Euro. Jnl of Applied Mathematics (2013), *vol.* 24, *pp.* 297–313.
doi:10.1017/S0956792512000411

Sharp condition for blow-up and global existence in a two species chemotactic Keller–Segel system in $\mathbb{R}^2$

ELIO ESPEJO[1], KARINA VILCHES[2] and CARLOS CONCA[2,3]

[1]*Department of Mathematics, Technion, 32000 Haifa, Israel*
email: *eespejo@techunix.technion.ac.il*
[2]*Departamento de Ingeniería Matemática (DIM) and Centro de Modelamiento Matemático (CMM), Universidad de Chile (UMI CNRS 2807), Casilla 170-3, Correo 3, Santiago, Chile*
email: *kvilches@dim.uchile.cl*
[3]*Institute for Cell Dynamics and Biotechnology: A Centre for Systems Biology, University of Chile, Santiago, Chile*
email: *cconca@dim.uchile.cl*

(*Received 12 September 2011; revised 1 November 2012; accepted 2 November 2012; first published online 7 December 2012*)

For the parabolic–elliptic Keller–Segel system in $\mathbb{R}^2$ it has been proved that if the initial mass is less than $8\pi/\chi$, a global solution exists, and in case the initial mass is larger than $8\pi/\chi$, blow-up happens. The case of several chemotactic species introduces an additional question: What is the analog for the critical mass obtained for the single species system? We find a threshold curve in the two species case that allows us to determine if the system is a blow-up or a global in time solution. No radial symmetry is assumed.

Key words: chemotaxis, multicomponent Keller–Segel model, sharp conditions

1 Introduction

The Keller–Segel model describes the aggregation of living organisms like cells, bacteria or amoebae. This is the simplest mechanism of aggregation. The most famous example in nature for this type of cell motion is the Dictyostelium discoideum or Slime mould; this amoeba was discovered in the first half of the 20th century. The slime mould is a unicellular organism that detect an extracellular signal and transforms it into an intracellular signal. These signal activates oriented cell movement towards a signal, this is an aggregation process. The signal is a chemical secreted by themselves and is called cyclic Adenosine Monophosphate (cAMP).

A classical mathematical model in chemotaxis was introduced by Keller and Segel in 1971 [12]. The Keller–Segel model is as follows:

$$\begin{aligned} u_t &= \nabla \cdot (\mu \nabla u - \chi u \nabla v) \quad && x \in \Omega, \quad t > 0, \\ v_t &= \gamma \Delta v - \beta v + \alpha u && x \in \Omega, \quad t > 0, \end{aligned} \tag{1}$$

where $u(x,t)$ is the cell density and $v(x,t)$ is the concentration of chemical at point x and time t subject to the homogeneous Neumann boundary conditions and positive initial data $u(x,0) = u_0$ and $v(x,0) = v_0$. In this model, χ is the chemotactic sensitivity, γ is the diffusion coefficient of the chemo-attractant, μ is the diffusion coefficient of cell density, β is the rate of consumption and α is the rate of production, all are positive parameters, and $\Omega \subset \mathbb{R}^N$ has smooth boundary $\partial\Omega$. It was conjectured by Childress and Percus [5] that in a two-dimensional domain there exists a critical number C such that if $\int u_0(x)dx < C$ then the solution exists globally in time, and if $\int u_0(x)dx > C$, then blow-up happens. For different versions of the Keller–Segel model, the conjecture has been essentially proved, finding the critical value $C = 8\pi/\chi$; for a complete review of this topic, we refer readers to [9, 10] and the references therein, and [2, 4, 11, 13, 15].

In the case of several chemotactic species, a new question arises, namely: *Is there a critical curve in the plane of initial masses $\theta_1\theta_2$ delimiting on one side global existence and blow-up on the other side?* This question was previously formulated by Wolansky in [16], and from Theorem 5 of this last paper we readily deduce the following result.

Theorem 1 *Consider the system*

$$\begin{aligned}\partial_t u_1 &= \mu\Delta u_1 - \chi_1\nabla\cdot(u_1\nabla v)\\ \partial_t u_2 &= \Delta u_2 - \chi_2\nabla\cdot(u_2\nabla v)\\ 0 &= \Delta v + u_1 + u_2 - v,\end{aligned}$$

along with Dirichtlet boundary conditions for v and initial radial data: $u_1(0,\cdot) = \varphi$, $u_2(0,\cdot) = \psi$, $v(0,\cdot) = \phi$, with $\varphi, \psi, \phi \geqslant 0$ on the two-dimensional disc of radius 1. Further, let θ_1, θ_2 be the total preserved masses of the chemotactic species. Assume further that

$$\frac{4\pi\mu\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2}(\theta_1+\theta_2)^2 > 0, \quad \theta_1 < 8\pi\mu/\chi_1, \qquad \theta_2 < 8\pi/\chi_2\,. \tag{2}$$

Then for $(u_1(0,\cdot), u_2(0,\cdot)) \in Y_N$ with

$$Y_N = \left\{u_1, u_2 : B(0) \to \mathbb{R}^+ : \int u_i = \theta_i, \quad \int_{B_1(0)} u_i \log u_i < \infty\right\},$$

there exists a global in time classical solution.

A natural question arises from this last result. What happens if inequalities (2) do not hold? Is it still possible to have global solutions? With regard to this question it is worth recalling here a result from Conca *et al.* [6], who considered the following system in the whole space in two dimensions:

$$\left.\begin{aligned}&\partial_t u_1 = \mu\Delta u_1 - \chi_1\nabla\cdot(u_1\nabla v), \qquad x\in\mathbb{R}^2, \quad t>0\\ &\partial_t u_2 = \Delta u_2 - \chi_2\nabla\cdot(u_2\nabla v), \qquad x\in\mathbb{R}^2, \quad t>0\\ &v(x,t) = -\tfrac{1}{2\pi}\int_{\mathbb{R}^2}\log|x-y|\,(u_1(y,t)+u_2(y,t))\,dy, \quad x\in\mathbb{R}^2, \quad t>0\\ &u_1(x,0) = u_{10}\geqslant 0,\ u_2(x,0) = u_{20}\geqslant 0, \quad x\in\mathbb{R}^2, \quad t>0\end{aligned}\right\}, \tag{3}$$

where u_1 and u_2 are the density variables for two different chemotaxis species and v is the chemoattractant, χ_1, χ_2, μ are positive constants and positive initial conditions u_{10}, u_{20} are

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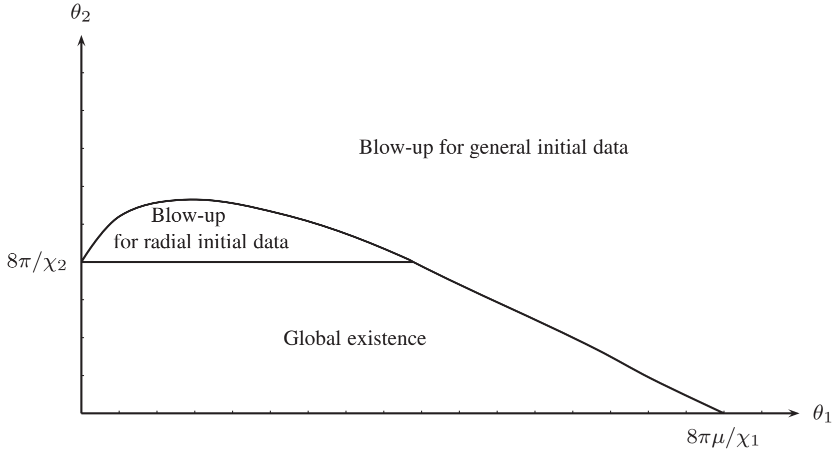

FIGURE 1. Regions of global existence in time and blow-up.

given. In their last paper it was proved that if θ_1, θ_2 satisfy *any* of the inequalities

$$\frac{4\pi\mu\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2}(\theta_1 + \theta_2)^2 < 0, \quad \theta_1 > \mu\frac{8\pi}{\chi_1}, \quad \theta_2 > \frac{8\pi}{\chi_2},$$

then system (3) can blow up. It was also proved in [6] that the inequalities

$$\theta_1 + \theta_2 < \frac{8\pi}{\chi_2}, \qquad \mu \geqslant 1$$
$$\theta_1 + \theta_2 < \frac{8\pi}{\chi_2}\mu, \quad \mu < 1$$

guarantee global existence.

In the present paper we aim to give a step further improving the results of global existence from [6] and to prove that even in the *non-radial case*, inequalities (2) guarantee global existence for system (3). In consequence, we give a generalization of the threshold number $8\pi/\chi$ for the classical parabolic–elliptic Keller–Segel system in $\mathbb{R}^2$ to a curve for the two species system. The global existence in time results of the present paper along with the blow-up results from [6] are summarised in Figure 1.

2 Preliminaries

Let us proceed formally to find a free energy functional for our system. First we write the equation for u_1 in (3) in the form

$$\partial_t u_1 = \nabla \cdot u_1 \nabla (\mu \log u_1 - \chi_1 v). \tag{4}$$

Next, we multiply both sides of (4) by $\mu \log u_1 - \chi_1 v$ and integrate to obtain

$$\int_{\mathbb{R}^2} u_{1t} (\mu \log u_1 - \chi_1 v)\, dx = \int_{\mathbb{R}^2} (\mu \log u_1 - \chi_1 v) \nabla \cdot u_1 \nabla (\mu \log u_1 - \chi_1 v)\, dx. \tag{5}$$

Then using mass conservation and integrating by parts, we see that (5) is equivalent to

$$\frac{d}{dt}\int_{\mathbb{R}^2}\mu u_1\log u_1 dx-\chi_1\int_{\mathbb{R}^2}u_{1t}v dx=-\int_{\mathbb{R}^2}u_1\left|\nabla\left(\mu\log u_1-\chi_1 v\right)\right|^2 dx. \tag{6}$$

Similarly,

$$\frac{d}{dt}\int_{\mathbb{R}^2}u_2\log u_2 dx-\chi_2\int_{\mathbb{R}^2}u_{2t}v dx=-\int_{\mathbb{R}^2}u_2\left|\nabla\left(\log u_2-\chi_2 v\right)\right|^2 dx. \tag{7}$$

Now we add $\frac{1}{\chi_1}$ (6) and $\frac{1}{\chi_2}$ (7) to obtain

$$\begin{aligned}&\frac{d}{dt}\left\{\int_{\mathbb{R}^2}\frac{\mu}{\chi_1}u_1\log u_1 dx+\frac{1}{\chi_2}\int_{\mathbb{R}^2}u_2\log u_2 dx\right\}-\int_{\mathbb{R}^2}\left(u_{1t}+u_{2t}\right)v dx\\ &\quad=-\int_{\mathbb{R}^2}u_1\left|\nabla\left(\mu\log u_1-\chi_1 v\right)\right|^2 dx-\int_{\mathbb{R}^2}u_2\left|\nabla\left(\log u_2-\chi_2 v\right)\right|^2 dx.\end{aligned} \tag{8}$$

We observe at this point that

$$\begin{aligned}\int_{\mathbb{R}^2}\left(u_{1t}+u_{2t}\right)v dx&=-\frac{1}{2\pi}\int_{\mathbb{R}^2}\left(u_1(x,t)+u_2(x,t)\right)_t\int_{\mathbb{R}^2}\log|x-y|\left(u_1(y,t)+u_2(y,t)\right)dydx\\ &=-\frac{1}{4\pi}\frac{d}{dt}\int_{\mathbb{R}^2\times\mathbb{R}^2}\left(u_1(x,t)+u_2(x,t)\right)\left(u_1(y,t)+u_2(y,t)\right)\log|x-y|\,dydx\\ &=\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}(u_1+u_2)v dx.\end{aligned} \tag{9}$$

In conclusion, we deduce from (8) and (9) that

$$\frac{d}{dt}\left\{\int_{\mathbb{R}^2}\frac{\mu}{\chi_1}u_1\log u_1 dx+\frac{1}{\chi_2}\int_{\mathbb{R}^2}u_2\log u_2 dx-\frac{1}{2}\int_{\mathbb{R}^2}(u_1+u_2)v dx\right\}\leqslant 0. \tag{10}$$

Result (10) motivates us to define the free energy functional for system (3) as

$$E(t):=\frac{\mu}{\chi_1}\int_{\mathbb{R}^2}u_1\log u_1 dx+\frac{1}{\chi_2}\int_{\mathbb{R}^2}u_2\log u_2 dx-\frac{1}{2}\int_{\mathbb{R}^2}u_1 v dx-\frac{1}{2}\int_{\mathbb{R}^2}u_2 v dx. \tag{11}$$

In order to give validity to our calculations, we suppose not only that $u_1,u_2\in C^0(\mathbb{R}^+,L^1(\mathbb{R}^2))\cap L^2((0,T);H^1(\mathbb{R}^2))$ but also that $u_1(1+|x|^2)$, $u_2(1+|x|^2)$, $u_1\log u_1$ and $u_2\log u_2$ are bounded in $L^\infty_{loc}(\mathbb{R}^+,L^1(\mathbb{R}^2))$. In addition, $\nabla\sqrt{u_1},\nabla\sqrt{u_2}\in L^1_{\mathrm{loc}}(\mathbb{R}^+,L^1(\mathbb{R}^2))$ and $\nabla v\in L^\infty_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R}^2)$.

Then we have that

$$\frac{d}{dt}E(t)=-\frac{1}{\chi_1}\int_{\mathbb{R}^2}u_1\left|\mu\nabla\log u_1-\nabla\chi_1 v\right|^2 dx-\frac{1}{\chi_2}\int_{\mathbb{R}^2}u_2\left|\nabla\log u_2-\nabla\chi_2 v\right|^2 dx\leqslant 0. \tag{12}$$

As a consequence of (12) and the Hardy–Littlewood–Sobolev (HLS) inequality [5, 9], the following entropy bound was obtained in [6].

Theorem 2 *If u_1 and u_2 are positive solutions of* (3) *on the interval* $[0,T)$ *and* $\chi_1\leqslant\chi_2$, *then we have the following entropy estimates:*

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- *If $\mu > 1$, then*

$$\left(1-\frac{M\chi_2}{8\pi}\right)\int_0^T\int_{\mathbb{R}^2}\left(\frac{1}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)\log\left(\frac{1}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)dxdt\leqslant C_T,$$

where C_T is a constant depending on T and $M=\theta_1+\theta_2$.

- *If $\mu \leqslant 1$, then*

$$\left(1-\frac{M\chi_2}{8\pi\mu}\right)\int_0^T\int_{\mathbb{R}^2}\left(\frac{1}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)\log\left(\frac{1}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)dxdt\leqslant \overline{C}_T,$$

where $\overline{C}_T$ is a constant depending on T and $M=\theta_1+\theta_2$.

Theorem 2 gives bounds for the entropy which is a key tool for the proof of global existence for system (3). In order to improve this last result, it would be desirable to use the HLS inequality for systems developed by Shafrir and Wolansky in [14]. However, as we will show in Section 2, a direct application of this tool to our system does not give the optimal result that we are looking for. We will show how an adequate introduction of some auxiliary parameters in (12) allows us to improve the result of global existence obtained in [6], namely we will show that if θ_1,θ_2 satisfy

$$\frac{4\pi\mu\theta_1}{\chi_1}+\frac{4\pi\theta_2}{\chi_2}-\frac{1}{2}(\theta_1+\theta_2)^2>0,\quad \theta_1<\mu\frac{8\pi}{\chi_1}\ ,\ \theta_2<\frac{8\pi}{\chi_2}$$

then global solutions in time exist. No kind of radial symmetry is assumed.

The most fundamental tool used through this paper is the logarithmic HLS's inequality for systems, which we proceed to recall now. Following the notation in [14] we define the space

$$\Gamma_M\left(\mathbb{R}^2\right)=\left\{\widetilde{\rho}=(\widetilde{\rho}_i)_{i\in I}:\widetilde{\rho}_i\geqslant 0,\int_{\mathbb{R}^2}\widetilde{\rho}_i\left|\log\widetilde{\rho}_i\right|dx<\infty,\right.$$
$$\left.\int_{\mathbb{R}^2}\widetilde{\rho}_i=M_i,\int_{\mathbb{R}^2}\widetilde{\rho}_i\log\left(1+|x|^2\right)<\infty,\forall i\in I\right\},$$

where $M=(M_i)_{i\in I}$ is given. Next we define the functional $F:\Gamma_M(\mathbb{R}^2)\to R$ by

$$F\left[\widetilde{\rho}\right]=\sum_{i\in I}\int_{\mathbb{R}^2}\widetilde{\rho}_i\log\widetilde{\rho}_i dx+\frac{1}{4\pi}\sum_{j,i\in I}a_{i,j}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\widetilde{\rho}_i(x)\log|x-y|\,\widetilde{\rho}_j(y)\,dxdy$$

and the polynomial

$$\Lambda_J(M)=8\pi\sum_{i\in J}M_i-\sum_{i,j\in J}a_{ij}M_iM_j,\qquad \forall\varnothing\neq J\subseteq I.$$

Then we have the following.

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Theorem 3 *Hardy–Litlewood–Sobolev's inequality for systems*
Let $A = (a_{ij})$ *a symmetric matrix such that* $a_{ij} \geqslant 0$ *for all* $i, j \in I$ *and* $M \in \mathbb{R}^n_+$. *Then:* $\Lambda_I(M) = 0$ *and*

$$\Lambda_J(M) \geqslant 0, \quad \textit{for all } J \subseteq I$$
$$\textit{if } \Lambda_J(M) = 0 \textit{ for some } J, \textit{ then } a_{ii} + \Lambda_{J\setminus\{i\}}(M) > 0, \quad \forall i \in J$$

are necessary and sufficient conditions for the boundedness from below of F *on* $\Gamma_M(\mathbb{R}^2)$. *There exists a minimizer* ρ *of* F *over* $\Gamma_M(\mathbb{R}^2)$ *if and only if*

$$\Lambda_I(M) = 0, \quad \textit{and} \quad \Lambda_J(M) > 0, \qquad \textit{for all } J \nsubseteq I$$

Proof See [30, Theorem 4]. □

3 Global existence

The first result of this section gives us bounds for entropy functionals. We achieve our aim through an appropriate use of the HLS inequality for systems, Theorem 3. The main idea of the proof reads as follows: Given that a direct application of the HLS inequality would allow us to get bounds *only on a curve* of the $\theta_1\theta_2$-plane for the entropies $\int_{\mathbb{R}^2} u_i(x,t)\log u_i(x,t)dx$, $i = 1, 2$, we introduce some parameters before applying the HLS inequality. This step will allows us 'to move', 'to shrink' and 'to dilate' this curve in such a way the the full region (18) is swept and therefore obtain estimate (19) in this region.

We suppose throughout this paper that

$$\left.\begin{array}{r} u_{10}, u_{20} \in L^1(\mathbb{R}^2, (1+|x|^2)dx), \\ u_{10}\log u_{10},\ u_{20}\log u_{20} \in L^1(\mathbb{R}^2, dx) \end{array}\right\}. \tag{13}$$

Lemma 4 (Lower bound for the entropy functionals) *Consider a non-negative weak solution of (3) such that* $u_i(1+|x|^2)$, $i = 1, 2$ *are bounded in* $L^\infty_{loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))$. *Then we have*

$$\int_{\mathbb{R}^2} u_i(x,t)\log u_i(x,t) \geqslant M\log M - M\log[\pi(1+t)] - C, \quad i = 1, 2.$$

Proof In the following, C will denote a generic constant. We have from [6, Theorem 1] that

$$\frac{d}{dt}\int_{\mathbb{R}^2}\left(\frac{\mu}{\chi_1}u_1(x,t) + \frac{1}{\chi_2}u_2(x,t)\right)|x|^2\,dx = \frac{4\theta_1}{\chi_1}\mu + \frac{4\theta_2}{\chi_2} - \frac{1}{2\pi}(\theta_1+\theta_2)^2. \tag{14}$$

We define

$$n := \frac{\mu}{\chi_1}u_1 + \frac{1}{\chi_2}u_2;$$

and

$$K := \frac{4\theta_1}{\chi_1}\mu + \frac{4\theta_2}{\chi_2} - \frac{1}{2\pi}(\theta_1+\theta_2)^2.$$

Thus, we obtain

$$\int_{\mathbb{R}^2} n(x,t)\,|x|^2\,dx = Kt + \int_{\mathbb{R}^2} n(x,0)\,|x|^2\,dx \leqslant C(1+t), \tag{15}$$

where $C := \max\{K, \int_{\mathbb{R}^2} n(x,0)\,|x|^2\,dx\}$. From the inequality $u_i \leqslant Cn$, where $i = 1,2$ and (15) we deduce that

$$\int_{\mathbb{R}^2} u_i(x,t)\,|x|^2\,dx \leqslant C(1+t), \quad i = 1,2.$$

Using the same idea presented in [4, Lemma 2.5], we observe that

$$\begin{aligned}\int_{\mathbb{R}^2} u_i\,(x,t) \log u_i\,(x,t) &\geqslant \tfrac{1}{1+t} \int_{\mathbb{R}^2} u_i\,(x,t)\,|x|^2 - C + \int_{\mathbb{R}^2} u_i\,(x,t) \log u_i\,(x,t)\\ &= \int_{\mathbb{R}^2} u_i\,(x,t) \log\left[\frac{u_i\,(x,t)}{e^{-\frac{|x|^2}{1+t}}}\right] - C.\end{aligned} \tag{16}$$

Let us now define the variable μ as

$$\mu\,(x,t) = \frac{1}{\pi\,(1+t)} \exp\left(-\frac{|x|^2}{1+t}\right).$$

We then obtain from (16) that

$$\begin{aligned}\int_{\mathbb{R}^2} u_i\,(x,t) \log u_i\,(x,t) &\geqslant \int_{\mathbb{R}^2} u_i\,(x,t) \log\left[\frac{u_i\,(x,t)}{\mu\,(x,t)}\right] dx - M \log\left[\pi\,(1+t)\right] - C\\ &= \int_{\mathbb{R}^2} \frac{u_i\,(x,t)}{\mu\,(x,t)} \log\left[\frac{u_i\,(x,t)}{\mu\,(x,t)}\right] \mu\,(x,t)\,dx - M \log\left[\pi\,(1+t)\right] - C,\end{aligned} \tag{17}$$

where $M = \frac{\mu}{\chi_1}\theta_1 + \frac{1}{\chi_2}\theta_2$. Using Jensen's inequality we get from (17) that

$$\int_{\mathbb{R}^2} u_i\,(x,t) \log u_i\,(x,t) \geqslant M \log M - M \log\left[\pi\,(1+t)\right] - C.$$

□

Theorem 5 (Upper bound for entropy functionals) *Consider a non-negative weak solution of (3) such that* $u_i(1+|x|^2)$, $u_i \log u_i$, $i = 1,2$ *are bounded in* $L^\infty_{loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))$. *If* (θ_1, θ_2) *satisfies*

$$\theta_1 < \frac{8\pi}{\chi_1}\mu; \quad \theta_2 < \frac{8\pi}{\chi_2}; \qquad 8\pi\left(\frac{\theta_1}{\chi_1}\mu + \frac{\theta_2}{\chi_2}\right) - (\theta_1+\theta_2)^2 > 0, \tag{18}$$

then we have

$$\int_{\mathbb{R}^2} u_i(x,t) \log u_i(x,t) dx \leqslant C, \tag{19}$$

where $i = 1,2$ *and* C *is a constant depending only on the parameters* θ_1 *and* $\theta_2, \mu, \chi_1, \chi_2$, $E(0)$.

Proof From (12) we have that

$$E\,(t) \leqslant E\,(0)\,, \qquad \forall t > 0.$$

In consequence, we have the following estimate:

$$
\begin{aligned}
&\frac{\mu}{\chi_1}\int_{\mathbb{R}^2} u_1(x,t)\log u_1(x,t)\,dx + \frac{1}{\chi_2}\int_{\mathbb{R}^2} u_2(x,t)\log u_2(x,t)\,dx \\
&\quad \leqslant E(0) - \frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u_1(x,t)\,u_1(y,t)\log|x-y|\,dxdy \\
&\quad - \frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u_1(x,t)\,u_2(y,t)\log|x-y|\,dxdy \\
&\quad - \frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u_2(x,t)\,u_1(y,t)\log|x-y|\,dxdy - \frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u_2(x,t)\,u_2(y,t)\log|x-y|\,dxdy.
\end{aligned}
$$

We introduce positive parameters a and b in the last inequality in the following way

$$
\begin{aligned}
&\frac{\mu}{\chi_1}\int_{\mathbb{R}^2} u_1(x,t)\log u_1(x,t)\,dx + \frac{1}{\chi_2}\int_{\mathbb{R}^2} u_2(x,t)\log u_2(x,t)\,dx \\
&\quad \leqslant E(0) - \frac{a^2}{\mu^2 4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\mu u_1(x,t)}{a}\frac{\mu u_1(y,t)}{a}\log|x-y|\,dxdy \\
&\quad - \frac{ab}{\mu 4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\mu u_1(x,t)}{a}\frac{u_2(y,t)}{b}\log|x-y|\,dxdy \\
&\quad - \frac{ab}{\mu 4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{u_2(x,t)}{b}\frac{\mu u_1(y,t)}{a}\log|x-y|\,dxdy \\
&\quad - \frac{b^2}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{u_2(x,t)}{b}\frac{u_2(y,t)}{b}\log|x-y|\,dxdy. \qquad (20)
\end{aligned}
$$

By doing so, we can now apply the HLS inequality for systems (Theorem 3) to the functions $\mu u_1/a$ and u_2/b in identity (20) getting that

$$
\begin{aligned}
&\frac{\mu}{\chi_1}\int_{\mathbb{R}^2} u_1(x,t)\log u_1(x,t) + \frac{1}{\chi_2}\int_{\mathbb{R}^2} u_2(x,t)\log u_2(x,t) \\
&\quad \leqslant E(0) - C + \int_{\mathbb{R}^2} \mu\frac{u_1(x,t)}{a}\log\left(\mu\frac{u_1(x,t)}{a}\right)dx + \int_{\mathbb{R}^2}\frac{u_2(x,t)}{b}\log\left(\frac{u_2(x,t)}{b}\right)dx,
\end{aligned}
$$

where the conditions for the existence of the constant C given by Theorem 3 are

$$
\begin{aligned}
\Lambda_{\{1\}}(M) &= 8\pi\mu\frac{\theta_1}{a} - a^2\left(\frac{\theta_1}{a}\right)^2 \geqslant 0; \\
\Lambda_{\{2\}}(M) &= 8\pi\frac{\theta_2}{b} - b^2\left(\frac{\theta_2}{b}\right)^2 \geqslant 0; \\
\Lambda_{\{1,2\}}(M) &= 8\pi\left(\mu\frac{\theta_1}{a} + \frac{\theta_2}{b}\right) - \left(a^2\frac{\theta_1}{a}\frac{\theta_1}{a} + 2ab\frac{\theta_1}{a}\frac{\theta_2}{b} + b^2\frac{\theta_2}{b}\frac{\theta_2}{b}\right) = 0.
\end{aligned}
$$

Equivalently,

$$
\left.\begin{array}{l}
\theta_1 \leqslant \mu\frac{8\pi}{a}, \qquad \theta_2 \leqslant \frac{8\pi}{b} \\
8\pi\left(\mu\frac{\theta_1}{a} + \frac{\theta_2}{b}\right) - (\theta_1+\theta_2)^2 = 0
\end{array}\right\}. \qquad (21)
$$

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In conclusion we have proved that condition (21) implies

$$\mu\left(\frac{1}{\chi_1}-\frac{1}{a}\right)\int_{\mathbb{R}^2}u_1(x,t)\log u_1(x,t)+\left(\frac{1}{\chi_2}-\frac{1}{b}\right)\int_{\mathbb{R}^2}u_2(x,t)\log u_2(x,t)$$
$$\leqslant E(0)-C+\frac{\theta_1\mu}{a}\log\frac{\mu}{a}+\frac{\theta_2}{b}\log\frac{1}{b}. \tag{22}$$

We have from Lemma 4 that the functionals $\int u_i\log u_i dx$ are bounded below for $i=1,2$. On the other hand, each of the coefficients of the entropy functionals in (22) are positive as long as $a>\chi_1$ and $b>\chi_2$. Then we take parameters a and b on the intervals (χ_1,∞) and (χ_2,∞) respectively We conclude that estimate (19) hold on region (18). □

Boundedness of entropies in the last theorem is the main tool that we will use to obtain the following result of global existence.

Theorem 6 (Global existence of weak solutions) *Under assumption (13) and*

$$8\pi\left(\frac{\theta_1}{\chi_1}\mu+\frac{\theta_2}{\chi_2}\right)-(\theta_1+\theta_2)^2>0, \tag{23}$$

$$\theta_1<\frac{8\pi}{\chi_1}\mu;\qquad \theta_2<\frac{8\pi}{\chi_2}, \tag{24}$$

system (3) has a global weak non-negative solution such that

$$(1+|x|^2+|\log u_i|)u_i\in L^\infty(0,T;L^1(\mathbb{R}^2))$$

and

$$-\frac{1}{\chi_1}\int\int_{[0,T]\times\mathbb{R}^2}u_1\left|\mu\nabla\log u_1-\nabla\chi_1 v\right|^2dx-\frac{1}{\chi_2}\int\int_{[0,T]\times\mathbb{R}^2}u_2\left|\nabla\log u_2-\nabla\chi_2 v\right|^2dx<\infty.$$

Before giving the proof, let us first give some explanation of this result. Inequality (23) corresponds to the interior of a rotated parabola in the plane $\theta_1\theta_2$. Choosing the parameters μ, χ_1 and χ_2 appropriately, condition (24) may be relevant or can be simply ignored. Next, Figure 2 illustrates the two possible cases:

More precisely we have that,

- if the parabola
$$8\pi\left(\frac{\theta_1}{\chi_1}\mu+\frac{\theta_2}{\chi_2}\right)-(\theta_1+\theta_2)^2=0 \tag{25}$$
intersects either of the lines $\theta_1=8\pi\mu/\chi_1$ or $\theta_2=8\pi/\chi_2$ in the first quadrant of the $\theta_1\theta_2$ plane (which happens exactly when $\chi_1<\mu\chi_2/2$ or $\chi_1>2\mu\chi_2$), then system (3) has global existence in time weak solutions as long as the initial masses satisfy inequalities (23) together with (24).
- However, if the parabola (25) does not intersect either of the lines $\theta_1=8\pi\mu/\chi_1$ or $\theta_2=8\pi/\chi_2$ (when $\mu\chi_2/2\leqslant\chi_1\leqslant 2\mu\chi_2$) in the first quadrant of the $\theta_1\theta_2$ plane, then inequality (23) is enough to guarantee that system (3) has a global in time weak solution.

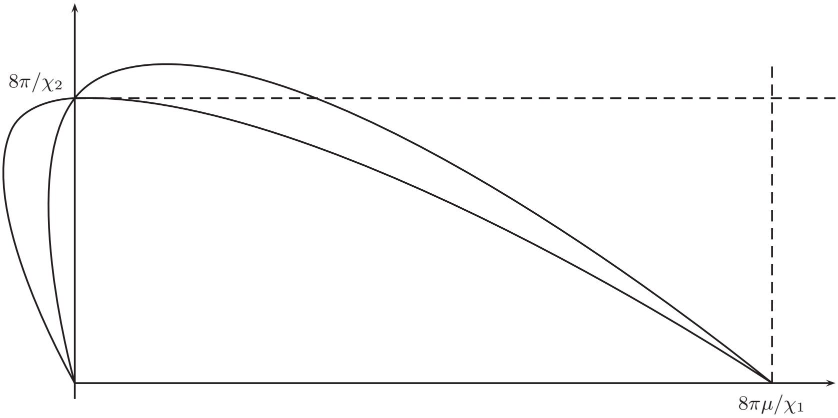

FIGURE 2. Two basic configurations of parabola (25).

On the other hand, we should point out that all of our results are formal so far. In order to make them rigorous, we should have a local existence result of smooth solutions. However, we will take another strategy which will allow us to obtain directly global existence in time of weak solutions with the corresponding mathematical rigour. In order to prove Theorem 6, we first modify the convolution kernel $k^0(z) = -\frac{1}{2\pi}\log|z|$ in (3) by truncating it around zero. This last will allow us to get a regularized version of system (3), which is rather easier to work. After proving the existence of global solutions of this last approximate problem, we look for uniform estimates of solutions and then pass to the limit that will give us the result of global existence we are looking for. After getting this result we recover properties such as mass conservation or the second moment formula by testing properly our weak solution. A similar technique was made in the one chemotaxis species case (see [4, 5]).

Proof (Sketch) For the reader's convenience, we divide the proof into four steps giving special attention where technical difficulties arise in comparison to the single species case.

Step 1. *Regularization of the system.* We define K^ϵ by $K^\epsilon(z) := K^1\left(\frac{z}{\epsilon}\right)$, where K^1 is a radial monotone non-decreasing smooth function satisfying

$$K^1(z) = \begin{cases} -\frac{1}{2\pi}\log|z| & \text{if } |z| \geqslant 4 \\ 0 & \text{if } |z| \leqslant 1 \end{cases}.$$

Assume also that

$$\left|\nabla K^1(z)\right| \leqslant \frac{1}{2\pi|z|}$$

$$K^1(z) \leqslant -\frac{1}{2\pi}\log|z|\,; \qquad -\Delta K^1(z) \geqslant 0; \qquad \forall z \in \mathbb{R}^2$$

for any $z \in \mathbb{R}^2$. Then we consider the following regularized version of system (3)

$$\begin{cases} \partial_t u_1^\epsilon = \Delta u_1^\epsilon - \chi_1 \nabla \cdot (u_1^\epsilon \nabla v^\epsilon), & t \geqslant 0, \quad x \in \mathbb{R}^2 \\ \partial_t u_2^\epsilon = \Delta u_2^\epsilon - \chi_2 \nabla \cdot (u_2^\epsilon \nabla v^\epsilon), & t \geqslant 0, \quad x \in \mathbb{R}^2, \\ v^\epsilon = K^\epsilon * \left(u_1^\epsilon + u_2^\epsilon\right), & t \geqslant 0, \quad x \in \mathbb{R}^2 \end{cases} \tag{26}$$

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which we interpret in the sense of distributions. Since $K^\varepsilon(z) = K^1(\frac{z}{\varepsilon})$, we also have

$$|\nabla K^\varepsilon(z)| = \frac{1}{\varepsilon}\left|\nabla K\left(\frac{z}{\varepsilon}\right)\right| \leqslant \frac{1}{\varepsilon}\frac{1}{2\pi|z/\varepsilon|} = \frac{1}{2\pi|z|}. \tag{27}$$

The proof of global solutions in $L^2(0,T;H^1(\mathbb{R}^2)\cap C\left(0,T;L^2(\mathbb{R}^2)\right))$ for system (26) with initial data in $L^2(\mathbb{R}^2)$ follows essentially the same lines as in [4, Proposition 2.8] and therefore we omit the proof here.

Step 2. A priori *estimates for the approximate solutions* u_1^ε, u_2^ε *and* v^ε.

Consider a solution $\left(u_1^\epsilon, u_2^\epsilon\right)$ of the regularized system. If

$$\theta_1 < \frac{8\pi}{\chi_1}\mu; \quad \theta_2 < \frac{8\pi}{\chi_2}; \qquad 8\pi\left(\frac{\theta_1}{\chi_1}\mu + \frac{\theta_2}{\chi_2}\right) - (\theta_1+\theta_2)^2 > 0,$$

then, uniformly as $\epsilon \to 0$, with bounds depending only upon $\int_{\mathbb{R}^2}(1+|x|^2)u_{i0}dx$ and $\int_{\mathbb{R}^2} u_{i0}\log u_{i0}dx$ with $i = 1,2$, we have the following estimates:

(i) The function $(x,t) \to |x|^2\left(u_1^\epsilon + u_2^\epsilon\right)$ is bounded in $L^\infty\left(\mathbb{R}^+_{loc}; L^1(\mathbb{R}^2)\right)$.

(ii) The functions $t \to \int_{\mathbb{R}^2} u_j^\epsilon(x,t)\log u_j^\epsilon(x,t)\,dx$ and $t \to \int_{\mathbb{R}^2} u_j^\epsilon(x,t)\,v^\epsilon(x,t)\,dx$ are bounded for $j = 1,2$.

(iii) The function $(x,t) \to u_j^\epsilon(x,t)\log\left(u_j^\epsilon(x,t)\right)$ is bounded in $L^\infty\left(\mathbb{R}^+_{loc}; L^1(\mathbb{R}^2)\right)$ for $j = 1,2$.

(iv) The function $(x,t) \to \nabla\sqrt{u_j^\epsilon(x,t)}$ is bounded in $L^2\left(\mathbb{R}^+_{loc}\times\mathbb{R}^2\right)$ for $j = 1,2$.

(v) The function $(x,t) \to u_j^\epsilon(x,t)$ is bounded in $L^2\left(\mathbb{R}^+_{loc}\times\mathbb{R}^2\right)$ for $j = 1,2$.

(vi) The function $(x,t) \to u_j^\epsilon(x,t)\,\Delta v^\epsilon(x,t)$ is bounded in $L^1\left(\mathbb{R}^+_{loc}\times\mathbb{R}^2\right)$ for $j = 1,2$.

(vii) The function $(x,t) \to \sqrt{u_j^\epsilon(x,t)}\nabla v^\epsilon(x,t)$ is bounded in $L^2\left(\mathbb{R}^+_{loc}\times\mathbb{R}^2\right)$ for $j = 1,2$.

The proof of estimates (i)–(vii) follows essentially the same steps as in the one species case and therefore we refer the reader to [4, Lema 2.11].

As a consequence of estimate (ii), the first two equations of system (3) have the hyper-contractivity property [4, Theorem 3.5], i.e. for any $1 < p < \infty$, there exists a continuous function $h_p^j : (0,T) \to \mathbb{R}$ such that $\left\|u_j^\epsilon(\cdot,t)\right\|_{L^p(\mathbb{R}^2)} \leqslant h_p^j(t)$, $j = 1,2$. Hence, $u_j^\epsilon \in L^\infty((\delta,T), L^p(\mathbb{R}^2))$, $p \in (1,\infty)$ for any $\delta \in (0,T)$. Therefore, we have the following result:

(viii) The function $(x,t) \to u_j^\epsilon(x,t)$ is bounded in $L^\infty((\delta,T), L^p(\mathbb{R}^2))$ for $j = 1,2$, $p > 1$.

Step 3. *Construction of a strong convergence subsequence in* L^p. To achieve our aim in this step we will apply the Aubin–Lions compactness lemma.

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First we get a uniform bound on $\|\nabla u_i^\epsilon\|_{L^2_{\rm loc}((\delta,T)\times\mathbb{R}^2)}$. We observe that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2}|u_i^\epsilon|^2\,dx &= -2\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\,dx + 2\chi_1\int_{\mathbb{R}^2}u_i^\epsilon\nabla u_i^\epsilon\cdot\nabla v^\epsilon dx \leqslant -2\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\,dx\\ &+2\chi_1\left(\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\right)^{1/2}\left(\int_{\mathbb{R}^2}|u_i^\epsilon|^2\,|\nabla v^\epsilon|^2\,dx\right)^{1/2}\leqslant -2\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\,dx\\ &+2\chi_1\left(\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\right)^{1/2}\left(\int_{\mathbb{R}^2}|u_i^\epsilon|^3\,dx\right)^{1/3}\left(\int_{\mathbb{R}^2}|\nabla v^\epsilon|^6\,dx\right)^{1/6}, \end{aligned}\tag{28}$$

where we have used the Hölder inequality in the last line. The classical Gagliardo–Nirenberg–Sobolev inequality along with the Calderon–Zigmund inequality allow us to conclude that

$$\left(\int_{\mathbb{R}^2}|\nabla v^\epsilon|^6\,dx\right)^{1/6}\leqslant C\left(\int_{\mathbb{R}^2}|\Delta v^\epsilon|^{3/2}\,dx\right)^{2/3}. \tag{29}$$

From inequalities (28) and (29) we deduce that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^2}|u_i^\epsilon|^2\,dx\\ &\leqslant -2\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\,dx + 2C\chi_1\left(\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\right)^{1/2}\left(\int_{\mathbb{R}^2}|u_i^\epsilon|^3\,dx\right)^{1/3}\left(\int_{\mathbb{R}^2}|\Delta v^\epsilon|^{3/2}\,dx\right)^{2/3}\\ &\leqslant -2\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\,dx + 2C\chi_1\left(\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\right)^{1/2}\left(\int_{\mathbb{R}^2}|u_i^\epsilon|^3\,dx\right)^{1/3}\\ &\quad\times\left(\left(\int_{\mathbb{R}^2}|u_1^\epsilon|^{3/2}\,dx\right)^{2/3}+\left(\int_{\mathbb{R}^2}|u_2^\epsilon|^{3/2}\,dx\right)^{2/3}\right).\end{aligned}$$

Integrating with respect to t and reordering last inequality, we now obtain

$$\begin{aligned}&2\int_\delta^T\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\,dxdt - 2C\chi_1\left\{\sup_{t\in[\delta,T]}\left(\int_{\mathbb{R}^2}|u_i^\epsilon|^3\,dx\right)^{1/3}\left(\sup_{t\in[\delta,T]}\left(\int_{\mathbb{R}^2}|u_1^\epsilon|^{3/2}\,dx\right)^{2/3}\right.\right.\\ &\left.\left.+\sup_{t\in[\delta,T]}\left(\int_{\mathbb{R}^2}|u_2^\epsilon|^{3/2}\,dx\right)^{2/3}\right)\right\}\int_\delta^T\left(\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\right)^{1/2}dt + \int_{\mathbb{R}^2}|u_i^\epsilon|^2\,dx - \int_{\mathbb{R}^2}|u_i^\epsilon(x,0)|^2\,dx\leqslant 0.\end{aligned}$$

We observe now that

$$\int_\delta^T\left(\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\,dx\right)^{1/2}dt\leqslant (T-\delta)^{1/2}\left(\int_0^T\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\,dxdt\right)^{1/2}.$$

Denoting by $X := \|\nabla u_i^\epsilon\|_{L^2_{loc}((\delta,T)\times\mathbb{R}^2)}$ and taking into account (viii), we conclude from last two estimates that for positive constants a, b and c we have that

$$aX^2 - bX + c\leqslant 0,$$

in consequence $X := \|\nabla u_i^\epsilon\|_{L^2_{loc}((\delta,T)\times\mathbb{R}^2)}$ is bounded, i.e there exists a constant C such that

$$\|\nabla u_i^\epsilon\|_{L^2_{loc}((\delta,T)\times\mathbb{R}^2)}\leqslant C. \tag{30}$$

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Now we obtain a bound for $\|du_i^\varepsilon/dt\|_{L^2((\delta,T);H^{-1}(\mathbb{R}^2))}$. First of all we notice that in the middle of the proof of estimation (30) we have proved that

$$\|u_i\nabla v^\epsilon\|_{L^2(\mathbb{R}^2)} \leqslant \left(\int_{\mathbb{R}^2} |u_i^\epsilon|^3\,dx\right)^{1/3}\left(\left(\int_{\mathbb{R}^2} |u_1^\epsilon|^{3/2}\,dx\right)^{2/3} + \left(\int_{\mathbb{R}^2} |u_2^\epsilon|^{3/2}\,dx\right)^{2/3}\right). \quad (31)$$

It follows from the last estimate and (viii) that for some constant C we have

$$\|u_i\nabla v^\epsilon\|_{L^2(\mathbb{R}^2)} \leqslant C. \quad (32)$$

Let $\phi \in H^1(\mathbb{R}^2)$, then we have

$$\begin{aligned}|\langle du_i^\varepsilon/dt,\phi\rangle| &= |\langle \Delta u_i - \nabla\cdot(u_i\nabla v^\epsilon),\phi\rangle| \leqslant |\langle\nabla u_i,\nabla\phi\rangle| + |\langle u_i\nabla v^\epsilon,\nabla\phi\rangle| \\ &\leqslant \|\nabla\phi\|_{L^2(\mathbb{R}^2)}\,\|\nabla u_i\|_{L^2(\mathbb{R}^2)} + \|\nabla\phi\|_{L^2(\mathbb{R}^2)}\,\|u_i\nabla v^\epsilon\|_{L^2(\mathbb{R}^2)}. \end{aligned} \quad (33)$$

Thus,

$$\|du_i^\varepsilon/dt\|_{H^{-1}(\mathbb{R}^2)} = \sup_{\|\phi\|_{H^1(\mathbb{R}^2)}=1} |\langle du_i^\varepsilon/dt,\phi\rangle| \leqslant \|\nabla u_i^\varepsilon\|_{L^2(\mathbb{R}^2)} + \|u_i^\varepsilon\nabla v^\epsilon\|_{L^2(\mathbb{R}^2)}.$$

From the last estimate and taking into account (30) and (32), it follows that

$$\|du_i^\varepsilon/dt\|_{L^2((\delta,T);H^{-1}(\mathbb{R}^2))} = \left(\int_\delta^T \|du_i^\varepsilon/dt\|^2_{H^{-1}(\mathbb{R}^2)}\,dt\right)^{1/2} \leqslant C. \quad (34)$$

Compactness: In order to apply the Aubin–Lions Lemma, we define the spaces $B_0 = H^1(\mathbb{R}^2)\cap\{f|\quad |x|^2 f \in L^1(\mathbb{R}^2)\}$, $B := L^2(\mathbb{R}^2)$ and $B_1 := B_0'$. Let $\{f_i\}$ be an arbitrary bounded sequence in B_0, then we have L^2equi-integrability at infinity (cf. [1, Corollary 5.3.1]) as the following account shows:

$$\begin{aligned}\int_{\{|x|>R\}} f_i^2dx &\leqslant \frac{1}{R}\int_{\{|x|>R\}}\left(|x|\,f_i^{1/2}\right)f_i^{3/2}dx \leqslant \frac{1}{R}\left(\int_{\{|x|>R\}}|x|^2 f_i dx\right)^{1/2}\left(\int_{\{|x|>R\}} f_i^3dx\right)^{1/2} \\ &\leqslant \frac{1}{R}\left(\int_{\mathbb{R}^2}|x|^2 f_i dx\right)^{1/2}\left(\int_{\mathbb{R}^2} f^3dx\right)^{1/2}.\end{aligned}$$

Thus,

$$\lim_{R\to+\infty}\int_{\{|x|>R\}} f_i^2dx = 0 \ \text{ uniformly with respect to } f_i. \quad (35)$$

From the Rellich–Kondrakov Theorem (cf. [1, Corollary 5.3.1] we obtain the compact inclusion

$$B_0 \hookrightarrow\hookrightarrow B.$$

Given that u_i^ε satisfies (30), (34) and (35), we can now invoke the Aubin–Lions–Simon theorem to conclude that u_i^ε has a subsequence that converge strongly in $L^2(\delta,T,B)$. Therefore, up to a subsequence we have that

$$u_i^\epsilon \to u_i \text{ a.e. in } \mathbb{R}^2\times[\delta,T]. \quad (36)$$

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We have also proved uniformly boundedness for $\|u_i^\epsilon\|_{L^p(\mathbb{R}^2)\times[\delta,T]}$, from this, estimation (36) and the Vitali theorem, we obtain

$$u_i^\epsilon \to u_i \ \text{ strongly in } L^p(\mathbb{R}^2 \times [0,T]) \text{ for } p \geqslant 1. \tag{37}$$

Step 4. *Pass to the limit*. We pass now to the limit in the weak sense to obtain our result of global existence. The most significant technical difficulty to show that u_1, u_2 solved (3) arise with the nonlinear terms. In order to prove that

$$u_i^\epsilon \nabla v^\epsilon \rightharpoonup u_i \nabla v, \text{in } D'(\mathbb{R}^+ \times \mathbb{R}^2), \tag{38}$$

we first notice that the expression $u_i^\epsilon |\nabla v^\epsilon|$ is integrable as estimate (vii) of part 2 along with the following estimate shows

$$\begin{aligned}\left(\int_{[0,T]\times\mathbb{R}^2} u_i^\epsilon |\nabla v^\epsilon|\, dxdt\right)^2 &= \left(\int_{[0,T]\times\mathbb{R}^2} \sqrt{u_i^\epsilon}\sqrt{u_i^\epsilon}\, |\nabla v^\epsilon|\, dxdt\right)^2 \\ &\leqslant \int_{[0,T]\times\mathbb{R}^2} u_i^\epsilon dxdt \int_{[0,T]\times\mathbb{R}^2} u_i^\epsilon |\nabla v^\epsilon|^2\, dxdt \leqslant \theta_i T \int_{[0,T]\times\mathbb{R}^2} u_i^\epsilon |\nabla v^\epsilon|^2\, dxdt.\end{aligned}$$

It follows that we can interpret $u_i^\epsilon \nabla v^\epsilon$ as an element of $\left(C_0^\infty\left(\mathbb{R}^+ \times \mathbb{R}^2\right)\right)'$ and therefore its divergence is defined.

In order to prove that $\|\nabla v^\varepsilon\|_{L^r(\mathbb{R}^n)} \leqslant C$ for $r > 2$, we recall the HLS inequality: For all $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, $1 < p, q < \infty$, such that $1/p + 1/q + \lambda/n = 2$ and $0 < \lambda < n$, there exists a constant $C = C(p, q, \lambda) > 0$ such that

$$\left|\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{1}{|x-y|^\lambda} f(x)g(y)dxdy\right| \leqslant C\, \|f\|_{L^p(\mathbb{R}^n)}\, \|g\|_{L^q(\mathbb{R}^n)}.$$

Taking the supremum over the ball $\|g\|_{L^q(\mathbb{R}^n)} = 1$ on both sides of the last inequality, we obtain

$$\left\|\int_{\mathbb{R}^n} \frac{1}{|x-y|^\lambda} f(x)dx\right\|_{L^{\frac{q}{q-1}}(\mathbb{R}^n)} \leqslant C\, \|f\|_{L^p(\mathbb{R}^n)}. \tag{39}$$

In particular

$$\left\|\int_{\mathbb{R}^n} \frac{1}{|x-y|} f(x)dx\right\|_{L^{\frac{q}{q-1}}(\mathbb{R}^2)} \leqslant C\, \|f\|_{L^p(\mathbb{R}^2)} \text{ where } 1 < p, q < \infty, \text{ and } 1/p + 1/q + 1/2 = 2.$$

Thus, we have that

$$\|\nabla v^\varepsilon\|_{L^r(\mathbb{R}^n)} = \|\nabla K^\varepsilon * (u_1^\varepsilon + u_2^\varepsilon)\|_{L^r(\mathbb{R}^n)} \tag{40}$$

$$\leqslant \left\|\frac{1}{2\pi}\int \frac{1}{|x-y|}(u_1^\varepsilon + u_2^\varepsilon)dx\right\|_{L^r(\mathbb{R}^n)} \leqslant C\left(\|u_1^\varepsilon\|_{L^p(\mathbb{R}^2)} + \|u_2^\varepsilon\|_{L^p(\mathbb{R}^2)}\right) \leqslant C, \tag{41}$$

where we have used step 2 (viii). From $r = \frac{q}{q-1}$ and $1/p + 1/q + 1/2 = 2$ we obtain that $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$. In addition, $p \in (1,2)$ implies that $r \in (2,\infty)$. We conclude that (up to a

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subsequence) $\nabla v^\varepsilon \rightharpoonup h$, where h is in L^r. In order to prove that actually $h = \nabla K * n$ we have to do some extra work yet. With this end in mind, we now propose to show that

$$\nabla v^\epsilon \to \nabla v \qquad a.e. \tag{42}$$

We have that

$$\begin{aligned} \nabla v^\epsilon - \nabla v = & -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2}\left((u_1^\epsilon + u_2^\epsilon) - (u_1+u_2)\right)(y,t)\,dy \\ & + \int_{|x-y|\leqslant 2\epsilon}\left(\frac{1}{\varepsilon}\nabla K^1\left(\frac{x-y}{\varepsilon}\right) + \frac{|x-y|}{2\pi\,|x-y|^2}\right)\left(u_1^\epsilon + u_2^\epsilon\right)(y,t)dy. \end{aligned} \tag{43}$$

We deduce from (37) and (39) that (up to a subsequence) the first integral in (43) converges to zero *a.e.* On the other hand, estimates (27) allows us to conclude that

$$\begin{aligned} & \left|\int_{|x-y|\leqslant 2\epsilon}\left(\frac{1}{\varepsilon}\nabla K^1\left(\frac{x-y}{\varepsilon}\right) + \frac{|x-y|}{2\pi\,|x-y|^2}\right)\left(u_1^\epsilon + u_2^\epsilon\right)(y,t)dy\right| \\ & \quad \leqslant \int_{|x-y|\leqslant 2\epsilon}\left(\frac{1}{\pi\,|x-y|}\right)\left(u_1^\epsilon + u_2^\epsilon\right)(y,t)dy. \end{aligned}$$

Last integral converges to 0 as $\varepsilon \to 0$, therefore we conclude (42).

We therefore obtain from [8, Prop. 2.46 (i)] that $\nabla v_\varepsilon \rightharpoonup \nabla K * n$ weakly in L^r for $r \geqslant 2$. Finally, we choose conjugate exponents $r = 4$ and $p = 4/3$ to conclude the convergence (38). □

4 Conclusions and open questions

It has been proved in this paper that system (3) has a threshold curve that determines global existence or blow-up. A more difficult task is to find out if the blow-up has to be simultaneous or not and also to describe the asymptotics near the blow-up time. A first step in this direction was given by Espejo *et al.* in [7], where it was shown that the blow-up has to be simultaneous in the radial case. Should it be the same in the general case? Or should it depend on more specific information on the initial data? With regard to this point it is worth recalling that according to [6] it is possible to have blow-up even in the case that the total moment

$$m(t) := \frac{\pi}{\chi_1}\int_{\mathbb{R}^2} u_1(x,t)\,|x|^2\,dx + \frac{\pi}{\chi_2}\int_{\mathbb{R}^2} u_2(x,t)\,|x|^2\,dx \tag{44}$$

is increasing, that is when we have

$$\frac{4\pi\mu\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2}(\theta_1+\theta_2)^2 > 0.$$

This opens up a new possibility: The density of one chemotactic species could be increasing meanwhile the other decreases. That is to say, the question of a simultaneous blow-up or not as well as a possible collapse mass separation could eventually not only depend on the radial symmetry of the initial data but also on the L^1 size of the initial data.

On the other hand, if the parabola

$$\frac{4\pi\mu\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2}(\theta_1 + \theta_2)^2 = 0 \tag{45}$$

intersects any of the lines

$$\theta_1 = \frac{8\pi}{\chi_1} \quad \text{or} \quad \theta_2 = \frac{8\pi}{\chi_2}, \tag{46}$$

it would be very interesting to study the behaviour of system (3) on this lines. Here it is worth recalling that the proof of convergence towards a delta function at $T = \infty$ in the one species case, when total mass is exactly $8\pi/\chi$, uses in a essential way that the second moment is preserved (see, for instance, [3]). In contrast, for the two species case, the rotated parabola (45) can intersect any of the lines (46) and then we obtain threshold lines on which the second moment is not preserved. A description of the asymptotic behaviour in this case seems to require rather different techniques to those used in the one species case.

Acknowledgments

We gratefully acknowledge prof. Benoît Perthame (Université Pierre et Marie Curie) for his help in preparation of this article.

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