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Advanced gain scheduled H_∞ controller for robotic manipulators

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SUMMARY

A conservatism-reduced design of a gain scheduled output feedback H_∞ controller for an n-joint rigid robotic manipulator, which integrates the varying-parameter rate without their feedback, is proposed. The robotic system is reduced to a 1inear parameter varying (LPV) form, which depends on the varying-parameter. By using a parameter-dependent Lyapunov function, the design of a controller, which satisfies the closed-loop H_∞ performance, is reduced to a solution of the parameterized linear matrix inequalities (LMIs) of parameter matrices. With a use of the concept of "multi-convexity", the solution of the infinite LMIs in the varying-parameter and its rate space is reduced to a solution of the finite LMIs for the vertex set. The proposed controller eliminates the feedback of the varying-parameter rate and fixes its upper boundary so that the conservatism of the controller design is reduced. Experimental results verify the effectiveness of the proposed design.

KEYWORDS: Robotic manipulator; Gain scheduled output feedback; H_∞ control; Reducing conservatism; Linear matrix inequality (LMI).

1. INTRODUCTION

A robotic system is a highly coupled, time-varying nonlinear system, where its dynamics characteristics change along with its geometrical characteristics and inertia. At the same time, there are dynamic uncertainty and external disturbance in a robotic motion. In order to guarantee that the robotic manipulator have good dynamic performance in the whole motion range, there are two objectives in the controller design, namely, (1) the disturbance attenuation and the robust stability, (2) the real-time adjustment of the controller dynamics along with thet robotic geometrical characteristics and inertia. The two objectives can be achieved by using the H_∞ synthesis technique[1] and the gain scheduled technique, respectively.[2] One approach integrating the above objectives is to linearize the robotic dynamics at different motion regions to obtain a piecewise linear system.[3–5] By using a robust control method local controllers are designed and then a global controller via interpretation is obtained. However, the system stability and performance in the whole varying-parameter range are not theoretically guaranteed. Another approach is the gain scheduled H_∞ control for a LPV system developed during the recent years. Though this approach is currently rarely used in robotic control, the LPV theory has a potential in designing a robotic gain scheduled H_∞ controller, since the robotic dynamics can be reduced to be a LPV form with the joint position functions as the varying parameter. A gain scheduled H_∞ controller with a large-scale stability and disturbance-rejection performance for an inverted pendulum is presented with the use of the LPV method in reference [6]. However, this controller has strong conservatism, since the varying-parameter rate are not considered in the controller design and there is thus no restriction for the varying-parameter rate so that they can be infinity, which is impractical. Also, it is difficult to realize it since it seeks a unique Lyapunov function in the whole varying-parameter range to guarantee a system H_∞ performance for all possible trajectories.

Starting from the concept of reducing the conservatism of the controller design, a new approach to designing a robotic gain scheduled output feedback H_∞ controller without a varying-parameter rate feedback is proposed in this article based on the parameter-dependent Lyapunov function with a combination of a varying-parameter rate. With an introduction of the concept of "multi-convexity", the controller design is reduced to a solution of finite LMIs, so that the controller design has the facility of a solution of a convex optimization. Though the varying-parameter rate is considered in the design, it is eliminated in the gain scheduled controller and, therefore, no varying-parameter rate feedback signal is needed.

2. THE LPV FORM OF A ROBOTIC SYSTEM

The dynamics equation of an n-joint rigid robotic manipulator is

$$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+g(q)=\tau \tag{1}$$

where $q\in R^n$ is the joint position vector, $M(q)\in R^{n\times n}$ is the inertia matrix, $C(q,\dot{q})\dot{q}\in R^n$ is the centrifugal and Couliaulis term, $g(q)\in R^n$ is the gravity term and $\tau\in R^n$ is the control torque.

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Suppose $x_1=q, x_2=\dot{q}$ and $X=(x_1\ x_2)^T$. (1) can be expressed in a state space as

$$\dot{X}=F(X)+G(X)\tau \tag{2}$$

where $F(X)=\begin{bmatrix}[0_{n\times n} & I_{n\times n}]X \\ -M^{-1}(X)(C(X)+g(X))\end{bmatrix}$

and $G(X)=\begin{bmatrix}0_{n\times 1} \\ M^{-1}(X)\end{bmatrix}$.

For the system (2), a varying-parameter $\rho(t)=[\rho_1(t),\ldots,\rho_l(t)]^T\in T\subset R^l(\rho(t)$ is written as ρ for simplicity in the following text) is selected with its vertex set being $\mathbf{V}:=\{\omega=(\omega_1,\ldots,\omega_l)^T:\omega_i\in\{\underline{\omega}_i,\overline{\omega}_i\},\underline{\omega}_i,\overline{\omega}_i\in R\}$ and the vertex set of the changing range of the varying-parameter rate $\dot{\rho}$ being $\Gamma:=\{v=(v_1,\ldots,v_l)^T:v_1\in\{\underline{v}_i,\overline{v}_i\},\underline{v}_i,\overline{v}_i\in R\}$. Assume that there is an equilibrium family set parameterized by the varying-parameter ρ, i.e. there is a continuous function $X_e(\rho):R^l\rightarrow R^n$ and $\tau_e(\rho):R^l\rightarrow R^n$ such that for all ρ, which has a vertex set $\mathbf{V}$ we have

$$0=F(X_e(\rho))+G(X_e(\rho))\tau_e(\rho) \tag{3}$$

where the varying-parameter ρ could be a function of system states, inputs, outputs or external signals. For a specified object, the selection of the varying-parameter ρ is not unique. The selection principle is that the selected ρ is able to reflect the dynamic characteristics of the original system. For every ρ at the equilibrium family (3), (2) can be reduced, after a Jacobian linearization, to

$$\dot{x}(t)=A(\rho)x(t)+B(\rho)u(t) \tag{4}$$

where $x(t)=X(t)-X_e(\rho)$, $A(\rho)=\dfrac{\partial}{\partial X}(F(X)+G(X)\tau)|_{X_e(\rho),\tau_e(\rho)}$

and $B(\rho)=G(X_e(\rho))$.

(4) is the LPV form of the robotic system. Since there are model errors, such as the high-frequency unmodeled part in robotic modeling and the dynamic uncertainty and external disturbance in robotic motion such as joint coupling, friction, and sensor and executor noise, an equivalent disturbance $w_1(t)$ for the model errors, dynamic uncertainty and external disturbance are added in (4). Suppose that the manipulator joint positions are taken as the system output vector $y(t)$, and $w_2(t)$ is the position measurement noise. The performance index $z(t)$ expresses the disturbance-rejection performance for disturbances $w_1(t)$ and $w_2(t)$. Thus (4) is expanded into the following LPV form

$$\begin{aligned}\dot{x}(t)&=A(\rho)x(t)+B_1(\rho)w_1(t)+B_2(\rho)u(t)\\ z(t)&=C_1(\rho)x(t)+u(t)\\ y(t)&=C_2(\rho)x(t)+w_2(t)\end{aligned} \tag{5}$$

where $B_1(\rho)=\begin{bmatrix}0_{n\times n}\\ I_{n\times n}\end{bmatrix}$, $B_2(\rho)=B(\rho)$ and $C_1(\rho)$ is the performance weighting matrix and $C_2(\rho)=[I_{n\times n}\quad 0_{n\times n}]$. In this article, $A(\rho)$ and $B_2(\rho)$ are the affine functions of the varying parameter ρ, i.e. $[A(\rho)\quad B_2(\rho)]=[A_0\quad B_{20}]+\sum_{i=1}^{l}\rho_i[A_i\quad B_{2i}]$.

Design the following full-order gain scheduled output feedback controller $K(\rho)$ for system (5):

$$K(\rho(t)):\begin{cases}\dot{x}_K(t)=A_K(\rho)x_K(t)+B_K(\rho)y(t)\\ u(t)=C_K(\rho)x_K(t)\end{cases} \tag{6}$$

Define $x_{cl}(t)=[x(t)\ \ x_K(t)]^T$, $w(t)=[w_1(t)\quad w_2(t)]^T$. Due to (5) and (6), the closed-loop system can be expressed as

$$\begin{cases}\dot{x}_{cl}(t)=A_{cl}(\rho)x_{cl}(t)+B_{cl}(\rho)w(t)\\ z(t)=C_{cl}(\rho)x_{cl}(t)\end{cases} \tag{7}$$

where

$$A_{cl}(\rho)=\begin{bmatrix}A(\rho) & B_2(\rho)C_K(\rho)\\ B_K(\rho C_2(\rho) & A_K(\rho)\end{bmatrix}, B_{cl}(\rho)=\begin{bmatrix}B_1(\rho) & 0\\ 0 & B_K(\rho)\end{bmatrix},$$

$C_{cl}(\rho)=[C_1(\rho)\quad C_K(\rho)]$.

For the closed-loop system (7), define its H_∞ performance.

Definition 1: For a given $\gamma>0$, if the closed-loop performance index $z(t)$ satisfies $\int_0^\infty z^T(t)z(t)dt<\gamma^2\int_0^\infty w^T(t)w(t)dt$ for any disturbance $w(t)$, the closed-loop system is claimed to possess a H_∞ performance γ.

The design of a gain scheduled feedback controller $K(\rho)$ by integrating the bounded varying-parameter rate is to be presented so that for any ρ in the varying-parameter set, the closed-loop system (7) has a H_∞ performance γ.

3. THE CONSERVATISM-REDUCED GAIN SCHEDULED H_∞ CONTROLLER SYNTHESIS

3.1. *Controller synthesis*

The sufficient condition for the H_∞ performance γ of the closed-loop system for any ρ is presented first.

Theorem 1: For the closed-loop system (7), if there exists a continuous symmetric positive-definite matrix function $P(\rho)$ satisfying

$$A_{cl}^T(\rho)P(\rho)+P(\rho)A_{cl}(\rho)+\frac{d}{dt}P(\rho)+\gamma^{-2}C_{cl}^T(\rho)C_{cl}(\rho)$$
$$+P(\rho)B_{cl}(\rho)B_{cl}^T(\rho)P(\rho)<0 \tag{8}$$

for $\rho\in T$, then the closed-loop system possesses a H_∞ performance γ.

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Proof: Define a parameter-dependent Lyapunov function $V(x_{cl},\rho)=x_{cl}^T P(\rho)x_{cl}$. Based on the selected parameter-dependent Lyapunov function $V(x_{cl},\rho)$, suppose

$$H=\dot{V}+y^{-2}z^Tz-w^Tw \qquad (9)$$

Let $\Pi=\begin{bmatrix}\gamma^{-2}I & 0\\ 0 & -I\end{bmatrix}$, (9) is reduced to

$$H=\dot{V}+\begin{bmatrix}z\\ w\end{bmatrix}^T\Pi\begin{bmatrix}z\\ w\end{bmatrix}$$

$$=\dot{x}_{cl}^TP(\rho)x_{cl}+x_{cl}^TP(\rho)\dot{x}_{cl}+x_{cl}^T\dot{P}(\rho)x_{cl}$$

$$+\begin{bmatrix}C_{cl}(\rho)x_{cl}\\ w\end{bmatrix}^T\Pi\begin{bmatrix}C_{cl}(\rho)x_{cl}\\ w\end{bmatrix}$$

$$=\begin{bmatrix}x_{cl}\\ w\end{bmatrix}^T\left(\begin{bmatrix}\dot{P}(\rho)+A_{cl}^T(\rho)P(\rho)+P(\rho)A_{cl}(\rho) & P(\rho)B_{cl}(\rho)\\ B_{cl}^T(\rho)P(\rho) & 0\end{bmatrix}\right.$$

$$\left.+\begin{bmatrix}C_{cl}(\rho) & 0\\ 0 & I\end{bmatrix}^T\Pi\begin{bmatrix}C_{cl}(\rho) & 0\\ 0 & I\end{bmatrix}\right)\begin{bmatrix}x_{cl}\\ w\end{bmatrix}$$

If $\begin{bmatrix}\dot{P}(\rho)+A_{cl}^T(\rho)P(\rho)+P(\rho)A_{cl}(\rho) & P(\rho)B_{cl}(\rho)\\ B_{cl}^T(\rho)P(\rho) & 0\end{bmatrix}$

$$+\begin{bmatrix}C_{cl}(\rho) & 0\\ 0 & I\end{bmatrix}^T\Pi\begin{bmatrix}C_{cl}(\rho) & 0\\ 0 & I\end{bmatrix}<0$$

i.e., $\begin{bmatrix}\dot{P}(\rho)+A_{cl}^T(\rho)P(\rho)+P(\rho)A_{cl}(\rho) & P(\rho)B_{cl}(\rho)\\ B_{cl}^T(\rho)P(\rho) & -I\end{bmatrix}$

$$+\begin{bmatrix}C_{cl}^T(\rho)\\ 0\end{bmatrix}\gamma^{-2}[C_{cl}(\rho)\quad 0]<0 \qquad (10)$$

From the Schur Complement (See the Appendix), (10) is equivalent to

$$\begin{bmatrix}\dot{P}(\rho)+A_{cl}^T(\rho)P(\rho)+P(\rho)A_{cl}(\rho) & P(\rho)B_{cl}(\rho) & C_{cl}^T(\rho)\\ B_{cl}^T(\rho)P(\rho) & -I & 0\\ C_{cl}(\rho) & 0 & -\gamma^2I\end{bmatrix}<0 \qquad (11)$$

From the Schur Complement again, (11) is equivalent to (8). Then $H<0$.

Since, (11) holds, $\dot{P}(\rho)+A_{cl}^T(\rho)P(\rho)+P(\rho)A_{cl}(\rho)<0$. Thus, $\dot{V}(x,\rho)=x^T(\dot{P}(\rho)+A_{cl}^T(\rho)P(\rho)+P(\rho)A_{cl}(\rho))x<0$, i.e. it is exponentially stable inside the closed-loop system. When (8) holds, integrating both sides of (9) we have

$$V(\infty)-V(0)+\gamma^{-2}\int_0^\infty z^Tzdt-\int_0^\infty w^Twdt<0.$$

Since $V(\infty)=V(0)=0$, the above equation, becomes $\gamma^{-2}\int_0^\infty z^Tzdt-\int_0^\infty w^Twdt<0$. Thus from Definition 1 we know the closed-loop system has a H_∞ performance γ.

Based on Theorem 1, the detailed method for gain scheduled controller synthesis is given below.

Theorem 2: For a given $\gamma>0$, if there exist positive-definite continuous differentiable parameter matrix functions $X(\rho)$ and $Y(\rho)$, which satisfy the following for all $\rho\in\mathrm{T}$:

$$\begin{bmatrix}S(\rho) & 0 & \gamma^{-1}B_1(\rho)\\ 0 & -I & 0\\ \gamma^{-1}B_1^T(\rho) & 0 & -I\end{bmatrix}<0 \qquad (12)$$

$$\begin{bmatrix}R(\rho) & X(\rho)B_1(\rho) & \gamma^{-1}C_1^T(\rho)\\ B_1^T(\rho)X(\rho) & -I & 0\\ \gamma^{-1}C_1(\rho) & 0 & -I\end{bmatrix}<0 \qquad (13)$$

$$\begin{bmatrix}X(\rho) & \gamma^{-1}I\\ \gamma^{-1}I & Y(\rho)\end{bmatrix}>0 \qquad (14)$$

where

$$S(\rho)=[A(\rho)-B_2(\rho)C_1(\rho)]Y(\rho)+Y(\rho)[A(\rho)-B_2(\rho)C_1(\rho)]^T$$
$$-B_2(\rho)B_2^T(\rho)-\dot{Y}(\rho)$$
$$R(\rho)=X(\rho)A(\rho)+A^T(\rho)X(\rho)-C_2^T(\rho)C_2(\rho)+\dot{X}(\rho)$$

then for any reversible matrix function $N(\rho)$, there exists the following gain scheduled controller $K(\rho)$:

$$F(\rho)=-[B_2^T(\rho)Y^{-1}(\rho)+C_1(p)]$$
$$M(\rho)=[I-\gamma^2Y(\rho)X(\rho)]N^{-T}(\rho)$$
$$L(\rho)=-X^{-1}(\rho)C_2^T(\rho)$$
$$C_K(\rho)=\gamma^2F(\rho)Y(\rho)M^{-T}(\rho)$$
$$B_K(\rho)=N^{-1}(\rho)X(\rho)L(\rho)$$
$$A_K(P)=-N^{-1}(\rho)\{\gamma^2X(\rho)[A(\rho)+L(\rho)C_2(\rho)$$
$$+B_2(\rho)F(\rho)]Y(\rho)+A^T(\rho)+[C_1^T(\rho)C_1(\rho)$$
$$+C_1^T(\rho)F(\rho)]Y(\rho)+X(\rho)B_1(\rho)B_1^T(\rho)$$
$$+\gamma^2\dot{X}(\rho)Y(\rho)+\dot{N}(\rho)M(\rho)\}M^{-T}(\rho) \qquad (15)$$

$$K(\rho)=\begin{bmatrix}A_K(\rho) & B_K(\rho)\\ C_K(\rho) & 0\end{bmatrix}$$

so that for all $\rho\in\mathrm{T}$, the closed-loop system has a H_∞ performance γ.

Proof: Define $M(\rho)=[I-\gamma^2Y(\rho)X(\rho)]N^{-T}(\rho)$. (16)

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Due to the Schur Complement, (14) is equivalent to: $X(\rho)-\gamma^{-2}Y(\rho)>0$

$$\text{i.e. } Y^{-1}(\rho)-\gamma^2 X(\rho)<0 \tag{17}$$

The above equation also denotes that $[I-\gamma^2X(\rho)Y(\rho)]Y^{-1}(\rho)$ is reversible and therefore $M(\rho)$ is reversible.

Define $P(\rho)=\begin{bmatrix} X(\rho) & N(\rho) \\ N^T(\rho) & -\gamma^2N^T(\rho)Y(\rho)M^{-T}(\rho)\end{bmatrix}$

and $P^{-1}(\rho)=\begin{bmatrix} \gamma^2Y(\rho) & M(\rho) \\ M^T(\rho) & -N^{-1}(\rho)X(\rho)M(\rho)\end{bmatrix}$

$$\begin{aligned}H(\rho)=&A_{cl}^T(\rho)P(\rho)+P(\rho)A_{cl}(\rho)+\dot{P}(\rho)+\gamma^{-2}C_{cl}^T(\rho)C_{cl}(\rho)\\ &+P(\rho)B_{cl}(\rho)B_{cl}^T(\rho)P(\rho)\end{aligned} \tag{18}$$

From Theorem 1, in order to let the closed-loop system (7) possess a H_∞ performance γ, we only need to have $P(\rho)>0$ and $H(\rho)<0$ for all $\rho\in$T.

Since $P(\rho)P^{-1}(\rho)=I$, we have $N(\rho)M^T(\rho)=I-\gamma^2X(\rho)Y(\rho)$.

$$\begin{aligned}\text{Since } & X(\rho)+N(\rho)[\gamma^2N^T(\rho)Y(\rho)M^{-T}(\rho)]^{-1}N^T(\rho)\\ &=X(\rho)+\gamma^{-2}N(\rho)M^T(\rho)Y^{-1}(\rho)\\ &=X(\rho)+\gamma^{-2}[I-\gamma^2X(\rho)Y(\rho)]Y^{-1}(\rho)\\ &=\gamma^{-2}Y^{-1}(\rho)>0\end{aligned} \tag{19}$$

and

$$\begin{aligned}-\gamma^2N^T(\rho)Y(\rho)M^{-T}(\rho)&=-\gamma^2N^T(\rho)Y(\rho)[I-\gamma^2X(\rho)Y(\rho)]^{-1}N(\rho)\\ &=-\gamma^2N^T(\rho)[Y^{-1}(\rho)-\gamma^2X(\rho)]^{-1}N(\rho)\end{aligned}$$

due to (17), we have $-\gamma^2N^T(\rho)Y(\rho)M^{-T}(\rho)>0.$ (20)

Therefore due to the Schur Complement, (19) and (20) are equivalent to $P(\rho)>0$.

In order to prove $H(\rho)<0$,

define $P_1(\rho)=\begin{bmatrix} \gamma^2Y(\rho) & I \\ M^T(\rho) & 0\end{bmatrix}$ and $P_2(\rho)=P(\rho)P_1(\rho)$

$$=\begin{bmatrix} I & X(\rho) \\ 0 & N^T(\rho)\end{bmatrix}.$$

Suppose $\overline{H}(\rho)=P_1^T(\rho)H(\rho)P_1(\rho)$. (21)

Since $P_1(\rho)$ is reversible, we have $H(\rho)<0$ if and only if $\overline{H}(\rho)<0$.

Substituting (18) into (21), we have

$$\begin{aligned}\overline{H}(\rho)=&P_1^T(\rho)A_{cl}^T(\rho)P_2(\rho)+P_2^T(\rho)A_{cl}(\rho)P_1(\rho)\\ &+P_1^T(\rho)\dot{P}(\rho)P_1(\rho)+\gamma^{-2}P_1^T(\rho)C_{cl}^T(\rho)C_{cl}(\rho)P_1(\rho)\\ &+P_2^T(\rho)B_{cl}(\rho)B_{cl}^T(\rho)P_2(\rho)\end{aligned} \tag{22}$$

Define $\overline{H}(\rho)=\begin{bmatrix} \overline{H}_{11}(\rho) & \overline{H}_{12}(\rho) \\ \overline{H}_{12}^T(\rho) & \overline{H}_{22}^T(\rho)\end{bmatrix}$

Substituting $P_1(\rho)$, $P_2(\rho)$, $A_{cl}(\rho)$, $B_{cl}(\rho)$ and $C_{cl}(\rho)$ into (22), we obtain

$$\begin{aligned}\overline{H}_{11}(\rho)=&\gamma^2[A(\rho)-B_2(\rho)C_1(\rho)]Y(\rho)+\gamma^2Y(\rho)[A(\rho)\\ &-B_2(\rho)C_1(\rho)]^T-\gamma^2B_2(\rho)B_2^T(\rho)+B_1(\rho)B_1^T(\rho)\\ &-\gamma^2\dot{Y}(\rho)\end{aligned}$$

By multiplying both sides by γ^{-2}, the above equation becomes

$$\begin{aligned}\gamma^2\overline{H}_{11}(\rho)=&[A(\rho)-B_2(\rho)C_1(\rho)]Y(\rho)+Y(\rho)[A(\rho)\\ &-B_2(\rho)C_1(\rho)]^T-B_2(\rho)B_2^T(\rho)\\ &+\gamma^{-2}B_1(\rho)B_1^T(\rho)-\dot{Y}(\rho)\end{aligned} \tag{23}$$

Due to Schur Complement, (12) is equivalent to

$$\begin{aligned}&[A(\rho)-B_2(\rho)C_1(\rho)]Y(\rho)+Y(\rho)[A(\rho)-B_2(\rho)C_1(\rho)]^T\\ &-B_2(\rho)B_2^T(\rho)+\gamma^{-2}B_1(\rho)B_1^T(\rho)-\dot{Y}(\rho)<0\end{aligned} \tag{24}$$

Substituting (24) into (23), we obtain $\gamma^{-2}\overline{H}_{11}(\rho)<0$ (25)

Similarly, we can derive

$$\begin{aligned}\overline{H}_{22}(\rho)=&X(\rho)A(\rho)+A^T(\rho)X(\rho)-C_2^T(\rho)C_2(\rho)\\ &+X(\rho)B_1(\rho)B_1^T(\rho)X(\rho)+\gamma^{-2}C_1^T(\rho)C_1(\rho)+\dot{X}(\rho)\end{aligned} \tag{26}$$

Due to Schur Complement, (13) is equivalent to

$$\begin{aligned}&X(\rho)A(\rho)+A^T(\rho)X(\rho)-C_2^T(\rho)C_2(\rho)+X(\rho)B_1(\rho)B_1^T(\rho)X(\rho)\\ &+\gamma^{-2}C_1^T(\rho)C_1(\rho)+\dot{X}(\rho)<0\end{aligned} \tag{27}$$

Substituting (27) into (26), we obtain $\overline{H}_{22}(\rho)<0$. (28)

From (22), we have

$$\begin{aligned}\overline{H}_{12}(\rho)=&\gamma^2X(\rho)[A(\rho)+L(\rho)C_2(\rho)+B_2(\rho)F(\rho)]Y(\rho)\\ &+N(\rho)A_K(\rho)M^T(\rho)+A^T(\rho)\\ &+[C_1^T(\rho)C_1(\rho)+C_1^T(\rho)F(\rho)]+X(\rho)B_1(\rho)B_1^T(\rho)\\ &+\gamma^2\dot{X}(\rho)Y(\rho)+\dot{N}(\rho)M^T(\rho)\end{aligned}$$

When $A_K(\rho)$ takes (15), $\overline{H}_{12}(\rho)=0$. Therefore, with a combination of (25) and (28), $\overline{H}(\rho)<0$, i.e., $\overline{H}(\rho)<0$. Thus, for all $\rho\in$T, the closed-loop system has a H_∞ performance γ.

Remark 1: If $x(\rho)$ and $Y(\rho)$ are restricted to be constant matrices in the varying parameter set **T**, Theorem 2 is reduced to be the LPV synthesis problem in references [6] and [7]. In other words, [6] and [7] are actually a special case of this article.

3.2. Elimination of the varying-parameter rate feedback in the controller

It is seen from (15), the gain scheduled controller needs the feedback signals of the varying-parameter rate, which are practically impossible to obtain. In this paper, the dependence of the controller on the varying-parameter rate is eliminated by an appropriate selection of $N(\rho)$.

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Corollary 1: In Theorem 2, if $N(\rho)$ satisfies

$$\frac{\partial N(\rho)}{\partial \rho_i} = -\gamma^2 \frac{\partial X(\rho)}{\partial \rho_i} Y(\rho)[I-\gamma^2 X(\rho)Y(\rho)]^{-1}N(\rho) \quad (i=1,\ldots,l) \tag{29}$$

then in the gain scheduled controller $K(\rho)$, after $A_K(\rho)$ is reconstructed as

$$\begin{aligned} A_K(\rho) = -N^{-1}(\rho)\{&\gamma^2 X(\rho)[A(\rho)+L(\rho)C_2(\rho) \\ &+B_2(\rho)F(\rho)]Y(\rho)+A^T(\rho)+[C_1^T(\rho)C_1(\rho) \\ &+C_1^T(\rho)F(\rho)]Y(\rho)+X(\rho)B_1(\rho)B_1^T(\rho)\}M^{-1}(\rho) \end{aligned} \tag{30}$$

the controller still can guarantee that, for all $\rho \in T$, the closed-loop system possesses a H_∞ performance γ.

Proof: In the proof of Theorem 2, suppose $A_K(\rho)$ takes (30). Then

$$\overline{H}_{12}(\rho) = \gamma^2 \dot{X}(\rho)Y(\rho) + \dot{N}(\rho)M^T(\rho)$$

Due to (16), $M^{-T}(\rho) = [I-\gamma^2 X(\rho)Y(\rho)]^{-1}N(\rho)$.

Substituting the above equation into (29), we have

$$\frac{\partial N(\rho)}{\partial \rho_i} = -\gamma^2 \frac{\partial X(\rho)}{\partial \rho_i} Y(\rho)M^{-1}(\rho) \quad (i=1,\ldots,l).$$

Then $\frac{\partial N(\rho)}{\partial \rho_1}\dot{\rho}_1 + \ldots + \frac{\partial N(\rho)}{\partial \rho_l}\dot{\rho}_l$

$$= -\gamma^2 \left[\frac{\partial X(\rho)}{\partial \rho_1}\dot{\rho}_1 + \ldots + \frac{\partial X(\rho)}{\partial \rho_l}\dot{\rho}_l\right] Y(\rho)M^{-T}(\rho)$$

i.e., $\dot{N}(\rho)M^T(\rho) = -\gamma^2 \dot{X}(\rho)Y(\rho)$.

Therefore $\overline{H}_{12}(\rho)=0$ and then $\overline{H}(\rho)<0$, i.e. $H(\rho)<0$. Thus, for all $\rho \in T$, the closed-loop system has a H_∞ performance γ.

Remark 2: For the case of scalar parameters, (29) is a first-order linear homogenous differential equation, which is easy to solve. For any given reversible initial condition N_0, assume that the reversible matrix $T(\rho, \rho_0)$ is the transmission matrix of the differential equation. Then for any ρ in the varying-parameter set, the solution of the differential equation can be expressed as

$$N(\rho) = T(\rho, \rho_0)N_0$$

4. CONTROLLER CALCULATION

Theorem 2 shows that along with the changes of the varying parameter ρ and its rate $\dot{\rho}$, infinite numbers of LMIs need to be solved in order to obtain the gain scheduled controller $K(\rho)$, which satisfies the requirements. This is impractical. One approach is to divide the parameter space into grids.[2] Suppose the grid set of parameter ρ *is* **G** and the grid set of parameter $\dot{\rho}$ is Γ. Then the solutions of (12)–(14) are reduced to be a problem of solving infinite LMIs at $G \times \Gamma$. If the grids are close, this approach offers an approximate solution. However, the computation load is big and $X(\rho)$ and $Y(\rho)$ cannot be adjusted continuously along with the varying-parameter ρ. In this article, $X(\rho)$ and $Y(\rho)$ are assumed to have a same structure as the system, i.e.,

$$X(\rho) = X_0 + \sum_{i=1}^{l} \rho_i X_i \text{ and } Y(\rho) = Y_0 + \sum_{i=1}^{l} \rho_i Y_i$$

With the above structure, (12) and (13) become a quadratic form of the varying parameter ρ. The concept of "multi convexity" is used in this article to obtain its solution. The following theorem gives the detailed method for controller calculation

Theorem 3: For any given $\gamma>0$, if there exist symmetric matrices $X_0, \ldots, X_l$ and $Y_0, \ldots, Y_l$ such that the following LMIs are feasible,

$$\begin{bmatrix} S(\omega,v) & 0 & \gamma^{-1}B_1 \\ 0 & -I & 0 \\ \gamma^{-1}B_1^T & 0 & -I \end{bmatrix} < 0 \qquad \forall(\omega,v) \in \mathbf{V} \times \Gamma \tag{31}$$

$$\begin{bmatrix} R(\omega,v) & X(\omega)B_1 & \gamma^{-1}C_1^T \\ B_1^T X(\omega) & -I & 0 \\ \gamma^{-1}C_1 & 0 & -I \end{bmatrix} < 0 \qquad \forall(\omega,v) \in \mathbf{V} \times \Gamma \tag{32}$$

$$\begin{bmatrix} X(\omega) & \gamma^{-1}I \\ \gamma^{-1}I & Y(\omega) \end{bmatrix} > 0 \qquad \forall \omega \in \mathbf{V} \tag{33}$$

$$[A_i - B_{2i}C_1]Y_i + Y_i[A_i - B_{2i}C_1]^T - B_{2i}B_{2i}^T \geq 0 \quad i=1,\ldots,l \tag{34}$$

$$X_iA_i + A_i^T X_i \geq 0 \qquad i=1,\ldots,l \tag{35}$$

where

$$\begin{aligned} S(\omega,v) = {}&[A(\omega) - B_2(\omega)C_1]Y(\omega) + Y(\omega)[A(\omega) - B_2(\omega)C_1]^T \\ &- B_2(\omega)B_2^T(\omega) - Y(v) + Y_0 \end{aligned}$$

$$R(\omega,v) = X(\omega)A(\omega) + A^T(\omega)X(\omega) - C_2^TC_2 + X(v) - X_0$$

then the controller $K(\rho)$ guarantees that the closed-loop system has a H_∞ performance γ for the whole changing range of ρ and $\dot{\rho}$.

Proof: For (12), fix $\dot{\rho}$ at its vertex and then $\dot{Y} = Y(v) - Y_0$ can be regarded as a constant term. Considering that $B_1(\rho)$ is a constant matrix, we know from the Lemma for the multi-convexity (See the Appendix) that as long as (31) and (34) hold for all vertices $\omega \in \mathbf{V}$, it is guaranteed that (12) holds at any point in the space of the varying-parameter ρ, which has a vertex set $\mathbf{V}$. The above case holds for any vertex of the space of the varying-parameter rate $\dot{\rho}$. Since $\dot{Y}(\rho) = \sum_{i=1}^{l} \dot{\rho}_i Y_i$,

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i.e. $\dot{Y}(\rho)$ and $\dot{\rho}$ have an affine relationship, then due to the convex feature, it can be further guaranteed that (12) holds in the whole space of the varying-parameter rate $\dot{\rho}$, which has a vertex set Γ. In conclusion, as long as (31) and (34) hold in the vertex set $(\omega,v)\in\mathbf{V}\times\Gamma$ of the varying-parameter ρ and its rate $\dot{\rho}$, it can be guaranteed that (12) holds in the whole range of the varying-parameter ρ and its rate $\dot{\rho}$.

Similarly, it can be proved that we only need to have (32) and (35) to hold in the vertex set $(\omega,v)\in\mathbf{V}\times\Gamma$ of the varying-parameter ρ and its rate $\dot{\rho}$ in order to have (13) to hold in the whole range of the varying-parameter ρ and its rate $\dot{\rho}$. Also, we only need to have (33) to hold in the vertex set $\omega\in\mathbf{V}$ of the varying-parameter ρ in order to have (14) to hold in the whole range of the varying-parameter ρ.

The feasible solution of Theorem 3 can be obtained easily from MatLab *LMI Control Toolbox*.[8]

5. EXPERIMENTAL RESULTS

In order to verify the effectiveness of the proposed method, experiments are done on a self-manufactured planar two joint direct-drive manipulator. Its dynamics equation is expressed as[9]

$$\begin{bmatrix} a & b\cos(\theta_2-\theta_1) \\ b\cos(\theta_2-\theta_1) & c \end{bmatrix}\cdot\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} + \begin{bmatrix} -b\dot{\theta}_2^2\sin(\theta_2-\theta_1) \\ b\dot{\theta}_1^2\sin(\theta_2-\theta_1)\end{bmatrix} = \begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} \tag{36}$$

where $a=5.6794\ kg\cdot m^2$, $b=1.4730\ kg\cdot{}^2$, $c=1.7985\ kg\cdot m^2$.

By using the Jacobian linearization method, the robotic dynamics equation (36) is linearized at the equilibrium point $X_e=(\theta_{1e}\ \ \theta_{2e}\ \ \dot{\theta}_{1e}\ \ \dot{\theta}_{2e})^T=(\theta_{1e}\ \ \theta_{2e}\ \ 0\ \ 0)^T$ and $\tau_e=(0\ \ 0)^T$:

$$\begin{bmatrix}\dot{\hat{\theta}}_1\\ \dot{\hat{\theta}}_2\\ \ddot{\hat{\theta}}_1\\ \ddot{\hat{\theta}}_2\end{bmatrix} = \begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}\hat{\theta}_1\\ \hat{\theta}_2\\ \dot{\hat{\theta}}_1\\ \dot{\hat{\theta}}_2\end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \dfrac{c}{ac-b^2\cos^2(\theta_{2e}-\theta_{1e})} & \dfrac{-b\cos(\theta_{2e}-\theta_{1e})}{ac-b^2\cos^2(\theta_{2e}-\theta_{1e})} \\ \dfrac{-b\cos(\theta_{2e}-\theta_{1e})}{ac-b^2\cos^2(\theta_{2e}-\theta_{1e})} & \dfrac{a}{ac-b^2\cos^2(\theta_{2e}-\theta_{1e})}\end{bmatrix}\begin{bmatrix}\hat{\tau}_1\\ \hat{\tau}_2\end{bmatrix} \tag{37}$$

where $\hat{\theta}_1=\theta_1-\theta_{1e}$, $\hat{\theta}_2=\theta_2-\theta_{2e}$, $\dot{\hat{\theta}}_1=\dot{\theta}_1-\dot{\theta}_{1e}=\dot{\theta}_{1e}$, $\dot{\hat{\theta}}_2=\dot{\theta}_2-\dot{\theta}_{2e}=\dot{\theta}_{2e}$, $\hat{\tau}_1=\tau_1-\tau_{1e}=\tau_1$ and $\hat{\tau}_2=\tau_2-\tau_{2e}=\tau_2$.

Since $ac>>b^2$, the state space expression for (37) is

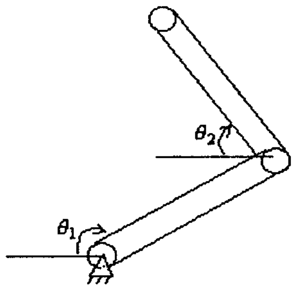

Fig. 1. A direct drive manipulator.

$$\dot{x}=Ax+Bu \tag{38}$$

where $A=\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$,

$$B=\begin{bmatrix}0&0\\0&0\\ \dfrac{1}{a} & -\dfrac{b}{ac}\cos(\theta_{2e}-\theta_{1e}) \\ -\dfrac{b}{ac}\cos(\theta_{2e}-\theta_{1e}) & \dfrac{1}{c}\end{bmatrix},$$

$x=(\hat{\theta}_1\ \ \hat{\theta}_2\ \ \dot{\hat{\theta}}_1\ \ \dot{\hat{\theta}}_2)^T$, $u=(\hat{\tau}_1\ \ \hat{\tau}_2)^T$.

It is seen from (38), matrix B is a linear function of $\cos(\theta_{2e}-\theta_{1e})$, and $\theta_{2e}-\theta_{1e}$ is the angle between joint 1 and joint 2 in Figure 1, which decides the dynamic characteristics of (38).[10] Practically, the measured values of θ_1 and θ_2 can be used as the equilibrium point for the linearization of the system (36). Thus, along with the changes of θ_1 and θ_2, (38) can be regarded as a continuous LPV system with respect to $\cos(\theta_2-\theta_1)$. In this section different values of $\cos(\theta_2-\theta_1)$ at the different angles are used to design the gain scheduled controller to improve the control performance. Define a varying parameter $\rho=\cos(\theta_2-\theta_1)$ and therefore $\rho=\dot{\theta}_1-\dot{\theta}_2)\sin(\theta_2-\theta_1)$. It is seen from Figure 1 that since $\theta_2-\theta_1\in[-\pi,0]$, $\rho\in[-1,1]$. Practically, $\dot{\theta}_1\in[-v_1,v_1]$, $\dot{\theta}_2\in[-v_2,v_2]$, $v_1=1\ rad/s$ and $v_2=1\ rad/s$, and therefore $\dot{\rho}\in[-(v_1+v_2),(v_1+v_2)]$. Thus, the vertex set of the changing range of the varying-parameter ρ and of the varying-parameter rate $\dot{\rho}$ is $\mathbf{V}:=\{\omega:\omega\in\{-1,1\}\}$ and $\Gamma:=\{v:v\in\{-(v_1+v_2),(v_1+v_2)\}\}$, respectively.

Equation (38) can be expanded to the form (5),where

$$A(\rho)=A,\ B_1(\rho)=\begin{bmatrix}0&0\\0&0\\1&0\\0&1\end{bmatrix}\text{ and }B_2(\rho)=B.$$

$C_1(\rho)$ is taken as $C_1(\rho)=\begin{bmatrix}1&0&1&0\\0&1&0&1\end{bmatrix}$

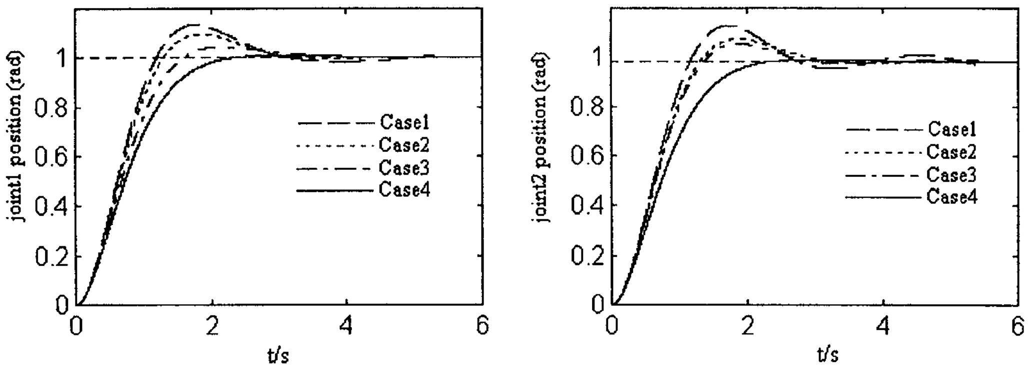

Fig. 2. Experimental results.

Table I. Control performance comparison.

	$X(\rho)$	$Y(\rho)$	γ
Case 1	$X_1=0$	$Y_1=0$	3.5632
Case 2	$X_1=0$	$Y_1\neq 0$	2.1546
Case 3	$X_1\neq 0$	$Y_1=0$	1.3768
Case 4	$X_1\neq 0$	$Y_1\neq 0$	0.8547

and

$C_2(\rho)=\begin{bmatrix}1&0&0&0\\0&1&0&0\end{bmatrix}$. $B_2(\rho)$ can be written as

$B_2(\rho)=B_{20}+\rho B_{21}$,

$$\text{where } B_{20}=\begin{bmatrix}0&0\\0&0\\\frac{1}{a}&0\\0&\frac{1}{c}\end{bmatrix},\ B_{21}=\begin{bmatrix}0&0\\0&0\\0&-\frac{b}{ac}\\-\frac{b}{ac}&0\end{bmatrix}.$$

In Theorem 3, $X(\rho)$ and $Y(\rho)$ are taken as $X(\rho)=X_0+\rho X_1$ and $Y(\rho)=Y_0+\rho Y_1$, respectively. After obtaining their solutions, the gain scheduled controller $K(\rho)$ without a varying-parameter rate feedback is obtained based on Section 3. $N(\rho)$ can be obtained based on Corollary 1 and Remark 2. Let $N_0=I_4$. In the below, four cases are discussed: (1) The varying parameter rate is not considered. Let $X_1=Y_1=0$, i.e., $\dot{\rho}$ is allowed to be an infinity; (2) Let $X_1=0$ only; (3) Let $Y_1=0$ only; (4) Let $X_1\neq 0$ and $Y_1\neq 0$. The H_∞, performances of the designed gain scheduled controllers for the four cases are shown in Table I.

It can be seen from Table I that the proposed method (i.e. Case 4) has the best H_∞ performance, while Case 1 gives the worst situation and therefore demonstrates a rather strong conservatism. Also, it can be seen that Cases 2 and 3 are better than Case 1. Figure 2 shows the step responses of the robotic manipulator by using the above four controllers, respectively. The experimental results verify Table I.

6. CONCLUSIONS

Considering the bounds of the varying-parameter rate, we have a conservatism-reduced treatment for the existing LPV theory; we apply the theory to robotic control so that robotic control performance is improved. The robotic system is first reduced to a LPV form affinely depending on the varying-parameter. In order to overcome the drawback of allowing the varying parameter rate to be infinitely large in the existing LPV theory, the design of the gain scheduled output feedback controller, which has a closed-loop H_∞ performance, is reduced to the solution of the parameter LMIs of $X(\rho)$ and $Y(\rho)$ by using a parameter-dependent Lyapunov function. $X(\rho)$ and $Y(\rho)$ are taken as an affine form of a same structure as that of the robotic LPV expression. The concept of "multi convexity" is used to reduce the solution of the infinite LMIs in the varying-parameter and its rate space to a solution of the finite LMIs for the vertex set. The computation then becomes feasible and the conservatism of the controller design is reduced, as verified by the experimental results. Though the varying-parameter rate is considered in the design, it is eliminated in the gain scheduled controller and therefore any feedback signal of the varying-parameter rate is not needed; this is also one of the features of this paper.

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APPENDIX 1

Schur Complement[10]

If a symmetric matrix $F=F^T\in R^{(n+m)\times(n+m)}$ can be expressed as $F=\begin{bmatrix} P & M \\ M^T & Q \end{bmatrix}$, where $Q\in R^{m\times m}$ is non-singular, then $F<0$ if and only if $Q<0$ and $P-MQ^{-1}M^T<0$, where $P-MQ^{-1}M^T$ is regarded as the Schur complement of Q.

Lemma: For the following quadratic function with respect to $\varsigma\in R^L$

$$f(\varsigma_1,\ldots,\varsigma_L)=\alpha_0+\sum_i \alpha_i\varsigma_i+\sum_{i<j}\beta_{ij}\varsigma_i\varsigma_j+\sum_i \gamma_i\varsigma_i^2$$

where ς changes in the supercube with a vertex set of $\mathbf{V}:=\{\omega=(\omega_1,\ldots,\omega_L)^T:\omega_1\in\{\underline{\varsigma_i},\overline{\varsigma_i}\}\}$. If $f(\omega)<0\,\forall\omega\in\mathbf{V}$ and $\gamma_i=\frac{1}{2}\frac{\partial^2 f}{\partial\varsigma_i^2}(\varsigma)\geq 0$, $i=1,\ldots,L$ hold, then for any ς in the supercube with a vertex set of $\mathbf{V}$, there is $f(\varsigma)<0$.

Proof: In order to guarantee that for any ς in the parameter space, there is $f(\omega)<0$, it is necessary to have $f(\omega)<0$ for any vertex $\omega\in\mathbf{V}$ in the parameter ς space.

Suppose for any i, $\gamma_i\geq 0$ and $f(\varsigma^*)$ is the global maximum at $\varsigma^*=(\varsigma_1^*,\ldots,\varsigma_L^*)^T$. If ς^* is not the vertex of the parameter space, there must exist certain i so that $\underline{\varsigma_i}<\varsigma_i^*<\overline{\varsigma_i}$. Let $\varsigma_j=\varsigma_j^*$, when $j\neq i$. Define the following quadratic polynomial

$$g(\varsigma_i):=f(\varsigma_1^*,\ldots,\varsigma_{i-1}^*,\varsigma_i,\varsigma_{i+1}^*,\ldots,\varsigma_L^*)=p+q\varsigma_i+\gamma_i\varsigma_i^2$$

Since $\gamma_i\geq 0$, $g(\varsigma_i)$ is convex. In other words, the maximum value in the range of $[\underline{\varsigma_i},\overline{\varsigma_i}]$ must be obtained at the ends, i.e.

$$g(\varsigma_i^*)\leq\max(g(\underline{\varsigma_i}),\,g(\overline{\varsigma_i})) \qquad (A1)$$

Also, (A1) is obtained based on the prior assumption of the validity of the following expression

$$g(\varsigma_i^*)\geq\max(g(\underline{\varsigma_i}),\,g(\overline{\varsigma_i})) \qquad (A2)$$

Thus, in order to have (A1) and (A2) holding at the same time, there must be $g(\varsigma_i^*)=\max(g(\underline{\varsigma_i}),\,g(\overline{\varsigma_i}))$. This shows that the maximum value of $g(\varsigma_i)$ is obtained at the end set. Repeat the above steps for every i. It can be seen that $f(\varsigma)$ reaches its maximum at certain vertices in the ς parameter space. Therefore, under the premise of $\gamma_i\geq 0$, if all the vertices $\omega\in\mathbf{V}$ satisfying $f(\omega)<0$, it can be guaranteed that for any ς in the supercube with a vertex set of $\mathbf{V}$, $f(\varsigma)<0$.