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Rigidity, universality, and hyperbolicity of renormalization for critical circle maps with non-integer exponents

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Abstract. We construct a renormalization operator which acts on analytic circle maps whose critical exponent α is not necessarily an odd integer $2n+1$, $n \in \mathbb{N}$. When $\alpha = 2n+1$, our definition generalizes cylinder renormalization of analytic critical circle maps by Yampolsky [Hyperbolicity of renormalization of critical circle maps. *Publ. Math. Inst. Hautes Études Sci.* **96** (2002), 1–41]. In the case when α is close to an odd integer, we prove hyperbolicity of renormalization for maps of bounded type. We use it to prove universality and $C^{1+\alpha}$-rigidity for such maps.

Key words: low-dimensional dynamics, renormalization, universality, rigidity, critical circle map
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1. *Introduction*

Renormalization theory of critical circle maps was developed in the early 1980s to explain universality phenomena in smooth families of circle homeomorphisms with one critical point, the so-called *critical circle maps*. By definition, a critical circle map is a C^3-homeomorphism of the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with a single critical point of an odd integer order $\alpha > 1$. A canonical example is given by *Arnold's family*, which is constructed as follows. For $\theta \in \mathbb{R}$, let

$$A_\theta(x) = x + \theta - \frac{1}{2\pi} \sin 2\pi x.$$

The maps A_θ are analytic orientation-preserving homeomorphisms $\mathbb{R} \to \mathbb{R}$ with critical points of cubic type at integer values of x. Since $A_\theta(x+1) = A_\theta(x) + 1$, it projects to a well-defined critical circle map f_θ. Of course, values of θ which differ by an integer produce the same maps f_θ, so it is natural to consider the parametric family $\theta \mapsto f_\theta$ with $\theta \in \mathbb{T}$.

To give an example of universality phenomena, let us describe the *golden-mean universality*. Let g_θ be a family of critical circle maps with critical exponent α, depending on a parameter $\theta \in \mathbb{T}$ which is smooth in θ and has the property $\partial \tilde{g}_\theta(x)/\partial\theta > 0$ for all $x \in \mathbb{R}$, where $\tilde{g}_\theta : \mathbb{R} \to \mathbb{R}$ is any lift of the family to a smooth family of homeomorphisms of the real line. Arnold's family is an example of such a family with $\alpha = 3$. Fix $\rho_* = (\sqrt{5} - 1)/2$, the inverse golden mean. This irrational number has a continued fraction expansion

$$\rho_* = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

(further on, we will abbreviate such expressions as $[1, 1, 1, \ldots]$ for typographical convenience). The nth convergent of this continued fraction is a rational number

$$p_n/q_n = \underbrace{[1, 1, 1, \ldots, 1]}_{n}.$$

Suppose that θ_* is a parameter for which the rotation number

$$\rho(f_{\theta_*}) = \rho_*.$$

It is not hard to see that there is a sequence of closed intervals I_n consisting of parameters θ for which $\rho(f_\theta) = p_n/q_n$, which converges to the parameter θ_*. The empirical observation in this case is that their lengths

$$|I_n| \sim a\delta^{-n}$$

for some $a > 0$ and $\delta > 1$. Moreover, δ is a universal constant—its value is the same for all families of critical circle maps g_θ with the same critical exponent α (so $\delta = \delta(\alpha)$).

In [**ÖRSS83**] and [**FKS82**], this universality phenomenon was translated into conjectural hyperbolicity of a renormalization transformation $\mathcal{R}$, a certain nonlinear operator acting on the space of commuting pairs, where commuting pairs are pairs of maps closely related to critical circle maps (see §§2.2 and 2.3 for precise definitions). The conjectures were fully developed in the works of Lanford [**Lan86**, **Lan87**, **Lan88**] and became known as *Lanford's program*.

Let us mention here the key steps of Lanford's program, while a detailed discussion can be found in [**Yam02**, **Yam03**]. The program consists of proving the following statements.

(1) *Global horseshoe attractor.* The renormalization operator $\mathcal{R}$ possesses a 'horseshoe' attractor in the space of commuting pairs equipped with a C^r-metric. The action of $\mathcal{R}$ on the attractor is topologically conjugated to the shift on the space of bi-infinite sequences of natural numbers. Moreover, the orbits of commuting pairs with the same irrational rotation number converge to the same orbit in the attractor.

(2) *Hyperbolicity of the attractor.* There exists a neighborhood of the attractor that admits a structure of an infinite-dimensional smooth manifold compatible with

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a C^r-metric. The horseshoe attractor is a hyperbolic invariant set of the renormalization operator with respect to this manifold structure, and the unstable direction at each point of the attractor is one dimensional.

Another important consequence of Lanford's program is *rigidity of critical circle mappings*. The question of rigidity in the context of circle homeomorphisms goes back Poincaré, who showed that every circle homeomorphism with an irrational rotation number is semi-conjugate to a rigid rotation. In the particular case of critical circle maps, Yoccoz [**Yoc84**] proved that any two such maps f_1, f_2 with the same irrational rotation number are topologically conjugated. This result can be thought of as an analog of Denjoy's theorem for circle diffeomorphisms. Later, Yoccoz showed that the conjugacy between critical circle maps f_1 and f_2 is quasisymmetric (see [**dFdM99**] for a modern exposition). Finally, Khanin and Teplinsky [**KT07**] proved that exponential convergence of the orbits of f_1 and f_2 under renormalization implies C^1-smoothness of the conjugacy. Furthermore, if f_1 and f_2 are of bounded combinatorial type (cf. Definition 2.1), then, according to [**dFdM99**], this conjugacy is $C^{1+\beta}$-smooth for some $\beta > 0$.

A key breakthrough in the study of renormalization of one-dimensional dynamical systems in general and of critical circle maps in particular was made by Sullivan [**Sul87**, **Sul92**], who introduced methods of holomorphic dynamics and Teichmüller theory into the subject. Extending the ideas of Sullivan, McMullen [**McM96**], Lyubich [**Lyu99**], de Faria [**dF99**], and de Faria and de Melo [**dFdM99**, **dFdM00**], the second author brought Lanford's program for analytic critical circle maps with $\alpha \in 2\mathbb{N} + 1$ to a successful completion in a series of works [**Yam99**, **Yam01**, **Yam02**, **Yam03**, **Yam17**]. In particular, in [**Yam02**, **Yam03**] he introduced a new renormalization transformation, known as the *cylinder renormalization operator* $\mathcal{R}_{\text{cyl}}$, and then showed that this operator has a hyperbolic horseshoe attractor with a one-dimensional unstable direction.

It is well known that successive renormalizations of a C^3-smooth map with a critical point of an odd order $\alpha > 1$ converge to a certain space of analytic maps (cf. [**dFdM99**]) and the above developments happened in the analytic realm. However, empirical observations of universality were made for families of differentiable homeomorphisms of the circle with the critical point at 0 of the form

$$\psi \circ q_\alpha \circ \phi, \tag{1}$$

where ϕ and ψ are local C^3-diffeomorphisms, $\phi(0) = 0$, and

$$q_\alpha(x) = x|x|^{\alpha-1} \quad \text{where } \alpha > 1 \tag{2}$$

even when α is not an odd integer. It is clear that such maps cannot be analytic at zero, which does not allow us to apply previously known analytic methods for maps of this type.

The above discussion naturally suggests the following definition: a homeomorphism $f : \mathbb{T} \to \mathbb{T}$ is called an *analytic critical circle map with critical exponent* α if it is analytic everywhere except possibly the critical point, in the neighborhood of which f can be represented in the form (1), where ϕ and ψ are analytic.

In this paper we complete Lanford's program for analytic critical circle maps of bounded combinatorial type with values of the critical exponent sufficiently close to odd integers greater than 1. In particular, we:

(1) extend the definition of renormalization operator $\mathcal{R}_{\mathrm{cyl}}$ in a suitable fashion to a space of maps with critical points of the type (1);

(2) extend the hyperbolicity results of [**Yam02**, **Yam03**] to maps with critical points of non-integer order, sufficiently close to odd integers greater than 1, and with bounded combinatorics;

(3) prove existence of a *global* horseshoe attractor for renormalizations of such maps.

As a corollary of the above results, we prove:

(4) universality for families of analytic critical circle maps with values of critical exponents sufficiently close to odd integers greater than 1;

(5) $C^{1+\beta}$-rigidity for such maps of bounded combinatorial type.

Let us mention that a very different approach to questions similar to (1)–(2) for renormalization of unimodal maps was suggested in [**CS**].

Before formulating our results, let us give a few useful definitions. Suppose that $\mathbf{B}$ is a complex Banach space whose elements are functions of the complex variable. Following the notation of [**Yam02**], let us say that the *real slice* of $\mathbf{B}$ is the real Banach space $\mathbf{B}^{\mathbb{R}}$ consisting of the real-symmetric elements of $\mathbf{B}$. If $\mathbf{X}$ is a Banach manifold modeled on $\mathbf{B}$ with the atlas $\{\Psi_\gamma\}$, we shall say that $\mathbf{X}$ is *real-symmetric* if $\Psi_{\gamma_1} \circ \Psi_{\gamma_2}^{-1}(U) \subset \mathbf{B}^{\mathbb{R}}$ for any pair of indices γ_1, γ_2 and any open set $U \subset \mathbf{B}^{\mathbb{R}}$ for which the above composition is defined. The *real slice of* $\mathbf{X}$ is then defined as the real Banach manifold $\mathbf{X}^{\mathbb{R}} = \bigcup_\gamma \Psi_\gamma^{-1}(\mathbf{B}^{\mathbb{R}}) \subset \mathbf{X}$ with an atlas $\{\Psi_\gamma\}$. An operator A defined on a subset $Y \subset \mathbf{X}$ is *real-symmetric* if $A(Y \cap \mathbf{X}^{\mathbb{R}}) \subset \mathbf{X}^{\mathbb{R}}$.

We prove the following theorems.

THEOREM 1.1. *For every positive integer $k > 0$, there exists a real-symmetric analytic Banach manifold $\mathbf{N}^{2k+1}$ containing a disjoint union of sets $\mathbf{C}^\alpha$, parameterized by $\alpha \in (2k+1-\epsilon, 2k+1+\epsilon)$, $\epsilon = \epsilon(k) > 0$, so that for each such α, the set $\mathbf{C}^\alpha \cap (\mathbf{N}^{2k+1})^{\mathbb{R}}$ is non-empty and consists of analytic critical circle maps with critical exponent α. The cylinder renormalization operator $\mathcal{R}_{\mathrm{cyl}}$ extends to a real-symmetric analytic operator on an open subset of $\mathbf{N}^{2k+1}$ and, if $f \in \mathbf{C}^\alpha$ for some parameter α and $\mathcal{R}_{\mathrm{cyl}}(f)$ is defined, then $\mathcal{R}_{\mathrm{cyl}}(f) \in \mathbf{C}^\alpha$.*

THEOREM 1.2. *For every positive integer $k > 0$, there exist a real-symmetric analytic Banach manifold $\mathbf{M}^{2k+1}$, an open interval $I_k \subset \mathbb{R}$ such that $2k+1 \in I_k$, and a family of real-symmetric analytic maps $i_\alpha : \mathbf{M}^{2k+1} \to \mathbf{C}^\alpha \subset \mathbf{N}^{2k+1}$ that analytically depend on $\alpha \in I_k$. In addition to that, there exists a family of real-symmetric analytic operators $\mathcal{R}_\alpha$ that analytically depend on $\alpha \in I_k$ and are defined on a certain open subset of $\mathbf{M}^{2k+1}$, so that the diagram*

$$\begin{array}{ccc} \mathbf{M}^{2k+1} & \xrightarrow{\mathcal{R}_\alpha} & \mathbf{M}^{2k+1} \\ \downarrow{i_\alpha} & & \downarrow{i_\alpha} \\ \mathbf{C}^\alpha & \xrightarrow{\mathcal{R}_{\mathrm{cyl}}} & \mathbf{C}^\alpha \end{array}$$

commutes whenever all maps are defined. Moreover, for any positive integer $B \geq 1$, the operator $\mathcal{R}_{2k+1}$ has a hyperbolic horseshoe attractor of type bounded by B with a one-dimensional unstable direction.

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The manifolds $\mathbf{M}^{2k+1}$ are 'large': the image $i_{2k+1}(\mathbf{M}^{2k+1})$ contains an open neighborhood of the renormalization horseshoe attractor in the space of critical cylinder maps with the exponent $2k+1$ constructed in [**Yam02**]. Furthermore, in a forthcoming paper, we will demonstrate that renormalizations of smooth circle homeomorphisms with a unique singularity of the form (1) converge to $i_\alpha(\mathbf{M}^{2k+1})$.

It is worth pointing out that the maps $i_\alpha : \mathbf{M}^{2k+1} \to \mathbf{C}^\alpha$ are not injective when α is rational.

As a consequence of Theorem 1.2, we derive the following result.

THEOREM 1.3. (Renormalization hyperbolicity) *For any pair of integers $B, k \geq 1$, there exists $\varepsilon > 0$ such that for any $\alpha \in (2k+1-\varepsilon, 2k+1+\varepsilon)$, the operator $\mathcal{R}_\alpha$ has a hyperbolic horseshoe attractor of type bounded by B with a one-dimensional unstable direction.*

Hyperbolicity of the attractors in all of the above theorems is understood in the sense of Definition 2.20, given in the next section.

In order to simplify the notation, we will provide proofs of Theorems 1.1–1.3 only for the case when $k = 1$. For $k > 1$, the proofs are identical, so for the remaining part of the paper we assume that in the above theorems the parameter k is always set to $k = 1$.

The above theorems establish the local hyperbolic structure of the horseshoe attractor of renormalization. What follows next are the global results.

For every $\alpha > 1$, we naturally define the space $\mathcal{A}^\alpha$ of analytic commuting pairs with the critical exponent α, so that renormalizations of analytic critical circle maps with singularity of the form (1) belong to $\mathcal{A}^\alpha$ (cf. Definitions 9.1 and 9.3). Our first result is the following.

THEOREM 1.4. (Global renormalization attractor) *For every $k, B \in \mathbb{N}$, there exists an open interval $J = J(k, B) \subset \mathbb{R}$ such that $2k+1 \in J$ and, for every $\alpha \in J$, the renormalization operator restricted to $\mathcal{A}^\alpha$ has a global horseshoe attractor $\mathcal{I}_B^\alpha \subset \mathcal{A}^\alpha$ of type bounded by B. Furthermore, renormalizations of any two commuting pairs $\zeta_1, \zeta_2 \in \mathcal{A}^\alpha$ with the same irrational rotation number of type bounded by B converge exponentially in C^r-metric for every non-negative r.*

In §9 we formulate and prove an expanded version of this theorem (cf. Theorem 9.6).

We will say that $\{f_t \mid t \in (-1, 1)\}$ is a C^1-smooth one-parameter family of analytic critical circle maps with critical exponent α if there exist neighborhoods $V_2 \subset V_1 \subset \mathbb{C}/\mathbb{Z}$ such that $\mathbb{T} \subset V_1$, $0 \in V_2$, and, for every t, the critical circle map f_t extends to an analytic function in $V_1 \setminus \{\operatorname{Re} z = 0\}$ and to an analytic (multiple-valued) function in $V_2 \setminus \{0\}$, where it can be represented as $f_t = \psi_t \circ q_\alpha \circ \phi_t$ for some conformal maps ϕ_t and ψ_t defined on V_2 and $q_\alpha(\phi_t(V_2))$, respectively. Furthermore, ϕ_t and ψ_t are required to be C^1-smooth in t and $\phi_t(0) = 0$.

As a first consequence of Theorem 1.4, we obtain a universality statement.

THEOREM 1.5. (Universality) *We adopt the notation of Theorem 1.4. Let ρ be an irrational rotation number of type bounded by B which is periodic under the Gauss map with period p:*

$$\rho = [r_0, r_1, \ldots, r_{p-1}, r_0, \ldots, r_{p-1}, \ldots].$$

There exists a positive integer $N = N(k)$ with the property that for every $\alpha \in J(k, B)$, there exists $\delta = \delta(\rho, \alpha) > 1$ such that the following holds. Suppose that $\{f_t \mid t \in (-1, 1)\}$ is a C^1-smooth one-parameter family of analytic critical circle maps with critical exponent α such that:

- *for every $x \in \mathbb{T}$, the derivative $(\partial/\partial t) f_t(x) > 0$;*
- *the rotation number $\rho(f_0) = \rho$.*

Let $I_m \to 0$ denote the sequence of closed intervals of parameters t such that the first Npm numbers in the continued fraction of $\rho(f_t)$ coincide with those of ρ. Then

$$|I_m| \sim a\delta^{-m} \quad \textit{for some } a > 0.$$

Furthermore, we prove the following rigidity result.

THEOREM 1.6. (Rigidity) *Every two analytic critical circle maps with the same irrational rotation number of type bounded by B and with the same critical exponent $\alpha \in \bigcup_{k \in \mathbb{N}} J(k, B)$ are $C^{1+\beta}$-conjugate, where $\beta > 0$ depends on B and α.*

Having stated the main results, let us also highlight several conceptual issues that we handled in the paper. In the case of analytic critical circle maps (that is, when $k \in \mathbb{N}$) there are two complementary approaches for defining renormalization. The 'classical' one, going back to [**ÖRSS83**, **FKS82**], is done in the language of commuting pairs (see Definition 2.2 below). In contrast, cylinder renormalization defined in [**Yam02**] acts on analytic maps of an annulus, which restrict to critical circle maps (critical cylinder maps). Neither the classical definition of an analytic critical commuting pair nor the definition of a critical cylinder map makes sense for the case when the singularity is of type (1) with $\alpha \notin 2\mathbb{N} + 1$, since in this situation the iterates of our circle map cannot be analytically continued to a neighborhood of the origin. We finesse this difficulty in §9 for the case of commuting pairs and in §3 for critical cylinder maps. Moreover, we give a new proof of the existence of fundamental domains for cylinder renormalization which works for locally analytic maps and does not require any global structure. Finally, in §6 we introduce a new framework which bridges the gap between the two definitions of renormalization and allows us to extend renormalization hyperbolicity results to $\alpha \notin 2\mathbb{N} + 1$.

The structure of the paper is as follows. In §2 we recall the relevant facts of renormalization theory of critical circle maps. The reader can find a more detailed introduction in [**Yam02**]. In §3 we introduce the functional spaces that are used in the construction of Banach manifolds from Theorems 1.1 and 1.2 and discuss their properties. In §4 we lay the ground for extending cylinder renormalization of [**Yam02**] to maps with non-odd-integer critical exponents and generalize the results of [**Yam02**] on the existence of fundamental crescents. In §5 we give a generalized definition of cylinder renormalization. Then in §6 we construct the family of renormalizations $\mathcal{R}_\alpha$ acting on $\mathbf{M}^{2k+1}$. In §7 we extend the renormalization hyperbolicity results of [**Yam02**, **Yam03**] to a convenient setting. In §8 we complete the proof of renormalization hyperbolicity for α close to odd integers and derive Theorems 1.1–1.3. Finally, in §9 we prove the global renormalization convergence, universality, and rigidity theorems (Theorems 1.4–1.6).

2. *Preliminaries*

2.1. *Critical circle maps.* To fix our ideas, we will always assume that the critical point of a critical circle map f is placed at $0 \in \mathbb{T}$.

Being a homeomorphism of the circle, a critical circle map has a well-defined rotation number $\rho(f)$. It is useful to represent $\rho(f)$ as a finite or infinite continued fraction

$$\rho(f) = [r_0, r_1, r_2, \ldots] \quad \text{with } r_i \in \mathbb{N}.$$

Note that an irrational number has a unique expansion as a continued fraction, so the following definition makes sense.

Definition 2.1. We will say that an irrational rotation number $\rho(f) = [r_0, r_1, r_2, \ldots]$ (or f itself) is of a type bounded by a positive constant B if $\sup_{n \geq 0} r_n \leq B$.

For convenience of representing rational rotation numbers, let us add the symbol ∞ with the convention $1/\infty = 0$. Then $0 = [\infty]$. For a non-zero rational rotation number $\rho(f)$, we will use the unique finite continued fraction expansion

$$\rho = [r_0, r_1, \ldots, r_m, \infty]$$

specified by the requirement $r_m > 1$. For a rotation number $\rho(f) = [r_0, r_1, r_2, \ldots]$ whose continued fraction contains at least $m + 1$ terms, we denote by p_m/q_m the mth convergent $p_m/q_m = [r_0, \ldots, r_{m-1}]$ of $\rho(f)$, written in the irreducible form.

If $f^{q_m}(0) \neq 0$, then we let

$$I_m = [0, f^{q_m}(0)] \tag{3}$$

denote the arc of the circle which does not contain $f^{q_m+1}(0)$. We note that $f^{q_m}(0)$ is a *closest return of* 0, that is, I_m contains no iterates $f^k(0)$ with $k < q_m$.

2.2. *Commuting pairs.* For two points $a, b \in \mathbb{R}$, by $[a, b]$ we will denote the closed interval with end points a, b without specifying their order.

Definition 2.2. Let $a, b \in \mathbb{R}$ be two real numbers, one of which is positive and the other negative, and consider two intervals $I_\eta = [a, 0]$ and $I_\xi = [b, 0]$. A commuting pair of class C^r, $r \geq 3$ (C^∞, analytic) acting on the intervals I_η, I_ξ is an ordered pair of maps $\zeta = (\eta, \xi)$, $\eta : I_\eta \to [a, b]$, $\xi : I_\xi \to [a, b]$ with the corresponding smoothness, such that the following properties hold:

(i) $a = \xi(0)$ and $b = \eta(0)$;

(ii) there exist C^3-smooth (C^∞, analytic) extensions $\hat{\eta} : \hat{I}_\eta \to \mathbb{R}$ and $\hat{\xi} : \hat{I}_\xi \to \mathbb{R}$ of η and ξ respectively to some intervals $\hat{I}_\eta \Supset I_\eta$ and $\hat{I}_\xi \Supset I_\xi$ such that $\hat{\eta}$ and $\hat{\xi}$ are orientation-preserving homeomorphisms of $\hat{I}_\eta$ and $\hat{I}_\xi$, respectively, onto their images and

$$\hat{\eta} \circ \hat{\xi} = \hat{\xi} \circ \hat{\eta},$$

where both compositions are defined;

(iii) $\hat{\eta}'(x) \neq 0$ for all $x \in \hat{I}_\eta \setminus \{0\}$, $\hat{\xi}'(y) \neq 0$ for all $y \in \hat{I}_\xi \setminus \{0\}$, and 0 is a cubic critical point for both maps;

(iv) $\xi \circ \eta(0) \in I_\eta$.

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Definition 2.3. An analytic commuting pair $\zeta = (\eta, \xi)$ will be called a *critical commuting pair*.

Given a commuting pair $\zeta = (\eta, \xi)$, we can identify the neighborhoods of the points $\eta(0)$ and $\xi \circ \eta(0)$ by the map $\hat{\xi}$ from property (ii) of Definition 2.2. As a result of this identification, the interval $[\eta(0), \xi \circ \eta(0)]$ is glued into a smooth one-dimensional compact manifold M diffeomorphic to a circle. It also follows from property (ii) of Definition 2.2 that the map

$$f_\zeta(x) = \begin{cases} \eta \circ \xi(x) & \text{if } x \in [\eta(0), 0], \\ \eta(x) & \text{if } x \in [\xi \circ \eta(0), 0] \end{cases}$$

projects to a smooth homeomorphism F_ζ of M to itself that has a unique critical point at the origin. Notice that there are many ways of putting an affine structure on the manifold M, which gives rise to the whole conjugacy class (the smoothness of the conjugacy is the same as that of the commuting pair) of critical circle maps that are conjugate to F_ζ.

2.3. *Renormalization of commuting pairs.*

Definition 2.4. The *height* $\chi(\zeta)$ of a commuting pair $\zeta = (\eta, \xi)$ is the positive integer r such that

$$0 \in [\eta^r(\xi(0)), \eta^{r+1}(\xi(0))).$$

If no such r exists, we set $\chi(\zeta) = \infty$.

If $\chi(\zeta) = r \neq \infty$ and $\eta^r(\xi(0)) \neq 0$, one can verify that

$$\hat{\zeta} = (\eta^r \circ \xi, \eta)$$

is a commuting pair acting on the intervals I_ξ, $[\eta^r(\xi(0)), 0]$. It is known as the *pre-renormalization* of ζ and denoted by $p\mathcal{R}\zeta$.

Definition 2.5. If $\chi(\zeta) = r \neq \infty$ and $\eta^r(\xi(0)) \neq 0$, then the renormalization $\mathcal{R}\zeta$ of a commuting pair $\zeta = (\eta, \xi)$ is defined to be the affine rescaling of the commuting pair $\hat{\zeta} = p\mathcal{R}\zeta$:

$$\mathcal{R}\zeta = (h \circ \eta^r \circ \xi \circ h^{-1}, h \circ \eta \circ h^{-1}),$$

where $h(x) = x/\eta(0)$. It is a commuting pair acting on the intervals $[0, 1]$, $h([\eta^r(\xi(0)), 0])$.

Definition 2.6. For a commuting pair ζ, define its *rotation number* $\rho(\zeta) \in [0, 1]$ to be equal to the continued fraction $[r_0, r_1, \ldots]$, where $r_n = \chi(\mathcal{R}^n \zeta)$. Here, as before, $1/\infty$ is understood as 0.

It is easy to see that the renormalization operator acts as a Gauss map on rotation numbers: if $\rho(\zeta) = [r, r_1, r_2, \ldots]$, then $\rho(\mathcal{R}\zeta) = [r_1, r_2, \ldots]$.

The following proposition can be easily verified.

PROPOSITION 2.7. *For a commuting pair ζ, the rotation number of the map F_ζ from §2.2 is equal to $\rho(\zeta)$.*

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2.4. *From critical circle maps to commuting pairs.* Consider a critical circle map f with rotation number $\rho(f) = [r_0, r_1, \ldots]$ whose continued fraction expansion contains at least $m+2$ terms. As before, set $p_m/q_m = [r_0, \ldots, r_{m-1}]$ and $p_{m+1}/q_{m+1} = [r_0, \ldots, r_m]$ and let $I_m = [0, f^{q_m}]$ and $I_{m+1} = [0, f^{q_{m+1}}]$ be as in (3). At the cost of a minor abuse of the notation, identify these intervals with their lifts to $(-1, 1) \subset \mathbb{R}$. The pair of maps

$$\zeta_m = (f^{q_{m+1}}, f^{q_m})$$

is a commuting pair acting on the intervals I_m, I_{m+1}. Note that, if for a given critical circle map f, the corresponding commuting pairs ζ_m and ζ_{m+1} are defined, then ζ_{m+1} is the pre-renormalization of ζ_m. This motivates the following definition.

Definition 2.8. For a critical circle map f and a positive integer m as above, we denote

$$p\mathcal{R}^m f \equiv \zeta_m$$

and we define $\mathcal{R}^m f$ to be the affine rescaling of the commuting pair ζ_m:

$$\mathcal{R}^m f = (h \circ f^{q_{m+1}} \circ h^{-1}, h \circ f^{q_m} \circ h^{-1}),$$

where $h(x) = x/f^{q_m}(0)$.

2.5. *Epstein class.* Given an open (possibly unbounded) interval $J \subset \mathbb{R}$, let $\mathbb{C}_J = \mathbb{C} \setminus (\mathbb{R} \setminus J)$ denote the plane slit along two rays. We will say that a *topological disk* is an open simply connected (not necessarily bounded) region in $\mathbb{C}$.

Definition 2.9. For $a \in \mathbb{R}$, $a \neq 0$, let I denote the interval $I = [0, a]$. The *Epstein class* $\mathbf{E}_I$ consists of all maps $g : I \to \mathbb{R}$ such that g is an orientation-preserving homeomorphisms of I onto its image $g(I) = J$ and there exists an open interval $\tilde{J} \supset J$ with the property that g extends to an analytic three-fold branched covering map of a topological disk $G \supset I$ onto the double-slit plane $\mathbb{C}_{\tilde{J}}$ with a single critical point at 0. Let $\mathbf{E}$ denote the union $\mathbf{E} = \bigcup_I \mathbf{E}_I$ over all non-degenerate intervals I.

Any map $g \in \mathbf{E}$ in the Epstein class can be decomposed as

$$g = Q_c \circ h, \tag{4}$$

where $Q_c(z) = z^3 + c$ and $h : I \to \mathbb{R}$ is an analytic map with $h(0) = 0$ that extends to a biholomorphism between G and the complex plane with six slits, which triple covers $\mathbb{C}_{\tilde{J}}$ under the cubic map $Q_c(z)$. In what follows, we will always assume that for an Epstein map $g : I \to \mathbb{R}$, and $J = g(I)$, the interval $\tilde{J} \supset J$ is the maximal (possibly unbounded) open interval such that g extends to a triple-branched covering of a topological disk G onto $\mathbb{C}_{\tilde{J}}$.

We say that two positive constants α and β are *K-commensurable* for some $K > 1$ if

$$K^{-1}\alpha \leq \beta \leq K\alpha.$$

Definition 2.10. For any $s \in (0, 1)$ and an interval $I = [0, a] \subset \mathbb{R}$, let $\mathbf{E}_{I,s} \subset \mathbf{E}_I$ be the set that consists of all $g \in \mathbf{E}_I$ such that both $|I|$ and $\mathrm{dist}(I, J)$ are s^{-1}-commensurable with $|J|$, the length of each component of $\tilde{J} \setminus J$ is at least $s|J|$, and $g'(a) \geq s$.

We recall that the distortion of a conformal map f in a domain $U \subset \mathbb{C}$ is defined as

$$\text{Distortion}_U(f) = \sup_{z,w\in U} \frac{|f'(z)|}{|f'(w)|}.$$

The following lemma was proved in [**Yam01**].

LEMMA 2.11. [**Yam01**, Lemma 2.13] *For any* $s \in (0, 1)$ *and an interval* $I = [0, a] \subset \mathbb{R}$, *there exists a domain* $O_{I,s} \supset I$ *such that for any* $g \in \mathbf{E}_{I,s}$, *the univalent map* h *in (4) is well defined in* $O_{I,s}$ *and has* $K(s)$*-bounded distortion in* $O_{I,s}$.

We will often refer to the space $\mathbf{E}$ as *the* Epstein class, and to each $\mathbf{E}_{I,s}$ as *an* Epstein class.

We say that a critical commuting pair $\zeta = (\eta, \xi)$ belongs to the (an) Epstein class if both of its maps do. Allowing some abuse of notation, we will denote the space of all commuting pairs from the Epstein class by $\mathbf{E}$. Similarly, for any $s \in (0, 1)$, by $\mathbf{E}_s$ we will denote the space of all commuting pairs (η, ξ) such that $\eta \in \mathbf{E}_{[0,1],s}$ and $\xi \in \mathbf{E}_{I,s}$, where $I = [0, \eta(0)]$. It immediately follows from the definitions that the following lemma holds.

LEMMA 2.12. *If a renormalizable commuting pair* ζ *is in the Epstein class, then the same is true for* $\mathcal{R}\zeta$.

The following lemma is a consequence of real *a priori* bounds (cf. [**dFdM99**]).

LEMMA 2.13. *For any* $s \in (0, 1)$, *there exist positive constants* $C > 1$ *and* $\lambda < 1$ *such that if* $\zeta \in \mathbf{E}_s$ *is* k*-times renormalizable and* $I^k_\zeta = I^k_\eta \cup I^k_\xi$ *denotes the domain of the commuting pair* $p\mathcal{R}^k\zeta$, *then*

$$|I^k_\zeta| < C\lambda^k.$$

2.6. *Carathéodory convergence.* Consider the set of all triples (U, u, f), where $U \subset \mathbb{C}$ is a topological disk, $u \in U$ is a marked point in U, and $f : U \to \mathbb{C}$ is an analytic map. Recall (cf. [**McM94**]) that the *Carathédory convergence* on this set is defined in the following way: $(U_n, u_n, f_n) \to (U, u, f)$ if and only if:

- $u_n \to u \in U$;
- for any Hausdorff limit point K of the sequence $\hat{\mathbb{C}}\backslash U_n$, U is the component of $\hat{\mathbb{C}}\backslash K$ containing u; and
- for all n sufficiently large, f_n converges to f uniformly on compact subsets of U.

For a topological disk $U \subset \mathbb{C}$, $U \neq \mathbb{C}$, and $u \in U$, let $R_{(U,u)} : \mathbb{D} \to U$ denote the inverse Riemann mapping with normalization $R_{(U,u)}(0) = u$, $R'_{(U,u)}(0) > 0$. By a classical result of Carathéodory, if neither U nor U_n is equal to $\mathbb{C}$, then the Carathédory convergence $(U_n, u_n, f_n) \to (U, u, f)$ is equivalent to the simultaneous convergence $R_{(U_n,u_n)} \to R_{(U,u)}$ and $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$ and U, respectively.

As before, for an Epstein map $g \in \mathbf{E}$, let $\tilde{J}$ be the maximal (possibly unbounded) open interval such that g extends to a triple-branched covering of a topological disk G onto $\mathbb{C}_{\tilde{J}}$. Notice that the pair $(g, \tilde{J})$ uniquely determines the topological disk G and hence we can associate to any $g \in \mathbf{E}$ the triple $(G, 0, g)$. In this way, the space $\mathbf{E}$ of all Epstein maps can be equipped with Carathéodory convergence.

Let us make a note of an important compactness property of $\mathbf{E}_{I,s}$.

LEMMA 2.14. [**Yam01**, Lemma 2.12] *For any $s \in (0, 1)$ and an interval $I = [0, a]$, where $a \in \mathbb{R}$, $a \neq 0$, the set $\mathbf{E}_{I,s}$ is sequentially compact with respect to Carathéodory convergence.*

The space of all commuting pairs from the Epstein class can be equipped with Carathéodory convergence defined as Carathéodory convergence of each of the maps forming the commuting pair. Now Lemma 2.14 implies the following statement.

PROPOSITION 2.15. *For any $s \in (0, 1)$, the set $\mathbf{E}_s$ of critical commuting pairs is sequentially compact with respect to Carathéodory convergence.*

Proof. By definition, the set $\mathbf{E}_s$ consists of all critical commuting pairs (η, ξ) such that $\eta \in \mathbf{E}_{[0,1],s}$ and $\xi \in \mathbf{E}_{I,s}$ for some interval I. According to Lemma 2.14, the set $\mathbf{E}_{[0,1],s}$ is sequentially compact. Since $\xi(0) = 1$, it follows from the definition of the class $\mathbf{E}_{I,s}$ that the number $|I|$ is $K(s)$-commensurable with 1, where $K(s)$ is some constant that depends on s. This implies that the set of all maps ξ such that $(\eta, \xi) \in \mathbf{E}_s$ is also sequentially compact. The rest of the proof follows easily. □

2.7. *Renormalization horseshoe.* The following theorem was proved by the second author in [**Yam01**]. It generalizes the results of de Faria [**dF99**] and de Faria and de Melo [**dFdM00**] for pairs of a bounded type. Consider the space of bi-infinite sequences

$$\Sigma = \{(\ldots, r_{-k}, \ldots, r_{-1}, r_0, r_1, \ldots, r_k, \ldots) \text{ with } r_i \in \mathbb{N}\}$$

equipped with the weak topology—the coarsest topology in which the coordinate projections are continuous. Denote by $\sigma : \Sigma \to \Sigma$ the right shift on this space:

$$\sigma : (r_k)_{-\infty}^{\infty} \mapsto (r_{k+1})_{-\infty}^{\infty}.$$

THEOREM 2.16. *There exist $s > 0$ and an $\mathcal{R}$-invariant set $\mathcal{I} \subset \mathbf{E}_s$ consisting of commuting pairs with irrational rotation numbers with the following properties. The action of $\mathcal{R}$ on $\mathcal{I}$ is topologically conjugate to the shift $\sigma : \Sigma \to \Sigma$:*

$$\iota \circ \mathcal{R} \circ \iota^{-1} = \sigma,$$

where $\iota : \mathcal{I} \to \Sigma$ is the conjugacy and, if

$$\zeta = \iota^{-1}(\ldots, r_{-k}, \ldots, r_{-1}, r_0, r_1, \ldots, r_k, \ldots),$$

then

$$\rho(\zeta) = [r_0, r_1, \ldots, r_k, \ldots].$$

The set $\mathcal{I}$ is sequentially pre-compact in the sense of Carathéodory convergence: its closure $\overline{\mathcal{I}} \subset \mathbf{E}$ is the attractor for renormalization. That is, for any analytic commuting pair $\zeta \in \mathbf{E}$ with irrational rotation number, we have

$$\mathcal{R}^n \zeta \to \overline{\mathcal{I}}$$

in the sense of Carathéodory convergence. Moreover, for any two commuting pairs $\zeta, \zeta' \in \mathbf{E}$ with equal irrational rotation numbers $\rho(\zeta) = \rho(\zeta')$, we have

$$\operatorname{dist}(\mathcal{R}^n \zeta, \mathcal{R}^n \zeta') \to 0 \tag{5}$$

for the uniform distance between analytic extensions of commuting pairs on compact sets.

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Remark 2.17. In [**dFdM00**], de Faria and de Melo extended the result of Theorem 2.16 by proving the convergence (5) for all (not necessarily Epstein) critical commuting pairs ζ, ζ' with equal irrational rotation numbers $\rho(\zeta) = \rho(\zeta')$. They also showed that for any integer $k \geq 0$, this convergence is exponential in C^k-metric on the real line. In addition to that, it was shown in [**GdM17**] and [**GMdM18**] that such exponential convergence in C^2-metric also holds for C^4-smooth commuting pairs in general and C^3-smooth commuting pairs with bounded combinatorics.

Furthermore, in [**Yam02**, **Yam03**], it was shown that the above attractor is hyperbolic in a suitable sense. More specifically, in [**Yam02**], the second author constructed a real-symmetric Banach manifold $\mathbf{W}$ of *critical cylinder maps* such that the elements of $\mathbf{W}^{\mathbb{R}}$ are analytic critical circle maps. Furthermore, he constructed a continuous projection $\pi_{\mathbf{W}}$ from an open neighborhood of $\mathcal{I}$ in $\mathbf{E}$ to $\mathbf{W}$, which, on the image, semi-conjugates $\mathcal{R}^k$ for a fixed $k \in \mathbb{N}$ to a real-symmetric analytic operator $\hat{\mathcal{R}}_{\mathrm{cyl}} : \mathcal{V} \to \mathbf{W}$ called the *cylinder renormalization* and defined on an open set $\mathcal{V}$ in $\mathbf{W}$:

$$\pi_{\mathbf{W}} \circ \mathcal{R}^k = \hat{\mathcal{R}}_{\mathrm{cyl}} \circ \pi_{\mathbf{W}}.$$

The projection $\pi_{\mathbf{W}}$ also satisfies the following properties.

PROPOSITION 2.18.

(i) *If $\pi_{\mathbf{W}}(\zeta_1) = \pi_{\mathbf{W}}(\zeta_2)$ for some ζ_1, ζ_2 from a neighborhood of $\mathcal{I}$ in $\mathbf{E}$, then the Epstein pairs $\mathcal{R}\zeta_1$ and $\mathcal{R}\zeta_2$ are conformally conjugate in a neighborhood of their intervals of definition.*

(ii) *Conversely, suppose that ζ_1 and ζ_2 are conformally conjugate in some neighborhoods of their intervals of definition and $\rho(\zeta_1) = \rho(\zeta_2) \notin \mathbb{Q}$. Then there exists $n \geq 0$ such that $\pi_{\mathbf{W}}(\mathcal{R}^n\zeta_1) = \pi_{\mathbf{W}}(\mathcal{R}^n\zeta_2)$.*

(iii) *The rotation number of the critical circle map $\pi_{\mathbf{W}}(\zeta)$ is equal to the image of $\rho(\zeta)$ under the Gauss map.*

Finally, we have the following.

THEOREM 2.19. [**Yam03**] *Denote $\hat{\mathcal{I}} = \pi_{\mathbf{W}}(\mathcal{I})$. Then the attractor $\hat{\mathcal{I}} \subset \mathcal{V}$ is a uniformly hyperbolic set for $\hat{\mathcal{R}}_{\mathrm{cyl}}$ with a complex one-dimensional unstable direction.*

It is important to note that the operator $\hat{\mathcal{R}}_{\mathrm{cyl}}$ is not injective on $\mathcal{V}$: the images under $\hat{\mathcal{R}}_{\mathrm{cyl}}$ of two analytic maps which are conformally conjugate in a sufficiently large neighborhood (see Proposition 2.18 or [**Yam02**] for details) coincide. Uniform hyperbolicity of $\hat{\mathcal{I}}$ is understood in the sense of Definition 2.20 below.

Before proceeding to Definition 2.20, we need to introduce the following notation: let U be an open set in a smooth Banach manifold $\mathbf{M}$. Consider a C^∞-smooth (not necessarily injective) mapping $f : U \to \mathbf{M}$ that possesses a forward invariant set $\Lambda \subset U$. Let $\hat{\Lambda}$ be the subset of the direct product $\Lambda^{\mathbb{N}}$ that consists of all histories in Λ, i.e.

$$\hat{\Lambda} = \{(x_i)_{i \leq 0} : x_i \in \Lambda;\ f(x_i) = x_{i+1}\}.$$

The metric on $\Lambda^{\mathbb{N}}$ is defined by

$$d((x_i), (y_i)) = \sum_{i \leq 0} 2^i \|x_i - y_i\|.$$

The restriction $f|_\Lambda$ lifts to a homeomorphism $\hat f : \hat\Lambda \to \hat\Lambda$ defined by $\hat f((x_i)) = (x_{i+1})$. The natural projection from $\hat\Lambda$ to Λ sends (x_i) to x_0, and the pullback under this projection of the tangent bundle $T_\Lambda\mathbf{M}$ is a tangent bundle on $\hat\Lambda$. We denote this tangent bundle by $T_{\hat\Lambda}$. Explicitly, an element of $T_{\hat\Lambda}$ is of the form $((x_i), v)$, where $(x_i) \in \hat\Lambda$ and $v \in T_{x_0}\mathbf{M}$. The differential Df naturally lifts to a map $D\hat f : T_{\hat\Lambda} \to T_{\hat\Lambda}$.

Definition 2.20. We say that the map f is hyperbolic on Λ if there exists a continuous splitting of the tangent bundle $T_{\hat\Lambda} = E^s \oplus E^u$ such that $D\hat f(E^{u/s}) = E^{u/s}$ and there exist constants $c > 0$ and $\lambda > 1$ such that for all $n \ge 1$,

$$\|D\hat f^n(v)\| \ge c\lambda^n\|v\|, \quad v \in E^u,$$
$$\|D\hat f^n(v)\| \le c^{-1}\lambda^{-n}\|v\|, \quad v \in E^s.$$

Note that the fiber of the stable bundle E^s at a point $(x_i) \subset \hat\Lambda$ depends only on the point $x_0 \subset \Lambda$.

We also notice that the set $\hat{\mathcal{I}}$ is not compact in $\mathbf{W}$ and its closure is not contained in $\mathcal{V}$. In particular, this means that, *a priori*, the existence of local stable/unstable manifolds is not guaranteed at every point of $\hat{\mathcal{I}}$. However, the following was shown in [**Yam03**].

THEOREM 2.21. *For any $f \in \hat{\mathcal{I}}$, there exists a local stable manifold $W^s(f) \subset \mathcal{V}$ passing through the point f. Moreover, $W^s(f) \cap \mathbf{W}^{\mathbb{R}}$ consists of all critical circle maps g that are sufficiently close to f and such that $\rho(g) = \rho(f)$.*

2.8. *Holomorphic commuting pairs and complex a priori bounds.* De Faria [**dF99**] introduced holomorphic commuting pairs to apply Sullivan's Riemann surface laminations technique to the renormalization of critical circle maps. They are suitably defined holomorphic extensions of critical commuting pairs which replace Douady–Hubbard polynomial-like maps [**DH85**]. A critical commuting pair $\zeta = (\eta|_{I_\eta}, \xi|_{I_\xi})$ extends to a *holomorphic commuting pair* $\mathcal{H}$ if there exist three simply connected $\mathbb{R}$-symmetric domains $D, U, V \subset \mathbb{C}$, whose intersections with the real line are denoted by $I_U = U \cap \mathbb{R}$, $I_V = V \cap \mathbb{R}$, $I_D = D \cap \mathbb{R}$, and a simply connected $\mathbb{R}$-symmetric Jordan domain Δ such that:

- $\bar D, \bar U, \bar V \subset \Delta$; $\bar U \cap \bar V = \{0\} \subset D$; the sets $U\backslash D$, $V\backslash D$, $D\backslash U$, and $D\backslash V$ are non-empty, connected, and simply connected; $I_\eta \subset I_U \cup \{0\}$, $I_\xi \subset I_V \cup \{0\}$;
- the sets $U \cap \mathbb{H}$, $V \cap \mathbb{H}$, $D \cap \mathbb{H}$ are Jordan domains;
- the maps η and ξ have analytic extensions to U and V, respectively, so that η is a conformal diffeomorphism of U onto $(\Delta\backslash\mathbb{R}) \cup \eta(I_U)$ and ξ is a conformal diffeomorphism of V onto $(\Delta\backslash\mathbb{R}) \cup \xi(I_V)$;
- the maps $\eta : U \to \Delta$ and $\xi : V \to \Delta$ can be further extended to analytic maps $\hat\eta : U \cup D \to \Delta$ and $\hat\xi : V \cup D \to \Delta$, so that the map $\nu = \hat\eta \circ \hat\xi = \hat\xi \circ \hat\eta$ is defined in D and is a three-fold branched covering of D onto $(\Delta\backslash\mathbb{R}) \cup \nu(I_D)$ with a unique critical point at zero.

We shall identify a holomorphic pair $\mathcal{H}$ with a triple of maps $\mathcal{H} = (\eta, \xi, \nu)$, where $\eta : U \to \Delta$, $\xi : V \to \Delta$, and $\nu : D \to \Delta$ (cf. Figure 1). We shall also call ζ the *commuting pair underlying* $\mathcal{H}$ and write $\zeta \equiv \zeta_{\mathcal{H}}$. When no confusion is possible, we will use the

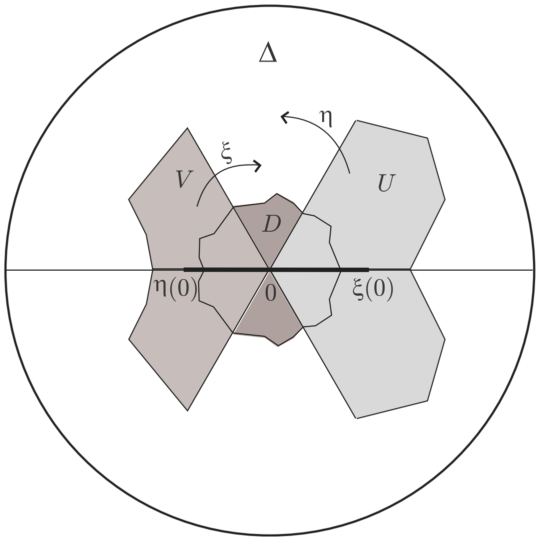

FIGURE 1. A holomorphic commuting pair.

same letters η and ξ to denote both the maps of the commuting pair $\zeta_{\mathcal{H}}$ and their analytic extensions to the corresponding domains U and V.

The sets $\Omega_{\mathcal{H}} = D \cup U \cup V$ and $\Delta \equiv \Delta_{\mathcal{H}}$ will be called *the domain* and *the range* of a holomorphic pair $\mathcal{H}$. We will sometimes write Ω instead of $\Omega_{\mathcal{H}}$ when this does not cause any confusion.

We can associate to a holomorphic pair $\mathcal{H}$ a piecewise-defined map $S_{\mathcal{H}} : \Omega \to \Delta$:

$$S_{\mathcal{H}}(z) = \begin{cases} \eta(z) & \text{if } z \in U, \\ \xi(z) & \text{if } z \in V, \\ \nu(z) & \text{if } z \in \Omega \backslash (U \cup V). \end{cases}$$

De Faria [**dF99**] calls $S_{\mathcal{H}}$ the *shadow* of the holomorphic pair $\mathcal{H}$.

If $\zeta_{\mathcal{H}}$ is renormalizable with height r, then the pre-renormalization of the corresponding holomorphic pair $\mathcal{H}$ is defined in an obvious fashion as a holomorphic commuting pair $p\mathcal{R}(\mathcal{H}) = (\eta^r \circ \xi, \eta, \eta^r \circ \nu)$ with range Δ and the following domains U', V', D':

- $V' = U$;
- $U' = (\xi^{-1} \circ \eta^{-(r-1)})(\tilde{U})$, where $\tilde{U} = (U \backslash \mathbb{R}) \cup (0, (\eta^{r-1} \circ \xi)(0))$;
- $D' = \hat{\eta}^{-1}(U')$, where $\hat{\eta}$ is the analytic extension of η to the domain $D \cup U$ and $\hat{\eta}^{-1}$ denotes the full pre-image.

The renormalization $\mathcal{R}(\mathcal{H})$ is, as usual, the linear rescaling of $p\mathcal{R}(\mathcal{H})$ sending $\eta(0)$ to 1. Clearly, $\mathcal{H}$ is renormalizable if and only if $\zeta_{\mathcal{H}}$ is renormalizable and we have

$$\zeta_{\mathcal{R}(\mathcal{H})} = \mathcal{R}(\zeta_{\mathcal{H}}).$$

We can naturally view a holomorphic pair $\mathcal{H}$ as three triples

$$(U, \xi(0), \eta), \ (V, \eta(0), \xi), \ (D, 0, \nu).$$

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We say that a sequence of holomorphic pairs converges in the sense of Carathéodory convergence if the corresponding triples do. We denote the space of triples equipped with this notion of convergence by **H**.

We let the modulus of a holomorphic commuting pair $\mathcal{H}$, which we denote by $\mathrm{mod}(\mathcal{H})$, to be the modulus of the largest annulus $A \subset \Delta$, which separates $\mathbb{C} \setminus \Delta$ from $\overline{\Omega}$.

Definition 2.22. For $\mu \in (0, 1)$, let $\mathbf{H}(\mu) \subset \mathbf{H}$ denote the space of holomorphic commuting pairs $\mathcal{H} : \Omega_{\mathcal{H}} \to \Delta_{\mathcal{H}}$ with the following properties:

(i) $\mathrm{mod}(\mathcal{H}) \geq \mu$;
(ii) $|I_\eta| = 1$, $|I_\xi| \geq \mu$, and $|\eta^{-1}(0)| \geq \mu$;
(iii) $\mathrm{dist}(\eta(0), \partial V_{\mathcal{H}})/\mathrm{diam}\, V_{\mathcal{H}} \geq \mu$ and $\mathrm{dist}(\xi(0), \partial U_{\mathcal{H}})/\mathrm{diam}\, U_{\mathcal{H}} \geq \mu$;
(iv) the domains $\Delta_{\mathcal{H}}$, $U_{\mathcal{H}} \cap \mathbb{H}$, $V_{\mathcal{H}} \cap \mathbb{H}$, and $D_{\mathcal{H}} \cap \mathbb{H}$ are $(1/\mu)$-quasidisks;
(v) $\mathrm{diam}(\Delta_{\mathcal{H}}) \leq 1/\mu$.

LEMMA 2.23. [**Yam01**, Lemma 2.17] *For each $\mu \in (0, 1)$, the space $\mathbf{H}(\mu)$ is sequentially compact.*

We say that a real commuting pair $\zeta = (\eta, \xi)$ with an irrational rotation number has *complex* a priori *bounds* if there exists $\mu > 0$ such that all renormalizations of $\zeta = (\eta, \xi)$ extend to holomorphic commuting pairs in $\mathbf{H}(\mu)$. The existence of complex *a priori* bounds is a key analytic issue of renormalization theory. Before proceeding with the theorem on complex bounds for critical circle maps, we need to give the following definition.

Definition 2.24. For a set $S \subset \mathbb{C}$ and $r > 0$, we let $N_r(S)$ stand for the r-neighborhood of S in $\mathbb{C}$. For each $r > 0$, we introduce a class $\mathcal{A}_r$ consisting of pairs (η, ξ) such that the following holds:

- η, ξ are real-symmetric analytic maps defined in the domains
$$N_r([0, 1]) \quad \text{and} \quad N_{r|\eta(0)|}([0, \eta(0)]),$$
respectively, and continuous up to the boundary of the corresponding domains;
- the pair
$$\zeta \equiv (\eta|_{[0,1]}, \xi|_{[0,\eta(0)]})$$
is a critical commuting pair.

For simplicity, if ζ is as above, we will write $\zeta \in \mathcal{A}_r$. But it is important to note that viewing our critical commuting pair ζ as an element of $\mathcal{A}_r$ imposes restrictions on where we are allowed to iterate it. Specifically, we view such ζ as undefined at any point $z \notin N_r([0, \xi(0)]) \cup N_r([0, \eta(0)])$ (even if ζ can be analytically continued to z). Similarly, when we talk about iterates of $\zeta \in \mathcal{A}_r$, we iterate the restrictions $\eta|_{N_r([0,\xi(0)])}$ and $\xi|_{N_r([0,\eta(0)])}$. In particular, we say that the first and second elements of $p\mathcal{R}\zeta = (\eta^r \circ \xi, \eta)$ are defined in the maximal domains, where the corresponding iterates are defined in the above sense.

For a domain $\Omega \subset \mathbb{C}$, we denote by $\mathbf{D}(\Omega)$ the complex Banach space of analytic functions in Ω, continuous up to the boundary of Ω and equipped with the sup-norm. Consider the mapping $i : \mathcal{A}_r \to \mathbf{D}(N_r([0, 1])) \times \mathbf{D}(N_r([0, 1]))$ defined by
$$i(\eta, \xi) = (\eta, h \circ \xi \circ h^{-1}) \quad \text{where } h(z) = z/\eta(0). \tag{6}$$

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The map i is injective, since $\eta(0)$ completely determines the rescaling h. Hence, this map induces a metric on $\mathcal{A}_r$ from the direct product of sup-norms on $\mathbf{D}(N_r([0, 1])) \times \mathbf{D}(N_r([0, 1]))$. This metric on $\mathcal{A}_r$ will be denoted by $\text{dist}_r(\cdot, \cdot)$.

THEOREM 2.25. (Complex bounds) *There exists a universal constant $\mu > 0$ such that the following holds. For every positive real number $r > 0$ and every pre-compact family $S \subset \mathcal{A}_r$ of critical commuting pairs, there exists $N = N(r, S) \in \mathbb{N}$ such that if $\zeta \in S$ is a $2n$-times renormalizable commuting pair, where $n \geq N$, then $p\mathcal{R}^n\zeta$ restricts to a holomorphic commuting pair $\mathcal{H}_n : \Omega_n \to \Delta_n$ with $\Delta_n \subset N_r(I_\eta) \cup N_r(I_\xi)$. Furthermore, the range Δ_n is a Euclidean disk and the appropriate affine rescaling of $\mathcal{H}_n$ is in $\mathbf{H}(\mu)$.*

Theorem 2.25 was first proved in [**Yam99**] for critical commuting pairs from the Epstein class. This proof was later adapted by de Faria and de Melo [**dFdM00**] to the case of non-Epstein critical commuting pairs. We note that both in [**Yam99**] and in [**dFdM00**] this theorem was formulated for a single critical circle map (or commuting pair) with an irrational rotation number; however, the uniformity of the estimates in a pre-compact family stated above is evident from the proofs.

3. *The functional spaces*

3.1. *The spaces of generalized analytic critical cylinder maps $\mathbf{C}_r$ and $\mathbf{C}_r^\alpha$.*

According to the standard terminology, an analytic critical circle map is a critical circle map which is analytic at every point of the circle including the critical point. This implies that the critical exponent is an odd integer not smaller than 3. In this section we generalize the class of analytic critical circle maps to include maps with other critical exponents.

For a fixed positive real number $0 < r < 1/2$, let $A_r \subset \mathbb{C}$ be the r-neighborhood of the interval $[0, 1]$ with the points 1 and 0 removed:

$$A_r = \{z \in \mathbb{C} \mid \text{dist}(z, [0, 1]) < r\} \backslash \{0, 1\}.$$

The universal cover $\tilde{A}_r$ of A_r can be identified with the space of all pairs (z, γ) such that $z \in A_r$ and γ is a homotopy class of paths in A_r which begin at $\frac{1}{2}$ and end at z. We define the surface U_r as the subset of the universal cover $\tilde{A}_r$, consisting of all pairs $(z, \gamma) \in \tilde{A}_r$, such that γ has a representative that enters the disks $\mathbb{D}_r(0)$ or $\mathbb{D}_r(1)$ no more than once and lies either entirely to the left or entirely to the right from the vertical line $\text{Re } z = 1/2$. The surface U_r inherits a complex analytic structure as well as the Euclidean distance from A_r via the projection of U_r onto the first coordinate. If f is an analytic function on U_r, then f can be viewed as a multiple-valued analytic function on A_r. We will write $f(z, \gamma)$ for the value of f at $(z, \gamma) \in U_r$ and we will write $f(z)$ when we view f as a multiple-valued function. If γ has a representative that lies in $A_r \backslash ((-\infty, 0) \cup (1, +\infty))$, we will shorten the notation for $f(z, \gamma)$ by writing $f(z, \gamma) = f(z)$ provided that this does not cause any confusion. By $\overline{U}_r$, we denote the completion of U_r with respect to the Euclidean distance.

Definition 3.1. For a positive real number $r > 0$, let $\mathbf{A}_r$ denote the space of all functions $f : \overline{U}_r \to \mathbb{C}$ that are analytic on U_r, continuous on $\overline{U}_r$, and satisfy the identity $f(0) = f(1)$. Equipped with the sup-norm, the space $\mathbf{A}_r$ is a complex Banach space.

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Let $\pi : \mathbb{C} \to \mathbb{C}/\mathbb{Z}$ be the natural projection. Every map $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ naturally lifts to a (non-uniquely defined) map $\tilde{f} : [0, 1] \to \mathbb{C}$ such that

$$\pi \circ \tilde{f} = f \circ \pi. \tag{7}$$

Definition 3.2. Given a positive real number $r > 0$, we define the set of generalized cylinder maps $\mathbf{C}_r$ to be the space of all continuous maps $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ such that any lift of f to a map of $[0, 1] \to \mathbb{C}$ has an analytic extension $\tilde{f} : U_r \to \mathbb{C}$ that is continuous on $\overline{U}_r$ and $\tilde{f}(1) = \tilde{f}(0) + 1$.

Let $V_r \subset \mathbb{C}/\mathbb{Z}$ be an equatorial neighborhood defined by

$$V_r \equiv \{z \in \mathbb{C} : |\mathrm{Im}\, z| < r\}/\mathbb{Z} \subset \mathbb{C}/\mathbb{Z}. \tag{8}$$

Let $f \in \mathbf{C}_r$ and let $\tilde{f} : U_r \to \mathbb{C}$ be as in Definition 3.2. Pushing $\tilde{f}$ down by the covering map $\pi : \mathbb{C} \to \mathbb{C}/\mathbb{Z}$, we obtain a multiple-valued map from V_r to $\mathbb{C}/\mathbb{Z}$, which restricts to f on the circle. In what follows, we will identify f with this particular multiple-valued analytic continuation. Whenever we iterate the multi-valued map f, we will specify the choice of the branches.

PROPOSITION 3.3. *For any positive real number $r > 0$, the set $\mathbf{C}_r$ has a structure of an affine complex Banach manifold modeled on $\mathbf{A}_r$.*

Proof. For any map $\tilde{f} : U_r \to \mathbb{C}$ which is a lift of a map $f \in \mathbf{C}_r$ via the relation (7), and for any map $g \in \mathbf{C}_r$ from a sufficiently small neighborhood of f, let $\tilde{g} : U_r \to \mathbb{C}$ be the unique lift of g via the same relation (7) such that the mapping $g \mapsto \tilde{g}$ is continuous in the uniform metric and takes f to $\tilde{f}$. Then for any such $\tilde{f}$ we can consider a local chart $\sigma_{\tilde{f}} : (\mathbf{C}_r, f) \to \mathbf{A}_r$ defined in a small open neighborhood of f in $\mathbf{C}_r$ by the relation

$$\sigma_{\tilde{f}} : g \mapsto \tilde{g} - \tilde{f}.$$

It is obvious that the transition maps $\sigma_{\tilde{f}_1} \circ \sigma_{\tilde{f}_2}^{-1}$ are affine and hence $\mathbf{C}_r$ is an affine complex Banach manifold modeled on $\mathbf{A}_r$. □

We denote by $\mathbf{C}_r^{\mathbb{R}}$ the real slice of the Banach manifold $\mathbf{C}_r$. It consists of all maps from $\mathbf{C}_r$ which map the circle $\mathbb{R}/\mathbb{Z}$ to itself.

Definition 3.4. Let $U \subset \mathbb{C}$ be a simply connected neighborhood of a point z_0 and consider a possibly multiple-valued analytic map $f : U \setminus \{z_0\} \to \mathbb{C}$. We say that f has a critical exponent $\alpha \in \mathbb{C}$ at z_0 if in a neighborhood of z_0 the map f can be represented as

$$f(z) = \psi((\phi(z))^{\alpha}),$$

where ϕ, ψ are locally conformal maps, $\phi(z_0) = 0$, and $z \mapsto z^{\alpha}$ is an appropriate branch of the power map.

Definition 3.5. Given a positive real number $r > 0$ and a complex number $\alpha \in \mathbb{C}$, $\alpha \neq 1$, we define the set of generalized critical cylinder maps $\mathbf{C}_r^{\alpha} \subset \mathbf{C}_r$ to be the set of all maps $f \in \mathbf{C}_r$ whose lift to the interval $[0, 1]$ has an analytic extension $\tilde{f} : U_r \to \mathbb{C}/\mathbb{Z}$ with critical exponent α at 0 and 1.

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Definition 3.6. When $\alpha = 3$ (or any other odd integer greater than 3), we define $\hat{\mathbf{C}}_r^\alpha \subset \mathbf{C}_r^\alpha$ to be the proper subset of $\mathbf{C}_r^\alpha$ that consists of all maps $f \in \mathbf{C}_r^\alpha$ which are analytic in V_r. In other words, all branches of the map f in the disk of radius r around zero coincide.

Remark 3.7. For any $\alpha \in 2\mathbb{N} + 1$, the set $\hat{\mathbf{C}}_r^\alpha$ is a submanifold of $\mathbf{C}_r$ (cf. [**Yam02**]).

3.2. *The spaces of critical triples* $\mathbf{P}_{U,t,h}$ *and* $\mathbf{P}_{U,t,h}^\alpha$. In what follows, we will find it convenient when perturbing an analytic critical circle map f to a map with a critical exponent $\alpha \neq 2n + 1$ to first decompose f into the form (1) and then perturb each of the terms in the decomposition separately. To lay the ground for this discussion, we introduce a suitable space of decompositions (1) below. We note that the main difficulty here is to define these decompositions globally in a neighborhood of the circle (rather than just in a neighborhood of the critical point), so that the space of decompositions is invariant under renormalizations.

Given a positive real number $\alpha > 1$, the function

$$p_{\alpha+} : \mathbb{C}\backslash\mathbb{R}^- \to \mathbb{C}$$

is defined as the branch of the map $z \mapsto z^\alpha$ which maps positive reals to positive reals. Similarly, we define the function

$$p_{\alpha-} : \mathbb{C}\backslash\mathbb{R}^+ \to \mathbb{C}$$

as the branch of the map $z \mapsto -(-z)^\alpha$ which maps negative reals to negative reals.

Remark 3.8. Functions $p_{\alpha+}$ and $p_{\alpha-}$ can be defined for any $\alpha \in \mathbb{C}$ by means of analytic continuation in the α-coordinate.

Given a positive real number $t > 0$, let I_t^+ denote the straight-line segment connecting the points $\frac{1}{2} + it$ and $\frac{1}{2} - it$. Similarly, let I_t^- denote the straight-line segment connecting the points $-\frac{1}{2} + it$ and $-\frac{1}{2} - it$.

Consider a simply connected domain U which contains both I_t^+ and I_t^- as well as the interval $[-\frac{1}{2}, \frac{1}{2}]$. For a complex number $\alpha \neq 0$ and a univalent analytic function $\phi : U \to \mathbb{C}$, let us assume that $\phi(I_t^+)$ and $\phi(I_t^-)$ lie in the domains of the functions $p_{\alpha+}$ and $p_{\alpha-}$, respectively. We define a closed curve $\gamma_{\phi,\alpha,t} \subset \mathbb{C}$ which consists of four pieces: $p_{\alpha+}(\phi(I_t^+))$, $p_{\alpha-}(\phi(I_t^-))$ and the two straight-line segments ℓ_1, ℓ_2 connecting $p_{\alpha+}(\phi(\frac{1}{2} + it))$ with $p_{\alpha-}(\phi(-\frac{1}{2} + it))$ and $p_{\alpha+}(\phi(\frac{1}{2} - it))$ with $p_{\alpha-}(\phi(-\frac{1}{2} - it))$, respectively.

Let us make the following simple observation.

PROPOSITION 3.9. *The curve $\gamma_{\phi,\alpha,t}$ is simple, closed, and bounds a quasidisk if and only if the following four conditions hold:*
- *the curves $p_{\alpha+}(\phi(I_t^+))$, $p_{\alpha-}(\phi(I_t^-))$ are simple and disjoint;*
- *the line segments ℓ_1, ℓ_2 are disjoint;*
- *the segments ℓ_1, ℓ_2 do not intersect with the curves $p_{\alpha+}(\phi(I_t^+))$, $p_{\alpha-}(\phi(I_t^-))$ except at the end points; and*
- *the curves $p_{\alpha+}(\phi(I_t^+))$, $p_{\alpha-}(\phi(I_t^-))$ are not tangent to ℓ_1, ℓ_2 at their intersection points.*

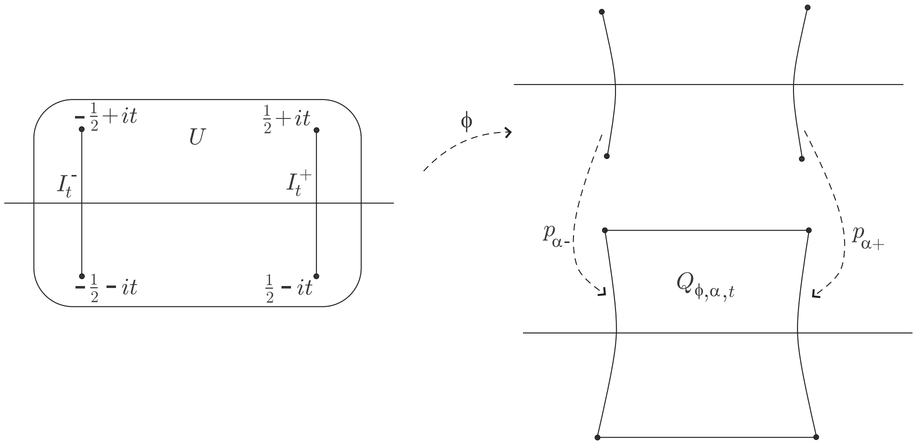

FIGURE 2. A domain $Q_{\phi,\alpha,t}$.

Definition 3.10. Suppose that $\gamma_{\phi,\alpha,t}$ is simple, closed, and bounds a quasidisk. Then we denote this quasidisk by $Q_{\phi,\alpha,t}$ (see Figure 2).

Definition 3.11. For a positive real number $t > 0$, a complex number $\alpha \neq 0$, and a Jordan domain $U \subset \mathbb{C}$ that contains the set $I_t^+ \cup I_t^- \cup [-\frac{1}{2}, \frac{1}{2}]$, by $\mathbf{B}_{U,t}^{\alpha}$ we denote the set of all analytic maps $\phi : U \to \mathbb{C}$ that satisfy the following properties:

(i) the map ϕ continuously extends to the closure $\overline{U}$ and $\phi(0) = 0$, $\phi'(0) = 1$;

(ii) the map ϕ is univalent in a neighborhood of $I_t^+ \cup I_t^- \cup [-\frac{1}{2}, \frac{1}{2}]$;

(iii) the quasidisk $Q_{\phi,\alpha,t}$ as well as the curves $p_{\alpha-}(\phi((-\frac{1}{2}, 0]))$ and $p_{\alpha+}(\phi([0, \frac{1}{2})))$ are defined and $p_{\alpha-}(\phi((-\frac{1}{2}, 0]))$, $p_{\alpha+}(\phi([0, \frac{1}{2}))) \subset Q_{\phi,\alpha,t}$;

(iv) every straight line through the origin intersects the boundary of the quasidisk $Q_{\phi,\alpha,t}$ transversally at exactly two points.

Let $U \subset \mathbb{C}$ be a simply connected domain containing the origin. By $\mathbf{A}_U$, we denote the space of all bounded analytic functions $\phi : U \to \mathbb{C}$ that are continuous up to the boundary and satisfy the properties $\phi(0) = 0$ and $\phi'(0) = 1$. We note that $\mathbf{A}_U$ equipped with the sup-norm forms an affine complex Banach space (the space $\mathbf{A}_U - \mathrm{id}$ is a complex Banach space).

The next statement follows from obvious continuity considerations.

PROPOSITION 3.12. *For every t, α, and U as in Definition 3.11, the set $\mathbf{B}_{U,t}^{\alpha}$ is an open subset of the affine Banach space $\mathbf{A}_U$ and hence it is a complex Banach manifold, modeled on $\mathbf{A}_U$.*

Proof. It follows from Proposition 3.9 that the quasidisk $Q_{\phi,\alpha,t}$ exists for all maps ϕ from a certain open subset of $\mathbf{A}_U$. All other properties of Definition 3.11 also define open subsets in $\mathbf{A}_U$. The set $\mathbf{B}_{U,t}^{\alpha}$ is the intersection of these open subsets and hence it is also open. □

Definition 3.13. We define the set $\mathbf{B}_{U,t}$ as the union of all pairs (α, ϕ) such that $\alpha \in \mathbb{C}$, $\alpha \neq 0$, and $\phi \in \mathbf{B}_{U,t}^{\alpha}$.

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Similarly to Proposition 3.12, considerations of continuity yield the following proposition.

PROPOSITION 3.14. *The set* $\mathbf{B}_{U,t}$ *is an open subset of the direct product of Banach spaces* $\mathbb{C} \times \mathbf{A}_U$ *and hence it is a complex Banach manifold, modeled on* $\mathbb{C} \times \mathbf{A}_U$.

For a positive real number $h > 0$, we denote by $\mathbf{U}_h$ the set of all 1-periodic functions that are defined and analytic in the strip $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < h\}$ and continuous up to the boundary of this strip. The set $\mathbf{U}_h$ equipped with the uniform norm is a complex Banach space. As before, for a positive real number h, we denote by V_h the h-neighborhood of the equatorial circle in $\mathbb{C}/\mathbb{Z}$ defined in (8).

Definition 3.15. For a fixed real number $h > 0$, we denote by $\mathbf{D}_h$ the set of all analytic functions $\psi : V_h \to \mathbb{C}/\mathbb{Z}$ such that the following properties hold:

- ψ continuously extends to the boundary of V_h;
- the image of the unit circle under ψ is homotopic to the equatorial circle $\mathbb{T} \subset \mathbb{C}/\mathbb{Z}$;
- ψ is univalent in a neighborhood of $\mathbb{T} \subset \mathbb{C}/\mathbb{Z}$.

We immediately notice the following result.

LEMMA 3.16. *Each set* $\mathbf{D}_h$ *has a structure of a complex Banach manifold modeled on* $\mathbf{U}_h$.

Proof. For a map $g \in \mathbf{U}_h$, the map

$$\tilde{f}(z) = g(z) + z$$

is a lift of some $f \in \mathbf{D}_h$ via the relation (7). The constructed correspondence $g \mapsto f$ is an affine covering map from $\mathbf{U}_h$ onto the space of all analytic functions $\psi : V_h \to \mathbb{C}/\mathbb{Z}$ that satisfy the first two properties of Definition 3.15. The last property of Definition 3.15 defines $\mathbf{D}_h$ as an open subset of the former space and hence $\mathbf{D}_h$ is an affine complex Banach manifold, modeled on $\mathbf{U}_h$. □

Definition 3.17. Given a neighborhood of the origin $U \subset \mathbb{C}$ with Jordan boundary and positive real numbers t, $h > 0$, by $\mathbf{P}_{U,t,h}$ we denote

$$\mathbf{P}_{U,t,h} = \mathbf{B}_{U,t} \times \mathbf{D}_h.$$

In other words, $\mathbf{P}_{U,t,h}$ is the set of all triples (α, ϕ, ψ) such that $\alpha \in \mathbb{C}$, $\alpha \neq 0$, $\phi \in \mathbf{B}^{\alpha}_{U,t}$, and $\psi \in \mathbf{D}_h$.

Definition 3.18. For a fixed $\alpha \in \mathbb{C}$ such that $\alpha \neq 0$, we denote by $\mathbf{P}^{\alpha}_{U,t,h} \subset \mathbf{P}_{U,t,h}$ the set of all elements from $\mathbf{P}_{U,t,h}$ whose first coordinate is equal to α. The set $\mathbf{P}^{\alpha}_{U,t,h}$ can be identified with the direct product $\mathbf{B}^{\alpha}_{U,t} \times \mathbf{D}_h$.

Finally, let us formulate an immediate corollary from Proposition 3.12, Proposition 3.14, and Lemma 3.16.

LEMMA 3.19. *The space* $\mathbf{P}_{U,t,h}$ *has the structure of a complex Banach manifold. For each* $\alpha \in \mathbb{C}$, $\alpha \neq 0$, *the space* $\mathbf{P}^{\alpha}_{U,t,h}$ *is a complex Banach submanifold of* $\mathbf{P}_{U,t,h}$ *of complex codimension* 1.

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3.3. *Generalized critical cylinder maps in* $\mathbf{P}_{U,t,h}$ *and* $\mathbf{P}^{\alpha}_{U,t,h}$. In this subsection we construct a correspondence between elements of $\mathbf{P}_{U,t,h}$ and generalized analytic critical cylinder maps and show that this correspondence is an analytic map (cf. Lemma 3.22).

Let $U \subset \mathbb{C}$ be a sufficiently large neighborhood of the origin and let $t > 0$ be a sufficiently small real number such that the space $\mathbf{B}_{U,t}$ is defined and non-empty. For every $(\alpha, \phi) \in \mathbf{B}_{U,t}$, we notice that the function

$$\nu(z) = p_{\alpha+}(\phi(\phi^{-1}(p_{\alpha-}^{-1}(z)) + 1))$$

is analytic and univalent in a neighborhood of the curve $p_{\alpha-}(\phi(I_t^-))$. (Here $p_{\alpha-}^{-1}$ denotes the inverse branch, for which $\phi^{-1} \circ p_{\alpha-}^{-1}$ maps $p_{\alpha-}(\phi(I_t^-))$ to I_t^-.) Because of that, the set $Q_{\phi,\alpha,t} \cup p_{\alpha-}(\phi(I_t^-)) \cup p_{\alpha+}(\phi(I_t^+))$, factored by the action of the map ν, can be viewed as a Riemann surface

$$A_{\phi,\alpha,t} = Q_{\phi,\alpha,t} \cup p_{\alpha-}(\phi(I_t^-)) \cup p_{\alpha+}(\phi(I_t^+))/\nu \tag{9}$$

conformally isomorphic to an annulus.

LEMMA 3.20. *Let $U \subset \mathbb{C}$ and $t > 0$ be such that the space $\mathbf{B}_{U,t}$ is defined and contains an element $(\alpha_0, \phi_0) \in \mathbf{B}_{U,t}$, where $\alpha_0 \in \mathbb{R}$ and ϕ_0 is real-symmetric. Then there exists a family of conformal maps $\pi_{\phi,\alpha,t} : Q_{\phi,\alpha,t} \to \mathbb{C}/\mathbb{Z}$ parameterized by $(\alpha, \phi) \in \mathbf{B}_{U,t}$ such that for every $(\alpha, \phi) \in \mathbf{B}_{U,t}$, the map $\pi_{\phi,\alpha,t}$ projects to a conformal map of the annulus $A_{\phi,\alpha,t}$, $\pi_{\phi,\alpha,t}(0) = 0$, and $\pi_{\phi,\alpha,t}$ is real-symmetric if $\alpha \in \mathbb{R}$ and ϕ is real-symmetric. Moreover, for every $z \in Q_{\phi,\alpha,t}$, the dependence of $\pi_{\phi,\alpha,t}(z)$ on (α, ϕ) is analytic.*

Proof. For every $(\alpha, \phi) \in \mathbf{B}_{U,t}$, two sides of the quasidisk $Q_{\phi,\alpha,t}$—the curves $p_{\alpha+}(\phi(I_t^+))$ and $p_{\alpha-}(\phi(I_t^-))$—are injective holomorphic images of fixed intervals I_t^+ and I_t^- and hence there exists a holomorphic motion

$$\chi_{\alpha,\phi} : p_{\alpha_0+}(\phi_0(I_t^+)) \cup p_{\alpha_0-}(\phi_0(I_t^-)) \to p_{\alpha+}(\phi(I_t^+)) \cup p_{\alpha-}(\phi(I_t^-))$$

over $(\alpha, \phi) \in \mathbf{B}_{U,t}$, given by the formula

$$\chi_{\alpha,\phi}(z) = \begin{cases} (p_{\alpha+} \circ \phi \circ \phi_0^{-1} \circ p_{\alpha_0+}^{-1})(z) & \text{if } z \in p_{\alpha_0+}(\phi_0(I_t^+)), \\ (p_{\alpha-} \circ \phi \circ \phi_0^{-1} \circ p_{\alpha_0-}^{-1})(z) & \text{if } z \in p_{\alpha_0-}(\phi_0(I_t^-)). \end{cases} \tag{10}$$

Since the other two sides of the quasidisk $Q_{\phi,\alpha,t}$ are straight-line segments whose end points move holomorphically, the holomorphic motion (10) extends to a holomorphic motion $\chi_{\alpha,\phi} : \partial Q_{\phi_0,\alpha_0,t} \to \partial Q_{\phi,\alpha,t}$. Finally, it follows from property (iii) of Definition 3.11 that zero belongs to $Q_{\phi,\alpha,t}$ and, according to property (iv) of Definition 3.11, the quasidisk $Q_{\phi,\alpha,t}$ is a star domain and hence the holomorphic motion $\chi_{\alpha,\phi}$ of the boundary of the quasidisk extends to a holomorphic motion

$$\chi_{\alpha,\phi} : \overline{Q}_{\phi_0,\alpha_0,t} \to \overline{Q}_{\phi,\alpha,t} \quad \text{with } \chi_{\alpha,\phi}(0) = 0 \text{ for all } \alpha, \phi \tag{11}$$

of the closure of the quasidisk over the whole space $\mathbf{B}_{U,t}$.

We construct the family of maps $\pi_{\phi,\alpha,t}$ in the following way: for $(\alpha, \phi) = (\alpha_0, \phi_0)$, we put $\pi_{\phi_0,\alpha_0,t}$ to be the unique conformal map with the property that $\pi_{\phi_0,\alpha_0,t}(0) = 0$ and $\pi_{\phi_0,\alpha_0,t}$ induces a conformal diffeomorphism between $A_{\phi_0,\alpha_0,t}$ and an annulus $A_0 \subset \mathbb{C}/\mathbb{Z}$ whose boundary consists of two circles parallel to the equator.

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For an arbitrary $(\alpha, \phi) \in \mathbf{B}_{U,t}$, we make the following construction: by the lambda lemma [**MSS83**, **Lyu84**], for each $(\alpha, \phi) \in \mathbf{B}_{U,t}$, the map $\chi_{\alpha,\phi}$ is quasiconformal. Pulling back the standard conformal structure on $Q_{\phi,\alpha,t}$ by the map $\chi_{\alpha,\phi} \circ \pi^{-1}_{\phi_0,\alpha_0,t}$, we obtain a conformal structure $\mu_{\alpha,\phi}$ on A_0. We extend the conformal structure $\mu_{\alpha,\phi}$ beyond the annulus A_0 by the standard conformal structure. We obtain a conformal structure $\mu_{\alpha,\phi}$ on the whole cylinder $\mathbb{C}/\mathbb{Z}$, and the dependence of $\mu_{\alpha,\phi}$ on (α, ϕ) is analytic. By the measurable Riemann mapping theorem, there exists a unique map $f_{\alpha,\phi} : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ that straightens the conformal structure $\mu_{\alpha,\phi}$ and fixes the point 0 and the two boundary points at infinity. Moreover, $f_{\alpha,\phi}$ depends analytically on $(\alpha, \phi) \in \mathbf{B}_{U,t}$. We define the map $\pi_{\phi,\alpha,t}$ as the composition

$$\pi_{\phi,\alpha,t} = f_{\alpha,\phi} \circ \pi_{\phi_0,\alpha_0,t} \circ \chi^{-1}_{\alpha,\phi}.$$

It follows from the construction that if $\alpha \in \mathbb{R}$ and ϕ is real-symmetric, then $\pi_{\phi,\alpha,t}$ is also real-symmetric. Analytic dependence of $\pi_{\phi,\alpha,t}$ on ϕ and α follows from analytic dependence of $f_{\alpha,\phi}$ and $\chi_{\alpha,\phi}$ on the parameters (cf. [**Lyu99**, Remark on p. 345]). □

Definition 3.21. For a positive real number $r > 0$, let $U_r^+ \subset U_r$ be the surface that consists of all pairs $(z, \gamma) \in U_r$ such that $z \in A_r \cap \{\mathrm{Re}\, z < 0.5\}$. Similarly, let U_r^- be the surface that consists of all pairs (z, γ) such that $(z + 1, \gamma) \in U_r$ and $z + 1 \in A_r \cap \{\mathrm{Re}\, z > 0.5\}$.

Let $U \subset \mathbb{C}$ be a neighborhood of the interval $[-1/2, 1/2]$. Assume that the analytic map $\phi : U \to \mathbb{C}$ is such that $\phi(0) = 0$ and $\phi'(0)$ is a non-negative and non-zero complex number. Then, for $\alpha \in \mathbb{C}$, the composition $p_{\alpha+} \circ \phi$ is defined on some interval $(0, \varepsilon) \subset \mathbb{R}$ to the right of the origin. Similarly, the composition $p_{\alpha-} \circ \phi$ is defined on some interval $(-\varepsilon, 0) \subset \mathbb{R}$ to the left of the origin. Let a positive real number $r > 0$ be sufficiently small, so that the projections of U_r^+, U_r^- onto the first coordinate are contained in U. Then the first composition can be extended to an analytic map on U_r^+ and the second composition can be extended to an analytic map on U_r^-. Let us denote the first map by $p_{\phi,\alpha+} : U_r^+ \to \mathbb{C}$ and the second map by $p_{\phi,\alpha-} : U_r^- \to \mathbb{C}$.

Now we fix a neighborhood $U \subset \mathbb{C}$ and a real number $t > 0$ that satisfy Lemma 3.20. We also fix a family of maps $\pi_{\phi,\alpha,t}$ whose existence is guaranteed by Lemma 3.20. For a positive real number $h > 0$, we consider the space $\mathbf{P}_{U,t,h}$ and, for an element $\tau = (\alpha, \phi, \psi) \in \mathbf{P}_{U,t,h}$, we define the map $g_\tau : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{C}/\mathbb{Z}$ according to the formula

$$g_\tau(z) = \begin{cases} \psi(\pi_{\phi,\alpha,t}(p_{\phi,\alpha-}(z))) & \text{if } z \in [-\frac{1}{2},\ 0], \\ \psi(\pi_{\phi,\alpha,t}(p_{\phi,\alpha+}(z))) & \text{if } z \in [0,\ \frac{1}{2}], \end{cases} \tag{12}$$

provided that the above compositions are defined.

It follows from the construction of the space $\mathbf{P}_{U,t,h}$ that the map g_τ projects to the continuous map of the circle $f_\tau : \mathbb{R}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ defined by the relation

$$f_\tau(\pi(z)) = g_\tau(z). \tag{13}$$

If, for some positive real number $r > 0$, the first and the second compositions in (12) are defined in the domains U_r^- and U_r^+, respectively, then, according to Definition 3.2, this implies that f_τ is a generalized critical cylinder map from $\mathbf{C}_r$. It also follows from (12) that the critical exponent of f_τ at zero is equal to α and hence $f_\tau \in \mathbf{C}_r^\alpha$.

LEMMA 3.22. *Assume that* $U \subset \mathbb{C}$ *and* $t > 0$ *satisfy the conditions of Lemma 3.20 and let* $\pi_{\phi,\alpha,t}$ *be a fixed family of maps from Lemma 3.20. Assume that for some* $r_0 > 0$, $h > 0$, *and* $\tau_0 \in \mathbf{P}_{U,t,h}$, *the first and the second compositions in (12) are defined in the domains* $U_{r_0}^-$ *and* $U_{r_0}^+$, *respectively. Then, for all positive real numbers* $r < r_0$, *there exists a neighborhood* $\mathcal{U}_r \subset \mathbf{P}_{U,t,h}$ *of* τ_0 *such that the correspondence* $\tau \mapsto f_\tau$ *is an analytic map from* $\mathcal{U}_r$ *to* $\mathbf{C}_r$.

Proof. It follows from continuous dependence of $f_\tau(z)$ on the parameters that since $f_{\tau_0} \in \mathbf{C}_{r_0}$, then for all $r < r_0$ there exists a neighborhood of τ_0 such that for every τ from that neighborhood, the map f_τ belongs to $\mathbf{C}_r$. We denote this neighborhood by $\mathcal{U}_r$. Analytic dependence of f_τ on the parameters follows from analytic dependence of the map $\pi_{\phi,\alpha,t}$ on α and ϕ (cf. Lemma 3.20). □

4. *Fundamental crescents*

4.1. *Fundamental crescent of an analytic map.*

The following discussion mirrors [**Yam02**]. Let $\Omega \subset \mathbb{C}$ be a domain and let $h : \Omega \to \mathbb{C}$ be an analytic map.

Definition 4.1. A simply connected Jordan domain $C \subset \mathbb{C}$ is called a *crescent domain* for h if the boundary of C is the union of two piecewise-smooth simple curves $l \subset \Omega$ and $h(l) \subset h(\Omega)$, the end points a, b of l are two distinct repelling fixed points of h, and $l \cap h(l) = \{a, b\}$.

Definition 4.2. A domain $C^o \subset \mathbb{C}$ is called a *fattening* of a crescent domain C as above if C^o is the union

$$C^o = C \cup W_l \cup h(W_l),$$

where $W_l \Subset \Omega$ is a neighborhood on which the map h is univalent, the arc l is contained in $\overline{W}_l$, so that $l \cap \partial W_l = \{a, b\}$, and $\overline{W}_l \cap h(\overline{W}_l) = \{a, b\}$.

Finally, we have the following definition.

Definition 4.3. A crescent domain C as above is called a *fundamental crescent* for h if it has a fattening C^o and the quotient of C^o by the action of the map h is a Riemann surface conformally isomorphic to the cylinder $\mathbb{C}/\mathbb{Z}$ (see Figure 3). We note that this Riemann surface is independent of the choice of a fattening C^o.

If C is a fundamental crescent for a map h, then the conformal isomorphism between the Riemann surface C^o/h and the cylinder $\mathbb{C}/\mathbb{Z}$ is uniquely determined up to the post-composition with a conformal automorphism of the cylinder $\mathbb{C}/\mathbb{Z}$. Hence, a lift of this isomorphism to the universal cover of the cylinder $\mathbb{C}/\mathbb{Z}$ is uniquely determined up to the post-composition with translations and multiplication by -1. This observation motivates the following definition.

Definition 4.4. Let $h : \Omega \to \mathbb{C}$ be an analytic map with a fundamental crescent C and its fattening $C^o = C \cup W_l \cup h(W_l)$. For a fixed base point $\omega \in C$, we denote by

$$\pi_{C^o}^{\omega} : C^o \to \mathbb{C}$$

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the unique conformal map which conjugates h with the unit shift $z \mapsto z+1$ in W_l and such that $\pi^{\omega}_{C^o}(\omega) = 0$. We also denote by $\tilde{\pi}^{\omega}_{C^o} : C \to \mathbb{C}/\mathbb{Z}$ the composition

$$\tilde{\pi}^{\omega}_{C^o} = \pi \circ \pi^{\omega}_{C^o}. \tag{14}$$

Finally, by π^{ω}_{C} and $\tilde{\pi}^{\omega}_{C}$ we denote the restrictions of $\pi^{\omega}_{C^o}$ and $\tilde{\pi}^{\omega}_{C^o}$ respectively to the fundamental crescent C:

$$\pi^{\omega}_{C} \equiv \pi^{\omega}_{C^o}|_C \quad \text{and} \quad \tilde{\pi}^{\omega}_{C} \equiv \tilde{\pi}^{\omega}_{C^o}|_C.$$

We note that the maps π^{ω}_{C} and $\tilde{\pi}^{\omega}_{C}$ depend only on the fundamental crescent C and are independent of the choice of its fattening C^o.

Remark 4.5. Whenever the base point is chosen to be equal to zero, $\omega = 0$, we will simplify the notation by writing π_{C^o}, $\tilde{\pi}_{C^o}$, π_C, and $\tilde{\pi}_C$ instead of $\pi^{\omega}_{C^o}$, $\tilde{\pi}^{\omega}_{C^o}$, π^{ω}_{C}, and $\tilde{\pi}^{\omega}_{C}$, respectively.

For a domain $\Omega \subset \mathbb{C}$, we denote by $\mathbf{D}(\Omega)$ the Banach space of analytic functions in Ω, continuous up to the boundary of Ω and equipped with the sup-norm. The following key lemma mirrors the statements proved in [**Yam02**].

LEMMA 4.6. *Let C be a fundamental crescent of a map $h \in \mathbf{D}(\Omega)$ with the boundary consisting of the curves l and $h(l)$ whose common end points are the points $a, b \in \Omega$. Let $C^o = C \cup W_l \cup h(W_l)$ be a fattening of C such that some other fattening $\hat{C}^o$ of C contains C^o and $\partial C^o \cap \partial \hat{C}^o = \{a, b\}$, and let $\omega \in C$ be a fixed base point. Then there exists an open neighborhood $\mathcal{W}(h) \subset \mathbf{D}(\Omega)$ of h such that every map $g \in \mathcal{W}(h)$ has a fundamental crescent C_g and a fattening C^o_g of C_g with the following properties.*

(i) *For $g = h$, we have $C_g = C$ and $C^o_g = C^o$, and the mappings*

$$g \mapsto C_g \quad \textit{and} \quad g \mapsto C^o_g$$

are continuous with respect to the Hausdorff metric on the image.

(ii) *If the map h, the fundamental crescent C, and its fattening C^o are real-symmetric, then C_g and C^o_g are also real-symmetric for all real-symmetric maps $g \in \mathcal{W}(h)$.*

(iii) *For all $g \in \mathcal{W}(h)$, the base point ω is contained in C_g. Moreover, for every $g_0 \in \mathcal{W}(h)$ and $z \in C^o_{g_0}$, the number $\pi^{\omega}_{C^o_g}(z)$ is defined for all $g \in \mathcal{W}(h)$ that are sufficiently close to g_0, and the dependence*

$$g \mapsto \pi^{\omega}_{C^o_g}(z)$$

is locally analytic in $g \in \mathbf{D}(\Omega)$.

Proof. We construct a neighborhood $\mathcal{W}(h)$ and the family of fattenings C^o_g that move holomorphically over $g \in \mathcal{W}(h)$. More precisely, we construct a holomorphic motion of a fattening C^o of the fundamental crescent

$$\chi_g : C^o \mapsto C^o_g \tag{15}$$

over a neighborhood $\mathcal{W}(h)$, so that

$$\chi_g(h(z)) = h(\chi_g(z)) \tag{16}$$

whenever both sides are defined. Then C_g is defined as

$$C_g = \chi_g(C).$$

The construction is made in two steps.

The first step consists of constructing the holomorphic motion χ_g of $\overline{W_l \cup h(W_l)}$ over some neighborhood $\mathcal{W}_1(h) \subset \mathbf{D}(\Omega)$, satisfying the equivariance relation (16). This construction is analogous to the construction made in [**Yam02**, Proposition 7.3]. We refer the reader to the proof of that statement and omit further details. In the second step, applying the theorem of Bers and Royden [**BR**], we extend the previously constructed holomorphic motion χ_g to a holomorphic motion (15) over a possibly smaller open neighborhood $\mathcal{W}(h)$ with an additional property that

$$\chi_g(\omega) = \omega.$$

The holomorphic motion (15) defines the family of fundamental crescents C_g and their fattenings C_g^o that continuously depend on the map $g \in \mathcal{W}(h)$ and hence satisfy property (i). It also follows from the construction that these families satisfy property (ii) (cf. [**Yam02**, Proposition 7.3]).

To prove property (iii) for the constructed family of fundamental crescents, we use a version of the argument given in [**Yam02**, Proposition 7.5]. Namely, by the lambda lemma [**MSS83**] all maps χ_g are quasiconformal on C^o. Even though the map $\tilde{\pi}^{\omega}_{C^o}$ is not univalent on C^o, the equivariance relation (16) implies that the pullback of the standard conformal structure on C_g^o by the map $\chi_g \circ (\tilde{\pi}^{\omega}_{C^o})^{-1}$ is well defined. This pullback provides a conformal structure on the cylinder $\mathbb{C}/\mathbb{Z}$ and the dependence of this conformal structure on g is analytic. By the measurable Riemann mapping theorem, there exists a unique map $f_g : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ that straightens this conformal structure and fixes the point 0 and the two boundary points at infinity. The composition $f_g \circ \tilde{\pi}^{\omega}_{C^o} \circ \chi_g^{-1}$ induces a conformal isomorphism between the Riemann surfaces C_g^o/g and $\mathbb{C}/\mathbb{Z}$, and $f_g \circ \tilde{\pi}^{\omega}_{C^o} \circ \chi_g^{-1}(\omega) = 0$. The uniqueness of such an isomorphism implies that

$$\tilde{\pi}^{\omega}_{C_g^o} = f_g \circ \tilde{\pi}^{\omega}_{C^o} \circ \chi_g^{-1}. \tag{17}$$

Finally, analytic dependence on the parameter in the measurable Riemann mapping theorem implies that f_g depends analytically on $g \in \mathcal{W}(h)$ and hence (17) implies that for a fixed z, the point $\tilde{\pi}^{\omega}_{C_g^o}(z)$ depends analytically on g. Now analytic dependence of $\pi^{\omega}_{C_g^o}(z)$ on g follows from (14). □

4.2. *Fundamental crescents of maps in* $\mathbf{C}_r$. We let V_r be as in (8).

Definition 4.7. A simple piecewise-smooth arc $l \subset V_r$ will be called a *separating arc* if l is an image of an arc $\tilde{l} \subset A_r \setminus ((-\infty, 0) \cup (1, +\infty))$ under the projection π and l connects two points strictly above and below the equator.

Note that since a separating arc l has to intersect the equator at least at one point, its pre-image $\tilde{l} \subset A_r \setminus ((-\infty, 0) \cup (1, +\infty))$ is uniquely determined.

Let $f \in \mathbf{C}_r$ be a generalized cylinder map. We denote by $\tilde{f} : U_r \to \mathbb{C}$ some fixed lift of f via the relation (7). If $l \subset V_r$ is a separating arc, then by f_l we denote the analytic continuation of f to a neighborhood of l, obtained as the projection of $\tilde{f}$ restricted to a neighborhood of the curve $\tilde{l}$. This specifies a branch of f in the case it is multiple valued in V_r. In particular, this analytic continuation satisfies the relation $\pi \circ \tilde{f} = f_l \circ \pi$. If both l and $f_l(l)$ are separating arcs, then, in order to simplify the notation, by f_l^2 we denote the composition $f_{f_l(l)} \circ f_l$. Similarly, by induction we can define $f_l^n = f_{f_l^{n-1}(l)} \circ f_l^{n-1}$ for any positive integer n, provided that $f_l^{n-1}(l)$ is a separating arc.

Definition 4.8. Let $f \in \mathbf{C}_r$. We say that a domain $C \subset V_r$ is *a fundamental crescent of f with period n* if the boundary ∂C contains a separating arc l with end points a, b, the map f_l^n is defined and univalent in a neighborhood of l, $\partial C = l \cup f_l^n(l)$, $l \cap f_l^n(l) = \{a, b\}$, and C is a fundamental crescent for the map f_l^n.

In the remaining part of the paper we will always require that $0 \in C$ and we set the base point $\omega = 0$.

4.3. *Fundamental crescents of holomorphic pairs.* In [**Yam02**], the existence of fundamental crescents was stated for all maps in the Epstein class. In this paper it will be more convenient for us to prove the corresponding statement for holomorphic commuting pairs. We start with some notation. Let $\mathcal{H} = (\eta, \xi, \nu)$ be a holomorphic pair with domains U, V, D and range Δ, as above. By Denjoy–Wolff theorem, there exists a unique point $p_\eta^+ \in \overline{\mathbb{H} \cap U}$, such that the iterates of η^{-1} converge to p_η^+ uniformly on compact subsets of $\mathbb{H} \cap \Delta$. We define the point $p_\eta^- \in \mathbb{C}$ as $p_\eta^- = \overline{p_\eta^+}$. If $p_\eta^+ \neq p_\eta^-$, then both p_η^+ and p_η^- are repelling fixed points of the map η. Otherwise, if $p_\eta^+ = p_\eta^-$, then p_η^+ lies on the real line and is a parabolic fixed point of the map η.

Definition 4.9. In the above notation, we say that a simply connected domain $C_\mathcal{H} \Subset \Delta$ is a fundamental crescent for the holomorphic pair $\mathcal{H}$, if the domain $C_\mathcal{H}$ is a real-symmetric fundamental crescent for the map η, such that $p_\eta^+, p_\eta^- \in \partial C_\mathcal{H}$, and $0 \in C_\mathcal{H}$ (see Figure 3).

For the remaining part of the paper we again set the basepoint $\omega = 0$, so that the projection

$$\pi_{C_\mathcal{H}^o} : C_\mathcal{H}^o \to \mathbb{C}$$

is the unique holomorphic map that conjugates η with the unit shift $z \mapsto z + 1$, and $\pi_{C_\mathcal{H}^o}(0) = 0$.

We notice that according to our definition of a holomorphic pair, the domains $U \cap \mathbb{H}$ and $\Delta \cap \mathbb{H}$ have Jordan boundary and hence the map $\eta^{-1}|_{\Delta \cap \mathbb{H}}$ continuously extends to the closure $\overline{\Delta \cap \mathbb{H}}$. In order to simplify further notation, we will write η^{-1} instead of $\eta^{-1}|_{\overline{\Delta \cap \mathbb{H}}}$.

The main result of this subsection is the following lemma.

LEMMA 4.10.

(i) *Let $\mathcal{H} = (\eta, \xi, \nu)$ be a holomorphic commuting pair such that the map η^{-1} does not have fixed points on the real line. Then there exists a fundamental crescent $C_\mathcal{H}$ for the holomorphic pair $\mathcal{H}$ such that $\pi_{C_\mathcal{H}}(C_\mathcal{H})$ contains the interval $(-1/2, 1/2) \subset \mathbb{R}$.*

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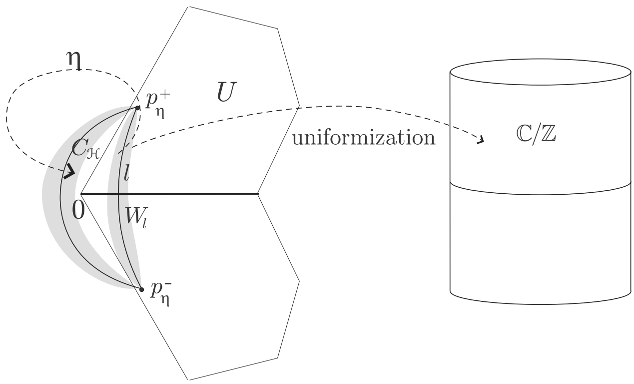

FIGURE 3. A fundamental crescent of a holomorphic pair $\mathcal{H}$.

(ii) *For every $s \in (0, 1)$, there exist a real number $a_s > 0$ and a domain $D_s \subset \mathbb{C}$ such that $[-1/2, 1/2] \subset D_s$ and, if $\mathcal{H} = (\eta, \xi, \nu)$ is the same as in part (i) of the lemma, and $\zeta_\mathcal{H} \in \mathbf{E}_s$, then the fundamental crescent $C_\mathcal{H}$ has a fattening $C^o_\mathcal{H}$ such that $D_s \subset \pi_{C^o_\mathcal{H}}(C^o_\mathcal{H})$, and $|\pi^{-1}_{C^o_\mathcal{H}}(1/2) - \pi^{-1}_{C^o_\mathcal{H}}(-1/2)| > a_s$.*

The proof of Lemma 4.10 is based on the following construction.

PROPOSITION 4.11. *Let $\mathcal{H} = (\eta, \xi, \nu)$ be a holomorphic commuting pair with range Δ such that the map η^{-1} does not have fixed points on the real line. Then there exists a unique conformal change of coordinates $\theta_\eta : \Delta \cap \mathbb{H} \to \mathbb{C}$ such that $\theta_\eta(p^+_\eta) = 0$, the map θ_η conjugates η^{-1} with the linear map $z \mapsto \lambda z$, where λ is the multiplier of the fixed point p^+_η for the map η^{-1}, and θ_η has a continuous extension to the boundary $\partial(\Delta \cap \mathbb{H})$.*

Proof. Since the map η does not have fixed points on the real line, the point p^+_η is an interior point of $\Delta \cap \mathbb{H}$ and hence is a globally attracting fixed point for the map η^{-1}. Then, by Koenig's theorem, there exists a conformal change of coordinates θ_η in a neighborhood of p^+_η which linearizes the map η^{-1}. Since the map η^{-1} is univalent on $\Delta \cap \mathbb{H}$, the local linearizing conformal chart around p^+_η can be extended to the whole domain $\Delta \cap \mathbb{H}$ by pullbacks under the dynamics of η^{-1}. Thus, we obtain the linearizing conformal chart defined on the whole domain $\Delta \cap \mathbb{H}$.

Since η does not have fixed points on the real line, this implies that there exists an integer $k > 0$ such that for every $z \in \partial(\Delta \cap \mathbb{H})$, the point $\eta^{-k}(z)$ is contained in $\Delta \cap \mathbb{H}$. This means that the linearizing chart θ_η extends to the boundary $\partial(\Delta \cap \mathbb{H})$ by finitely many pullbacks and hence θ_η has a continuous extension to $\partial(\Delta \cap \mathbb{H})$. □

Assume that $\mathcal{H} = (\eta|_U, \xi|_V, \nu|_D)$ is the same as in Proposition 4.11. Since η^{-1} does not have fixed points on the real line, the forward iterates of the interval $I_U = U \cap \mathbb{R}$ under the map η^{-1} converge to the point p^+_η. Notice that $\eta(I_U) \cup I_U$ is an interval on the real

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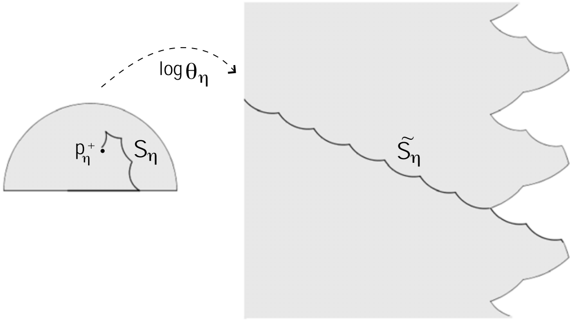

FIGURE 4. The curve S_η and its image under the log-linearizing coordinate.

line; hence, the set

$$S_\eta = \{p_\eta^+\} \bigcup \left(\bigcup_{j=-1}^{\infty} \eta^{-j}(\overline{I_U}) \right)$$

is a simple curve and the point p_η^+ is one of its end points.

Passing to logarithmic coordinates $\log \theta_\eta$ in the image $\theta_\eta(\Delta \cap \mathbb{H})$, we obtain a domain $W \subset \mathbb{C}$ that is invariant under the shifts by $2\pi i$ and contains a half-plane $\{z \in \mathbb{C} \mid \operatorname{Re} z < x_0\}$ for some $x_0 \in \mathbb{R}$ (see Figure 4). The map $\phi_\eta : W \to \Delta \cap \mathbb{H}$ defined by $\phi_\eta(z) = \theta_\eta^{-1}(e^z)$ is a covering map from W onto $(\Delta \cap \mathbb{H}) \setminus \{p_\eta^+\}$. It follows from Proposition 4.11 that the map ϕ_η semi-conjugates the map η^{-1} with the shift $z \mapsto z + w_\eta$, where $w_\eta \in \mathbb{C}$ is such that $\operatorname{Re} w_\eta < 0$ and $-\pi < \operatorname{Im} w_\eta \le \pi$.

Let $\tilde{S}_\eta \subset \overline{W}$ be an arbitrary fixed lift of the curve S_η to $\overline{W}$ under the covering map ϕ_η. In particular, it follows from the previous observation that the simple curve $\tilde{S}_\eta$ is invariant under the shift by w_η and a part of this curve belongs to the boundary of W. The curve $\tilde{S}_\eta$ splits the domain W into two subdomains and we let $W_\eta \subset W$ be the one of these subdomains whose boundary contains $\tilde{S}_\eta$ (see Figure 4). Further, let $p_0 \in \tilde{S}_\eta$ be the point that projects to zero under the covering map: $\phi_\eta(p_0) = 0$.

PROPOSITION 4.12. *Let $\mathcal{H} = (\eta, \xi, \nu)$ be the same as in Proposition 4.11. Then there exists a unique conformal map $\tau : W_\eta \to \mathbb{C}$ such that $\tau(z + w_\eta) = \tau(z) - 1$ for all $z \in W_\eta$, the map τ continuously extends to the boundary curve $\tilde{S}_\eta$, so that $\tau(\tilde{S}_\eta) \subset \mathbb{R}$ and $\tau(p_0) = 0$, and there exists a simple curve $\gamma \subset \tau(W_\eta)$ that does not have common points with its shift by* 1 *and whose projection onto the imaginary axis is either $\{ix \mid x > 0\}$ or $\{ix \mid x < 0\}$. Furthermore, the curve $\phi_\eta(\tau^{-1}(\gamma))$ is simple and has an end point at p_η^+.*

Proof. Since the domain W contains a half-plane $\{z \in \mathbb{C} \mid \operatorname{Re} z < x_0\}$ for some $x_0 \in \mathbb{R}$, this means that there exists an infinite ray $l \subset W_\eta$ with the end point in $\tilde{S}_\eta$ such that l is not parallel to the vector w_η and the projection of l onto $\mathbb{R}$ is an infinite interval of the form $(-\infty, a]$. Let $l' \subset W_\eta$ be the shift of l by w_η. Since l is not parallel to w_η, the rays l and l' have no common points. Let $C_l \subset W_\eta$ be the subdomain of W_η bounded by the

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curves l, l', and $\tilde{S}_\eta$. Again, since l is not parallel to w_η, the set $C_l \cup l \cup l'$, factored by the action of the shift by w_η, is a Riemann surface conformally isomorphic to a punctured disk.

Let $p_0' \in \tilde{S}_\eta$ be the unique point such that $p_0' \in \partial C_l$ and $p_0' - p_0 = k w_\eta$, where $k \in \mathbb{Z}$. Then there exists a unique conformal map τ defined in the relative neighborhood of C_l in W_η such that $\tau(z + w_\eta) = \tau(z) - 1$ whenever both sides are defined, τ continuously extends to $\tilde{S}_\eta \cap \partial W_\eta$, $\tau(\tilde{S}_\eta \cap \partial W_\eta) \subset \mathbb{R}$, and $\tau(p_0') = -k$. Finally, we extend the map τ to the whole domain W_η by pullbacks or pushforwards under the dynamics.

Since the rays l and l' have an empty intersection, the curve $\tau(l)$ does not have common points with its shift by 1. Since the set $C_l \cup l \cup l'$, factored by the action of the shift by w_η, is a Riemann surface conformally isomorphic to a punctured disk, the simple curve $\tau(l)$ has an unbounded projection onto the imaginary axis. Finally, we set $\gamma = \tau(l)$ and we notice that since the projection of l onto $\mathbb{R}$ is an interval of the form $(-\infty, a]$, the curve $\phi_\eta(\tau^{-1}(\gamma)) = \phi_\eta(l)$ is simple and has an end point at p_η^+. □

Proof of Lemma 4.10. (i) Let $\tau : W_\eta \to \mathbb{C}$ be the conformal map provided by Proposition 4.12 and let $\gamma \subset \tau(W_\eta)$ be the curve from the same proposition. Then there exists a straight-line segment in $\tau(W_\eta)$ that connects the point 1/2 with the curve γ. The point at which this line segment intersects the curve γ splits γ into two pieces. Let $l \subset \tau(W_\eta)$ be the curve obtained as the union of the unbounded piece of γ and the line segment. We define $l' \subset \tau(W_\eta)$ as the shift of l by -1. It follows from Proposition 4.12 that the image under the map $\phi_\eta \circ \tau^{-1}$ of the subdomain of $\tau(W_\eta)$ bounded by the curves l and l' is the upper or lower half of a fundamental crescent for $\mathcal{H}$. The other half of this fundamental crescent is obtained by reflection about the real axis.

(ii) We fix the real number $s \in (0, 1)$. All constants in the following proof will depend only on the number s and not on a particular commuting pair $\zeta_{\mathcal{H}} \in \mathbf{E}_s$, unless otherwise stated. We split the proof into the following simple steps.

Step 1. For every $\zeta_{\mathcal{H}} = (\eta, \xi) \in \mathbf{E}_s$, we have $\eta \in \mathbf{E}_{[0,1],s}$, which implies that there exist two constants $C_1 > c_1 > 0$ such that $c_1 \le |\eta(0)| \le C_1$ (cf. Definition 2.10).

Step 2. Lemma 2.11 together with Step 1 implies that there exists a constant $c_2 > 0$ such that $c_2 \le |\eta^{-1}(0)| \le 1$.

Step 3. There exists a constant $r > 0$ such that the map $\pi_{C_{\mathcal{H}}}$ analytically extends to $N_r([r + \eta(0), \eta^{-1}(0)])$, where $N_r(S)$ is the complex r-neighborhood of the set $S \subset \mathbb{C}$.

Proof of Step 3. It follows from Proposition 4.12 that the map $\pi_{C_{\mathcal{H}}}$ analytically extends to $\Delta \backslash S_\eta$. Now Lemma 2.11, Step 2, and injectivity of η^{-1} imply that $\eta^{-3}(\Delta \cap \mathbb{H})$ has an empty intersection with $N_r([r + \eta(0), \eta^{-1}(0)])$ for some fixed r. In particular, this means that $S_\eta \cap N_r([r + \eta(0), \eta^{-1}(0)]) = (\eta(0), \eta^{-1}(0) + r)$, so the map $\pi_{C_{\mathcal{H}}}$ analytically extends to $N_r([r + \eta(0), \eta^{-1}(0)]) \cap \mathbb{H}$. Finally, $\pi_{C_{\mathcal{H}}}$ extends to the whole neighborhood $N_r([r + \eta(0), \eta^{-1}(0)])$ according to the reflection principle.

Step 4. Let $z_{-1/2} \in [0, \eta^{-1}(0)]$ be the point for which $\pi_{C_{\mathcal{H}}}(z_{-1/2}) = -1/2$. Then there exists a constant $c_3 > 0$ such that $z_{-1/2} \ge c_3$.

Proof of Step 4. Since the interval $\pi_{C_{\mathcal{H}}}([0, z_{-1/2}])$ has a fixed length equal to 1/2, Step 2, Step 3, and the Koebe distortion theorem imply that if the interval $[0, z_{-1/2}]$ is too small, then the interval $\pi_{C_{\mathcal{H}}}([0, \eta^{-1}(0)])$ is larger than the unit interval, which is a contradiction.

Step 5. Consider the point $z_{1/2} = \eta(z_{-1/2})$. It follows from Lemma 2.11 and Step 4 that there exists a constant $c_4 > 0$ such that $z_{1/2} - \eta(0) > c_4$.

Step 6. It follows from Step 1, Step 2, and Step 4 that the interval $[z_{1/2}, z_{-1/2}]$ is commensurable with a unit interval. Now, since the interval $\pi_{C_{\mathcal{H}}}([z_{1/2}, z_{-1/2}])$ has a fixed (unit) length, Step 5, Step 3, and the Koebe distortion theorem imply that $\pi_{C_{\mathcal{H}}}(N_r([r + \eta(0), \eta^{-1}(0)]))$ contains a fixed domain D_s such that $[-1/2, 1/2] \subset D_s$. If D_s is sufficiently small, we can always choose a fattening $C^o_{\mathcal{H}}$ of $C_{\mathcal{H}}$ so that $D_s \subset \pi_{C^o_{\mathcal{H}}}(C^o_{\mathcal{H}})$.

Existence of the constant a_s immediately follows from Step 4. Namely, we can put $a_s = c_3$. □

4.4. *Holomorphic commuting pairs and critical circle maps. Choice of constants.* For a positive integer $B \in \mathbb{N}$, consider the space $\Sigma_B \subset \Sigma$ of all bi-infinite sequences of positive integers that are not greater than B.

Definition 4.13. For a positive integer $B > 0$, let $\mathcal{I}_B \subset \mathbf{E}$ be the set of all commuting pairs from $\mathcal{I}$ that are images of Σ_B under the map ι (cf. Theorem 2.16). Let us denote by $\mathcal{K} \subset \mathcal{I}$ the union

$$\mathcal{K} = \bigcup_{B \in \mathbb{N}} \mathcal{I}_B.$$

It follows from Theorem 2.16 that for every integer $B > 0$, the set $\mathcal{I}_B$ is sequentially compact and has a topological structure of a Cantor set.

LEMMA 4.14. *Let $\mu > 0$ be the universal constant from Theorem 2.25. Then there exists a continuous embedding $\zeta \mapsto \mathcal{H}_\zeta$ from $\mathcal{K}$ to $\mathbf{H}(\mu)$ such that for every critical commuting pair $\zeta \in \mathcal{K}$, the image $\mathcal{H}_\zeta = \mathcal{H} = (\eta, \xi, \nu)$ is its holomorphic pair extension, and that the inverse map $\eta^{-1} = \eta^{-1}|_{\overline{\Delta_{\mathcal{H}} \cap \mathbb{H}}}$ does not have fixed points on the real line.*

Proof. Let $s > 0$ be the same as in Theorem 2.16. It follows from Theorem 2.16 that for every $\zeta \in \mathcal{K}$, there exists an infinite sequence of commuting pairs $\zeta_{-1}, \zeta_{-2}, \ldots \in \mathbf{E}_s$ such that $\mathcal{R}^k \zeta_{-k} = \zeta$ for all $k \in \mathbb{N}$. It follows from Lemma 2.11 and Proposition 2.15 that $\mathbf{E}_s$ is a compact subset of $\mathcal{A}_r$ for some $r > 0$. Then Theorem 2.25 implies that ζ extends to a holomorphic pair $\mathcal{H} = (\eta, \xi, \nu)$ from $\mathbf{H}(\mu)$. Theorem 2.25 also implies that if η^{-1} has a fixed point on the real line, then this point is a periodic point of a critical commuting pair $\zeta_{-k} \in \mathcal{K}$ for some $k \in \mathbb{N}$. However, the latter is not possible, since ζ_{-k} has an irrational rotation number.

Now, for each $\zeta \in \mathcal{K}$, the constructed correspondence $\zeta \mapsto \mathcal{H}$ can be extended to a continuous map from a sufficiently small neighborhood of ζ in $\mathbf{E}$ to the space $\mathbf{H}(\mu)$ preserving the property that the map η^{-1} does not have fixed points on $\mathbb{R}$. Finally, the set $\mathcal{K} \subset \mathbf{E}$ is a countable union of nested topological Cantor sets. Hence, it can be covered by a countable collection of small enough open disjoint neighborhoods in each of which we can choose the holomorphic pair extension $\mathcal{H}_\zeta$ in such a way that it depends continuously on ζ. Continuity of the inverse map $\mathcal{H}_\zeta \mapsto \zeta$ follows easily. □

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Definition 4.15. For every positive integer $B > 0$, we denote by $\tilde{\mathcal{I}}_B \subset \mathbf{H}(\mu)$ the image of $\mathcal{I}_B$ under the map

$$\zeta \mapsto \mathcal{H}_\zeta.$$

Similarly, we denote by $\tilde{\mathcal{K}} \subset \mathbf{H}(\mu)$ the image of $\mathcal{K}$ under the same map.

LEMMA 4.16. *For every holomorphic pair $\mathcal{H} \in \tilde{\mathcal{K}}$, there exists a fundamental crescent $C_\mathcal{H}$ with a fattening $C^o_\mathcal{H}$ such that the mapping $\mathcal{H} \mapsto C^o_\mathcal{H}$ is continuous on $\tilde{\mathcal{K}}$. Moreover, there exist two Jordan domains $\tilde{U} \Subset U_1 \subset \mathbb{C}$ such that*

$$[-1/2, 1/2] \subset \tilde{U}$$

and, for every $\mathcal{H} \in \tilde{\mathcal{K}}$ and the corresponding fattening $C^o_\mathcal{H}$, the map $\pi^{-1}_{C^o_\mathcal{H}}$ is defined and univalent on the domain U_1 and

$$|\pi^{-1}_{C^o_\mathcal{H}}(1/2) - \pi^{-1}_{C^o_\mathcal{H}}(-1/2)| > a > 0 \tag{18}$$

for some universal constant $a \in \mathbb{R}$.

Proof. First, we notice that $\mathcal{K} \subset \mathbf{E}_s$ for the fixed value of $s > 0$ from Theorem 2.16. Now it follows from Lemmas 4.14 and 4.10 that for every holomorphic pair $\mathcal{H} \subset \tilde{\mathcal{K}}$, we can choose a fundamental crescent $C_\mathcal{H}$ that has a fattening $C^o_\mathcal{H}$ such that a certain fixed domain $U_1 \supset [-1/2, 1/2]$ is compactly contained in $\pi_{C^o_\mathcal{H}}(C^o_\mathcal{H})$ and condition (18) holds. According to Lemma 4.6, this fundamental crescent and its fattening may be chosen to depend locally continuously on $\mathcal{H} \in \mathbf{H}(\mu)$. Since $\tilde{\mathcal{K}} \subset \mathbf{H}(\mu)$ is a countable union of nested Cantor sets, the global continuous dependence $\mathcal{H} \mapsto C^o_\mathcal{H}$ is obtained in the same way as in Lemma 4.14.

Finally, we choose the domain $\tilde{U}$ so that $\tilde{U} \Subset U_1$ and $[-1/2, 1/2] \subset \tilde{U}$, which completes the proof. □

Let $\mathcal{H} \in \tilde{\mathcal{K}}$ be a holomorphic commuting pair with a fundamental crescent $C_\mathcal{H}$ provided by Lemma 4.16. For a point $z \in C_\mathcal{H}$, whose iterates under $\mathcal{H}$ eventually return to $C_\mathcal{H}$, we define $R_{C_\mathcal{H}}(z)$ to be the first return of z to $C_\mathcal{H}$ under the dynamics of $\mathcal{H}$. Then, according to [**Yam02**, Proposition 7.9], there exists an equatorial neighborhood in the cylinder $\mathbb{C}/\mathbb{Z}$ such that for every $\mathcal{H} \in \tilde{\mathcal{K}}$, the map

$$f_\mathcal{H} \equiv \tilde{\pi}_{C_\mathcal{H}} \circ R_{C_\mathcal{H}} \circ \tilde{\pi}^{-1}_{C_\mathcal{H}} \tag{19}$$

is defined and analytic in that neighborhood.

LEMMA 4.17. *There exists a positive real number $\tilde{t} > 0$ such that for every holomorphic pair $\mathcal{H} \in \tilde{\mathcal{K}}$, the map $\phi_\mathcal{H} = \pi'_{C_\mathcal{H}}(0) \cdot \pi^{-1}_{C_\mathcal{H}}$ is contained in the space $\mathbf{B}^3_{\tilde{U},\tilde{t}}$, where $\tilde{U}$ is the same as in Lemma 4.16.*

Proof. According to Lemma 4.16, the family of maps

$$\{\pi^{-1}_{C^o_\mathcal{H}}|_{U_1} : \mathcal{H} \in \tilde{\mathcal{K}}\}$$

is defined and since, by Definition 2.22, the image $\pi^{-1}_{C^o_\mathcal{H}}(U_1) \subset \Delta_\mathcal{H}$ is uniformly bounded, this family is normal. According to (18), every map π^{-1} from the closure of this family is non-constant and hence univalent and there exists a positive number $t > 0$ such that $\pi'(0) \cdot \pi^{-1} \in \mathbf{B}^3_{\tilde{U},t}$. Now the existence of $\tilde{t} > 0$, satisfying the conditions of Lemma 4.17, follows from compactness arguments. □

LEMMA 4.18. *Let $\tilde{U}$ and $\tilde{t}$ be the same as in Lemmas 4.16 and 4.17, and fix a family of maps $\pi_{\phi,\alpha,\tilde{t}} : Q_{\phi,\alpha,\tilde{t}} \to \mathbb{C}/\mathbb{Z}$, $(\alpha, \phi) \in \mathbf{B}_{\tilde{U},\tilde{t}}$ provided by Lemma 3.20. Then there exist positive real numbers $r_1 > 0$ and $h_1 > 0$ such that for every $\mathcal{H} \in \tilde{\mathcal{K}}$ and the corresponding map $\phi_{\mathcal{H}}$ from Lemma 4.17, there exists a unique real-symmetric map $\psi_{\mathcal{H}} \in \mathbf{D}_{h_1}$ with the property that the composition*

$$z \mapsto \psi_{\mathcal{H}}(\pi_{\phi_{\mathcal{H}},\alpha,\tilde{t}}((\phi_{\mathcal{H}}(z))^3)) \tag{20}$$

is defined in the domain $\{z \in \mathbb{C} \mid |\mathrm{Re}\, z| < 1/2;\ |\mathrm{Im}\, z| < r_1\}$ and coincides with the map

$$z \mapsto (\tilde{\pi}_{C_{\mathcal{H}}} \circ R_{C_{\mathcal{H}}} \circ \pi^{-1}_{C_{\mathcal{H}}})(z)$$

on that domain.

Proof. Since, for every $n \in \mathbb{N}$, every $\zeta \in \mathcal{K}$ has a pre-image $\mathcal{R}^{-n}\zeta \in \mathcal{K}$, it follows from Theorem 2.16 that each map η, ξ, ν from the holomorphic pair $\mathcal{H}_\zeta = (\eta, \xi, \nu)$ can be represented in the form

$$z \mapsto g(z^3),$$

where g is a conformal map. Since $R_{C_{\mathcal{H}}}$ is a composition of such maps and is a first return map to the neighborhood of the origin, it can also be represented in the same form. Thus, we get the following identity:

$$(\tilde{\pi}_{C_{\mathcal{H}}} \circ R_{C_{\mathcal{H}}} \circ \pi^{-1}_{C_{\mathcal{H}}})(z) = \tilde{\pi}_{C_{\mathcal{H}}}(g((\pi^{-1}_{C_{\mathcal{H}}}(z))^3)),$$

where g is a conformal map that depends on $R_{C_{\mathcal{H}}}$. On the other hand, we have

$$\psi_{\mathcal{H}}(\pi_{\phi_{\mathcal{H}},\alpha,\tilde{t}}((\phi_{\mathcal{H}}(z))^3)) = \psi_{\mathcal{H}}(\pi_{\phi_{\mathcal{H}},\alpha,\tilde{t}}((\pi'_{C_{\mathcal{H}}}(0))^3 \cdot (\pi^{-1}_{C_{\mathcal{H}}}(z))^3)).$$

Now, canceling the identical cubic parts, we get the following equation that should be satisfied by $\psi_{\mathcal{H}}$:

$$\tilde{\pi}_{C_{\mathcal{H}}}(g(z)) = \psi_{\mathcal{H}}(\pi_{\phi_{\mathcal{H}},\alpha,\tilde{t}}((\pi'_{C_{\mathcal{H}}}(0))^3 \cdot z)). \tag{21}$$

Since all maps in this equation are conformal, the equation uniquely determines the map $\psi_{\mathcal{H}}$ on the domain, where appropriate compositions are defined.

Finally, since the map $f_{\mathcal{H}}$ from (19) is also defined in a fixed equatorial neighborhood, independent of $\mathcal{H} \in \tilde{\mathcal{K}}$, the left-hand side of (21) is defined in some fixed neighborhood. Now the existence of positive numbers r_1 and h_1 is proved using a compactness argument, similar to the one used in the proof of Lemma 4.17. □

Remark 4.19. In particular, Lemma 4.18 implies that for every $\mathcal{H} \in \tilde{\mathcal{K}}$, we have the inclusion

$$\tau = (3, \phi_{\mathcal{H}}, \psi_{\mathcal{H}}) \in \mathbf{P}_{\tilde{U},\tilde{t},h_1}$$

and the map f_τ defined in (13) belongs to the class $\mathbf{C}_{r_1}$ and coincides with the map $f_{\mathcal{H}}$ from (19).

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Definition 4.20. For a positive integer $B > 0$, by $\hat{\mathcal{I}}_B \subset \mathbf{C}^3_{r_1}$ we denote the image of the set $\tilde{\mathcal{I}}_B \subset \mathbf{H}(\mu)$ under the map $\mathcal{H} \mapsto f_{\mathcal{H}}$ given by (19). Similarly, by $\hat{\mathcal{K}} \subset \mathbf{C}^3_{r_1}$ we denote the image of the set $\tilde{\mathcal{K}} \subset \mathbf{H}(\mu)$ under the same map.

Definition 4.21. Let the constants $\tilde{t}$, h_1, and the domain $\tilde{U}$ be the same as in Lemma 4.17, Lemma 4.18, and Lemma 4.16. For a positive integer $B > 0$, by $\check{\mathcal{I}}_B \subset \mathbf{P}_{\tilde{U},\tilde{t},h_1}$ we denote the image of $\tilde{\mathcal{I}}_B$ under the map

$$\mathcal{H} \mapsto (3, \phi_{\mathcal{H}}, \psi_{\mathcal{H}}),$$

where $\phi_{\mathcal{H}}$ and $\psi_{\mathcal{H}}$ are the same as in Lemma 4.17 and Lemma 4.18, respectively. Similarly, by $\check{\mathcal{K}} \subset \mathbf{P}_{\tilde{U},\tilde{t},h_1}$ we denote the image of the set $\tilde{\mathcal{K}} \subset \mathbf{H}(\mu)$ under the same map.

We recall that if positive numbers r and h are such that $r < r_1$ and $h < h_1$, then there are natural inclusions $\mathbf{C}_{r_1} \subset \mathbf{C}_r$ and $\mathbf{P}_{\tilde{U},\tilde{t},h_1} \subset \mathbf{P}_{\tilde{U},\tilde{t},h}$; hence, we can view the sets $\hat{\mathcal{K}}$ and $\check{\mathcal{K}}$ as subsets of $\mathbf{C}_r$ and $\mathbf{P}_{\tilde{U},\tilde{t},h}$, respectively.

LEMMA 4.22. *There exist positive constants $\tilde{h}$, $\tilde{r}$, and an open set $\check{\mathcal{U}} \subset \mathbf{P}_{\tilde{U},\tilde{t},\tilde{h}}$ such that $0 < \tilde{h} < h_1$, $0 < \tilde{r} < r_1$, $\check{\mathcal{K}} \Subset \check{\mathcal{U}}$, and the mapping*

$$\tau \mapsto f_\tau$$

is an analytic map from $\check{\mathcal{U}} \subset \mathbf{P}_{\tilde{U},\tilde{t},\tilde{h}}$ to $\mathbf{C}_{\tilde{r}}$.

Proof. Fix a constant $\tilde{h} < h_1$. According to Lemma 4.18, for every $\tau = (3, \phi_{\mathcal{H}}, \psi_{\mathcal{H}})$ from the closure of $\check{\mathcal{K}}$, the map $\psi_{\mathcal{H}}$ is defined on the annulus V_{h_1}. If we view $\psi_{\mathcal{H}}$ as the same map but restricted to a smaller annulus $V_{\tilde{h}}$, then, according to Lemma 4.18 and the Koebe distortion theorem, the composition (20) is guaranteed to be defined in the set $\{z \in \mathbb{C} \mid |\mathrm{Re}\, z| < 1/2;\ |\mathrm{Im}\, z| < \hat{r}\}$, where $\hat{r}$ is some constant such that $0 < \hat{r} \le r_1$. Finally, we set $\tilde{r} = \hat{r}/2$. Then Lemma 3.22 implies the existence of an open set $\check{\mathcal{U}} \subset \mathbf{P}_{\tilde{U},\tilde{t},\tilde{h}}$ with the required properties. □

For future reference, we formulate the following lemma.

LEMMA 4.23. *There exists a positive real number $r > 0$ such that every critical commuting pair $\zeta = (\eta|_{I_\eta}, \xi|_{I_\xi}) \in \mathcal{K}$ belongs to the class $\mathcal{A}_r$ from Definition 2.24 and, if $\mathcal{H} = \mathcal{H}_\zeta \in \tilde{\mathcal{K}}$, then*

$$\tilde{\pi}_{C^o_{\mathcal{H}}}(C^o_{\mathcal{H}} \cap (N_r(I_\eta) \cup N_{r|\eta(0)|}(I_\xi))) \subset V_{\tilde{r}}.$$

Furthermore, there exists $K \in \mathbb{N}$ with the property that for every positive integer $k \ge K$, and for every $\zeta \in \mathcal{K}$, the critical commuting pair $p\mathcal{R}^k\zeta$ restricts (in the sense of Theorem 2.25) to a holomorphic commuting pair $\mathcal{H}_k : \Omega_k \to \Delta_k$ such that $\Delta_k \Subset C_{\mathcal{H}} \cap (N_r(I_\eta) \cup N_{r|\eta(0)|}(I_\xi))$, and $\mathcal{H}_k$ is an affine rescaling of a holomorphic pair from $\tilde{\mathcal{K}}$.

Proof. Existence of the number r that satisfies the first part of the lemma follows easily from Lemma 2.11, Lemma 4.16, and the Koebe distortion theorem.

Let s and μ be the same as in Theorem 2.16 and Theorem 2.25, respectively. Since $\mathcal{K} \subset \mathbf{E}_s$ and $\mathcal{K} \subset \mathcal{A}_r$, the set $\mathcal{K}$ is pre-compact in $\mathcal{A}_r$; hence, by Theorem 2.25, there exists $N = N(r, \mathcal{K})$ such that for every $\zeta \in \mathcal{K}$, the pre-renormalization $p\mathcal{R}^N\zeta$ restricts (in the sense of Theorem 2.25) to some holomorphic pair $\mathcal{G}_\zeta$ whose appropriate affine rescaling lies in $\mathbf{H}(\mu)$ and the range of $\mathcal{G}_\zeta$ is compactly contained in $C_{\mathcal{H}} \cap (N_r(I_\eta) \cup N_{r|\eta(0)|}(I_\xi))$. Then we note that the range of holomorphic pairs $\mathcal{R}^k(\mathcal{G}_\zeta)$ is the same for all $k = 0, 1, 2, \ldots$; hence, Lemma 2.13 and properties (ii) and (v) of Definition 2.22 imply that there exists a positive integer $L \in \mathbb{N}$ with the property that for every $k \geq L$ and $\zeta \in \mathcal{K}$, if $\mathcal{G}$ is a holomorphic pair extension of $p\mathcal{R}^{k+N}\zeta$ such that an appropriate affine rescaling of $\mathcal{G}$ lies in $\mathbf{H}(\mu)$, then $\mathcal{G}$ is a restriction of the holomorphic pair $p\mathcal{R}^k(\mathcal{G}_\zeta)$. This implies that the range of $\mathcal{G}$ is compactly contained in $C_{\mathcal{H}} \cap (N_r(I_\eta) \cup N_{r|\eta(0)|}(I_\xi))$ and $\mathcal{G}$ is a restriction of $p\mathcal{R}^{k+N}\zeta$ in the sense of Theorem 2.25.

We finish the proof by putting $K = N + L$. Now the choice of holomorphic pairs $\mathcal{H}_k$ is possible, since $\mathcal{R}(\mathcal{K}) = \mathcal{K}$. □

5. *Renormalization in* $\mathbf{C}_r$

5.1. *Renormalization with respect to a fundamental crescent.*

We let V_r be as in (8). Assume that $f \in \mathbf{C}_r$ for some real number $r \in \mathbb{R}$, $0 < r < 0.5$. Then we can restrict f to an analytic (single-valued) map $\breve{f} : V_r \setminus \{\operatorname{Re} z = 0\} \to \mathbb{C}/\mathbb{Z}$.

Definition 5.1. Let $f \in \mathbf{C}_r$ be a generalized critical cylinder map and let $C_f \subset V_r$ be its fundamental crescent of period n. For a point $z \in C_f$ such that $\{\breve{f}^j(f(z))\}_{j\in\mathbb{N}} \cap C_f \neq \varnothing$, we define $R_{C_f}(z)$ to be the first return of $f(z)$ to C_f under the map $\breve{f}$. In general, R_{C_f} is a multiple-valued map, since f is also multiple-valued.

Definition 5.2. Given a generalized critical cylinder map $f \in \mathbf{C}_r$ and its fundamental crescent $C_f \subset V_r$ of period n, let us say that f is renormalizable with respect to the fundamental crescent C_f if there exists a positive real number $r' > 0$ such that the composition

$$\hat{f} = \tilde{\pi}_{C_f} \circ R_{C_f} \circ \tilde{\pi}_{C_f}^{-1}$$

belongs to $\mathbf{C}_{r'}$.

We will say that the generalized critical cylinder map $\hat{f} \in \mathbf{C}_{r'}$ is the *renormalization of* f *with respect to the fundamental crescent* C_f and we will denote it by

$$\hat{f} = \mathcal{R}_{C_f} f.$$

The following lemma is easy to verify.

LEMMA 5.3. *If $f \in \mathbf{C}_r^\alpha$ for some $\alpha \in \mathbb{C}$, and f is renormalizable with respect to a fundamental crescent C_f, then $\mathcal{R}_{C_f} f \in \mathbf{C}_{r'}^\alpha$. In other words, renormalization does not change the critical exponent at zero.*

5.2. *Cylinder renormalization operator* $\mathcal{R}_{\mathrm{cyl}}$.

Let μ be as in Theorem 2.25 and let $\hat{\mathcal{K}} \subset \mathbf{C}_{\tilde{r}}^3$ be the same as in Definition 4.20. We also fix $r_1 > \tilde{r}$ as in Lemmas 4.18 and 4.22 and let $\tilde{U}$ be as in Lemma 4.16.

Remark 5.4. It follows from real *a priori* bounds (cf. Lemma 2.13) and the Koebe distortion theorem that there exists a positive integer $L > 0$ with the property that for every

$f \in \hat{\mathcal{K}}$ and for every real-symmetric conformal map $h : \tilde{U} \to \mathbb{C}$ such that $h([-1/2, 1/2])$ is contained in the domain of the commuting pair $p\mathcal{R}^L f$, we have

$$|h'(0)| < 1/3.$$

Set N to be the smallest even integer greater than $\max(K, L)$, where K is the same as in Lemma 4.23 and L is the same as in Remark 5.4.

Remark 5.5. The fact that the number N is even will be important in §6.

LEMMA 5.6. *There exist a positive real number $r_2 > \tilde{r}$, an open neighborhood $U_2 \subset \mathbb{C}$ with $\tilde{U} \Subset U_2$, and an open set $\mathcal{U} \subset \mathbf{C}_{\tilde{r}}$ such that $\hat{\mathcal{K}} \subset \mathcal{U}$ and, for each $f \in \mathcal{U}$, there exists a choice of a fundamental crescent $C_f \subset V_{\tilde{r}}$ and its fattening $C_f^o \subset V_{\tilde{r}}$ such that both C_f and C_f^o depend continuously on $f \subset \mathcal{U}$ and the following holds:*

(i) *C_f has period q_{N-1}, where q_{N-1} is the denominator of the $(N-1)$st convergent of $\rho(f)$, written in the irreducible form;*

(ii) *the dependence $f \mapsto \pi_{C_f^o}(z)$ is analytic for any fixed $z \in C_f^o$;*

(iii)

$$\mathcal{R}_{C_f} f \in \mathbf{C}_{r_2},$$

the map $\pi_{C_f^o}^{-1}$ is defined and univalent on U_2, and $|(\pi_{C_f^o}^{-1})'(0)| < 1/2$;

(iv) *for $\zeta \in \mathcal{K}$, $\mathcal{H} = \mathcal{H}_\zeta \in \tilde{\mathcal{K}}$, and $f = f_{\mathcal{H}} \in \hat{\mathcal{K}}$, the fundamental crescent C_f and its fattening C_f^o satisfy the relations*

$$C_f = \tilde{\pi}_{C_{\mathcal{H}}}(C_{\mathcal{H}_1}), \quad C_f^o = \tilde{\pi}_{C_{\mathcal{H}}}(C_{\mathcal{H}_1}^o),$$

where $\mathcal{H}_1$ is a linear rescaling of the holomorphic pair $\mathcal{H}_{\mathcal{R}^N\zeta}$ and $C_{\mathcal{H}_1}$ and $C_{\mathcal{H}_1}^o$ are linear rescalings of $C_{\mathcal{H}_{\mathcal{R}^N\zeta}}$ and $C_{\mathcal{H}_{\mathcal{R}^N\zeta}}^o$, respectively.

Proof. For an arbitrary $f = f_{\mathcal{H}} \in \hat{\mathcal{K}}$ with the corresponding holomorphic commuting pair $\mathcal{H} \in \tilde{\mathcal{K}}$ and the underlying critical commuting pair $\zeta \in \mathcal{K}$, the renormalization $\mathcal{R}^N\zeta$ belongs to $\mathcal{K}$ and hence it extends to a holomorphic commuting pair $\mathcal{H}_{\mathcal{R}^N\zeta} \in \mathbf{H}(\mu)$. Let $\mathcal{H}_1$ be the holomorphic pair extension of the pre-renormalization $p\mathcal{R}^N\zeta$ such that $\mathcal{H}_1$ is a linear rescaling of $\mathcal{H}_{\mathcal{R}^N\zeta}$. Let $C_{\mathcal{H}}$ be the fundamental crescent of $\mathcal{H}$ from Lemma 4.16. From Lemma 4.23, we can see that $\Delta_{\mathcal{H}_1} \Subset C_{\mathcal{H}}$ and the projection $\pi_{C_{\mathcal{H}}}$ conjugates $\mathcal{H}_1$ with a holomorphic pair extension $\mathcal{H}_2$ of $p\mathcal{R}^{N-1}(f)$ such that $\Delta_{\mathcal{H}_2} \Subset V_{\tilde{r}}$ and the correspondence $f \mapsto \mathcal{H}_2$ extends to a continuous map from a neighborhood of $f \in \hat{\mathbf{C}}_r^3$ to $\mathbf{H}$.

Let $C_{\mathcal{H}_1}$ and $C_{\mathcal{H}_1}^o$ be the fundamental crescent and its fattening that are linear rescalings of the fundamental crescent $C_{\mathcal{H}_{\mathcal{R}^N\zeta}}$ and the corresponding fattening $C_{\mathcal{H}_{\mathcal{R}^N\zeta}}^o$, provided by Lemma 4.16. Then we set

$$C_f = \tilde{\pi}_{C_{\mathcal{H}}}(C_{\mathcal{H}_1}) \quad \text{and} \quad C_f^o = \tilde{\pi}_{C_{\mathcal{H}}}(C_{\mathcal{H}_1}^o).$$

We notice that our construction satisfies property (iv) of the lemma. Now it follows from the above argument that C_f is a fundamental crescent for $\mathcal{H}_2$; hence, according to Definition 4.8, it is also a fundamental crescent for $f \in \hat{\mathcal{K}}$ of period q_{N-1} and C_f^o is

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a fattening of C_f. Moreover, since $\pi_{C_{\mathcal{H}}}$ is a conjugacy between $\mathcal{H}_1$ and $\mathcal{H}_2$, we have $\mathcal{R}_{C_f} f = f_{\mathcal{H}_1} \in \mathbf{C}_{r_1}$, where r_1 is the same as in Lemma 4.18.

Now Lemma 4.6 implies that the correspondence $f \mapsto C_f^o$ extends to a continuous map on some neighborhood of $f \in \mathbf{C}_{\tilde{r}}$, satisfying property (ii). Choose $r_2 \in \mathbb{R}$ and $U_2 \subset \mathbb{C}$ so that $\tilde{r} < r_2 < r_1$ and $\tilde{U} \Subset U_2 \Subset U_1$. Then, according to Lemma 4.16, Remark 4.19, Remark 5.4, and a continuity argument, by possibly shrinking the neighborhood of f, we can ensure that property (iii) also holds. Finally, in order to construct the open set $\mathcal{U}$, we use the fact that the set $\hat{\mathcal{K}}$ is a countable union of nested Cantor sets: the rest of the proof goes in the same way as the proof of Lemma 4.14. □

Definition 5.7. Let $\mathcal{U} \subset \mathbf{C}_{\tilde{r}}$ be the open set constructed in Lemma 5.6. Then we define the *cylinder renormalization operator* $\mathcal{R}_{\text{cyl}} : \mathcal{U} \to \mathbf{C}_{\tilde{r}}$ as

$$\mathcal{R}_{\text{cyl}}(f) = \mathcal{R}_{C_f} f,$$

where C_f is the canonical fundamental crescent for $f \in \mathcal{U}$, constructed in Lemma 5.6.

Our definition extends the definition given by the second author in [**Yam02**] to a wider class of analytic maps.

Definition 5.8. By $\hat{\mathcal{R}}_{\text{cyl}}$, we denote the restriction of the cylinder renormalization operator $\mathcal{R}_{\text{cyl}}$ to the subset $\mathcal{U} \cap \hat{\mathbf{C}}^3_{\tilde{r}}$, where $\hat{\mathbf{C}}^3_{\tilde{r}} \subset \mathbf{C}^3_{\tilde{r}}$ is the set from Definition 3.6.

In the following proposition, we summarize some basic properties of the cylinder renormalization operator.

PROPOSITION 5.9.

(i) *The cylinder renormalization operator* $\mathcal{R}_{\text{cyl}}$ *is a real-symmetric analytic operator* $\mathcal{U} \to \mathbf{C}_{\tilde{r}}$.

(ii) *For every critical commuting pair* $\zeta \in \mathcal{K}$, *we have*

$$\mathcal{R}_{\text{cyl}} f_{\mathcal{H}_\zeta} = f_{\mathcal{H}_{(\mathcal{R}^N \zeta)}} \quad \text{where } f_{\mathcal{H}_{(\mathcal{R}^N \zeta)}} \text{ is as in (19).}$$

(iii) $\mathcal{R}_{\text{cyl}}(\hat{\mathcal{I}}_B) = \hat{\mathcal{I}}_B$ *for every* $B \in \mathbb{N}$.

(iv) *For every complex number* $\alpha \in \mathbb{C}$ *sufficiently close to* 3, *we have*

$$\mathcal{R}_{\text{cyl}}(\mathcal{U} \cap \mathbf{C}^\alpha_{\tilde{r}}) \subset \mathbf{C}^\alpha_{\tilde{r}}.$$

(v) $\hat{\mathcal{R}}_{\text{cyl}}(\mathcal{U} \cap \hat{\mathbf{C}}^3_{\tilde{r}}) \subset \hat{\mathbf{C}}^3_{\tilde{r}}$.

(vi) *At every point* $f \in \mathcal{U} \cap \hat{\mathbf{C}}^3_{\tilde{r}}$, *the differential* $D_f \hat{\mathcal{R}}_{\text{cyl}} : T_f \hat{\mathbf{C}}^3_{\tilde{r}} \to T_{\mathcal{R}_{\text{cyl}} f} \hat{\mathbf{C}}^3_{\tilde{r}}$ *is a compact operator.*

Proof. The analyticity part of the statement (i) follows from Lemma 5.6. Real symmetry is evident from the construction. Properties (ii), (iii), and (v) were established in [**Yam02**]. Finally, (iv) follows from Lemmas 5.6 and 5.3.

By Montel's theorem, a bounded set in the tangent space $T_{\hat{\mathcal{R}}_{\text{cyl}} f} \hat{\mathbf{C}}^3_{r_2}$ is normal and hence pre-compact in $T_{\hat{\mathcal{R}}_{\text{cyl}} f} \hat{\mathbf{C}}^3_{\tilde{r}}$. Since differential is a bounded operator, property (vi) follows from this and property (iii) of Lemma 5.6 (cf. [**Yam02**, Proposition 9.1]). □

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6. *Renormalization in* $\mathbf{P}_{U,t,h}$

Let the maps $p_{\phi,\alpha+}$ and $p_{\phi,\alpha-}$ be defined in the same way as in §3 after Definition 3.21. The following technical statement is straightforward to verify.

PROPOSITION 6.1. *Let the maps $p_{\phi,\alpha+}$ and $p_{\phi,\alpha-}$ be as above and let $c \in \mathbb{C}$ be a non-zero complex number such that* $\operatorname{Re} c > 0$. *Assume that* $\operatorname{Re} \phi'(0) > 0$. *Then*

$$p_{\phi,\alpha+} \equiv \frac{1}{p_{\alpha+}(c)} p_{c\phi,\alpha+} \quad \textit{and} \quad p_{\phi,\alpha-} \equiv \frac{1}{p_{\alpha+}(c)} p_{c\phi,\alpha-}.$$

LEMMA 6.2. *Let r_2 and U_2 be the same as in Lemma 5.6. There exist a positive real number $h_2 > \tilde{h}$ and an open set $\tilde{\mathcal{U}} \subset \mathbf{P}_{\tilde{U},\tilde{t},\tilde{h}}$ such that $\check{\mathcal{K}} \subset \tilde{\mathcal{U}}$ and, for every $\tau = (\alpha, \phi, \psi) \in \tilde{\mathcal{U}}$, the following properties hold:*

(i) *the generalized critical cylinder map f_τ from (13) is contained in the open set $\mathcal{U}$ from Definition 5.7 and, hence, has the canonical fundamental crescent C_{f_τ} with the fattening $C^o_{f_\tau}$, constructed in Lemma 5.6;*

(ii) $\operatorname{Re} \pi'_{C^o_{f_\tau}}(0) > 0$;

(iii) *the map $\tilde{\phi} = \pi'_{C_{f_\tau}}(0) \cdot \phi \circ \pi^{-1}_{C^o_{f_\tau}}$ is defined in U_2;*

(iv) *$\tilde{\phi} \in \mathbf{B}^{\alpha}_{U_3,\tilde{t}}$ for every domain U_3 such that $\tilde{U} \subseteq U_3 \Subset U_2$;*

(v) *there exists a unique map $\tilde{\psi} \in \mathbf{D}_{h_2}$ such that the following identity holds in the domain $\{z \in \mathbb{C} \mid |\operatorname{Re} z| < 1/2;\ |\operatorname{Im} z| < r_2\}$:*

$$\tilde{\pi}_{C_{f_\tau}} \circ R_{C_{f_\tau}} \circ \pi^{-1}_{C_{f_\tau}} \equiv \tilde{\psi} \circ \pi_{\tilde{\phi},\alpha,\tilde{t}} \circ p_{\tilde{\phi},\alpha\pm},$$

where $\{\pi_{\phi,\alpha,\tilde{t}} \mid (\alpha, \phi) \in \mathbf{B}_{\tilde{U},\tilde{t}}\}$ is the same family of maps as in Lemma 4.18 and both sides of the identity are multiple-valued functions.

Proof. Let $h_1 > \tilde{h}$ be the same as in Lemmas 4.18 and 4.22 and choose $h_2 \in \mathbb{R}$ so that $\tilde{h} < h_2 < h_1$. According to Lemma 4.22, we can choose $\tilde{\mathcal{U}}$ so that $\check{\mathcal{K}} \subset \tilde{\mathcal{U}}$, and (i) holds. For $\tau \in \tilde{\mathcal{U}}$, let C_{f_τ} and $C^o_{f_\tau}$ be the canonical fundamental crescent and its fattening, constructed in Lemma 5.6. Then it follows from Lemma 5.6 that for all $\tau \in \tilde{\mathcal{U}}$, the map $\tilde{\phi} = \pi'_{C_{f_\tau}}(0) \cdot \phi \circ \pi^{-1}_{C^o_{f_\tau}}$ is defined in the domain U_2, which proves property (iii).

For $\tau = (3, \phi_{\mathcal{H}}, \psi_{\mathcal{H}}) \in \check{\mathcal{K}}$, where $\phi_{\mathcal{H}}$ and $\psi_{\mathcal{H}}$ are the same as in Lemmas 4.17 and 4.18, it follows from part (iv) of Lemma 5.6 that

$$\pi'_{C_{f_\tau}}(0) \cdot \phi_{\mathcal{H}} \circ \pi^{-1}_{C^o_{f_\tau}} = \pi'_{C_{\mathcal{H}_1}}(0) \cdot \pi^{-1}_{C^o_{\mathcal{H}_1}}, \tag{22}$$

where $\mathcal{H}_1$ is a linear rescaling of a holomorphic pair from $\check{\mathcal{K}}$. Now Lemma 4.17 implies that $\pi'_{C_{\mathcal{H}_1}}(0) \cdot \pi^{-1}_{C^o_{\mathcal{H}_1}} \in \mathbf{B}^3_{\tilde{U},\tilde{t}}$ and, because of property (iii), by continuity considerations, shrinking $\tilde{\mathcal{U}}$, if necessary, we can make property (iv) hold for all $\tau \in \tilde{\mathcal{U}}$.

Since the constant N is even, this implies that for $\tau \in \check{\mathcal{K}}$, the number $\pi'_{C_{f_\tau}}(0)$ is positive. Again, by continuity, shrinking $\tilde{\mathcal{U}}$, if necessary, we guarantee that property (ii) holds.

Now the composition $\tilde{\pi}_{C_{f_\tau}} \circ R_{C_{f_\tau}} \circ \pi^{-1}_{C_{f_\tau}}$ can be naturally represented as

$$\tilde{\pi}_{C_{f_\tau}} \circ R_{C_{f_\tau}} \circ \pi^{-1}_{C_{f_\tau}} = g \circ p_{\phi \circ \pi^{-1}_{C_{f_\tau}},\alpha\pm},$$

where g is a conformal map on some sufficiently large domain. By Proposition 6.1 and property (ii), we have

$$p_{\phi\circ\pi^{-1}_{C_{f_\tau}},\alpha\pm} = \frac{1}{p_{\alpha+}(\pi'_{C_{f_\tau}}(0))}\cdot p_{\tilde{\phi},\alpha\pm}$$

and hence

$$\tilde{\pi}_{C_{f_\tau}}\circ R_{C_{f_\tau}}\circ\pi^{-1}_{C_{f_\tau}} = g_1\circ p_{\tilde{\phi},\alpha\pm}$$

for some conformal map g_1. Now, if $f_\tau\in\hat{\mathcal{K}}$, then, by Lemma 4.18, we know that both sides of the above identity are defined in the domain $\{z\in\mathbb{C}\mid |\mathrm{Re}\,z|<1/2;\ |\mathrm{Im}\,z|<r_1\}$ and the map g_1 can be represented as

$$g_1 = \tilde{\psi}\circ\pi_{\tilde{\phi},\alpha,\tilde{t}},$$

where $\tilde{\psi}\in\mathcal{D}_{h_1}$ for h_1 from Lemma 4.18. Using a continuity argument and possibly shrinking the neighborhood $\tilde{\mathcal{U}}$ again, we guarantee that the above representation of the map g_1 with $\tilde{\psi}\in\mathcal{D}_{h_2}$ holds for every $\tau\in\tilde{\mathcal{U}}$. □

Definition 6.3. We define the renormalization operator $\mathcal{R}_{\mathbf{P}}:\tilde{\mathcal{U}}\to\mathbf{P}_{\tilde{U},\tilde{t},\tilde{h}}$ by the relation

$$\mathcal{R}_{\mathbf{P}}(\alpha,\phi,\psi) = (\alpha,\tilde{\phi},\tilde{\psi}),$$

where $\tilde{\phi}$ and $\tilde{\psi}$ are the same as in properties (iii) and (v) of Lemma 6.2.

In the following proposition, we summarize some basic properties of the renormalization operator $\mathcal{R}_{\mathbf{P}}$.

PROPOSITION 6.4.

(i) *The operator $\mathcal{R}_{\mathbf{P}}$ is real-symmetric and analytic.*

(ii) *For every $\tau\in\tilde{\mathcal{U}}$, we have*

$$\mathcal{R}_{\mathrm{cyl}}f_\tau = f_{\mathcal{R}_{\mathbf{P}}\tau}.$$

(iii) *For every $B\in\mathbb{N}$, let the map $\Psi:\mathcal{I}_B\to\check{\mathcal{I}}_B$ be defined by the relation*

$$\Psi(\zeta) = (3,\phi_{\mathcal{H}_\zeta},\psi_{\mathcal{H}_\zeta}),$$

where $\phi_{\mathcal{H}_\zeta}$ and $\psi_{\mathcal{H}_\zeta}$ are the same as in Lemma 4.17 and Lemma 4.18, respectively. Then Ψ is a homeomorphism between $\mathcal{I}_B$ and $\check{\mathcal{I}}_B$ that conjugates $\mathcal{R}^N|_{\mathcal{I}_B}$ and $\mathcal{R}_{\mathbf{P}}|_{\check{\mathcal{I}}_B}$. In particular, $\check{\mathcal{I}}_B$ is an invariant horseshoe for the operator $\mathcal{R}_{\mathbf{P}}$.

(iv) *For every $\tau\in\tilde{\mathcal{U}}$, the differential $D_\tau\mathcal{R}_{\mathbf{P}}$ is a compact operator.*

Proof. Properties (i) and (ii) are evident from the construction of the operator $\mathcal{R}_{\mathbf{P}}$.

In property (iii), we will first prove that Ψ is a homeomorphism. Indeed, if $\zeta\in\mathcal{I}_B$ and $\tau=(3,\phi_{\mathcal{H}_\zeta},\psi_{\mathcal{H}_\zeta})=\Psi(\zeta)$, then f_τ together with the local coordinate $\phi_{\mathcal{H}_\zeta}$ around the origin completely determines the critical commuting pair $\mathcal{R}\zeta\in\mathcal{I}_B$. Since, according to Theorem 2.16, the operator $\mathcal{R}$ is bijective on $\mathcal{I}_B$, the inverse map Ψ^{-1} is defined. Continuity of Ψ and its inverse follows from the definition of Ψ. Now computation (22) and the fact that $\mathcal{H}_1$ is a linear rescaling of a holomorphic pair from $\check{\mathcal{K}}$ (cf. Lemma 5.6, part (iv)) yield the conjugacy statement of property (iii).

The proof of property (iv) is analogous to the proof of (vi) from Proposition 5.9 and is based on the fact that according to Lemma 6.2, $\mathcal{R}_{\mathbf{P}}(\tilde{\mathcal{U}})\subset\mathbf{P}_{U_3,\tilde{t},h_2}$ for some domain $U_3\Supset\tilde{U}$ and, by Montel's theorem, a bounded set in $T_{\mathcal{R}_{\mathbf{P}}\tau}\mathbf{P}_{U_2,\tilde{t},h_2}$ is pre-compact in $T_{\mathcal{R}_{\mathbf{P}}\tau}\mathbf{P}_{\tilde{U},\tilde{t},\tilde{h}}$. □

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Now we are ready to prove the hyperbolicity results for the renormalization operator $\mathcal{R}_{\mathbf{P}}$.

Definition 6.5. For a complex number $\alpha \in \mathbb{C}$ close to 3, we denote by $\mathcal{R}_{\mathbf{P},\alpha}$ the restriction of the operator $\mathcal{R}_{\mathbf{P}}$ on the set $\tilde{\mathcal{U}} \cap \mathbf{P}^{\alpha}_{\tilde{U},\tilde{t},\tilde{h}}$.

7. *Hyperbolicity of renormalization for odd integer α*

The goal of this section is to establish the following theorem.

THEOREM 7.1. *For every $B \in \mathbb{N}$, the set $\check{\mathcal{I}}_B$ is a uniformly hyperbolic invariant set for the operator $\mathcal{R}_{\mathbf{P},3}$ with a complex one-dimensional unstable direction. Moreover, for every $\tau \in \check{\mathcal{I}}_B$ and the corresponding local stable manifold $W^s(\tau)$, the set $W^s(\tau) \cap (\mathbf{P}^3_{\tilde{U},\tilde{t},\tilde{h}})^{\mathbb{R}}$ consists of all $\omega \in (\mathbf{P}^3_{\tilde{U},\tilde{t},\tilde{h}})^{\mathbb{R}}$ that are sufficiently close to τ and such that $\rho(f_\tau) = \rho(f_\omega)$.*

Before giving a proof of Theorem 7.1, we formulate its immediate corollary.

COROLLARY 7.2. *For every positive integer $B > 0$, there exists an open interval $I = I(B) \subset \mathbb{R}$ such that $3 \in I$ and, for every $\alpha \in I$, the operator $\mathcal{R}_{\mathbf{P},\alpha} : \tilde{\mathcal{U}} \cap \mathbf{P}^{\alpha}_{\tilde{U},\tilde{t},\tilde{h}} \to \mathbf{P}^{\alpha}_{\tilde{U},\tilde{t},\tilde{h}}$ has a hyperbolic horseshoe attractor $\check{\mathcal{I}}^{\alpha}_B \subset (\mathbf{P}^{\alpha}_{\tilde{U},\tilde{t},\tilde{h}})^{\mathbb{R}}$ of type bounded by B. The action of $\mathcal{R}_{\mathbf{P},\alpha}$ on $\check{\mathcal{I}}^{\alpha}_B$ is topologically conjugate to the shift $\sigma^N : \Sigma_B \to \Sigma_B$:*

$$\kappa_\alpha \circ \mathcal{R}_{\mathbf{P},\alpha} \circ \kappa_\alpha^{-1} = \sigma^N$$

and, if

$$\tau = \kappa_\alpha^{-1}(\ldots, r_{-k}, \ldots, r_{-1}, r_0, r_1, \ldots, r_k, \ldots),$$

then

$$\rho(f_\tau) = [r_0, r_1, \ldots, r_k, \ldots].$$

Moreover, for every $\tau \in \check{\mathcal{I}}^{\alpha}_B$, the corresponding local stable manifold $W^s(\tau)$ has complex codimension 1 and, if $\omega \in W^s(\tau) \cap (\mathbf{P}^{\alpha}_{\tilde{U},\tilde{t},\tilde{h}})^{\mathbb{R}}$, then $\rho(f_\tau) = \rho(f_\omega)$.

Remark 7.3. In Theorem 9.6, we will strengthen Corollary 7.2 by showing that for every $\tau \in \check{\mathcal{I}}^{\alpha}_B$, the set $W^s(\tau) \cap (\mathbf{P}^{\alpha}_{\tilde{U},\tilde{t},\tilde{h}})^{\mathbb{R}}$ consists of all $\omega \in (\mathbf{P}^{\alpha}_{\tilde{U},\tilde{t},\tilde{h}})^{\mathbb{R}}$ that are sufficiently close to τ and such that $\rho(f_\tau) = \rho(f_\omega)$.

Proof of Corollary 7.2. For all α sufficiently close to 3, the operator $\mathcal{R}_{\mathbf{P},\alpha}$ can be thought of as an operator acting on a neighborhood of $\mathbf{P}^3_{\tilde{U},\tilde{t},\tilde{h}}$ by the correspondence $(3, \phi, \psi) \mapsto (3, \tilde{\phi}, \tilde{\psi})$, where $\tilde{\phi}$ and $\tilde{\psi}$ are such that $\mathcal{R}_{\mathbf{P},\alpha}(\alpha, \phi, \psi) = (\alpha, \tilde{\phi}, \tilde{\psi})$. Then this operator is a real-symmetric analytic perturbation of $\mathcal{R}_{\mathbf{P},3}$ in a small neighborhood of the attractor $\check{\mathcal{I}}_B$ of type bounded by B. Since this is a hyperbolic horseshoe attractor, it survives for α close to 3.

For every $\tau \in \check{\mathcal{I}}^{\alpha}_B$, the corresponding local stable manifold $W^s(\tau)$ must have complex codimension 1 and, if $\omega \in (\mathbf{P}^{\alpha}_{\tilde{U},\tilde{t},\tilde{h}})^{\mathbb{R}}$ is such that the combinatorics of f_τ and f_ω are asymptotically different, then because of part (ii) of Proposition 6.4 the point ω does not belong to $W^s(\tau)$. □

We will prove Theorem 7.1 at the end of this section. Now let us start by formulating a theorem which is a special case of Theorems 2.19 and 2.21.

THEOREM 7.4. *For any positive integer $B > 0$, the set $\hat{\mathcal{I}}_B \subset \hat{\mathbf{C}}^3_r$ is a uniformly hyperbolic invariant set for the operator $\hat{\mathcal{R}}_{\rm cyl}$ with a one-dimensional unstable direction. Moreover, for any $f \in \hat{\mathcal{I}}$ and the corresponding local stable manifold $W^s(f) \subset \hat{\mathbf{C}}^3_r$, the intersection $W^s(f) \cap (\hat{\mathbf{C}}^3_r)^{\mathbb{R}}$ consists of all critical circle maps g that are sufficiently close to f and such that $\rho(g) = \rho(f)$.*

We proceed with the following construction: according to part (iii) of Lemma 5.6, there exists a real number $R > 0$ such that for every $g \in \mathcal{U}$ the map $\pi^{-1}_{C^o_g}$ is defined in the disk $\mathbb{D}_{2R}$ and

$$\pi^{-1}_{C^o_g}(\mathbb{D}_R) \subset \mathbb{D}_{2R/3}. \tag{23}$$

We denote by $\mathbf{L}$ the space that consists of all bounded analytic functions $\phi : \mathbb{D}_R \to \mathbb{C}$ that are continuous up to the boundary and satisfy the properties $\phi(0) = 0$ and $\phi'(0) = 1$. The space $\mathbf{L}$ equipped with the sup-norm forms an affine complex Banach space ($\mathbf{L} - \mathrm{Id}$ is a complex Banach space).

Definition 7.5. For each cylinder renormalizable map $g \in \mathcal{U}$, we define an affine operator $A_g : \mathbf{L} \to \mathbf{L}$ by the formula

$$A_g(\phi) = \pi'_{C_g}(0) \cdot \phi \circ \pi^{-1}_{C^o_g},$$

where C_g is the canonical fundamental crescent provided by Lemma 5.6.

Condition (23) implies that the operator A_g is well defined for every $g \in \mathcal{U}$ and it follows from Lemma 5.6 that A_g analytically depends on the parameter g.

Definition 7.6. We set $\mathcal{V} = \mathcal{U} \cap \hat{\mathbf{C}}^3_r$ and we define the operator $\tilde{\mathcal{R}} : \mathbf{L} \times \mathcal{V} \to \mathbf{L} \times \hat{\mathbf{C}}^3_{\tilde{r}}$ as the skew product

$$\tilde{\mathcal{R}}(\phi, g) = (A_g(\phi), \hat{\mathcal{R}}_{\rm cyl}\, g).$$

The following lemma is an immediately corollary from (23) and Proposition 5.9, part (vi).

LEMMA 7.7. *For every $\kappa \in \mathbf{L} \times \mathcal{V}$, the differential $D_\kappa \tilde{\mathcal{R}}$ is a compact operator.*

We denote by $\mathcal{J} \subset \mathbf{L} \times \mathcal{V}$ the image of the set $\tilde{\mathcal{K}}$ under the map

$$\mathcal{H} \mapsto (\pi'_{C_{\mathcal{H}}}(0) \cdot \pi^{-1}_{C^o_{\mathcal{H}}}, f_{\mathcal{H}}). \tag{24}$$

Note that the projection of the set $\mathcal{J}$ onto the second coordinate is the set $\hat{\mathcal{K}}$. Similarly, for every positive integer $B > 0$, we denote by $\mathcal{J}_B \subset \mathbf{L} \times \mathcal{V}$ the image of $\tilde{\mathcal{I}}_B$ under the map (24).

LEMMA 7.8. *For every $B \in \mathbb{N}$, the set $\mathcal{J}_B$ is a forward invariant set for the operator $\tilde{\mathcal{R}}$.*

Proof. For $\zeta \in \mathcal{I}_B$ and the corresponding $\mathcal{H} = \mathcal{H}_\zeta \in \tilde{\mathcal{I}}_B$, let $\mathcal{H}_1 \subset \tilde{\mathcal{I}}_B$ be the holomorphic pair $\mathcal{H}_1 = \mathcal{H}_{\mathcal{R}^N \zeta}$. Then, for $(\pi'_{C_\mathcal{H}}(0) \cdot \pi^{-1}_{C^o_\mathcal{H}}, f_\mathcal{H}) \in \mathcal{J}_B$, we have

$$\begin{aligned}\tilde{\mathcal{R}}(\pi'_{C_\mathcal{H}}(0) \cdot \pi^{-1}_{C^o_\mathcal{H}}, f_\mathcal{H}) &= (A_{f_\mathcal{H}}(\pi'_{C_\mathcal{H}}(0) \cdot \pi^{-1}_{C^o_\mathcal{H}}), f_{\mathcal{H}_1}) \\ &= (\pi'_{C_{\mathcal{H}_1}}(0) \cdot \pi^{-1}_{C^o_{\mathcal{H}_1}}, f_{\mathcal{H}_1}) \subset \mathcal{J}_B.\end{aligned}$$

(We notice that the last equality follows from (22).) □

For maps $\phi_0 \in \mathbf{L}$ and $g \in \mathcal{V}$, one can consider a sequence of iterates $\{\tilde{\mathcal{R}}^k(\phi_0, g)\}$. Restricting to the first coordinate, this corresponds to the sequence of functions $\phi_0, \phi_1 = A_g(\phi_0), \phi_2 = A_{\hat{\mathcal{R}}_{\text{cyl}} g}(\phi_1), \phi_3 = A_{\hat{\mathcal{R}}^2_{\text{cyl}} g}(\phi_2), \ldots, \phi_{k+1} = A_{\hat{\mathcal{R}}^k_{\text{cyl}} g}(\phi_k), \ldots$. For simplicity of notation, we will write $\phi_k = A^k_g(\phi_0)$, provided that ϕ_k is defined.

LEMMA 7.9. *There exist a positive real number $\nu < 1$ and a positive integer $M > 0$ such that for any $(m \geq M)$-times cylinder renormalizable $g \in \mathcal{V}$ and $\phi_1, \phi_2 \in \mathbf{L}$, we have $\|A^m_g(\phi_1) - A^m_g(\phi_2)\| \leq \nu \|\phi_1 - \phi_2\|$, where $\|\phi\| = \sup_{z \in \mathbb{D}_R} |\phi(z)|$.*

Proof. Assume that $\|\phi_1 - \phi_2\| = cR^2$ for some positive real number $c > 0$. Then for all $z \in \mathbb{D}_R$ we have the inequality

$$|\phi_1(z) - \phi_2(z)| \leq c|z|^2.$$

We notice that if $g \in \mathcal{V}$ is n-times cylinder renormalizable, then there exists a sequence $C_n \Subset \cdots \Subset C_1 \subset V_{\tilde{r}}$ of n nested fundamental crescents for g such that $\mathcal{R}^m_{\text{cyl}} g = \mathcal{R}_{C_m} g$ and

$$A^m_g(\phi) = \pi'_{C_m}(0) \cdot \phi \circ \pi^{-1}_{C^o_m}$$

for $m \leq n$. By part (iii) of Lemma 5.6, there exists a positive integer $M > 0$ such that for all $m \geq M$, if $g \in \mathcal{V}$ is m-times cylinder renormalizable, then for the corresponding fundamental crescent C_m we have $|(\pi^{-1}_{C_m})'(0)| < 1/17$.

Since $\pi^{-1}_{C_m}$ is defined in the disk $\mathbb{D}_{2R}$, then again from the Koebe distortion theorem it follows that

$$|\pi^{-1}_{C^o_m}(z)| \leq 4|(\pi^{-1}_{C_m})'(0)| \cdot |z|$$

for all $z \in \mathbb{D}_R$. Combining the above estimates for all $z \in \mathbb{D}_R$, we obtain

$$\begin{aligned}|[A^m_g(\phi_1)](z) - [A^m_g(\phi_2)](z)| &\leq |\pi'_{C_m}(0)| \cdot |\phi_1(\pi^{-1}_{C^o_m}(z)) - \phi_2(\pi^{-1}_{C^o_m}(z))| \\ &\leq |\pi'_{C_m}(0)| \cdot c \cdot 16|(\pi^{-1}_{C_m})'(0)|^2 \cdot |z|^2 \leq \tfrac{16}{17} c(R)^2.\end{aligned}$$

Thus,

$$\|A^m_g(\phi_1) - A^m_g(\phi_2)\| \leq \tfrac{16}{17}\|\phi_1 - \phi_2\|.$$ □

LEMMA 7.10. *For every $B \in \mathbb{N}$, the set $\mathcal{J}_B$ is a uniformly hyperbolic invariant set for the operator $\tilde{\mathcal{R}}$ with a one-dimensional unstable direction. Moreover, the stable manifold of every point $(\phi, f) \in \mathcal{J}_B$ is $\mathbf{L} \times W^s(f)$, where $W^s(f)$ is the stable manifold of f for the cylinder renormalization $\hat{\mathcal{R}}_{\text{cyl}}$.*

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Proof. Since $\mathbf{L}$ is an affine Banach manifold, the tangent space to each of its points can be naturally identified with the same Banach space, which we denote by $\mathbf{K}$. Let $E^s \subset T_{\hat{\mathcal{I}}_B} \hat{\mathbf{C}}^3_{\tilde{r}}$ be the stable bundle of the hyperbolic set $\hat{\mathcal{I}}_B$ for the operator $D\hat{\mathcal{R}}_{\text{cyl}}$. Then the set

$$\tilde{E}^s = \mathbf{K} \times E^s$$

is an invariant subbundle of the tangent bundle $T_{\mathcal{J}_B}(\mathbf{L} \times \hat{\mathbf{C}}^3_{\tilde{r}})$ for the operator $D\tilde{\mathcal{R}}$. It follows from Definition 7.6 that the action of $D\tilde{\mathcal{R}}$ on the second factor of $\tilde{E}^s$ is independent from the first factor. Hence, the differential $D\tilde{\mathcal{R}}$ uniformly contracts in the second factor of $\tilde{E}^s$.

Now, by compactness of $\mathcal{J}_B$, the derivative of $A_g(\phi)$ with respect to g is uniformly bounded over all $(\phi, g) \in \mathcal{J}_B$ and, by Lemma 7.9, the derivative of $A_g(\phi)$ with respect to ϕ uniformly contracts. Together with uniform contraction in the second factor, this implies that the differential $D\tilde{\mathcal{R}}$ contracts in the first factor of $\tilde{E}^s$ as well. Thus, $\tilde{E}^s$ is the stable subbundle for the differential $D\tilde{\mathcal{R}}$.

In a similar way, one shows that if E^u is the unstable subbundle (in the sense of Definition 2.20) of the hyperbolic set $\hat{\mathcal{I}}_B$ for the operator $D\hat{\mathcal{R}}_{\text{cyl}}$, then $\tilde{E}^u = \mathbf{0} \times E^u$ is the unstable bundle for the operator $D\tilde{\mathcal{R}}$. Since, according to Theorem 7.4, the fibers of E^u are one dimensional, so are the fibers of $\tilde{E}^u$. Finally, by dimension count we see that there are no neutral directions and hence the set $\mathcal{J}_B$ is uniformly hyperbolic. □

Proof of Theorem 7.1. The analytic map $\Phi : \mathbf{P}^3_{\tilde{U},\tilde{t},\tilde{h}} \to \mathbf{L} \times \hat{\mathbf{C}}^3_{\tilde{r}}$ defined by the relation

$$\Phi : \tau = (3, \phi, \psi) \mapsto (\phi, f_\tau)$$

is injective, since the maps ϕ and f_τ determine the map ψ (cf. Lemma 4.18). Now we notice that the map Φ maps $\check{\mathcal{I}}_B$ to $\mathcal{J}_B$ and conjugates the operator $\mathcal{R}_{\mathbf{P},3}$ with the operator $\tilde{\mathcal{R}}|_{\Phi(\tilde{\mathcal{U}})}$ on the neighborhood of the invariant set $\check{\mathcal{I}}_B$. Then Lemma 7.10 and Theorem 7.4 imply the statement of the theorem. □

8. *Proofs of Theorems 1.1–1.3*

Now we are ready to prove Theorems 1.1–1.3. As mentioned in the Introduction, we prove these theorems with the parameter k set to $k = 1$. For $k > 1$ the proofs are identical.

Proof of Theorem 1.1. We set $\mathbf{N}^3 = \mathbf{C}_{\tilde{r}}$ and define $\mathcal{R}_{\text{cyl}}$ as in Definition 5.7. Then $\mathbf{C}^\alpha = \mathbf{C}^\alpha_{\tilde{r}}$, for all complex α in a suitably small neighborhood of 3, and Theorem 1.1 follows from Proposition 5.9. □

Proof of Theorem 1.2. We set $\mathbf{M}^3$ to be an open subset of $\mathbf{P}^3_{\tilde{U},\tilde{t},\tilde{h}} \cap \check{\mathcal{U}}$, where $\check{\mathcal{U}}$ is the same as in Lemma 4.22, such that the closure of $\mathbf{M}^3$ in $\mathbf{P}^3_{\tilde{U},\tilde{t},\tilde{h}}$ is contained in $\check{\mathcal{U}}$ and $\check{\mathcal{K}} \subset \mathbf{M}^3$. Then for all $\alpha \in \mathbb{C}$ which lie in a small open neighborhood of 3, the map i_α can be defined by

$$i_\alpha : (\phi, \psi) \mapsto f_\tau,$$

where $\tau = (\alpha, \phi, \psi) \in \mathbf{P}^\alpha_{\tilde{U},\tilde{t},\tilde{h}}$. It follows from Lemma 4.22 that the map i_α is analytic and analytically depends on α. Finally, for all α close to 3, we define $\mathcal{R}_\alpha(\phi, \psi) = (\tilde{\phi}, \tilde{\psi})$, where $\tilde{\phi}$ and $\tilde{\psi}$ are such that $\mathcal{R}_{\mathbf{P}}(\alpha, \phi, \psi) = (\alpha, \tilde{\phi}, \tilde{\psi})$ whenever $\mathcal{R}_{\mathbf{P}}(\alpha, \phi, \psi)$ is defined.

The required properties of the operator $\mathcal{R}_\alpha$ follow from Proposition 6.4 and Theorem 7.1. Analyticity of the map i_α follows from Lemma 3.22. □

Proof of Theorem 1.3. The proof immediately follows from Corollary 7.2. □

9. *Global attractor, universality, and rigidity*

9.1. *Generalized critical commuting pairs.* We start by generalizing the notion of a critical commuting pair to the case of an arbitrary real critical exponent. In order to make this generalization, we need to substitute the commutation relation with a more general condition. The following definition generalizes the class $\mathcal{A}_r$ from Definition 2.24.

Definition 9.1. For positive real numbers $r, \alpha > 0$, the class $\mathcal{A}_r^\alpha$ consists of all pairs of real-symmetric analytic maps $\zeta = (\eta, \xi)$ such that the following holds:

(i) the maps η and ξ are analytic on the sets $N_r([0, 1])\backslash(-\infty, 0]$ and $N_{r|\eta(0)|}(I_\xi)\backslash[0, +\infty)$, respectively, where $I_\xi = [b, 0] \subset (-\infty, 0]$ and $N_r(S) \subset \mathbb{C}$ stands for the r-neighborhood of a set S in $\mathbb{C}$; furthermore, η and ξ extend continuously to the boundary of their corresponding domains (here we distinguish the upper and lower boundaries of slits);

(ii) the maps η and ξ can be locally represented as

$$\eta = \psi_\eta \circ p_{\alpha+} \circ \phi_\zeta \quad \text{and} \quad \xi = \psi_\xi \circ p_{\alpha-} \circ \phi_\zeta,$$

where ψ_η, ψ_ξ, and ϕ_ζ are real-symmetric conformal maps in a neighborhood of the origin and $\phi_\zeta(0) = 0$, $\phi'_\zeta(0) = 1$;

(iii) the compositions $\nu^+ \equiv \xi \circ \eta$ and $\nu^- \equiv \eta \circ \xi$ are defined in some neighborhood of the origin and can be represented in it as

$$\nu^+ = \rho_\zeta \circ p_{\alpha+} \circ \phi_\zeta \quad \text{and} \quad \nu^- = \rho_\zeta \circ p_{\alpha-} \circ \phi_\zeta,$$

where ρ_ζ is a real-symmetric conformal map in a neighborhood of the origin;

(iv) for every $z \in (0, 1]$, we have $\eta'(z) > 0$ and, for every $z \in I_\xi\backslash\{0\}$, we have $\xi'(z) > 0$; furthermore, $\eta(0) = b < 0$, $\xi(0) = 1$, and $\eta(\xi(0)) = \xi(\eta(0)) \in I_\eta$.

If a pair of maps $\zeta = (\eta, \xi)$ belongs to $\mathcal{A}_r^\alpha$ for some $r, \alpha > 0$, then we say that ζ is a *critical commuting pair with critical exponent* α. We also note that $\mathcal{A}_r \subset \mathcal{A}_r^3$.

Definition 9.2. For a positive real $r > 0$ and a set $J \subset (0, +\infty)$, we define

$$\mathcal{A}_r^J = \bigcup_{\alpha \in J} \mathcal{A}_r^\alpha.$$

Similarly to the case with the class $\mathcal{A}_r$, the map (6) induces a metric on $\mathcal{A}_r^J$ from the sup-norm on

$$\mathbf{D}(N_r([0, 1])\backslash(-\infty, 0])) \times \mathbf{D}(N_r([0, 1])\backslash(-\infty, 0])).$$

We will denote this metric by $\operatorname{dist}_r(\cdot, \cdot)$, since it coincides with the previously defined metric on $\mathcal{A}_r$.

Renormalization of pairs from $\mathcal{A}_r^J$ is defined in exactly the same way as renormalization of ordinary critical commuting pairs (cf. Definition 2.5) and is determined by the dynamics on the real line. In particular, the nth pre-renormalization $p\mathcal{R}^n\zeta$ of $\zeta = (\eta, \xi) \in \mathcal{A}_r^J$ can

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be viewed as a pair whose elements are certain compositions of η and ξ. We will say that the domains of the maps that constitute $p\mathcal{R}^n\zeta$ are the maximal domains in $\mathbb{C}$, where the corresponding compositions are defined.

Definition 9.3. We let $\mathcal{A}^\alpha$ (or $\mathcal{A}^J$) denote the set of all real-analytic pairs (η, ξ) such that an analytic extension of (η, ξ) belongs to $\mathcal{A}_r^\alpha$ (or $\mathcal{A}_r^J$) for some $r > 0$.

We think of critical commuting pairs ζ from $\mathcal{A}^\alpha$ or $\mathcal{A}^J$ and their renormalizations $\mathcal{R}\zeta$ as pairs of maps defined only on the real line, though having analytic extensions to some domains in $\mathbb{C}$.

It is convenient for our purposes to introduce a smaller class of critical commuting pairs.

Definition 9.4. For positive real numbers $r, \alpha > 0$, the class $\mathcal{B}_r^\alpha \subset \mathcal{A}_r^\alpha$ consists of all $\zeta = (\eta, \xi) \in \mathcal{A}_r^\alpha$ such that the following holds:

(i) the maps η and ξ can be represented as

$$\eta = \psi_\eta \circ p_{\alpha+} \quad \text{and} \quad \xi = \psi_\xi \circ p_{\alpha-},$$

where ψ_η and ψ_ξ are conformal diffeomorphisms of $p_{\alpha+}(N_r([0, 1])\setminus(-\infty, 0))$ and $p_{\alpha-}(N_{r|\eta(0)|}(I_\xi)\setminus(0, +\infty))$, respectively;

(ii) the compositions $\nu^+ \equiv \xi \circ \eta$ and $\nu^- \equiv \eta \circ \xi$ are defined in the domains $\mathbb{D}_{\hat{r}}\setminus(-\infty, 0)$ and $\mathbb{D}_{\hat{r}}\setminus(0, +\infty)$, respectively, where $\hat{r} = \min(r, r|\eta(0)|)$, and there exists a conformal diffeomorphism $\rho_\zeta : \mathbb{D}_{\hat{r}^\alpha} \to \mathbb{C}$ such that $\nu^+ = \rho_\zeta \circ p_{\alpha+}$ and $\nu^- = \rho_\zeta \circ p_{\alpha-}$.

Definition 9.5. We let $\mathcal{B}^\alpha$ (or $\mathcal{B}^J$) denote the set of all real-analytic pairs (η, ξ) such that an analytic extension of (η, ξ) belongs to $\mathcal{B}_r^\alpha$ (or $\mathcal{B}_r^J$) for some $r > 0$.

The following theorem is an expanded version of Theorem 1.4. As before, we define $\Sigma_B = \{1, \ldots, B\}^{\mathbb{Z}}$.

THEOREM 9.6. (Global renormalization attractor) *For every $k, B \in \mathbb{N}$, there exist an open interval $J = J(k, B) \subset \mathbb{R}$ and a positive real number $r = r(k)$ such that $2k + 1 \in J$ and, for every $\alpha \in J$, there exists an $\mathcal{R}$-invariant set $\mathcal{I}_B^\alpha \subset \mathcal{B}_r^\alpha$ with the following properties.*

(i) *The action of $\mathcal{R}$ on $\mathcal{I}_B^\alpha$ is topologically conjugate to the shift $\sigma : \Sigma_B \to \Sigma_B$:*

$$\iota_\alpha \circ \mathcal{R} \circ \iota_\alpha^{-1} = \sigma$$

and, if

$$\zeta = \iota_\alpha^{-1}(\ldots, r_{-k}, \ldots, r_{-1}, r_0, r_1, \ldots, r_k, \ldots),$$

then

$$\rho(\zeta) = [r_0, r_1, \ldots, r_k, \ldots].$$

(ii) *For every $\zeta \in \mathcal{A}^\alpha$ with an irrational rotation number of type bounded by B, there exists $M \in \mathbb{N}$ such that for all $m \geq M$ the renormalizations $\mathcal{R}^m\zeta$ belong to $\mathcal{A}_r^\alpha$ and, for every $\zeta' \in \mathcal{I}_B^\alpha$ with $\rho(\zeta) = \rho(\zeta')$, we have*

$$\operatorname{dist}_r(\mathcal{R}^m\zeta, \mathcal{R}^m\zeta') \leq C\lambda^m \tag{25}$$

for some constants $C > 0$, $\lambda \in (0, 1)$ that depend only on B and α.

We give a proof of Theorem 9.6 in §9.4. We note that Theorem 1.4 is an immediate corollary of Theorem 9.6.

9.2. *Generalized holomorphic commuting pairs and complex bounds.*

Definition 9.7. Let $\zeta = (\eta, \xi)$ be either an element of some class $\mathcal{B}^\alpha$ or a linear rescaling of such an element. We say that ζ extends to a *holomorphic commuting pair with critical exponent* α if there exist domains $U, V, D, \Delta \subset \mathbb{C}$ that satisfy the same conditions as the corresponding domains in the definition of a holomorphic commuting pair (cf. §2.8) and the following properties hold:

(i) η and ξ extend to analytic maps $\hat{\eta} : (U \cup D)\setminus(-\infty, 0] \to \mathbb{C}$ and $\hat{\xi} : (V \cup D)\setminus[0, +\infty) \to \mathbb{C}$, respectively, and

$$\hat{\eta} = \psi_\eta \circ p_{\alpha+} \quad \text{and} \quad \hat{\xi} = \psi_\xi \circ p_{\alpha-},$$

where ψ_η and ψ_ξ are real-symmetric conformal diffeomorphisms of $p_{\alpha+}(U \cup D\setminus(-\infty, 0))$ and $p_{\alpha-}(V \cup D\setminus(0, +\infty))$, respectively;

(ii) $\hat{\eta}|_U$ is a conformal diffeomorphism of U onto $(\Delta\setminus\mathbb{R}) \cup \eta(I_U)$, where $I_U = U \cap \mathbb{R}$, and $\hat{\xi}|_V$ is a conformal diffeomorphism of V onto $(\Delta\setminus\mathbb{R}) \cup \eta(I_V)$, where $I_V = V \cap \mathbb{R}$;

(iii) the compositions $\nu^+ = \hat{\xi} \circ \hat{\eta}$ and $\nu^- = \hat{\eta} \circ \hat{\xi}$ are defined on the sets $D\setminus(-\infty, 0)$ and $D\setminus(0, +\infty)$, respectively, and can be represented in these sets as $\nu^+ = \psi \circ p_{\alpha+}$ and $\nu^- = \psi \circ p_{\alpha-}$, where ψ is some conformal real-symmetric map; furthermore, we have $\mathbb{H} \cap \nu^+(D\setminus(-\infty, 0)) = \mathbb{H} \cap \nu^-(D\setminus(0, +\infty)) = \mathbb{H} \cap \Delta$.

We let $\mathbf{H}^\alpha$ denote the space of all holomorphic commuting pairs with critical exponent α. It follows from the above definition that $\mathbf{H}^3 \subset \mathbf{H}$, where $\mathbf{H}$ is the same as in §2.8.

For a subset $J \subset (2, +\infty)$, we let $\mathbf{H}^J$ be the space of all holomorphic commuting pairs with critical exponent α in the set J. This space can be equipped with the Carathéodory convergence in exactly the same way as the space of ordinary holomorphic pairs $\mathbf{H}$, namely, by viewing an element of $\mathbf{H}^J$ as three triples $(U, \xi(0), \eta)$, $(V, \eta(0), \xi)$, $(D, 0, \nu^+)$. Similarly, we define the modulus of a holomorphic pair $\mathcal{H} = (\eta, \xi) \in \mathbf{H}^J$ in the same way as for an ordinary holomorphic pair from $\mathbf{H}$. We denote the modulus of $\mathcal{H}$ by $\mathrm{mod}(\mathcal{H})$ and we say that the domain Δ in the above definition is the range of a holomorphic pair $\mathcal{H}$ and $\zeta_{\mathcal{H}} \in \mathcal{B}^J$ is the commuting pair underlying $\mathcal{H}$. Finally, we define the pre-renormalization $p\mathcal{R}(\mathcal{H})$ and the renormalization $\mathcal{R}(\mathcal{H})$ of a holomorphic pair $\mathcal{H}$ with critical exponent α in the same way as we defined renormalization of ordinary holomorphic pairs. In particular, we note that the ranges of $\mathcal{H}$ and $p\mathcal{R}(\mathcal{H})$ are the same.

Definition 9.8. For $\mu \in (0, 1)$ and a closed set $J \subset (2, +\infty)$, we let $\mathbf{H}^J(\mu) \subset \mathbf{H}^J$ be the space of all holomorphic commuting pairs $\mathcal{H} = (\eta, \xi) \in \mathbf{H}^J$ with corresponding domains $D_{\mathcal{H}}, U_{\mathcal{H}}, V_{\mathcal{H}}, \Delta_{\mathcal{H}}$ such that all properties (i)–(v) from Definition 2.22 hold.

LEMMA 9.9. *For each* $\mu \in (0, 1)$ *and a closed set* $J \subset (2, +\infty)$*, the space* $\mathbf{H}^J(\mu)$ *is sequentially compact. Furthermore, if a sequence of holomorphic pairs* $\{\mathcal{H}_k \mid k \in \mathbb{N}\} \subset \mathbf{H}^J(\mu)$ *with corresponding critical exponents* α_k *converges to a holomorphic pair with critical exponent* α*, then* $\lim_{k\to\infty} \alpha_k = \alpha$.

Proof. Exactly the same argument as in the proof of Lemma 2.23 (cf. [**Yam01**, Lemma 2.17]) shows that every sequence of holomorphic pairs from $\mathbf{H}^J(\mu)$ has a

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subsequence that converges to a holomorphic pair that satisfies properties (i)–(v) from Definition 2.22. In order to complete the proof, we need to show the convergence of critical exponents.

Consider a sequence $\mathcal{H}_1, \mathcal{H}_2, \ldots \in \mathbf{H}^J(\mu)$ of holomorphic pairs with corresponding critical exponents $\alpha_1, \alpha_2, \ldots$ and let U_k, V_k, D_k denote the corresponding domains of the maps that constitute the holomorphic pair $\mathcal{H}_k$. We note that because of property (iv) from Definition 2.22, Carathéodory convergence of these domains is equivalent to Hausdorff convergence of $\overline{U_k \cap \mathbb{H}}$, $\overline{V_k \cap \mathbb{H}}$, and $\overline{D_k \cap \mathbb{H}}$. Furthermore, properties (ii) and (iv) from Definition 2.22 imply that all domains D_k contain a disk of some fixed positive radius, centered at zero. Then it follows from property (i) of Definition 9.7 that all boundaries ∂U_k contain a straight-line segment of fixed positive length with one end point at 0, forming angle π/α_k with the ray $[0, +\infty)$. Now convergence of the domains U_k implies convergence of the critical exponents α_k. □

The following theorem was proved by the second author in **[Yam99]** for critical commuting pairs in the Epstein class with $\alpha = 3$; the proof was later generalized in **[dFdM00]** to remove the Epstein class condition. The proof extends to the case of a general $\alpha > 2$ *mutatis mutandis*.

THEOREM 9.10. (Complex bounds) *For every bounded set $J \subset [2, +\infty)$, there exists a constant $\mu = \mu(J) > 0$ such that the following holds. For every positive real number $r > 0$ and every pre-compact family $S \subset \mathcal{B}_r^J$ of critical commuting pairs, there exists $K = K(r, S) \in \mathbb{N}$ such that if $\zeta \in S$ is a $2n$-times renormalizable commuting pair, where $n \geq K$, then $p\mathcal{R}^n\zeta$ restricts to a holomorphic commuting pair $\mathcal{H}_n$ with range $\Delta_n \subset N_r(I_\eta) \cup N_r(I_\xi)$. Furthermore, the range Δ_n is a Euclidean disk and the appropriate affine rescaling of $\mathcal{H}_n$ is in $\mathbf{H}^J(\mu)$.*

9.3. *The maps between $\mathcal{B}_r^\alpha$ and $\mathbf{P}^\alpha_{\tilde{U},\tilde{t},\tilde{h}}$.* Let $\tilde{\mathcal{U}}_0$ denote the set of all $\tau \in \tilde{\mathcal{U}} \cup (\mathbf{P}_{\tilde{U},\tilde{t},\tilde{h}})^{\mathbb{R}}$ for which $\mathcal{R}^2_{\mathbf{P}}(\tau)$ is defined. Then this implies that the rotation number of the generalized critical circle map f_τ has at least $2N$ elements in its continued fraction expansion. Let $p\mathcal{R}^m f_\tau = (f_\tau^{q_{m+1}}, f_\tau^{q_m})$ be the mth pre-renormalization of the circle map f_τ considered in the sense of commuting pairs (cf. Definition 2.8). Then one can consider a map $\Phi : \tilde{\mathcal{U}}_0 \to \mathcal{B}^{(0,+\infty)}$ defined by

$$\Phi : \tau = (\alpha, \phi, \psi) \mapsto (h \circ \phi \circ f_\tau^{q_{2N}} \circ \phi^{-1} \circ h^{-1}, h \circ \phi \circ f_\tau^{q_{2N-1}} \circ \phi^{-1} \circ h^{-1}),$$

where $h(x) = x/\phi(f_\tau^{q_{2N-1}}(\phi^{-1}(0)))$. In other words, $\Phi(\tau)$ is the $(2N-1)$st pre-renormalization $p\mathcal{R}^{2N-1} f_\tau$, taken in the ϕ-coordinate and then rescaled so that the first map in the pair acts on the interval $[0, 1]$.

For $\tau_1 = (\alpha_1, \phi_1, \psi_1)$, $\tau_2 = (\alpha_2, \phi_2, \psi_2) \in \mathbf{P}_{\tilde{U},\tilde{t},\tilde{h}}$, we define

$$\operatorname{dist}_{\mathbf{P}}(\tau_1, \tau_2) = |\alpha_1 - \alpha_2| + \sup_{z \in \tilde{U}} |\phi_1(z) - \phi_2(z)| + \inf_{\tilde{\psi}_1, \tilde{\psi}_2} \left(\sup_{|\operatorname{Im} z| < \tilde{h}} |\tilde{\psi}_1(z) - \tilde{\psi}_2(z)| \right),$$

where the infimum is taken over all $\tilde{\psi}_1$, $\tilde{\psi}_2$ that are lifts of ψ_1 and ψ_2 respectively via the projection (7). We note that $\mathrm{dist}_{\mathbf{P}}(\cdot, \cdot)$ is a metric on the Banach manifold $\mathbf{P}_{\tilde{U},\tilde{t},\tilde{h}}$, induced by the norm in the Banach space that $\mathbf{P}_{\tilde{U},\tilde{t},\tilde{h}}$ is modeled on.

The following two lemmas establish a relation between the two metrics $\mathrm{dist}_r(\cdot, \cdot)$ and $\mathrm{dist}_{\mathbf{P}}(\cdot, \cdot)$ in a neighborhood of the attractors $\mathcal{I}_B$ and $\check{\mathcal{I}}_B$, respectively.

LEMMA 9.11. *For any $B \in \mathbb{N}$ and any closed interval $J \subset I(B)$, where $I(B)$ is the interval from Corollary 7.2, there exist positive constants $C = C(B, J) > 0$, $r = r(B, J) > 0$, and an open set $\tilde{\mathcal{U}}_{B,J} \subset \tilde{\mathcal{U}}_0$ such that the following holds:*
(i)

$$\bigcup_{\alpha \in J} \check{\mathcal{I}}_B^{\alpha} \subset \tilde{\mathcal{U}}_{B,J};$$

(ii) *for every $\tau \in \tilde{\mathcal{U}}_{B,J}$, the critical commuting pair $\Phi(\tau)$ analytically extends to a pair from $\mathcal{B}_r^J$ and, for any $\tau_1, \tau_2 \in \tilde{\mathcal{U}}_{B,J}$, we have $\mathrm{dist}_{\mathbf{P}}(\tau_1, \tau_2) \geq C\, \mathrm{dist}_r(\Phi(\tau_1), \Phi(\tau_2))$;*
(iii) *for every $\tau = (\alpha, \phi, \psi) \in \tilde{\mathcal{U}}_{B,J}$ with $\alpha \in J$, the sequence of iterates*

$$\mathcal{R}_{\mathbf{P}}(\tau), \mathcal{R}_{\mathbf{P}}^2(\tau), \mathcal{R}_{\mathbf{P}}^3(\tau), \ldots$$

either eventually leaves the set $\tilde{\mathcal{U}}_{B,J}$ or stays in it forever and converges to the attractor $\check{\mathcal{I}}_B^{\alpha}$.

Proof. The set $\bigcup_{\alpha\in J} \check{\mathcal{I}}_B^{\alpha}$ is compact and hence there exists a positive real number $r > 0$ such that for every $\tau \in \bigcup_{\alpha\in J} \check{\mathcal{I}}_B^{\alpha}$, the critical commuting pair $\Phi(\tau)$ analytically extends to a pair from $\mathcal{B}_{2r}^J$ and we can choose an open set $\mathcal{V}_{B,J} \subset \tilde{\mathcal{U}}_0$ so that $\bigcup_{\alpha\in J} \check{\mathcal{I}}_B^{\alpha} \subset \mathcal{V}_{B,J}$ and, for any $\tau \in \mathcal{V}_{B,J}$, the pair $\Phi(\tau)$ analytically extends to a pair from $\mathcal{B}_r^J$.

Let i be the map defined in (6). Since, according to Lemma 4.22, the map f_τ analytically depends on τ, it follows from the definition of the map Φ that the composition $i \circ \Phi$ is an analytic map from the neighborhood $\mathcal{V}_{B,J} \subset \mathbf{P}_{\tilde{U},\tilde{t},\tilde{h}}$ to the Banach space $\mathbf{D}(N_r([0, 1])\backslash(-\infty, 0]) \times \mathbf{D}(N_r([0, 1])\backslash(-\infty, 0])$.

The derivative of an analytic operator is locally bounded, which means that there exist a constant $c > 0$ and an open set $\tilde{\mathcal{V}}_{B,J} \subset \mathcal{V}_{B,J}$ such that $\bigcup_{\alpha\in J} \check{\mathcal{I}}_B^{\alpha} \subset \tilde{\mathcal{V}}_{B,J}$ and $\|D_\tau(i \circ \Phi)\| < c$ for all $\tau \in \tilde{\mathcal{V}}_{B,J}$.

Since the set $\bigcup_{\alpha\in J} \check{\mathcal{I}}_B^{\alpha}$ is compact, the set $\Phi(\bigcup_{\alpha\in J} \check{\mathcal{I}}_B^{\alpha})$ is also compact and hence it has a finite diameter with respect to the Banach norm. Finally, we choose an open set $\tilde{\mathcal{U}}_{B,J} \subset \tilde{\mathcal{V}}_{B,J}$ so that $\bigcup_{\alpha\in J} \check{\mathcal{I}}_B^{\alpha} \subset \tilde{\mathcal{U}}_{B,J}$, the set $\Phi(\tilde{\mathcal{U}}_{B,J})$ has a finite diameter $D > 0$, and, if $\tau_1, \tau_2 \in \tilde{\mathcal{U}}_{B,J}$, $\mathrm{dist}_{\mathbf{P}}(\tau_1, \tau_2) < \varepsilon$, for some sufficiently small $\varepsilon > 0$, then τ_1 and τ_2 can be connected by the shortest geodesic contained in $\tilde{\mathcal{V}}_{B,J}$. Now it is easy to check that for any $\tau_1, \tau_2 \in \tilde{\mathcal{U}}_{B,J}$ and $C = \min(c, \varepsilon/D)$, the inequality

$$\mathrm{dist}_{\mathbf{P}}(\tau_1, \tau_2) \geq C\|i(\Phi(\tau_1)) - i(\Phi(\tau_2))\|$$

holds, which immediately implies property (ii) of the lemma.

Finally, it follows from Corollary 7.2 that shrinking the open set $\tilde{\mathcal{U}}_{B,J}$ if necessary, we can guarantee that property (iii) of the lemma also holds. □

LEMMA 9.12. *Fix $B \in \mathbb{N}$ and $\mu = \mu(I)$, where $I = I(B)$ is the interval from Corollary 7.2 and $\mu(I)$ is the same as in Theorem 9.10. Then there exist a set $\mathcal{W}_B \subset \mathcal{B}^I$ and a map $\Psi : \mathcal{W}_B \to \mathbf{P}_{\tilde{U},\tilde{t},\tilde{h}}$ such that the following properties hold:*

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(i) *$\mathcal{W}_B$ is the projection of an open neighborhood $\tilde{\mathcal{W}}_B \subset \mathbf{H}^I(\mu)$ under the map $\mathcal{H} \mapsto \zeta_{\mathcal{H}}$ and $\tilde{\mathcal{I}}_B \subset \tilde{\mathcal{W}}_B$;*

(ii) *for every integer $m \geq 0$, the following holds whenever both sides are defined:*

$$\Psi(\mathcal{R}^{mN}(\Phi(\tau))) = \mathcal{R}_{\mathbf{P}}^{m+2}\tau;$$

(iii) *for every integer $m \geq 0$, the following holds whenever both sides are defined:*

$$\Phi(\mathcal{R}_{\mathbf{P}}^{m}(\Psi(\zeta))) = \mathcal{R}^{N(m+2)}\zeta;$$

(iv) *there exists a closed interval $J \subset I$ such that $3 \in J$ lies in the interior of J and*

$$\Psi(\mathcal{W}_B) \subset \tilde{\mathcal{U}}_{B,J},$$

where $\tilde{\mathcal{U}}_{B,J}$ is the set from Lemma 9.11;

(v) *there exist two real constants $s > 0$ and $C = C(B, \alpha) > 0$ such that every $\zeta \in \mathcal{W}_B$ analytically extends to a pair from $\mathcal{B}_s^I$ and, for any $\zeta_1, \zeta_2 \in \mathcal{W}_B$ with critical exponent α, we have $\operatorname{dist}_s(\zeta_1, \zeta_2) \geq C \operatorname{dist}_{\mathbf{P}}(\Psi(\zeta_1), \Psi(\zeta_2))$.*

Proof. Lemmas 4.16–4.18 provide a continuous mapping

$$\tilde{\Psi} : \mathcal{H} \mapsto (3, \phi_{\mathcal{H}}, \psi_{\mathcal{H}})$$

from $\tilde{\mathcal{K}} \subset \mathbf{H}^3(\mu)$ to $\mathbf{P}_{\tilde{U},\tilde{t},\tilde{h}}$. According to Lemma 4.6, the fundamental crescent $C_{\mathcal{H}}$ of a holomorphic pair $\mathcal{H}$ and its fattening $C_{\mathcal{H}}^o$ can be chosen to depend locally continuously with respect to $\mathcal{H} \in \mathbf{H}^I(\mu)$. Then, since $\tilde{\mathcal{K}}$ is a countable union of nested topological Cantor sets, the standard argument from the proofs of the above-mentioned lemmas shows that the map $\tilde{\Psi}$ continuously extends to some open set $\tilde{\mathcal{W}} \subset \mathbf{H}^I(\mu)$ such that $\tilde{\mathcal{K}} \subset \tilde{\mathcal{W}}$ and, for every $\mathcal{H} \in \tilde{\mathcal{W}}$ with $\tilde{\Psi}(\mathcal{H}) = (\alpha, \phi_{\mathcal{H}}, \psi_{\mathcal{H}})$, the identity

$$\tilde{\pi}_{C_{\mathcal{H}}} \circ R_{C_{\mathcal{H}}}^{\pm} \circ \pi_{C_{\mathcal{H}}}^{-1} \equiv \psi_{\mathcal{H}} \circ \pi_{\phi_{\mathcal{H}},\alpha,\tilde{t}} \circ p_{\phi_{\mathcal{H}},\alpha\pm} \tag{26}$$

holds in the domain $\{z \in \mathbb{C} \mid |\operatorname{Re} z| < 1/2; |\operatorname{Im} z| < r_2\}$, where $r_2 > 0$ is the same as in Lemma 5.6, the family of maps $\{\pi_{\phi,\alpha,\tilde{t}} \mid (\alpha, \phi) \in \mathbf{B}_{\tilde{U},\tilde{t}}\}$ is the same as in Lemma 4.18, and $R_{C_{\mathcal{H}}}^{+}(z)$, $R_{C_{\mathcal{H}}}^{-}(z)$ are the first returns of $\hat{\eta}(z)$ and $\hat{\xi}(z)$ respectively to the fundamental crescent $C_{\mathcal{H}}$ under the dynamics of $\mathcal{H}$. (Here $\hat{\eta}$ and $\hat{\xi}$ are the same as in Definition 9.7.)

We note that if $\mathcal{W} \subset \mathcal{B}^I$ is the projection of the set $\tilde{\mathcal{W}}$ under the map $\mathcal{H} \mapsto \zeta_{\mathcal{H}}$ and $h : \mathcal{W} \to \tilde{\mathcal{W}}$ is some continuous lift of $\mathcal{W}$ with respect to this projection, then the map

$$\Psi = \tilde{\Psi} \circ h$$

defined on the set $\mathcal{W}$ satisfies properties (ii) and (iii) of the lemma. Furthermore, for every $B \in \mathbb{N}$, we can choose an open subset $\tilde{\mathcal{W}}_B \subset \tilde{\mathcal{W}}$ such that $\tilde{\mathcal{I}}_B \subset \tilde{\mathcal{W}}_B$ and define $\mathcal{W}_B \subset \mathcal{W}$ as the projection of this set under the map $\mathcal{H} \mapsto \zeta_{\mathcal{H}}$, so that the restriction of Ψ to $\mathcal{W}_B$ satisfies property (iv).

Now we will show that, possibly after shrinking the set $\tilde{\mathcal{W}}_B$ and correspondingly its projection $\mathcal{W}_B$, we can establish property (v).

First of all, since every holomorphic pair $\mathcal{H} \subset \tilde{\mathcal{K}}$ restricts to a pair from $\mathcal{B}_{2s}^3$ for some constant $s > 0$ independent of $\mathcal{H}$, by shrinking the open set $\tilde{\mathcal{W}}_B$ if necessary, we can guarantee that every $\mathcal{H} \subset \tilde{\mathcal{W}}_B$ restricts to a pair from $\mathcal{B}_s^I$.

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Now we note that for every $\mathcal{H} = (\eta, \xi) \in \tilde{\mathcal{W}}_B$ and the corresponding triple $(\alpha, \phi_{\mathcal{H}}, \psi_{\mathcal{H}}) = \tilde{\Psi}(\mathcal{H})$, the map $\phi_{\mathcal{H}}$ is completely determined by the map η. Furthermore, according to Lemma 4.6, at every such $\eta : U_{\mathcal{H}} \to \Delta_{\mathcal{H}}$, the correspondence $\eta \mapsto \phi_{\mathcal{H}}$ extends to a locally analytic map from a neighborhood in $\mathbf{D}(\hat{U}_{\mathcal{H}})$ to $\mathbf{D}(\tilde{U})$, where $\hat{U}_{\mathcal{H}}$ is some open set, compactly contained in $U_{\mathcal{H}}$. Analyticity of this map implies local boundedness of its derivative and the Koebe distortion theorem implies that for all $\mathcal{H}_0 = (\eta_0, \xi_0)$ from some neighborhood of $\mathcal{H}$ in $\tilde{\mathcal{W}}_B$, we have

$$\sup_{z \in N_s([0,1])} |\eta_0(z)| \geq c_1 \sup_{z \in \hat{U}_{\mathcal{H}}} |\eta_0(z)|$$

for some universal constant $c_1 > 0$. Now, arguing in the same way as at the end of the proof of Lemma 9.11, using the above inequality and analyticity of the map $\eta \mapsto \phi_{\mathcal{H}}$ from $\mathbf{D}(\hat{U}_{\mathcal{H}})$ to $\mathbf{D}(\tilde{U})$, we conclude that possibly after shrinking the neighborhood $\tilde{\mathcal{W}}_B$, the following conditions are satisfied: $\tilde{\mathcal{I}}_B \subset \tilde{\mathcal{W}}_B$ and, for every $\mathcal{H}_1, \mathcal{H}_2 \in \tilde{\mathcal{W}}_B$, the inequality

$$\operatorname{dist}_s(\mathcal{H}_1, \mathcal{H}_2) \geq c_2 \sup_{z \in \tilde{U}} |\phi_{\mathcal{H}_1}(z) - \phi_{\mathcal{H}_2}(z)|$$

holds for some constant $c_2 > 0$, independent from $\mathcal{H}_1$ and $\mathcal{H}_2$.

Similarly to how it is done in the proof of Lemma 4.18, we can represent the left-hand side of (26) as

$$(\tilde{\pi}_{C_{\mathcal{H}}} \circ R^{\pm}_{C_{\mathcal{H}}} \circ \pi^{-1}_{C_{\mathcal{H}}})(z) = \tilde{\pi}_{C_{\mathcal{H}}}(g(p_{\phi_{\mathcal{H}},\alpha\pm}(z))),$$

where g is a conformal map that is a composition of a linear rescaling by $p_{\alpha-}(1/\pi'_{C_{\mathcal{H}}}(0))$, the map $\rho_{\zeta_{\mathcal{H}}}$ from property (ii) of Definition 9.4, and a finite number of the maps η. Combining this with (26), we get

$$\psi_{\mathcal{H}} \equiv \tilde{\pi}_{C_{\mathcal{H}}} \circ g \circ \pi^{-1}_{\phi_{\mathcal{H}},\alpha,\tilde{t}},$$

so $\psi_{\mathcal{H}}$ analytically depends on $\phi_{\mathcal{H}} \in \mathbf{B}^{\alpha}_{\tilde{U},\tilde{t}}$, $\rho_{\zeta_{\mathcal{H}}} \in \mathbf{D}(\mathbb{D}_{\hat{r}^{\alpha}})$, and $\eta \in \mathbf{D}(N_s([0,1]) \setminus (-\infty, 0])$; hence, it has a locally bounded derivative with respect to each of them. Finally, applying the argument from the end of the proof of Lemma 9.11, shrinking the open set $\tilde{\mathcal{W}}_B$ and its projection $\mathcal{W}_B$ again if necessary, we obtain property (v) of the lemma while property (i) remains true. □

9.4. *Global attractor.* Now we are ready to give a proof of Theorem 9.6.

Proof of Theorem 9.6. We will give a proof for the case $k = 1$. For other $k \in \mathbb{N}$ the proof is identical.

Let $I \subset \mathbb{R}$ be the interval from Corollary 7.2. Then, for every $\alpha \in I$, we define

$$\mathcal{I}^{\alpha}_B = \Phi(\check{\mathcal{I}}^{\alpha}_B).$$

It follows from Lemma 4.23 and the choice of the constant N, made in §5.2, that for all α from some open interval $J_1 \subset \mathbb{R}$ with $3 \in J_1$, every element of $\Phi(\check{\mathcal{I}}^{\alpha}_B)$ extends to a holomorphic commuting pair from $\tilde{\mathcal{W}}_B$, where $\tilde{\mathcal{W}}_B$ is the same as in property (i) of Lemma 9.12. Thus, for all $\alpha \in J_1$, we have

$$\mathcal{I}^{\alpha}_B \subset \mathcal{W}_B.$$

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Furthermore, since according to Corollary 7.2 the operator $\mathcal{R}_{\mathbf{P}}$ is bijective on $\check{\mathcal{I}}^{\alpha}_B$, property (ii) of Lemma 9.12 implies that the map Φ provides a bijection, and hence a homeomorphism, between $\check{\mathcal{I}}^{\alpha}_B$ and $\mathcal{I}^{\alpha}_B$. This homeomorphism, composed with the map κ_α from Corollary 7.2, induces a homeomorphism

$$\iota_\alpha : \mathcal{I}^{\alpha}_B \to \Sigma_B.$$

First, we will prove part (ii) of Theorem 9.6 only for $\zeta \in \mathcal{B}^{\alpha}$. We start with a proposition.

PROPOSITION 9.13. *Let $\mu = \mu(\overline{I})$ be the same as in Theorem 9.10. Then there exist an open interval $J_2 \subset I$ and a positive integer $L \in \mathbb{N}$ such that $3 \in J_2$ and, if a commuting pair $\zeta \in \mathcal{B}^{J_2}$ with an irrational rotation number of type bounded by B extends to a holomorphic commuting pair from $\mathbf{H}^{J_2}(\mu)$, then $\mathcal{R}^L\zeta \in \mathcal{W}_B$.*

Proof. It follows from compactness of $\mathbf{H}^3(\mu)$ and real *a priori* bounds [**Her86**, **Yoc84**] that there exists a positive integer $K > 0$ such that for every $\mathcal{H} \in \mathbf{H}^3(\mu)$ with $\rho(\zeta_{\mathcal{H}})$ of type bounded by B and every $k \geq K$, the range of the renormalization $\mathcal{R}^k(\mathcal{H})$ contains a disk of radius $2/\mu$. Then, since the set $\mathcal{I}^3_B$ is a global attractor of type bounded by B (cf. Remark 2.17) and the class $\mathbf{H}^3(\mu)$ is sequentially compact, there exists a positive integer $L > K$ such that for every $\mathcal{H} \in \mathbf{H}^3(\mu)$ with $\rho(\zeta_{\mathcal{H}})$ of type bounded by B, the renormalization $\mathcal{R}^L(\mathcal{H})$ restricts to a holomorphic pair $\mathcal{G} \in \tilde{\mathcal{W}}_B$, where $\tilde{\mathcal{W}}_B \subset \mathbf{H}^I(\mu)$ is an open set from property (i) of Lemma 9.12.

Finally, by continuity and sequential compactness of $\mathbf{H}^{\overline{I}}(\mu)$, we conclude the existence of an open interval J_2 with $3 \in J_2$ such that for every $\mathcal{H} \in \mathbf{H}^{J_2}(\mu)$ with $\rho(\zeta_{\mathcal{H}})$ of type bounded by B, the renormalization $\mathcal{R}^L(\mathcal{H})$ restricts to a holomorphic pair from $\tilde{\mathcal{W}}_B$. This completes the proof of the proposition. □

We define $J = J_1 \cap J_2$ and we set $\hat{r} = \min\{s, r(B, \overline{J})\}$, where s is the same as in property (v) of Lemma 9.12 and $r(B, \overline{J})$ is the same as in Lemma 9.11. Assume that $\zeta \in \mathcal{B}^{\alpha}$ for some $\alpha \in J$ and $\rho(\zeta)$ is of type bounded by B. Then, according to the complex bounds (Theorem 9.10), there exists an integer $K > 0$ such that for all $k \geq K$, the renormalization $\mathcal{R}^k\zeta$ extends to a holomorphic commuting pair from $\mathbf{H}^J(\mu)$. Then Proposition 9.13 implies that $\mathcal{R}^{k+L}\zeta \in \mathcal{W}_B$. Now it follows from property (v) of Lemma 9.12 that $\mathcal{R}^{k+L}\zeta$ extends to a commuting pair from $\mathcal{B}^{\alpha}_{\hat{r}}$ and, finally, exponential convergence (25) for any $r \leq \hat{r}$ follows from properties (iii)–(v) of Lemma 9.12 together with Lemma 9.11 and Corollary 7.2.

Now we will prove part (ii) of Theorem 9.6 for all $\zeta \in \mathcal{A}^{\alpha}$. Assume that $\zeta \in \mathcal{A}^{\alpha}$ has an irrational rotation number of type bounded by B. Then, according to the real *a priori* bounds [**Her86**, **Yoc84**], there exists a positive integer $K > 0$ such that the dynamical intervals of $p\mathcal{R}^K\zeta$ are contained in the domain of the map ϕ_ζ from Definition 9.1. Thus, there exists a conformal map ϕ_0 with $\phi_0(0) = 0$ and $\phi_0'(0) = 1$ such that $\mathcal{R}^K\zeta = (\phi_0^{-1} \circ \eta_0 \circ \phi_0, \phi_0^{-1} \circ \xi_0 \circ \phi_0)$, where $\zeta_0 = (\eta_0, \xi_0) \in \mathcal{B}^{\alpha}$.

Now it is easy to check that for every $n \in \mathbb{N}$, we have

$$\mathcal{R}^{K+n}\zeta = (\phi_n^{-1} \circ \eta_n \circ \phi_n, \phi_n^{-1} \circ \xi_n \circ \phi_n),$$

where $\zeta_n = (\eta_n, \xi_n) = \mathcal{R}^n\zeta_0$ and ϕ_n can be defined recurrently by

$$\phi_n(z) = \frac{1}{\eta_{n-1}(0)}\phi_{n-1}(\eta_{n-1}(0)z). \tag{27}$$

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Since $\zeta_0 \in \mathcal{B}^\alpha$, the above argument implies that all sufficiently high renormalizations of ζ_0 lie in $\mathcal{B}^\alpha_{\hat{r}}$ and exponential convergence (25) with $r \le \hat{r}$ holds for these renormalizations. At the same time, real bounds and (27) imply that the domains of the maps ϕ_n increase exponentially and the maps ϕ_n converge to the identity map exponentially fast in sup-norm on any compact set. This implies that all sufficiently high renormalizations of $\mathcal{R}^K \zeta$ belong to the class $\mathcal{A}^\alpha_{\hat{r}/2}$ and satisfy the exponential convergence condition (25) for all $r \le \hat{r}/2$. This completes the proof of part (ii) of Theorem 9.6.

Finally, we give a proof of part (i) of Theorem 9.6. It follows from the construction of the map Φ that this map conjugates the operators $\mathcal{R}_{\mathbf{P},\alpha}$ and $\mathcal{R}^N$ on the sets $\check{\mathcal{I}}^\alpha_B$ and $\mathcal{I}^\alpha_B$, respectively. This implies that the set $\mathcal{R}(\mathcal{I}^\alpha_B)$ is an invariant set for the operator $\mathcal{R}^N$. Then it follows from part (ii) of Theorem 9.6 that $\mathcal{R}(\mathcal{I}^\alpha_B) \subset \mathcal{I}^\alpha_B$. This means that the composition $\iota_\alpha \circ \mathcal{R} \circ \iota_\alpha^{-1}$ is defined for all $\alpha \in J$ and depends continuously on α. Then, since Σ_B is a totally disconnected space, this composition must be independent from α. Now part (i) of Theorem 9.6 follows from the fact that for $\alpha = 3$, this composition is the shift σ. □

9.5. *Proof of universality (Theorem 1.5).* Let B, N be as before. Denote by $\theta \in \mathcal{I}^\alpha_B$ the periodic orbit of $\mathcal{R}$ given by

$$\mathcal{R}^{lp} f_0 \to \theta$$

(Theorem 9.6). Let l be sufficiently large, so that for some real number $t_1 > 0$ and for all $t \in [-t_1, t_1]$, the domain of definition of the pre-renormalization $p\mathcal{R}^l f_t \equiv (\tilde{\eta}_t, \tilde{\xi}_t)$ is contained in V_2. It follows from complex bounds (Theorem 9.10) that, possibly after increasing l and decreasing $t_1 > 0$, the commuting pairs

$$\zeta_t \equiv (h_t \circ \phi_t \circ \tilde{\eta}_t \circ \phi_t^{-1} \circ h_t^{-1},\ \ h_t \circ \phi_t \circ \tilde{\xi}_t \circ \phi_t^{-1} \circ h_t^{-1}),$$

where $h_t(z) = z/\phi_t(\tilde{\xi}_t(\phi_t^{-1}(0)))$, belong to $\mathcal{B}^\alpha$ and extend to holomorphic pairs from $\mathbf{H}^\alpha(\mu)$ for all $t \in [-t_1, t_1]$. Then, according to Proposition 9.13, after further increasing l and decreasing $t_1 > 0$, we can make sure that $\zeta_t \in \mathcal{W}_B$ for all $t \in [-t_1, t_1]$. Then $\tau_t \equiv \Psi(\zeta_t)$ is defined for all $t \in [-t_1, t_1]$ and τ_0 belongs to the local stable manifold of $\Psi(\theta)$.

It was shown in [**Yam02**, Lemma 9.3] that the condition

$$\inf_{x \in \mathbb{T}} \frac{\partial}{\partial t} f_t(x) > 0$$

implies that the tangent vector to the family $\{f_{\tau_t}\}$ at $t = 0$ lies inside an invariant cone field $\mathcal{C}$ defined on the tangent bundle to the set of all infinitely cylinder renormalizable $f \in \mathbf{C}^\alpha_{\tilde{r}}$. Furthermore, it was shown in [**Yam02**] that this cone field is expanded by $D\mathcal{R}_{\mathrm{cyl}}$. Pulling back $\mathcal{C}$ by the analytic mapping $\tau \mapsto f_\tau$ gives a cone field $\mathcal{C}'$ defined on a subset of the tangent bundle of $\mathbf{P}^\alpha_{\tilde{U},\tilde{t},\tilde{h}}$. It follows from Proposition 6.4 that $\mathcal{C}'$ is invariant and expanding under the map $D\mathcal{R}_{\mathbf{P},\alpha}$. Since the tangent vector to the family $\{\tau_t\}$ at $t = 0$ lies in $\mathcal{C}'$, the expanding property of $\mathcal{C}'$ implies that the family $\{\tau_t\}$ intersects the local stable manifold $W^s_{\mathrm{loc}}(\Psi(\theta))$ transversely. Now the statement of Theorem 1.5 follows immediately, with δ being equal to the absolute value of the unstable eigenvalue of the linearization of $\mathcal{R}^p_{\mathbf{P},\alpha}$ at $\Psi(\theta)$.

9.6. *Proof of $C^{1+\beta}$-rigidity (Theorem 1.6).* The proof repeats identically the argument from [**dFdM99**]. We show that exponential convergence of renormalizations in the case of bounded combinatorics implies $C^{1+\beta}$-rigidity.

For a critical circle map $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ with an irrational rotation number, let I_m be as in (3) and define $I_m^j = f^j(I_m)$. The dynamical partition of level n, associated to f, is defined as

$$\mathcal{P}_n(f) = \{I_{n-1}, I_{n-1}^1, \ldots, I_{n-1}^{q_n-1}\} \cup \{I_n, I_n^1, \ldots, I_N^{q_{n-1}-1}\}.$$

The intervals from $\mathcal{P}_n(f)$ partition the circle modulo the end points and $\mathcal{P}_{n+1}(f)$ is a refinement of the partition $\mathcal{P}_n(f)$.

According to the real *a priori* bounds [**Her86**, **Yoc84**], the length of a maximal interval in $\mathcal{P}_n(f)$ goes to zero as $n \to \infty$. This implies that, given two critical circle maps f_1, f_2 with the same irrational rotation number, there exists a unique homeomorphism $h : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ that maps $0 \mapsto 0$ and conjugates f_1 with f_2. In particular, for any $n \in \mathbb{N}$, this homeomorphism h is an isomorphism between the dynamical partitions $\mathcal{P}_n(f_1)$ and $\mathcal{P}_n(f_2)$.

The following proposition is a reformulated version of [**dFdM99**, Proposition 4.3].

PROPOSITION 9.14. *Let f be a critical circle map with an irrational rotation number of type bounded by B and let $h : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a homeomorphism. If there exist constants $C > 0$ and $0 < \lambda < 1$ such that*

$$\left| \frac{|I|}{|J|} - \frac{|h(I)|}{|h(J)|} \right| \le C\lambda^n \tag{28}$$

for each pair of adjacent atoms $I, J \in \mathcal{P}_n(f)$ for all $n \ge 0$, then h is a $C^{1+\beta}$-diffeomorphism for some $\beta > 0$ that depends only on λ and B.

The estimate from [**dFdM99**, §4.4], together with Theorem 9.6, implies the following result.

PROPOSITION 9.15. *In the above notation, suppose that the rotation number of f_1 and f_2 is of the type bounded by $B \in \mathbb{N}$ and the critical exponent of f_1 and f_2 lies in the union $\bigcup_{k\in\mathbb{N}} J(k, B)$. Then there exist positive real numbers $C > 0$, $0 < \sigma, \lambda < 1$ such that for every $n \in \mathbb{N}$, the following holds. Let $I, J \in \mathcal{P}_n(f_1)$ be two adjacent atoms of the nth partition, both of which belong to the domain of the pre-renormalization $p\mathcal{R}^{n-[\sigma n]} f_1$. Then (28) holds.*

Finally, possibly after increasing the constants C and $\lambda < 1$, the Koebe distortion theorem together with real *a priori* bounds readily implies that (28) holds without the assumption that the atoms $I, J \in \mathcal{P}_n(f_1)$ belong to the domain of the pre-renormalization $p\mathcal{R}^{n-[\sigma n]} f_1$. Thus, the homeomorphism h satisfies the conditions of Proposition 9.14 and hence it is a $C^{1+\beta}$-diffeomorphism. This completes the proof of Theorem 1.6.

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