, 1, 2, \cdots, n \tag{25}$$

- If all poles of the generalized orthonormal bases are located at $\zeta_i = \zeta\ i = 1, 2, \cdots, n$ (where all of these are real), then the collection of Laguerre orthonormal bases are obtained as:

$$B_i(z) = \frac{\sqrt{1-|\zeta|^2}}{z-\zeta}\left(\frac{1-\zeta z}{z-\zeta}\right)^i \quad |\zeta| < 1 \quad i = 0, 1, 2, \cdots, n \tag{26}$$

- If two complex conjugate poles are used for the generalized orthonormal bases, then the Kautz orthogonal unit bases are obtained:

$$B_i(z) = \begin{cases} \dfrac{\sqrt{(1-\alpha^2)(1-\gamma^2)}}{z^2-\alpha(\gamma+1)z+\gamma}\left(\dfrac{\gamma z^2-\alpha(\gamma+1)z+1}{z^2-\alpha(\gamma+1)z+\gamma}\right)^{\frac{i-1}{2}} & \text{for} \quad i = 2k-1 \\ & \quad\ k = 1, 2, \cdots, n \\ \dfrac{\sqrt{(1-\gamma^2)}(z-\alpha)}{z^2-\alpha(\gamma+1)z+\gamma}\left(\dfrac{\gamma z^2-\alpha(\gamma+1)z+1}{z^2-\alpha(\gamma+1)z+\gamma}\right)^{\frac{i}{2}} & \text{for} \quad i = 2k \end{cases} \tag{27}$$

The above relations are valid for $|\alpha| < 1, |\gamma| < 1$.

3.2. *Transfer from time-discrete into time-continuous spaces*

In order to approximate a predefined transfer function in time-discrete space into a time-continuous space, the Tustin transformation can be used.

3.2.1. Tustin transformation. The Tustin transformation is a mapping from z plane (plane of complex digits) into s plane such that the time-discrete system is transferred into the time-continuous system. This is the mapping or transformation that transfers the transfer function $H_a(s)$ defined in the continuous LTI systems into transferred function in time-discrete space systems $H_d(z)$. In overall, this is the mapping that transfers the locations over imaginary axis in s plane, i.e. $\{s|\mathrm{Re}(s) = 0, j\omega\}$, on a unit circle in z plane, i.e. $|z| = 1$. Therefore, the following transformation can be used to transfer from the time-discrete systems space into the time-continuous systems space:

$$s \leftarrow \frac{2}{T}\frac{z-1}{z+1} \tag{28}$$

Thus, based on the defined bases in the time-discrete systems space as the bases for the parameter Youla, it is appropriate to use the Tustin transformation to define the orthogonal bases in time-continuous systems space. For this purpose and by using the following transformation, it can be written as (Table I):

$$z = \frac{s+a}{s-a} \quad \Leftrightarrow \quad s = a\frac{z+1}{z-1} \tag{29}$$

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Table I. The orthonormal bases in the time-discrete and time-continuous spaces.

	Z-Plane (time-discrete space)	S-Plane (time-continuous space)
FIR Basis	$B_i(z) = \left(\frac{1}{z}\right)^i \quad i = 0, 1, 2, \cdots, n$	$B_i(s) = \left(\frac{s-a}{s+a}\right)^i \quad a > 0 \quad i = 0, 1, 2, \cdots, n$
Laguerre Basis	$B_i(z) = \frac{\sqrt{1-\lvert a\rvert^2}}{z-a}\left(\frac{1-az}{z-a}\right)^i \quad \lvert a\rvert < 1 \; i = 0, 1, 2, \cdots, n$	$B_i(s) = \frac{\sqrt{2a}}{s+a}\left(\frac{s-a}{s+a}\right)^i \quad a > 0 \quad i = 0, 1, 2, \cdots, n$
Kautz Basis	$B_i(z) = \begin{cases} \frac{\sqrt{(1-\alpha^2)(1-\gamma^2)}}{z^2-\alpha(\gamma+1)z+\gamma}\left(\frac{\gamma z^2-\alpha(\gamma+1)z+1}{z^2-\alpha(\gamma+1)z+\gamma}\right)^{i-1} & \text{for } i = 2k-1 \\ \frac{\sqrt{(1-\gamma^2)}(z-\alpha)}{z^2-\alpha(\gamma+1)z+\gamma}\left(\frac{\gamma z^2-\alpha(\gamma+1)z+1}{z^2-\alpha(\gamma+1)z+\gamma}\right)^{i-1} & \text{for } i = 2k \end{cases}$ $k = 1, 2, \cdots, n, \; \lvert\alpha\rvert < 1, \quad \lvert\gamma\rvert < 1$	$B_i(s) = \begin{cases} \frac{\sqrt{2a}\,s}{s^2+as+b}\left(\frac{s^2-as+b}{s^2+as+b}\right)^{i-1} & \text{for } i = 2k-1 \\ \frac{\sqrt{2ab}}{s^2+as+b}\left(\frac{s^2-as+b}{s^2+as+b}\right)^{i-1} & \text{for } i = 2k \end{cases}$ $k = 1, 2, \cdots, n$
Generalized Basis	$B_i(z) = \frac{\sqrt{1-\lvert a_i\rvert^2}}{z-a_i}\prod_{k=0}^{i-1}\frac{1-a_k^* z}{z-a_k}$	$B_i(s) = \frac{\sqrt{2\mathrm{Re}(a_i)}}{s+a_i}\prod_{k=0}^{i-1}\frac{s-a_k^*}{s+a_k}$

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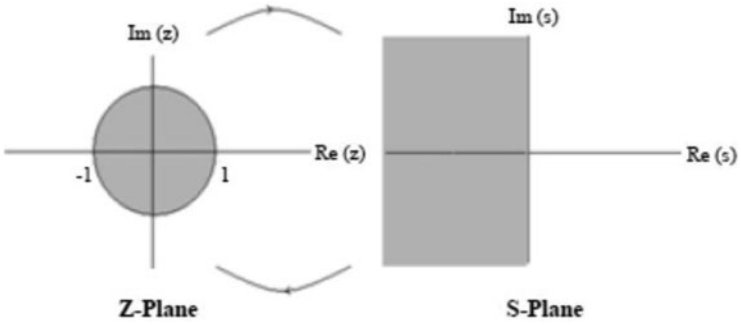

Fig. 2. The sustained stability over the two planes of s, z using the Tustin transformation.

The Tustin transformation sustain stability and orthogonal feature in the two time-discrete and time-continuous spaces by mapping all points inside the unit circle in the z plane $\{z||z| < 1\}$ into the corresponding collection of points on the left-hand side of the imaginary axis in s plane $\{s|\mathrm{Re}(s) < 0\}$. Thus, the Tustin transformation is an isomorphism transformation as shown in Fig. 2.

3.2.2. Performance characteristic of the system in time domain.

- **Characteristics of time response of the system**

One of the objectives in this section is to keep a closed-loop system closer to a second order system with appropriate time response. Considering the overshoot parameters, the rise time, the settling time, and the steady-state error as characteristics for the system time response, it is required to assess the conditions that lead to appropriate values for each parameter.

- **Overshoot**

In order that the closed-loop system obtains a certain overshoot for a step input, it is necessary that the maximum value of response for the closed-loop system due to the step input does not exceed from the expected value. By defining the upper limit of response ($b_u(t)$) to reach a certain overshoot, it is important that the norm of the difference between the time response of desirable upper limit and the output response of the system for the step input is always greater than zero; i.e.:

$$\left\| b_u(t) - L^{-1}\{T(s)R(s)\}(t) \right\|_\infty \geq 0 \quad \Rightarrow \quad \left\| b_u(t) - L^{-1}\{(N(X + MQ))R(s)\}(t) \right\|_\infty \geq 0 \tag{30}$$

- **The rise time, the settling time, and the steady-state error**

In order to obtain a desirable time response for a closed-loop system, in addition to control the system overshoot, other parameters of time response such as the rise time, the settling time, and the steady-state error must also be improved. Therefore, to control simultaneously these characteristics, the lower limit of the time response $b_l(t)$ is used such that by merging the output of the closed-loop system to the lower limit of time response, the aforementioned characteristics will approach some desirable quantities. Controlling the aforementioned characteristics will evidently result in choosing the lower limits. Thus, the lower limit of time response should be realized by considering the rise time, the settling time, and the steady-state time. Examining the effects of each parameter, it is clear that the lower limit of the time response will impose restriction on the time response of the closed-loop system such that the system in a time period of T_r seconds (equals to the desirable time rise) will converge from 0.1 of the steady response to 0.9 of that. In addition to this limit, the closed-loop system reaches 0.98 of the steady response after T_s seconds (equals to the desirable settling time). Moreover, to obtain a desirable steady-state error for the closed-loop system, the lower limit reaches to the steady response of the unit step input. The allowed region for the optimization problem is assigned with the region between the upper and lower limits such that the lower limit represents the time response characteristics including the rise time, the settling time, and the steady-state error and

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Table II. The performance characteristics of the system in two domains of frequency and time.

The objective function	Time domain
The characteristic of time response of the system (overshoot)	$\left\| b_u(t) - L^{-1}\{T(s)R(s)\}(t) \right\|_\infty \geq 0 \Rightarrow \left\| b_u(t) - L^{-1}\{(N(X+MQ))R(s)\}(t) \right\|_\infty \geq 0$
The characteristic of time response of the system (rise time, settling time, and steady state error)	$\left\| L^{-1}\{T(s)R(s)\}(t) - b_l(t) \right\|_\infty \geq 0 \Rightarrow \left\| L^{-1}\{(N(X+MQ))R(s)\}(t) - b_l(t) \right\|_\infty \geq 0$

the upper limit represents the boundary for the overshoot. Thus, to include the rise time characteristic, the settling time, and the steady-state error for the system, the following relation is considered as the constraint for the optimization problem:

$$\left\| L^{-1}\{T(s)R(s)\}(t) - b_l(t) \right\|_\infty \geq 0 \Rightarrow \left\| L^{-1}\{(N(X+MQ))R(s)\}(t) - b_l(t) \right\|_\infty \geq 0 \quad (31)$$

The above conditions are summarized for frequency and time domains in Table II.

4. Stabilizing Controller Implementation

The aim of the control process, for a system with negative unit feedback, is to design the Youla stabilizing controller to achieve the sought objectives. The main aim in this design is to minimize the norm of output response difference of the closed-loop system using the Youla stabilizing controller and obtain the second order output from system when exerting a step input. In other words, the aim is to merge the closed-loop system to the second order system with the desirable time response characteristic. For this purpose, the chosen Youla parameter $Q(s)$ plays an important role in this design. To solve the controlled problem and to design ultimately the Youla stabilizing controller, the optimization processes should take into account the specification of the following two control subjects:

- Adopting the Youla parameter bases (selecting the location of poles of the bases)
- Calculating coefficients for each basis

4.1. Adopting the Youla parameter

By defining the general problem of designing the Youla stabilizing controller, in the form of the optimization problem of linear programming type, it turns to adopting the Youla parameter. For selecting this parameter, linear combination of the bases are used to satisfy the condition $Q(s) \in RH^\infty$. Therefore, by examining published literature,[35–37] it can be concluded that a suitable way for choosing this parameter is to use the collection of orthonormal bases expressed in the previous

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sections as follow:

$$
\begin{aligned}
&q_i(s) = \left(\frac{s-a}{s+a}\right)^i && a>0 \quad i=0,1,2,\cdots,n \quad \text{FIR Basis}\\
&q_i(s) = \frac{\sqrt{2a}}{s+a}\left(\frac{s-a}{s+a}\right)^i && a>0 \quad i=0,1,2,\cdots,n \quad \text{Laguerre Basis}\\
&q_i(s) = \begin{cases} \dfrac{\sqrt{2a}\,s}{s^2+as+b}\left(\dfrac{s^2-as+b}{s^2+as+b}\right)^{i-1} & \text{for} \quad i=2k-1 \\ & \qquad k=1,2,\cdots,n \quad \text{Kautz Basis} \\ \dfrac{\sqrt{2ab}}{s^2+as+b}\left(\dfrac{s^2-as+b}{s^2+as+b}\right)^{i-1} & \text{for} \quad i=2k \end{cases}\\
&q_i(s) = \frac{\sqrt{2\mathrm{Re}(a_i)}}{s+a_i}\prod_{k=0}^{i-1}\frac{s-a_k^*}{s+a_k} && i=1,2,\cdots,n \quad \text{Generalized Basis}
\end{aligned}
\tag{32}
$$

4.1.1. Adopting the location of poles for the orthonormal bases. Consider that the aim of the control process by choosing the Youla parameter is to minimize the following objective function:

$$\|y(t) - y_d(t)\|_\infty \tag{33}$$

Therefore, it can be said that for minimizing the above function, $y(t)$ (actual output of the closed-loop system) must approach to $y_d(t)$ (desired output of the second order system) and since the desired state occurs according to the second order system's response to the step input, it is necessary that the closed-loop system behaves similar to the second order system. Note that the system greater than second order behaves similar to the second order system when additional poles of the system are far away from the real parts of the dominant poles of the second order system. Thus, the first rule for adopting the poles of the Youla parameter is that these poles must be located far away from the real parts of the dominant poles $y_d(t)$. Moreover, the Youla parameter bases can be chosen such that the order of closed-loop system is also reduced. For this reason, considering the transformation function of the closed-loop system $T(s) = N(s)[X(s) + M(s)Q(s)]$, the poles of Youla parameter should be equal to the poles of the stabilizing transfer function $N(s)X(s)$. Therefore, for selecting the optimum poles of the Youla parameter bases as appear on the poles of the transfer function $N(s)X(s)$ (which must be stable because $N(s), X(s) \in RH^\infty$) which are adequately away from the real part of the dominant poles, $y_d(t)$should be selected.

4.2. Calculating the coefficients for each base

For designing the Youla stabilizing controller, it is necessary to define the problem in the form of an optimization problem to be solved by optimization techniques. Thus, by considering the described objective in the controlled process, the problem can be designed in this way. The objective function of the optimization problem can be defined by minimizing the norm of the difference of the output of the closed-loop system with Youla stabilizing controller and the output response of the second order system. The desired characteristic of the time response of the system can be defined by constraint in the optimization problem. The real response of the system is the actual output of the system to the input step while the desired response generates conditions in which the system output to the step input is given as shown in Fig. 3 such that the output overshoot ($\%OS$) is less than %20 and the peak time (T_p) is also 14 seconds. By employing the aforementioned conditions, the desired response can be recognized as the response of the second order system, with the below transfer function, to the step input as:

$$\frac{y_d(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100$$

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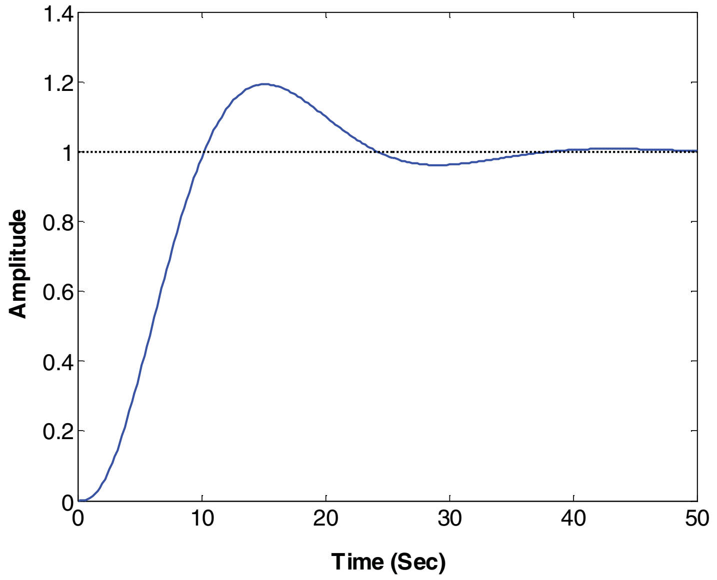

Fig. 3. The desired output response to step input.

$$T_P = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{34}$$

To ensure that the desired response is sufficiently stable and its dominant poles are sufficiently located on the left-hand side of the imaginary axis, the term $(s+\gamma)$ is also added in denominator of the transfer function of the desired output where γ is varied from 5 to 6 times larger than the poles of the second order system. Hence, the desired output of the transfer function to step input is expressed by:

$$\frac{y_d(s)}{R(s)} = \frac{\omega_n^2}{\left(s^2+2\zeta\omega_n s+\omega_n^2\right)(s+\gamma)} = \frac{0.06357}{\left(s^2+0.2299s+0.06357\right)(s+1)} \tag{35}$$

By introducing the objective function as:

$$\begin{aligned}&\text{Min} \quad \|y(t)-y_d(t)\|_\infty\\ &\text{Min} \quad \left\| L^{-1}\{T(s)R(s)\}(t) - L^{-1}\left\{\left[\frac{\omega_n^2}{\left(s^2+2\zeta\omega_n s+\omega_n^2\right)(s+\gamma)}\right]R(s)\right\}(t)\right\|_\infty\end{aligned} \tag{36}$$

Where $T(s)$ represents the transfer function of the closed-loop system and is equal to $N(X+MQ)$. To determine Q, the problem is transformed into the optimization domain such that the problem variables to be determined using the optimized techniques.

$$Min \left\| \mathrm{L}^{-1}\{N(X+MQ)R(s)\}(t) - \mathrm{L}^{-1}\left\{\left[\frac{\omega_n^2}{\left(s^2+2\zeta\omega_n s+\omega_n^2\right)(s+\gamma)}\right]R(s)\right\}(t)\right\|_\infty \tag{37}$$

By simplifying the term inside$\|\ldots\|$, it can be written as:

$$\text{Min} \quad \left\| L^{-1}\left\{\left[NX - \frac{\omega_n^2}{\left(s^2+2\zeta\omega_n s+\omega_n^2\right)(s+\gamma)}\right]R(s)\right\}(t) + L^{-1}\{NMQR(s)\}(t)\right\|_\infty \tag{38}$$

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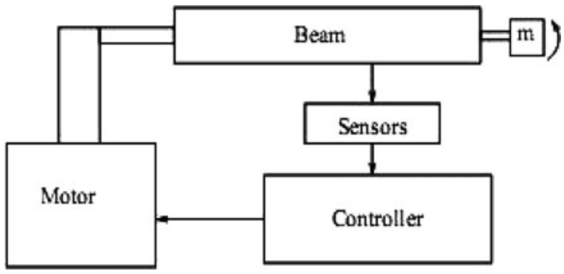

Fig. 4. Block diagram for the flexible arm with lumped mass at tip.

By equating $Q(s) = \sum_{r=1}^{n} \beta_r q_r$ where q_r and β_r are bases and coefficients of each base respectively, it holds:

$$\text{Min} \quad \left\| \mathrm{L}^{-1}\left\{\left[NX - \frac{\omega_n^2}{\left(s^2+2\zeta\omega_n s+\omega_n^2\right)(s+\gamma)}\right]R(s)\right\}(t) + L^{-1}\left\{NM\sum_{r=1}^{n}\beta_r q_r R(s)\right\}(t)\right\|_\infty \tag{39}$$

The first term in the above equation is the response of system inside [. . .] to step input and is the specific column vector; however, the second term is the system response $NM\sum_{r=1}^{n}\beta_r q_r$ to step input and is a matrix with n column (considering n bases for the Youla parameter) where each column corresponds to a unit base in RH^∞ space. Hence, the expression (39) can be written as:

$$\text{Min} \quad \|b - Ax\|_\infty \tag{40}$$

Where x is the column vector of unknowns and in fact the coefficients of the bases in RH^∞ space and:

$$b = L^{-1}\left\{\left[NX - \frac{\omega_n^2}{\left(s^2+2\zeta\omega_n s+\omega_n^2\right)(s+\gamma)}\right]R(s)\right\} \tag{41}$$

$$A = -L^{-1}\{NM\beta R(s)\}(t) \qquad \sum_{r=1}^{n}\beta_r q_r = \beta_{1\times n}x_{n\times 1}$$

In conclusion, the design of the stabilizing controller of the closed-loop system is changed to the optimization problem of the above form, aimed at finding the problem variables and the coefficients of the Youla parameter.

5. Design of Youla Stabilizing Controller for a Flexible Arm With Lumped Tip Mass

Extracting the state-space equations for the flexible arm is the required condition to arrive in designing the controller and to introduce fully the control method; the Youla controller shown in Fig. 4 can be designed for the flexible arm with the lumped tip mass in single-input single-output (SISO) state.

Taking motor armature voltage as the input control and ignoring applied torque to hub as the system disturbance (unwanted input) and considering the fact that the system output is in the form of angular variation of the arm, the state-space equations of the flexible arm in single-input single-output

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state is given by ref. [38]:

$$\frac{d}{dt}\begin{Bmatrix} i \\ Z_1 \\ Z_2 \end{Bmatrix} = \underbrace{\left[\begin{array}{c|c} A_{m11} & A_{m12} \\ \hline \frac{1}{2}\left(A_{m21} - \left(0|M^{-1}K\right)\right) & \frac{1}{2}\left(A_{m22}\right) \end{array}\right]}_{A}{}_{(2n+3)\times\times(2n+3)} \begin{Bmatrix} i \\ Z_1 \\ Z_2 \end{Bmatrix} + \underbrace{\left[\begin{array}{c} B_{m11} \\ \hline \frac{1}{2}\left(B_{m21}\right) \end{array}\right]}_{B}{}_{(2n+3)\times 1} V$$

$$Y = \theta$$

$$Y = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}}_{C}{}_{1\times(2n+3)} \begin{Bmatrix} i \\ Z_1 \\ Z_2 \end{Bmatrix} \tag{42}$$

Where

$$A_{m11} = \left[\begin{array}{cc} -\frac{R}{L} & [0]_{1\times(n+1)} \\ [0]_{(n+1)\times 1} & [0]_{(n+1)\times(n+1)} \end{array}\right]_{(n+2)\times(n+2)} \quad \text{and} \quad A_{m12} = \left[\begin{array}{cc} -\frac{NK_m}{L} & [0]_{1\times n} \\ \hline [I]_{(n+1)\times(n+1)} & \end{array}\right]_{(n+2)\times(n+1)}$$

$$A_{m21} = \begin{bmatrix} \frac{NK_m}{J_e} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}_{(n+1)\times(n+2)} \quad \text{and} \quad A_{m22} = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}_{(n+1)\times(n+1)}$$

$$B_{m11} = \begin{bmatrix} \frac{1}{L} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(n+2)\times 1} \quad \text{and} \quad B_{m21} = [0]_{(n+1)\times 1}$$

$$K = \frac{EI}{h^4}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 6 & -4 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & -4 & 6 & -4 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & -4 & 6 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 1 & -4 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & -2 & 1 \end{bmatrix}_{(n+1)\times(n+1)},$$

$$M = \begin{bmatrix} (I_H + I_b + I_t) & \rho x_1 & \rho x_2 & \rho x_3 & \rho x_4 & \rho x_5 & \ldots & \rho x_{n-3} & \rho x_{n-2} & \rho x_{n-1} & \rho x_n + M_t(r+l) \\ \rho x_1 & \rho & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ \rho x_2 & 0 & \rho & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ \rho x_3 & 0 & 0 & \rho & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho x_{n-2} & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \rho & 0 & 0 \\ \rho x_{n-1} + \frac{J_t}{h^2} & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & \rho + \frac{J_t}{h^2} & -\frac{J_t}{h^3} \\ \rho x_n + \frac{M_t(r+l)}{h} + \frac{J_t}{h^2} & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & -\frac{J_t}{h^3} & \rho + \frac{M_t}{h} + \frac{J_t}{h^3} \end{bmatrix}_{(n+1)\times(n+1)}$$

$$Z_1 = [\theta, y_1, y_2, \ldots, y_n]^T \quad \text{and} \quad Z_2 = \dot{Z}_1$$

As it is clear from demonstration of the state space, the dimension of matrices D, C, B, A are $1 \times 1, 1 \times (2n+3), (2n+3) \times 1, (2n+3) \times (2n+3)$, respectively. The oncoming figures and tables are based on the available parameters in Tables III and IV.

By attaining the initial conditions (stabilization and observability) for designing the stabilizing Youla controller, the problem is transferred from state equations space (time domain) into the transfer function space (frequency domain) so that we can use the governing equations for determining Youla parameterization to design an stabilized controller.

For designing the Youla stabilizing controller with the transfer function $K(s) = \frac{X+MQ}{Y-NQ}$, first the transfer functions $(N, M, X, Y) \in RH^{\infty}$ must be determined. For this purpose and by considering the explained methodology, the matrices H, F must be found so that the matrices $(A - BF)$ and $(A - BF)$ become stable based on the discussed optimization conditions. Since (A, B) are stabilizable

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Table III. Parameters of the DC motor.

Parameter	Value	Parameter	Value
$R(\Omega)$	1.2	$J(Kg\,m^2)$	0.02
$L(H)$	0.5	N	12
$J_m(Kg\,m^2)$	8×10^{-4}	$K_m(^{Nm}/_{A})$	0.5

Table IV. Key parameters of the flexible robot system.

Parameter	Value	Parameter	Value
$l(m)$	1	$J_t(Kg\,m^2)$	0.002
$w(m)$	0.01	$M_t(Kg)$	0.5
$b(m)$	0.02	$r(m)$	0.075
$EI(Nm^2)$	2000	$I_H(Kg\,m^2)$	0.0174
$\rho(^{kg}/_{m})$	1	$h(m)$	0.02

and (A, C) are observable, the gain matrices H, F can be determined through the presented methods in previous sections to stabilize four transfer functions (N, M, X, Y). Following these methods and considering the conditions in sections 2.1 and 2.2 for the gain matrices H, F, we have:

$$F = \begin{bmatrix} 2.0839 & 1.7321 & -7.2529 & -12.5792 & -17.9589 & -23.3937 - 42.6102 \\ 0.2859 & 0.1017 & 0.1777 & 0.2568 & 0.3379 & 0.5994 \end{bmatrix}^T_{(1\times 13)}$$

$$H = \begin{bmatrix} -81.9148 & 8.5791 & -0.2571 & -0.0224 & -1.0755 & 1.61 & -2.8097 \\ 35.3003 & -2.2527 & -0.2322 & -9.8352 & 13.5730 & -24.0424 \end{bmatrix}^T_{(1\times 13)} \tag{43}$$

By inspecting the eigenvalues of the matrices $(A - BF)$ and $(A - HC)$, it is observed that all of the eigenvalues are located on the left-hand side of the imaginary axis. Now with using the relations for the matrices (N, M, X, Y), it can be determined as:

$$\begin{aligned} \mathrm{N}(s) &= \frac{44.38}{s^3+6.568\,s^2+279\,s+76.87} \\ \mathrm{M}(s) &= \frac{s^3+2.4\,s^2+266.3\,s}{s^3+6.568\,s^2+279\,s+76.87} \end{aligned} \tag{44}$$

$$\begin{aligned} \mathrm{X}(s) &= \frac{\begin{array}{l} -54.9\,s^{12}+887.6\,s^{11}-7.986\times 10^{8}\,s^{10}+1.251\times 10^{10}\,s^{9}-3.31\times 10^{8}\,s^{8} \\ +5.015\times 10^{16}\,s^{7}-3.67\times 10^{21}\,s^{6}+5.297\times 10^{22}\,s^{5}-3.165\times 10^{26}\,s^{4}+4.211\times 10^{27}\,s^{3} \\ -1.091\times 10^{30}\,s^{2}-5.486\times 10^{29}\,s+1.714\times 10^{31} \end{array}}{\begin{array}{l} s^{13}+10.98\,s^{12}+1.455\times 10^{7}\,s^{11}+1.585\times 10^{8}\,s^{10}+6.032\times 10^{13}\,s^{9}+6.503\times 10^{14}\,s^{8} \\ +6.69\times 10^{19}\,s^{7}+7.082\times 10^{20}\,s^{6}+5.807\times 10^{24}\,s^{5}+5.951\times 10^{25}\,s^{4} \\ +2.32\times 10^{28}\,s^{3}+1.474\times 10^{29}\,s^{2}+5.882\times 10^{30}\,s+9.897\times 10^{30} \end{array}} \\ Y(s) &= \frac{\begin{array}{l} s^{13}+15.15\,s^{12}+1.455\times 10^{7}s^{11}+2.192\times 10^{8}\,s^{10}+6.032\times 10^{13}\,s^{9}+9.017\times 10^{14}\,s^{8} \\ +6.691\times 10^{19}\,s^{7}+9.87\times 10^{20}\,s^{6}+5.81\times 10^{24}\,s^{5}+8.364\times 10^{25}s^{4}+2.346\times 10^{28}\,s^{3} \\ -1.091\times 10^{30}\,s^{2}-5.486\times 10^{29}s+1.714\times 10^{31} \end{array}}{\begin{array}{l} s^{13}+10.98\,s^{12}+1.455\times 10^{7}\,s^{11}+1.585\times 10^{8}\,s^{10}+6.032\times 10^{13}\,s^{9}+6.503\times 10^{14}\,s^{8} \\ +6.69\times 10^{19}\,s^{7}+7.082\times 10^{20}\,s^{6}+5.807\times 10^{24}\,s^{5}+5.951\times 10^{25}\,s^{4} \\ +2.32\times 10^{28}\,s^{3}+1.474\times 10^{29}\,s^{2}+5.882\times 10^{30}\,s+9.897\times 10^{30} \end{array}} \end{aligned} \tag{45}$$

As discussed earlier, the transfer functions $M, N$and $Y, X$are not unique solutions. Hence, any multiplier of these transfer functions can also be a solution for the first subdivided problem. To

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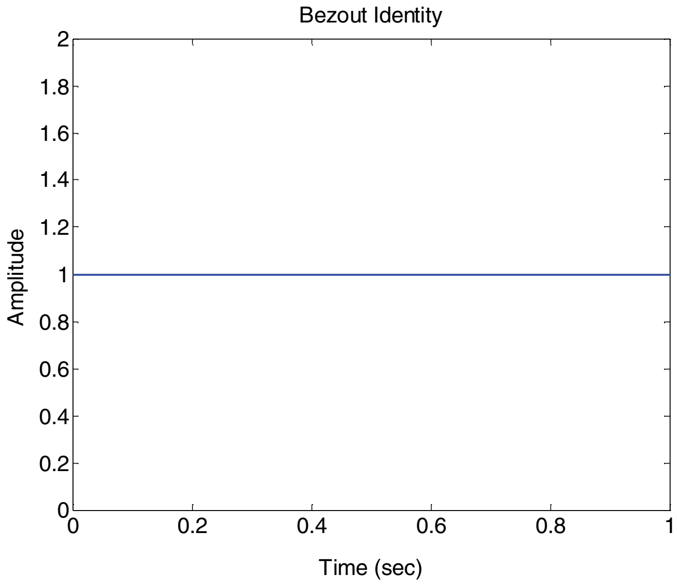

Fig. 5. Assessing equality of Bezout Identity for $k = 1$ $NX + MY = 1$.

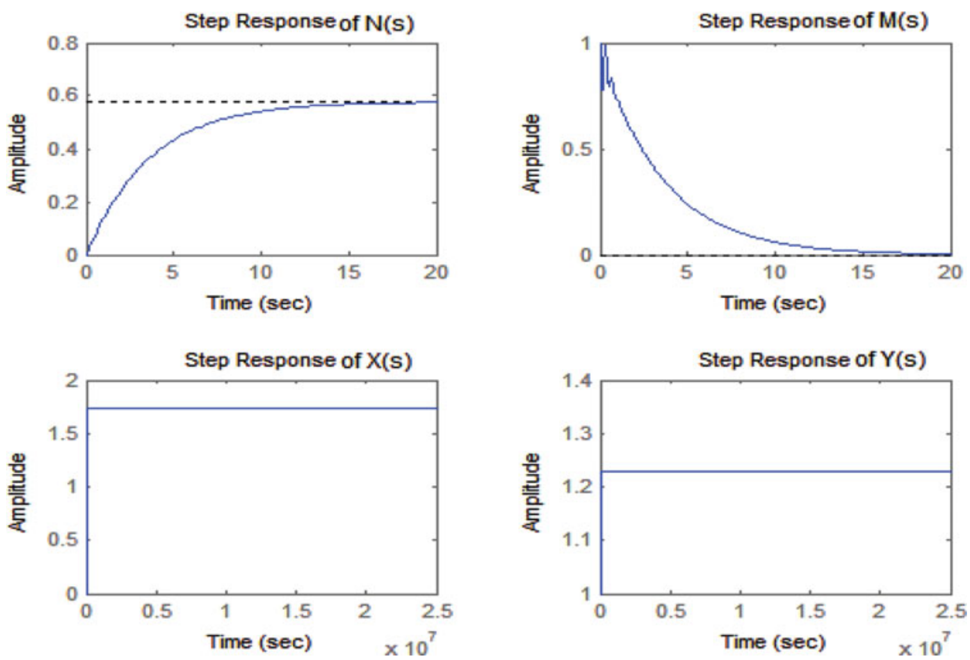

Fig. 6. Response to step input of the transfer function for $k = 1$.

examine correctness of the above transfer functions and their belonging to RH^{∞}space, the response of these to step input and the Bezout Identity can be assessed as shown in Figs. 5 and 6:

Acquiring the transfer functions $(N, M, X, Y) \in RH^{\infty}$ and for designing the Youla stabilizing controller, the Q parameter must be obtained. For this aim as described in previous sections, the orthonormal bases can be used as bases for $Q(s)$. To determine the Youla parameter, the problem is expressed as the optimization problem in linear programming, i.e. refs. [39, 40]:

$$\text{Minimize} \quad \|b - Ax\|_{\infty} = \text{Min}\left\{Max\left\{|r_1|, |r_2|, |r_3|, \cdots, |r_m|\right\}\right\}$$

$$LP \quad \text{form } of \ l_{\infty} \quad \text{norm}$$

$$\text{Min} \quad t$$

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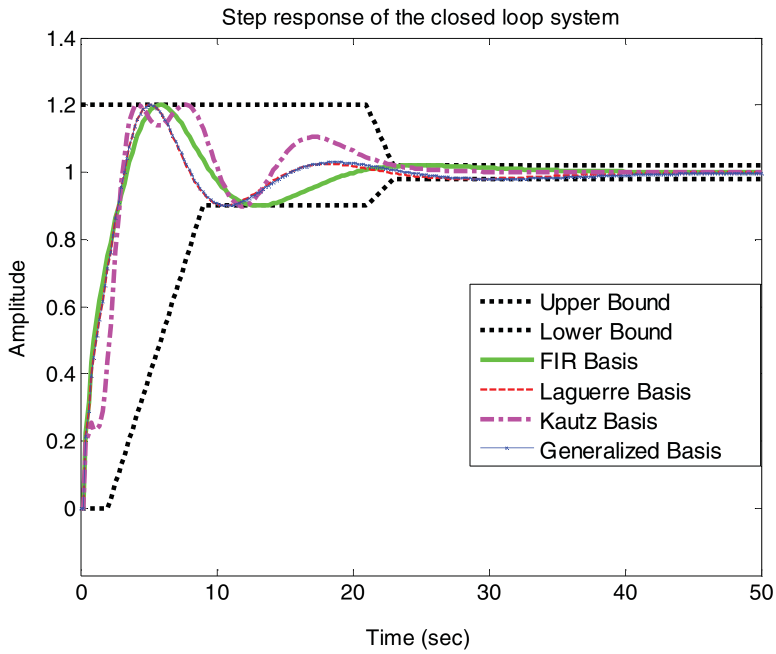

Fig. 7. The output of the closed-loop system to step input with the stabilizing Youla controller using orthonormal bases.

$$s.to \quad -t \leq b_i - \sum_{j=1}^{m} a_{ij} x_j \leq t \quad i = 1, 2, \cdots, n \tag{46}$$

$$\sum_{j=1}^{m} a_{ij} x_j \leq b_u(t) - \underset{t\in[0,t_f]}{L^{-1}\{N(s)\,X(s)\,R(s)\}(t)}$$

$$\sum_{j=1}^{m} a_{ij} x_j \leq \underset{t\in[0,t_f]}{L^{-1}\{N(s)\,X(s)\,R(s)\}(t)} - b_l(t)$$

The objective function in the problem of linear programming is to minimize infinity norm of the vector $b - Ax$ so that it result for every variable in the minimum of the objective function and the two last terms in the constraint equations represent characteristics of the desired time response (overshoot, rise time, settling time and the steady-state error) for the closed-loop system.

5.1. Results of design of Youla stabilizing controller based on orthonormal bases

The problem of designing controller has led to the problem of optimization with linear programming and large number of constraints that must employ the interior point method to derive the following solutions tabulated in Tables V to VIII for FIR, Laguerre, Kautz, and the orthonormal bases, respectively. The output of the closed-loop system to step input with the stabilizing Youla controller using these orthonormal bases are compared in Fig. 7.

6. Conclusion

In this paper, the Youla-Kucera parameterization method to control of planar manipulator with a single flexible link was used. This method of solution is very suitable because instead of finding the matrix of the transfer function, it is only required to determine the Youla parameter. To remedy with the infinite dimensional set of Youla-Kucera parameters, a new procedure based on orthonormal bases was presented here. In order to approximate transfer functions of orthonormal bases in time-discrete

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Table V. Results for the design of stabilizing Youla controller with orthonormal FIR bases.

Location of pole of the bases	-0.4
Number of bases	5
Youla parameter $Q(s)$	$Q(s) = \frac{2.875\,s^{10}+8.473\,s^{9}+13.32\,s^{8}+14.33\,s^{7}+10.79\,s^{6}+5.55\,s^{5}+1.908\,s^{4}+0.4265\,s^{3}+0.05919\,s^{2}+0.004648s+0.0001627}{s^{10}+4\,s^{9}+7.2\,s^{8}+7.68\,s^{7}+5.376\,s^{6}+2.58\,s^{5}+0.8602\,s^{4}+0.1966\,s^{3}+0.02949\,s^{2}+0.002621\,s+0.0001049}$
Coefficient of bases of the Youla parameter	$\left[2.0806\ \ -0.2302\ \ -0.4035\ \ 0.8917\ \ 0.5361\right]^{T}$
Number of iteration in the Interior Point method	19
Value of the objective function in the optimization problem $\lVert y(t) - y_d(t) \rVert_\infty$	0.007171

Table VI. Results for the design of stabilizing Youla controller with orthonormal Laguerre base s.

Location of pole of the bases	−0.4
Number of bases	6
Youla parameter $Q(s)$	$Q(s) = \frac{2.882\,s^{15}+14.15\,s^{14}+34.58\,s^{13}+55.95\,s^{12}+66.21\,s^{11}+59.52\,s^{10}+41.17\,s^{9}+21.98\,s^{8}+9.04\,s^{7}+2.847\,s^{6}+0.6785\,s^{5}+0.1198\,s^{4}+0.01515\,s^{3}+0.001291\,s^{2}+6.61\times 10^{-5}s+1.528\times 10^{-6}}{s^{15}+6\,s^{14}+16.8\,s^{13}+29.12\,s^{12}+34.94\,s^{11}+30.75\,s^{10}+20.5\,s^{9}+10.54\,s^{8}+4.217\,s^{7}+1.312\,s^{6}+0.3149\,s^{5}+0.05725\,s^{4}+0.007634\,s^{3}+0.0007046\,s^{2}+4.027\times 10^{-5}s+1.074\times 10^{-6}}$
Coefficient of bases of the Youla parameter	$\left[2.8816\ \ -0.7186\ \ -0.9323\ \ -1.3115\ \ -0.4986\ \ -0.0532\right]^T$
Number of iteration in the Interior Point method	17
Value of the objective function in the optimization problem $\|y(t) - y_d(t)\|_\infty$	0.007171

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Table VII. Results for the design of stabilizing Youla controller with orthonormal Kautz bases.

Location of pole of the bases	$\left[-0.8+0.5i \;\; -0.8-0.5i\right]$
Number of bases	6
Youla parameter $Q(s)$	$Q(s)=\dfrac{\begin{array}{l}2.887\,s^{30}+66.18\,s^{29}+742.5\,s^{28}+5427\,s^{27}+2.906\times10^{4}\,s^{26}+1.215\times10^{5}\,s^{25}\\+4.134\times10^{5}\,s^{24}+1.175\times10^{6}\,s^{23}+2.845\times10^{6}s^{22}+5.952\times10^{6}\,s^{21}+1.087\times10^{7}\,s^{20}\\+1.748\times10^{7}\,s^{19}+2.484\times10^{7}\,s^{18}+3.136\times10^{7}\,s^{17}+3.522\times10^{7}\,s^{16}+3.522\times10^{7}\,s^{15}\\+3.137\times10^{7}\,s^{14}+2.484\times10^{7}\,s^{13}+1.745\times10^{7}\,s^{12}+1.083\times10^{7}\,s^{11}+5.906\times10^{6}\,s^{10}\\+2.813\times10^{6}\,s^{9}+1.159\times10^{6}\,s^{8}+4.085\times10^{5}\,s^{7}+1.211\times10^{5}\,s^{6}+2.955\times10^{4}\,s^{5}\\+5746\,s^{2}+848.3\,s^{3}+87.81\,s^{2}+5.485s+0.1426\end{array}}{\begin{array}{l}s^{30}+24\,s^{29}+282.2\,s^{28}+2163\,s^{27}+1.214\times10^{4}\,s^{26}+5.312\times10^{4}\,s^{25}+1.885\times10^{5}\,s^{24}\\+5.566\times10^{5}\,s^{23}+1.394\times10^{6}\,s^{22}+3\times10^{6}\,s^{21}+5.609\times10^{6}\,s^{20}+9.177\times10^{6}\,s^{19}\\+1.322\times10^{7}\,s^{18}+1.682\times10^{7}\,s^{17}+1.897\times10^{7}\,s^{16}+1.899\times10^{7}\,s^{15}+1.688\times10^{7}\,s^{14}\\+1.332\times10^{7}\,s^{13}+9.316\times10^{6}\,s^{12}+5.758\times10^{6}\,s^{11}+3.132\times10^{6}\,s^{10}+1.491\times10^{6}\,s^{9}\\+6.165\times10^{5}\,s^{8}+2.191\times10^{5}\,s^{7}+6.603\times10^{4}\,s^{6}+1.656\times10^{4}\,s^{5}+3369\,s^{4}\\+534.2\,s^{3}+62.02\,s^{2}+4.695s+0.1741\end{array}}$
Coefficient of bases of the Youla parameter	$\left[2.8872 \;\; -1.8983 \;\; -0.9615 \;\; 0.2650 \;\; -0.1293 \;\; -0.1067\right]^{T}$
Number of iteration in the Interior Point method	17
Value of the objective function in the optimization problem $\lVert y(t)-y_d(t)\rVert_\infty$	0.099089

Table VIII. Results for the design of stabilizing Youla controller with orthonormal generalized bases.

Location of pole of the bases	$\begin{bmatrix} -0.2 & -0.3 & -0.4 & -0.5 & -0.6 \end{bmatrix}$
Number of bases	6
Youla parameter $Q(s)$	$Q(s) = \dfrac{\begin{array}{l} 2.888\,s^{15}+11.23\,s^{14}+22.56\,s^{13}+30.64\,s^{12}+30.31\,s^{11}+22.31\,s^{10}+12.31\,s^{9} \\ +5.121\,s^{8}+1.611\,s^{7}+0.3827\,s^{6}+0.06811\,s^{5}+0.008933\,s^{4}+0.0008379\,s^{3} \\ +5.322\times 10^{-5}\,s^{2}+2.055\times 10^{-6}s+3.659\times 10^{-8} \end{array}}{\begin{array}{l} s^{15}+5\,s^{14}+11.55\,s^{13}+16.35\,s^{12}+15.87s^{11}+11.18\,s^{10}+5.91\,s^{9}+2.386\,s^{8} \\ +0.7421\,s^{7}+0.1778\,s^{6}+0.03254\,s^{5}+0.004469\,s^{4}+0.000446\,s^{3} \\ +3.052\times 10^{-5}\,s^{2}+1.281\times 10^{-6}\,s+2.488\times 10^{-8} \end{array}}$
Coefficient of bases of the Youla parameter	$\begin{bmatrix} 2.8880 & -0.5084 & -0.5565 & -1.2992 & -1.0417 & -0.2329 \end{bmatrix}^{T}$
Number of iteration in the Interior Point method	18
Value of the objective function in the optimization problem $\Vert y(t) - y_d(t) \Vert_\infty$	0.005496

space into a time-continuous space, the Tustin transformation was used. The aim of the control process was to design the Youla stabilizing controller to achieve certain overshoot, rise time, settling time, and steady-state error. Therefore, LOR method was employed in order to optimize controller gains and the problem of designing a controller was replaced by an optimization problem. The interior point method was used to solve the optimization problem. As shown for the case study, the proposed method is very efficient for time-continuous problems. This method allows to fully utilizing the original method developed by Boyd to be extended by taking into account some multi-criteria constraints in time and frequency domains, simultaneously. The output of the closed-loop system to step input with the stabilizing Youla controller using four orthonormal bases was presented and showed that all of orthonormal bases have similar results.

Nomenclature

$\theta(t)$ = Reference angular motion
y_i = Transverse vibration components of beam
J_e = Effective inertia of load shaft of a DC motor
N = Gear ratio of motor
V = Armature voltage
i = Armature current
R = Resistance of armature circuit
L = Inductance of armature circuit
T_m = Produced torque by a DC motor
J = Moment of inertia of load
b = Height of link
l = Length of link
w = Width of link
EI = Stiffness of the link
ρ = Specific mass per unit of length of beam
I_H = Moment of inertia of the hub
r = Radius of the hub
M_t = Tip mass
J_t = Moment of inertia of tip mass
J_m = Inertia of rotor of a DC motor
M = System mass matrix
K = System stiffness matrix
h = Step size

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