Nagoya Math. J., **225** (2017), 100–151
DOI 10.1017/nmj.2016.36

# STABLE MODELS OF LUBIN–TATE CURVES WITH LEVEL THREE

NAOKI IMAI AND TAKAHIRO TSUSHIMA

**Abstract.** We construct a stable formal model of a Lubin–Tate curve with level three, and study the action of a Weil group and a division algebra on its stable reduction. Further, we study a structure of cohomology of the Lubin–Tate curve. Our study is purely local and includes the case where the characteristic of the residue field of a local field is two.

## Introduction

Let $K$ be a nonarchimedean local field with a finite residue field $k$ of characteristic $p$. Let $\mathfrak{p}$ be the maximal ideal of the ring of integers $\mathcal{O}_K$ of $K$. Let $n$ be a natural number. We write $\mathrm{LT}(\mathfrak{p}^n)$ for the Lubin–Tate curve with full level $n$ as a deformation space of formal $\mathcal{O}_K$-modules by quasi-isogenies. Let $D$ be the central division algebra over $K$ of invariant $1/2$. Let $\ell$ be a prime number different from $p$. We write $\mathbf{C}$ for the completion of an algebraic closure of $K$. Then, the groups $W_K$, $\mathrm{GL}_2(K)$ and $D^\times$ act on

$$\varinjlim_m H^1_c(\mathrm{LT}(\mathfrak{p}^m)_{\mathbf{C}}, \overline{\mathbb{Q}}_\ell),$$

and these actions partially realize the local Langlands correspondence and the local Jacquet–Langlands correspondence for $\mathrm{GL}_2$. The realization of the local Langlands correspondence was proved by global automorphic methods in [Ca]. Since Lubin–Tate curves are purely local objects, it is desirable to have a purely local proof which only makes use of the geometry of Lubin–Tate curves.

We put

$$K_1(\mathfrak{p}^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_K) \;\middle|\; c \equiv 0,\ d \equiv 1 \bmod \mathfrak{p}^n \right\}.$$

Received May 6, 2015. Revised February 8, 2016. Accepted February 21, 2016.
2010 Mathematics subject classification. Primary 11G20; Secondary 11F80.

Let $\mathrm{LT}_1(\mathfrak{p}^n)$ be the Lubin–Tate curve with level $K_1(\mathfrak{p}^n)$ as a deformation space of formal $\mathcal{O}_K$-modules by quasi-isogenies. Then, the cohomology group

$$H^1_{\mathrm{c}}(\mathrm{LT}_1(\mathfrak{p}^n)_{\mathbf{C}}, \overline{\mathbb{Q}}_\ell) = \left(\varinjlim_m H^1_{\mathrm{c}}(\mathrm{LT}(\mathfrak{p}^m)_{\mathbf{C}}, \overline{\mathbb{Q}}_\ell)\right)^{K_1(\mathfrak{p}^n)}$$

will give representations of $W_K$ and $D^\times$ that correspond to smooth irreducible representations of $\mathrm{GL}_2(K)$ with conductor less than or equal to $n$. The purpose of this paper is to study this cohomology in the case $n = 3$. We note that 3 is the smallest conductor of a two-dimensional representation of $W_K$ which cannot be written as an induction of a character. Such a representation is called a primitive representation.

Our method is purely local and geometric. In fact, we construct a stable model of the connected Lubin–Tate curve $\mathbf{X}_1(\mathfrak{p}^3)$ with level $K_1(\mathfrak{p}^3)$ by using the theory of semistable coverings (cf. [CM, Section 2.3]). Our study includes the case where $p = 2$, and in this case, primitive Galois representations of conductor 3 appear in the cohomology of $\mathbf{X}_1(\mathfrak{p}^3)$. It gives a geometric understanding of a realization of the primitive Galois representations.

Our method of the calculation of the stable reduction is similar to that in [CM]. In [CM], Coleman and McMurdy calculate the stable reduction of the modular curve $X_0(p^3)$ under the assumption $p \geqslant 13$. The calculation of the stable reductions in the modular curve setting is equivalent to that in the Lubin–Tate setting where $K = \mathbb{Q}_p$. As for the calculation of the stable reduction of the modular curve $X_1(p^n)$, it is given in [DR] if $n = 1$.

We explain the contents of this paper. In Section 1, we recall a definition of the connected Lubin–Tate curve, and study the action of a division algebra in a general setting. In Section 2, we study the cohomology of Lubin–Tate curves as representations of $\mathrm{GL}_2(K)$ by purely local methods. By this result, we can calculate the genus of some Lubin–Tate curves. In Section 3, we construct a stable covering of the connected Lubin–Tate curve with level $K_1(\mathfrak{p}^2)$, which is used to study a covering of $\mathbf{X}_1(\mathfrak{p}^3)$.

In Section 4, we define several affinoid subspaces $\mathbf{Y}_{1,2}$, $\mathbf{Y}_{2,1}$ and $\mathbf{Z}^0_{1,1}$ of $\mathbf{X}_1(\mathfrak{p}^3)$, and calculate their reductions. Let $k^{\mathrm{ac}}$ be the residue field of $\mathbf{C}$. We put $q = |k|$ and

$$\mathcal{S}_1 = \begin{cases} \mu_{2(q^2-1)}(k^{\mathrm{ac}}) & \text{if } q \text{ is odd,} \\ \mu_{q^2-1}(k^{\mathrm{ac}}) & \text{if } q \text{ is even.} \end{cases}$$

The reductions of $\mathbf{Y}_{1,2}$ and $\mathbf{Y}_{2,1}$ are isomorphic to the affine curve defined by $x^q y - x y^q = 1$. This affine curve has genus $q(q-1)/2$, and is called the

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

Deligne–Lusztig curve for $\mathrm{SL}_2(\mathbb{F}_q)$ or the Drinfeld curve. Here, the genus of a curve means the genus of the smooth compactification of the normalization of the curve. The reduction $\overline{\mathbf{Z}}^0_{1,1}$ of $\mathbf{Z}^0_{1,1}$ is isomorphic to the affine curve defined by $Z^q + X^{q^2-1} + X^{-(q^2-1)} = 0$. This affine curve has genus 0 and singularities at $X \in \mathcal{S}_1$.

Next, we analyze tubular neighborhoods $\{\mathcal{D}_\zeta\}_{\zeta \in \mathcal{S}_1}$ of the singular points of $\overline{\mathbf{Z}}^0_{1,1}$. If $q$ is odd, $\mathcal{D}_\zeta$ is a basic wide open space with the underlying affinoid $\mathbf{X}_\zeta$. See [CM, 2B] for the precise definition of a basic wide open space. Roughly speaking, it is a smooth geometrically connected one-dimensional rigid space which contains an affinoid such that the reduction of the affinoid is irreducible and has at worst ordinary double points as singularities, and the complement of the affinoid is a disjoint union of open annuli. The reduction of $\mathbf{X}_\zeta$ is isomorphic to the Artin–Schreier affine curve of degree 2 defined by $z^q - z = w^2$. This affine curve has genus $(q-1)/2$.

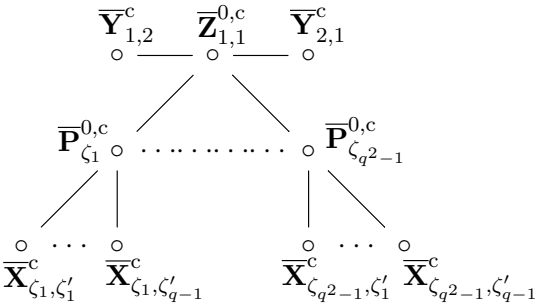
On the other hand, if $q$ is even, it is harder to analyze $\mathcal{D}_\zeta$, because the space $\mathcal{D}_\zeta$ is not a basic wide open space. First, we find an affinoid $\mathbf{P}^0_\zeta$. The reduction $\overline{\mathbf{P}}^0_\zeta$ of $\mathbf{P}^0_\zeta$ has genus 0 and singular points parametrized by $\zeta' \in k^\times$. Second, we analyze the tubular neighborhoods of singular points of $\overline{\mathbf{P}}^0_\zeta$. As a result, we find an affinoid $\mathbf{X}_{\zeta,\zeta'}$, whose reduction $\overline{\mathbf{X}}_{\zeta,\zeta'}$ is isomorphic to the affine curve defined by $z^2 + z = w^3$. The smooth compactification of this curve is the unique supersingular elliptic curve over $k^{\mathrm{ac}}$, whose $j$-invariant is 0, and its cohomology gives a primitive Galois representation. By using these affinoid spaces, we construct a covering $\mathcal{C}_1(\mathfrak{p}^3)$ of $\mathbf{X}_1(\mathfrak{p}^3)$.

In Section 5, we calculate the action of $\mathcal{O}_D^\times$ on the reductions of the affinoid spaces in $\mathbf{X}_1(\mathfrak{p}^3)$, where $\mathcal{O}_D$ is the ring of integers of $D$. In Section 6, we calculate an action of a Weil group on the reductions. In the case where $q$ is even, we construct an $\mathrm{SL}_2(\mathbb{F}_3)$-Galois extension of $K^{\mathrm{ur}}$, and show that the Weil action on $\overline{\mathbf{X}}_{\zeta,\zeta'}$ up to translations factors through the Weil group of the constructed extension. For such a Galois extension, see also [Weil, 31].

In Section 7, we show that the covering $\mathcal{C}_1(\mathfrak{p}^3)$ is semistable. To show this, we calculate the summation of the genera of the reductions of the affinoid spaces in $\mathbf{X}_1(\mathfrak{p}^3)$, and compare it with the genus of $\mathbf{X}_1(\mathfrak{p}^3)$. Using the constructed semistable model, we study a structure of cohomology of $\mathbf{X}_1(\mathfrak{p}^3)$.

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

The dual graph of the semistable reduction of $\mathbf{X}_1(\mathfrak{p}^3)$ in the case where $q$ is even is the following:




where $\mu_{q^2-1}(k^{\mathrm{ac}}) = \{\zeta_1, \ldots, \zeta_{q^2-1}\}$, $k^{\times} = \{\zeta'_1, \ldots, \zeta'_{q-1}\}$ and $X^{\mathrm{c}}$ denotes the smooth compactification of the normalization of $X$ for a curve $X$ over $k^{\mathrm{ac}}$. The constructed semistable model is in fact stable, except in the case where $q = 2$. If $q = 2$, we get the stable model by blowing down some $\mathbb{P}^1$-components.

The realization of the local Jacquet–Langlands correspondence in cohomology of Lubin–Tate curves was proved in [Mi] by a purely local method. Therefore, the remaining essential part of the study of the realization of the local Langlands correspondence is to study actions of Weil groups and division algebras. In the paper [IT3], we give a purely local proof of the realization of the local Langlands correspondence for representations of conductor three using the result of this paper.

Finally, we mention some recent progress on related topics according to a suggestion of a referee. In [Wein], Weinstein constructs semistable models of Lubin–Tate curves for arbitrary level in the case where the residue characteristic is not equal to two using Lubin–Tate perfectoid spaces. In [IT4] and [IT5], some of our results in this paper are generalized to arbitrary dimensional cases for Lubin–Tate perfectoid spaces. In [IT6], we construct an affinoid in the two-dimensional Lubin–Tate space such that the cohomology of the reduction of the affinoid realizes representations that are a bit more ramified than the epipelagic representations.

**Notation**

In this paper, we use the following notation. Let $K$ be a nonarchimedean local field. Let $\mathcal{O}_K$ denote the ring of integers of $K$, and let $k$ denote the residue field of $K$. Let $p$ be the characteristic of $k$. We fix a uniformizer $\varpi$ of $K$. Let $q = |k|$. We fix an algebraic closure $K^{\mathrm{ac}}$ of $K$. For any finite extension $F$ of $K$ in $K^{\mathrm{ac}}$, let $G_F$ denote the absolute Galois group of $F$, let $W_F$ denote

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

the Weil group of $F$, and let $I_F$ denote the inertia subgroup of $W_F$. The completion of $K^{\mathrm{ac}}$ is denoted by $\mathbf{C}$. Let $\mathcal{O}_{\mathbf{C}}$ be the ring of integers of $\mathbf{C}$, and let $k^{\mathrm{ac}}$ be the residue field of $\mathbf{C}$. For an element $a \in \mathcal{O}_{\mathbf{C}}$, we write $\bar{a}$ for the image of $a$ by the reduction map $\mathcal{O}_{\mathbf{C}} \to k^{\mathrm{ac}}$. Let $v(\cdot)$ denote the valuation of $\mathbf{C}$ such that $v(\varpi) = 1$. Let $K^{\mathrm{ur}}$ denote the maximal unramified extension of $K$ in $K^{\mathrm{ac}}$. The completion of $K^{\mathrm{ur}}$ is denoted by $\widehat{K}^{\mathrm{ur}}$. For $a, b \in \mathbf{C}$ and a rational number $\alpha \in \mathbb{Q}_{\geqslant 0}$, we write $a \equiv b \,(\mathrm{mod}\ \alpha)$ if we have $v(a-b) \geqslant \alpha$, and $a \equiv b \,(\mathrm{mod}\ \alpha+)$ if we have $v(a-b) > \alpha$. For a curve $X$ over $k^{\mathrm{ac}}$, we denote by $X^c$ the smooth compactification of the normalization of $X$, and the genus of $X$ means the genus of $X^{\mathrm{c}}$. For an affinoid $\mathbf{X}$, we write $\overline{\mathbf{X}}$ for its reduction. The category of sets is denoted by **Set**. For a representation $\tau$ of a group, the dual representation of $\tau$ is denoted by $\tau^*$. We take rational powers of $\varpi$ compatibly as needed.

## §1. Preliminaries

### 1.1 The universal deformation

Let $\Sigma$ denote a formal $\mathcal{O}_K$-module of dimension 1 and height 2 over $k^{\mathrm{ac}}$, which is unique up to isomorphism. Let $n$ be a natural number. We define $K_1(\mathfrak{p}^n)$ as in the introduction. In the following, we define the connected Lubin–Tate curve $\mathbf{X}_1(\mathfrak{p}^n)$ with level $K_1(\mathfrak{p}^n)$.

Let $\mathcal{C}$ be the category of Noetherian complete local $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}$-algebras with residue field $k^{\mathrm{ac}}$. For $A \in \mathcal{C}$, a formal $\mathcal{O}_K$-module $\mathcal{F} = \mathrm{Spf}\, A[[X]]$ over $A$ and an $A$-valued point $P$ of $\mathcal{F}$, the corresponding element of the maximal ideal of $A$ is denoted by $x(P)$. We consider the functor

$$\mathcal{A}_1(\mathfrak{p}^n) : \mathcal{C} \to \mathbf{Set}; \qquad A \mapsto [(\mathcal{F}, \iota, P)],$$

where $\mathcal{F}$ is a formal $\mathcal{O}_K$-module over $A$ with an isomorphism $\iota : \Sigma \simeq \mathcal{F} \otimes_A k^{\mathrm{ac}}$ and $P$ is a $\varpi^n$-torsion point of $\mathcal{F}$ such that

$$\prod_{a \in \mathcal{O}_K/\varpi^n\mathcal{O}_K} \big(X - x([a]_{\mathcal{F}}(P))\big) \,\Big|\, [\varpi^n]_{\mathcal{F}}(X)$$

in $A[[X]]$. This functor is represented by a regular local ring $\mathcal{R}_1(\mathfrak{p}^n)$ by [Dr, Section 4.B) Lemma]. We write $\mathfrak{X}_1(\mathfrak{p}^n)$ for $\mathrm{Spf}\, \mathcal{R}_1(\mathfrak{p}^n)$. Its generic fiber is denoted by $\mathbf{X}_1(\mathfrak{p}^n)$, which we call the connected Lubin–Tate curve with level $K_1(\mathfrak{p}^n)$. The space $\mathbf{X}_1(\mathfrak{p}^n)$ is a rigid analytic curve over $\widehat{K}^{\mathrm{ur}}$. We can define the Lubin–Tate curve $\mathrm{LT}_1(\mathfrak{p}^n)$ with level $n$ by changing $\mathcal{C}$ to be the category of $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}$-algebras where $\varpi$ is nilpotent, and $\iota$ to be a quasi-isogeny

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

$\Sigma \otimes_{k^{\mathrm{ac}}} A/\varpi A \to \mathcal{F} \otimes_A A/\varpi A$. We consider $\mathrm{LT}_1(\mathfrak{p}^n)$ as a rigid analytic curve over $\widehat{K}^{\mathrm{ur}}$.

The ring $\mathcal{R}_1(1)$ is isomorphic to the ring of formal power series $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}[[u]]$. We simply write $\mathcal{B}(1)$ for $\mathrm{Spf}\, \mathcal{O}_{\widehat{K}^{\mathrm{ur}}}[[u]]$. Let $B(1)$ denote an open unit ball such that $B(1)(\mathbf{C}) = \{u \in \mathbf{C} \mid v(u) > 0\}$. The generic fiber of $\mathcal{B}(1)$ is equal to $B(1)$. Then, the space $\mathbf{X}_1(1)$ is identified with $B(1)$. Let $\mathcal{F}^{\mathrm{univ}}$ denote the universal formal $\mathcal{O}_K$-module over $\mathfrak{X}_1(1)$.

In this subsection, we choose a parametrization of $\mathfrak{X}_1(1) \simeq \mathcal{B}(1)$ such that the universal formal $\mathcal{O}_K$-module has a simple form. Let $\mathcal{F}$ be a formal $\mathcal{O}_K$-module of dimension 1 over a flat $\mathcal{O}_K$-algebra $R$. For a nontrivial invariant differential $\omega$ on $\mathcal{F}$, a logarithm of $\mathcal{F}$ means a unique isomorphism $F : \mathcal{F} \xrightarrow{\sim} \mathbb{G}_a$ over $R \otimes K$ with $dF = \omega$ (cf. [GH, 3]). In the following, we always take an invariant differential $\omega$ on $\mathcal{F}$ so that a logarithm $F$ has the following form:

$$F(X) = X + \sum_{i \geqslant 1} f_i X^{q^i} \quad \text{with } f_i \in R \otimes K.$$

Let $F(X) = \sum_{i \geqslant 0} f_i X^{q^i} \in K[[u, X]]$ be the universal logarithm over $\mathcal{O}_K[[u]]$. By [GH, (5.5), (12.3), Proposition 12.10], the coefficients $\{f_i\}_{i \geqslant 0}$ satisfy $f_0 = 1$ and $\varpi f_i = \sum_{0 \leqslant j \leqslant i-1} f_j v_{i-j}^{q^j}$ for $i \geqslant 1$, where $v_1 = u$, $v_2 = 1$ and $v_i = 0$ for $i \geqslant 3$. Hence, we have the following:

$$\begin{aligned} &f_0 = 1, \qquad f_1 = \frac{u}{\varpi}, \qquad f_2 = \frac{1}{\varpi}\left(1 + \frac{u^{q+1}}{\varpi}\right), \\ &f_3 = \frac{1}{\varpi^2}\left(u + u^{q^2} + \frac{u^{q^2+q+1}}{\varpi}\right), \qquad \cdots . \end{aligned} \tag{1.1}$$

By [GH, Proposition 5.7] or [Ha, 21.5], if we set

$$\mathcal{F}^{\mathrm{univ}}(X, Y) = F^{-1}(F(X) + F(Y)), \quad [a]_{\mathcal{F}^{\mathrm{univ}}}(X) = F^{-1}(aF(X)) \tag{1.2}$$

for $a \in \mathcal{O}_K$, it is known that these power series have coefficients in $\mathcal{O}_K[[u]]$ and define the universal formal $\mathcal{O}_K$-module $\mathcal{F}^{\mathrm{univ}}$ over $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}[[u]]$ of dimension 1 and height 2 with logarithm $F(X)$. We have the following approximation formula for $[\varpi]_{\mathrm{u}}(X)$.

LEMMA 1.1. *We have the following congruence:*

$$[\varpi]_{\mathcal{F}^{\mathrm{univ}}}(X) \equiv \varpi X + uX^q + X^{q^2} - \frac{u}{\varpi}\{(uX^q + X^{q^2})^q - u^q X^{q^2} - X^{q^3}\} \mod(\varpi^2 X^q, u\varpi X^q, \varpi X^{q^2}, X^{q^3+1}).$$

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

*Proof.* This follows from a direct computation using the relation $F([\varpi]_{\mathcal{F}^{\mathrm{univ}}}(X)) = \varpi F(X)$ and (1.1). □

In the following, $\mathcal{F}^{\mathrm{univ}}$ means the universal formal $\mathcal{O}_K$-module with the identification $\mathfrak{X}_1(1) \simeq \mathcal{B}(1)$ given by (1.2), and we simply write $[a]_{\mathrm{u}}$ for $[a]_{\mathcal{F}^{\mathrm{univ}}}$. The reduction of (1.2) gives a simple model of $\Sigma$ such that

(1.3)
$$X +_\Sigma Y = X + Y, \qquad [\zeta]_\Sigma(X) = \bar{\zeta}X \quad \text{for } \zeta \in \mu_{q-1}(\mathcal{O}_K), \quad [\varpi]_\Sigma(X) = X^{q^2}.$$

We put

$$\mathfrak{A}_n = \mathcal{O}_{\widehat{K}^{\mathrm{ur}}}[[u, X_n]]/\big([\varpi^n]_{\mathrm{u}}(X_n)/[\varpi^{n-1}]_{\mathrm{u}}(X_n)\big).$$

Then, there is a natural identification

$$\mathfrak{X}_1(\mathfrak{p}^n) \simeq \operatorname{Spf} \mathfrak{A}_n \tag{1.4}$$

that is compatible with the identification $\mathfrak{X}_1(1) \simeq \mathcal{B}(1)$. The Lubin–Tate curve $\mathbf{X}_1(\mathfrak{p}^n)$ is identified with the generic fiber of the right-hand side of (1.4). We set $X_i = [\varpi^{n-i}]_{\mathrm{u}}(X_n)$ for $1 \leqslant i \leqslant n-1$. We write $\mathfrak{X}(1)$ for $\mathfrak{X}_1(1)$.

### 1.2 Action of a division algebra on $\mathfrak{X}_1(\mathfrak{p}^n)$

Let $D$ be the central division algebra over $K$ of invariant $1/2$. We write $\mathcal{O}_D$ for the ring of integers of $D$. In this subsection, we recall the left action of $\mathcal{O}_D^\times$ on the space $\mathfrak{X}_1(\mathfrak{p}^n)$.

Let $K_2$ be the unramified quadratic extension of $K$. Let $k_2$ be the residue field of $K_2$, and let $\sigma \in \mathrm{Gal}(K_2/K)$ be the nontrivial element. The ring $\mathcal{O}_D$ has the following description: $\mathcal{O}_D = \mathcal{O}_{K_2} \oplus \varphi\mathcal{O}_{K_2}$, with $\varphi^2 = \varpi$ and $a\varphi = \varphi a^\sigma$ for $a \in \mathcal{O}_{K_2}$. We define an action of $\mathcal{O}_D$ on $\Sigma$ by $\zeta(X) = \bar{\zeta}X$ for $\zeta \in \mu_{q^2-1}(\mathcal{O}_{K_2})$ and $\varphi(X) = X^q$. Then, this gives an isomorphism $\mathcal{O}_D \simeq \mathrm{End}(\Sigma)$ by [GH, Proposition 13.10].

Let $d = d_1 + \varphi d_2 \in \mathcal{O}_D^\times$, where $d_1 \in \mathcal{O}_{K_2}^\times$ and $d_2 \in \mathcal{O}_{K_2}$. By the definition of the action of $\mathcal{O}_D$ on $\Sigma$, we have

$$d(X) \equiv \bar{d}_1 X + (\bar{d}_2 X)^q \mod (X^{q^2}). \tag{1.5}$$

We take a lifting $\tilde{d}(X) \in \mathcal{O}_{K_2}[[X]]$ of $d(X) \in k_2[[X]]$. Let $\mathcal{F}_{\tilde{d}}$ be the formal $\mathcal{O}_K$-module defined by

$$\mathcal{F}_{\tilde{d}}(X, Y) = \tilde{d}\big(\mathcal{F}^{\mathrm{univ}}(\tilde{d}^{-1}(X), \tilde{d}^{-1}(Y))\big), \qquad [a]_{\mathcal{F}_{\tilde{d}}}(X) = \tilde{d}\big([a]_{\mathrm{u}}(\tilde{d}^{-1}(X))\big)$$

for $a \in \mathcal{O}_K$. Then, we have an isomorphism

$$\tilde{d} : \mathcal{F}^{\mathrm{univ}} \xrightarrow{\sim} \mathcal{F}_{\tilde{d}}; \qquad (u, X) \mapsto (u, \tilde{d}(X)).$$

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

By [GH, Proposition 14.7], the formal $\mathcal{O}_K$-module $\mathcal{F}_{\tilde{d}}$ with

$$\Sigma \xrightarrow{d^{-1}} \Sigma \xrightarrow{\iota} \mathcal{F}^{\mathrm{univ}} \otimes k^{\mathrm{ac}} \xrightarrow{\tilde{d}\otimes k^{\mathrm{ac}}} \mathcal{F}_{\tilde{d}} \otimes k^{\mathrm{ac}}$$

gives an isomorphism

$$d : \mathfrak{X}(1) \to \mathfrak{X}(1), \tag{1.6}$$

which is independent of the choice of a lifting $\tilde{d}$, such that there is the unique isomorphism

$$j : d^*\mathcal{F}^{\mathrm{univ}} \xrightarrow{\sim} \mathcal{F}_{\tilde{d}}; \qquad (u, X) \mapsto (u, j(X))$$

satisfying $j(X) \equiv X \bmod (\varpi, u)$, where $d^*\mathcal{F}^{\mathrm{univ}}$ denotes the pullback of $\mathcal{F}^{\mathrm{univ}}$ over $\mathfrak{X}(1)$ by the map (1.6). Hence, we have

$$[\varpi]_{d^*\mathcal{F}^{\mathrm{univ}}}(j^{-1}(X)) = j^{-1}([\varpi]_{\mathcal{F}_{\tilde{d}}}(X)). \tag{1.7}$$

On the other hand, we have the following isomorphism:

$$d^*\mathcal{F}^{\mathrm{univ}} \xrightarrow{\sim} \mathcal{F}^{\mathrm{univ}}; \qquad (u, X') \mapsto (d(u), X').$$

Furthermore, we consider the following isomorphism under the identification (1.4):

$$\psi_d : \mathfrak{X}_1(\mathfrak{p}^n) \longrightarrow \mathfrak{X}_1(\mathfrak{p}^n); \qquad (u, X_n) \mapsto \big(d(u), j^{-1}(\tilde{d}(X_n))\big), \tag{1.8}$$

which depends only on $d$ as in [GH, Proposition 14.7]. We put

$$d^*(X) = j^{-1}(\tilde{d}(X)).$$

We define a left action of $d$ on $\mathfrak{X}_1(\mathfrak{p}^n)$ by

$$[(\mathcal{F}, \iota, P)] \mapsto [(\mathcal{F}, \iota \circ d^{-1}, P)].$$

Then, this action coincides with $\psi_d$ by the definition.

By (1.5), we have

$$\tilde{d}^{-1}(X) = d_1^{-1}X - d_1^{-(q+1)}d_2^q X^q \mod (\varpi, X^{q^2}) \tag{1.9}$$

in $\mathcal{O}_{K_2}[[X]]$. We use the following lemma later to compute the $\mathcal{O}_D^\times$-action on the stable reduction of $\mathbf{X}_1(\mathfrak{p}^3)$.

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

LEMMA 1.2. *We assume $v(u) = 1/(2q)$. Let $d = d_1 + \varphi d_2 \in \mathcal{O}_D^\times$. We set $u' = d(u)$. We change variables as $u = \varpi^{1/(2q)}\tilde{u}$ and $u' = \varpi^{1/(2q)}\tilde{u}'$. Then, we have the following:*

$$u' \equiv d_1^{-(q-1)} u(1 + d_1^{-q} d_2 u) \mod (\varpi, u^3), \tag{1.10}$$

$$j^{-1}(X) \equiv X + d_1^{-q} d_2 u X \mod (\varpi, u^2 X, u X^2). \tag{1.11}$$

*Proof.* We set $d^{-1} = d_1' + \varphi d_2'$. Then,

$$d_1' \equiv d_1^{-1}, \qquad d_2' \equiv -d_1^{-(q+1)} d_2 \pmod 1 .$$

First, we prove (1.10). If $v(u) = 1/(2q)$, the function $w(u)$ in [GH, (25.11)] is well approximated by a function $\varpi u(\varpi + u^{q+1})^{-1}$. By [GH, (25.13)], we have

$$\begin{aligned} \frac{\varpi u'}{\varpi + u'^{q+1}} &\equiv \frac{d_1'^q \varpi u(\varpi + u^{q+1})^{-1} + \varpi d_2'^q}{d_2' \varpi u(\varpi + u^{q+1})^{-1} + d_1'} \\ &\equiv \frac{\varpi u(d_1 - d_2^q u^q)}{d_1^q(\varpi + u^{q+1}) - d_2 \varpi u} \pmod{1+} . \end{aligned}$$

Hence, we acquire the following by $u = \varpi^{1/(2q)}\tilde{u}$ and $u' = \varpi^{1/(2q)}\tilde{u}'$:

$$\frac{\tilde{u}'}{\tilde{u}'^{q+1} + \varpi^{(q-1)/(2q)}} \equiv \frac{\tilde{u}(d_1 - \varpi^{1/2} d_2^q \tilde{u}^q)}{d_1^q \tilde{u}^{q+1} + \varpi^{(q-1)/(2q)} d_1^q - \varpi^{1/2} d_2 \tilde{u}} \quad \left(\mod \frac{1}{2}+\right) . \tag{1.12}$$

By taking an inverse of the congruence (1.12), we obtain

$$\begin{aligned} (\tilde{u}' - d_1^{-(q-1)}\tilde{u})^q \equiv \varpi^{(q-1)/(2q)} &\left( \frac{\tilde{u}' - d_1^{-(q-1)}\tilde{u}}{d_1^{-(q-1)}\tilde{u}\tilde{u}'} \right) \\ &+ \varpi^{1/2}(d_1^{q-2} d_2^q \tilde{u}^{2q} - d_1^{-1} d_2) \quad \left(\mod \frac{1}{2}+\right) . \end{aligned} \tag{1.13}$$

Now, we set $\tilde{u}' - d_1^{-(q-1)}\tilde{u} = \varpi^{1/(2q)} x$. By substituting this into (1.13) and dividing it by $\varpi^{1/2}$, we obtain

$$(x - d_1^{1-2q} d_2 \tilde{u}^2)^q \equiv d_1^{2q-2} \tilde{u}^{-2} (x - d_1^{1-2q} d_2 \tilde{u}^2) \pmod{0+} .$$

Since $x$ is an analytic function of $\tilde{u}$, a congruence $x \equiv d_1^{1-2q} d_2 \tilde{u}^2 \pmod{0+}$ must hold. Hence, we have

$$\tilde{u}' \equiv d_1^{-(q-1)} \tilde{u}(1 + \varpi^{1/(2q)} d_1^{-q} d_2 \tilde{u}) \quad \left(\mod \frac{1}{2q}+\right)$$

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

using $\tilde{u}' - d_1^{q-1}\tilde{u} = \varpi^{1/(2q)}x$. This implies (1.10), because $u'$ is an analytic function of $u$.

By Lemma 1.1, (1.7) and (1.9), we have

$$u'j^{-1}(X)^q \equiv j^{-1}(ud_1^{-(q-1)}X^q) \mod (\varpi, X^{q^2}).$$

Hence, the assertion (1.11) follows from (1.10) and $j^{-1}(X) \equiv X \mod (\varpi, u)$. □

## §2. Cohomology of Lubin–Tate curve

Let $\ell$ be a prime number different from $p$. We take an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. Let $\mathrm{LT}(\mathfrak{p}^n)$ be the Lubin–Tate curve with full level $n$ over $\widehat{K}^{\mathrm{ur}}$ (cf. [Da, 3.2]). We put

$$H^i_{\mathrm{LT},\varpi} = \varinjlim_n H^i_{\mathrm{c}}\big((\mathrm{LT}(\mathfrak{p}^n)/\varpi^{\mathbb{Z}})_{\mathbf{C}}, \overline{\mathbb{Q}}_\ell\big)$$

for any nonnegative integer $i$, where $\mathrm{LT}(\mathfrak{p}^n)/\varpi^{\mathbb{Z}}$ denotes the quotient of $\mathrm{LT}(\mathfrak{p}^n)$ by the action of $\varpi^{\mathbb{Z}} \subset D^\times$. Then, we can define an action of $\mathrm{GL}_2(K) \times D^\times \times W_K$ on $H^i_{\mathrm{LT},\varpi}$ for a nonnegative integer $i$ (cf. [Da, 3.2, 3.3]).

We write $\mathrm{Irr}(D^\times, \overline{\mathbb{Q}}_\ell)$ for the set of isomorphism classes of irreducible smooth representations of $D^\times$ over $\overline{\mathbb{Q}}_\ell$, and $\mathrm{Disc}(\mathrm{GL}_2(K), \overline{\mathbb{Q}}_\ell)$ for the set of isomorphism classes of irreducible discrete series representations of $\mathrm{GL}_2(K)$ over $\overline{\mathbb{Q}}_\ell$. Let

$$\mathrm{JL} : \mathrm{Irr}(D^\times, \overline{\mathbb{Q}}_\ell) \to \mathrm{Disc}(\mathrm{GL}_2(K), \overline{\mathbb{Q}}_\ell)$$

be the local Jacquet–Langlands correspondence. We denote by LJ the inverse of JL. For an irreducible smooth representation $\pi$ of $\mathrm{GL}_2(K)$, let $\omega_\pi$ denote the central character of $\pi$. We write St for the Steinberg representation of $\mathrm{GL}_2(K)$.

The following fact is well known as a corollary of the Deligne–Carayol conjecture. Here, we give a purely local proof of this fact.

PROPOSITION 2.1. *We have isomorphisms*

$$H^1_{\mathrm{LT},\varpi} \simeq \bigoplus_\pi \pi^{\oplus 2\dim \mathrm{LJ}(\pi)} \oplus \bigoplus_\chi (\mathrm{St} \otimes (\chi \circ \det)),$$

$$H^2_{\mathrm{LT},\varpi} \simeq \bigoplus_\chi (\chi \circ \det)$$

*as representations of* $\mathrm{GL}_2(K)$, *where* $\pi$ *runs through irreducible cuspidal representations of* $\mathrm{GL}_2(K)$ *such that* $\omega_\pi(\varpi) = 1$, *and* $\chi$ *runs through characters of* $K^\times$ *satisfying* $\chi(\varpi^2) = 1$.

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

*Proof.* First, we show the second isomorphism. Let $\mathbf{X}(\mathfrak{p}^n)$ be the connected Lubin–Tate curve with full level $n$ over $\widehat{K}^{\mathrm{ur}}$ (cf. [St2, 2.1]). We put

$$H^2_{\mathbf{X}} = \varinjlim_n H^2_c\big(\mathbf{X}(\mathfrak{p}^n)_{\mathbf{C}}, \overline{\mathbb{Q}}_\ell\big),$$
$$\mathrm{GL}_2(K)^0 = \{g \in \mathrm{GL}_2(K) \mid \det g \in \mathcal{O}_K^\times\}.$$

Then, $\mathrm{GL}_2(K)^0$ acts on $H^2_{\mathbf{X}}$. By [St2, Theorem 4.4(i)], we have

$$H^2_{\mathbf{X}} \simeq \bigoplus_\chi (\chi \circ \det) \tag{2.1}$$

as representations of $\mathrm{GL}_2(\mathcal{O}_K)$, where $\chi$ runs through characters of $\mathcal{O}_K^\times$. Let $H$ be the kernel of $\mathrm{GL}_2(K)^0 \to \mathrm{Aut}(H^2_{\mathbf{X}})$. Then, $H = \mathrm{SL}_2(K)$, because a normal subgroup of $\mathrm{GL}_2(K)^0$ containing $\mathrm{SL}_2(\mathcal{O}_K)$ is $\mathrm{SL}_2(K)$ by [De, Lemme 2.2.5(iii)]. Hence, we see that (2.1) is an isomorphism as representations of $\mathrm{GL}_2(K)^0$. The second isomorphism follows from this, because we have

$$H^2_{\mathrm{LT},\varpi} \simeq \text{c-Ind}_{\mathrm{GL}_2(K)^0}^{\mathrm{GL}_2(K)/\varpi^{\mathbb{Z}}} H^2_{\mathbf{X}}.$$

Next, we show the first isomorphism. By [Mi, Definition 6.2 and Theorem 6.6], the cuspidal part of $H^1_{\mathrm{LT},\varpi}$ is

$$\bigoplus_\pi \pi^{\oplus 2\dim \mathrm{LJ}(\pi)}.$$

Here, we note that the characteristic of a local field is assumed to be zero in [Mi], but the same proof works in the equal characteristic case. By [Far2, Théorème 4.3] and the Faltings–Fargues isomorphism (cf. [Fal] and [FGL]), we see that the noncuspidal part of $H^1_{\mathrm{LT},\varpi}$ is the Zelevinsky dual of $H^2_{\mathrm{LT},\varpi}$. Therefore, we have the first isomorphism. □

## §3. Stable covering of Lubin–Tate curve with level two

In this section, we construct a stable covering of $\mathbf{X}_1(\mathfrak{p}^2)$. Let $(u, X_2)$ be the parameter of $\mathbf{X}_1(\mathfrak{p}^2)$ given by the identification (1.4).

Let $\mathbf{Y}_{1,1}$, $\mathbf{W}_0$, $\mathbf{W}_{k^\times}$, $\mathbf{W}_{\infty,1}$, $\mathbf{W}_{\infty,2}$ and $\mathbf{W}_{\infty,3}$ be the subspaces of $\mathbf{X}_1(\mathfrak{p}^2)$ defined by the following conditions.

$$\mathbf{Y}_{1,1} : v(u) = \frac{1}{q+1}, \qquad v(X_1) = \frac{q}{q^2-1}, \qquad v(X_2) = \frac{1}{q(q^2-1)}.$$

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

$$\mathbf{W}_0 : 0 < v(u) < \frac{1}{q+1}, \quad v(X_1) = \frac{1-v(u)}{q-1}, \qquad v(X_2) = \frac{1-qv(u)}{q(q-1)}.$$

$$\mathbf{W}_{k^\times} : 0 < v(u) < \frac{1}{q+1}, \quad v(X_1) = \frac{1-v(u)}{q-1}, \qquad v(X_2) = \frac{v(u)}{q(q-1)}.$$

$$\mathbf{W}_{\infty,1} : 0 < v(u) < \frac{q}{q+1}, \quad v(X_1) = \frac{v(u)}{q(q-1)}, \qquad v(X_2) = \frac{v(u)}{q^3(q-1)}.$$

$$\mathbf{W}_{\infty,2} : v(u) \geqslant \frac{q}{q+1}, \quad v(X_1) = \frac{1}{q^2-1}, \qquad v(X_2) = \frac{1}{q^2(q^2-1)}.$$

$$\mathbf{W}_{\infty,3} : \frac{1}{q+1} < v(u) < \frac{q}{q+1}, \quad v(X_1) = \frac{1-v(u)}{q-1}, \qquad v(X_2) = \frac{1-v(u)}{q^2(q-1)}.$$

We put

$$\mathbf{W}_\infty = \mathbf{W}_{\infty,1} \cup \mathbf{W}_{\infty,2} \cup \mathbf{W}_{\infty,3}.$$

Note that we have

$$\mathbf{X}_1(\mathfrak{p}^2) = \mathbf{Y}_{1,1} \cup \mathbf{W}_0 \cup \mathbf{W}_{k^\times}^1 \cup \mathbf{W}_\infty.$$

PROPOSITION 3.1. *The Lubin–Tate curve* $\mathbf{X}_1(\mathfrak{p}^2)$ *is a basic wide open space with underlying affinoid* $\mathbf{Y}_{1,1}$. *Further,* $\mathbf{W}_0$ *and* $\mathbf{W}_\infty$ *are open annuli, and* $\mathbf{W}_{k^\times}$ *is a disjoint union of* $q-1$ *open annuli.*

*Proof.* This is proved in [IT1] by direct calculations without cohomological arguments. Here, we sketch another proof based on arguments in this paper.

First, we note that $\mathcal{X}_1(1)$ is a good formal model of $\mathbf{X}_1(1)$. Then, we can show that $\mathbf{X}_1(\mathfrak{p})$ is isomorphic to an open annulus by a cohomological argument as in the proof of Theorem 7.14 using the natural level-lowering map $\mathbf{X}_1(\mathfrak{p}) \to \mathbf{X}_1(1)$.

Next, we can see that the reduction of $\mathbf{Y}_{1,1}$ is isomorphic to the affine curve defined by $x^q y - xy^q = 1$ by a calculation as in the proof of Proposition 4.2 (cf. [IT1, Section 3.1]). Then, we can prove the claim by a similar argument to that above using the natural level-lowering map $\mathbf{X}_1(\mathfrak{p}^2) \to \mathbf{X}_1(\mathfrak{p})$. □

## §4. Reductions of affinoid spaces in $\mathbf{X}_1(\mathfrak{p}^3)$

### 4.1 Definitions of several subspaces in $\mathbf{X}_1(\mathfrak{p}^3)$

In this subsection, we define several subspaces of $\mathbf{X}_1(\mathfrak{p}^3)$. Let $(u, X_3)$ be the parameter of $\mathbf{X}_1(\mathfrak{p}^3)$ given by the identification (1.4).

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

Let $\mathbf{Y}_{1,2}$, $\mathbf{Y}_{2,1}$ and $\mathbf{Z}^0_{1,1}$ be the subspaces of $\mathbf{X}_1(\mathfrak{p}^3)$ defined by the following conditions.

$$\mathbf{Y}_{1,2} : v(u) = \frac{1}{q+1}, \quad v(X_1) = \frac{q}{q^2-1}, \qquad v(X_2) = \frac{1}{q(q^2-1)},$$
$$v(X_3) = \frac{1}{q^3(q^2-1)}.$$
$$\mathbf{Y}_{2,1} : v(u) = \frac{1}{q(q+1)}, \quad v(X_1) = \frac{q^2+q-1}{q(q^2-1)}, \qquad v(X_2) = \frac{1}{q^2-1},$$
$$v(X_3) = \frac{1}{q^2(q^2-1)}.$$
$$\mathbf{Z}^0_{1,1} : v(u) = \frac{1}{2q}, \quad v(X_1) = \frac{2q-1}{2q(q-1)}, \qquad v(X_2) = \frac{1}{2q(q-1)},$$
$$v(X_3) = \frac{1}{2q^3(q-1)}.$$

We write down the following possible cases for $(u, X_1, X_2)$:

$$(1) \quad 0 < v(u) < \frac{1}{q+1}, \quad v(X_1) = \frac{1-v(u)}{q-1}, \qquad v(X_2) = \frac{1-qv(u)}{q(q-1)};$$
$$(2) \quad 0 < v(u) < \frac{1}{q+1}, \quad v(X_1) = \frac{1-v(u)}{q-1}, \qquad v(X_2) = \frac{v(u)}{q(q-1)};$$
$$(3) \quad v(u) = \frac{1}{q+1}, \quad v(X_1) = \frac{q}{q^2-1}, \qquad v(X_2) = \frac{1}{q(q^2-1)};$$
$$(4) \quad 0 < v(u) < \frac{q}{q+1}, \quad v(X_1) = \frac{v(u)}{q(q-1)}, \qquad v(X_2) = \frac{v(u)}{q^3(q-1)};$$
$$(5) \quad v(u) \geqslant \frac{q}{q+1}, \quad v(X_1) = \frac{1}{q^2-1}, \qquad v(X_2) = \frac{1}{q^2(q^2-1)};$$
$$(6) \quad \frac{1}{q+1} < v(u) < \frac{q}{q+1}, \quad v(X_1) = \frac{1-v(u)}{q-1}, \qquad v(X_2) = \frac{1-v(u)}{q^2(q-1)}.$$

(4.1)

Next, we consider the following possible cases for $(X_2, X_3)$:

$$(1') \quad v(X_3^{q^2}) = v(X_2) < v(uX_3^q), \qquad (2') \quad v(uX_3^q) = v(X_2) < v(X_3^{q^2}),$$
$$(3') \quad v(X_2) > v(X_3^{q^2}) = v(uX_3^q), \qquad (4') \quad v(X_2) = v(X_3^{q^2}) = v(uX_3^q).$$

(4.2)

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

LEMMA 4.1. *For $2 \leqslant i \leqslant 6$ in (4.1) and $2' \leqslant j' \leqslant 4'$ in (4.2), the case $i$ and $j'$ does not happen.*

*Proof.* This is an easy exercise. □

Let $\mathbf{W}_{i,j'}$ be the subspace of $\mathbf{X}_1(\mathfrak{p}^3)$ defined by the conditions $1 \leqslant i \leqslant 6$ in (4.1) and $1' \leqslant j' \leqslant 4'$ in (4.2). We note that $\mathbf{W}_{3,1'} = \mathbf{Y}_{1,2}$ and $\mathbf{W}_{1,4'} = \mathbf{Y}_{2,1}$. Let $\mathbf{W}^+_{1,1'}$ and $\mathbf{W}^-_{1,1'}$ be the subspaces of $\mathbf{W}_{1,1'}$ defined by $1/(2q) < v(u) < 1/(q+1)$ and $1/(q(q+1)) < v(u) < 1/(2q)$ respectively.

### 4.2 Reductions of the affinoid spaces $\mathbf{Y}_{1,2}$ and $\mathbf{Y}_{2,1}$

In this subsection, we compute the reductions of the affinoid spaces $\mathbf{Y}_{1,2}$ and $\mathbf{Y}_{2,1}$. The reductions of $\mathbf{Y}_{2,1}$ and $\mathbf{Y}_{1,2}$ are isomorphic to the affine curve defined by $x^q y - xy^q = 1$. These curves have genus $q(q-1)/2$.

PROPOSITION 4.2. *The reduction of $\mathbf{Y}_{1,2}$ is isomorphic to the affine curve defined by $x^q y - xy^q = 1$.*

*Proof.* We change variables as $u = \varpi^{1/(q+1)}\tilde{u}$, $X_1 = \varpi^{q/(q^2-1)}x_1$, $X_2 = \varpi^{1/(q(q^2-1))}x_2$ and $X_3 = \varpi^{1/(q^3(q^2-1))}x_3$. By Lemma 1.1, we have

$$\tilde{u} \equiv -x_1^{-(q-1)}, \qquad x_1 \equiv \tilde{u}x_2^q + x_2^{q^2}, \qquad x_2 \equiv x_3^{q^2} \pmod{0+}. \tag{4.3}$$

Then, we have $\tilde{u} = -x_1^{-(q-1)} + F_0(\tilde{u}, x_1)$ for some function $F_0(\tilde{u}, x_1)$ satisfying $v(F_0(\tilde{u}, x_1)) > v(\tilde{u})$. Substituting $\tilde{u} = -x_1^{-(q-1)} + F_0(\tilde{u}, x_1)$ into $F_0(\tilde{u}, x_1)$ and repeating it, we see that $\tilde{u}$ is written as a function of $x_1$. Similarly, by $x_2 \equiv x_3^{q^2} \pmod{0+}$, we can see that $x_2$ is written as a function of $x_1$ and $x_3$. By (4.3), we acquire

$$1 \equiv \frac{x_3^{q^4}}{x_1} - \frac{x_3^{q^3}}{x_1^q} \pmod{0+}. \tag{4.4}$$

By setting $1 + x_1^{-1}x_3^{q^2} = x_3^{q^3}t_1^{-1}$ and substituting this into (4.4), we obtain $t_1^q \equiv x_1 \pmod{0+}$ and hence $(1 + x_3^q t_1^{-1})^q \equiv x_3^{q^3}t_1^{-1} \pmod{0+}$. By setting $1 + x_3^q t_1^{-1} = x_3^{q^2}t_2^{-1}$, we obtain $t_2^q \equiv t_1 \pmod{0+}$. Hence,

$$(1 + x_3 t_2^{-1})^q \equiv x_3^{q^2}t_2^{-1} \pmod{0+}.$$

Finally, by setting $x = x_3$ and $1 + x_3 t_2^{-1} = x_3^q y$, we acquire $y^q \equiv t_2^{-1} \pmod{0+}$. Hence, we have $x^q y - xy^q \equiv 1 \pmod{0+}$. Note that

$$x = x_3, \quad y = \frac{x_1(1 + x_3^{q(q^2-1)} + x_3^{(q+1)(q^2-1)}) + x_3^{q^2}}{x_1 x_3^{q^3+q^2-1}}, \tag{4.5}$$

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

which we will use later. □

We put

$$\gamma_i = \varpi^{(q-1)/(2q^i)}$$

for $1 \leqslant i \leqslant 4$. We choose an element $c_0$ such that $c_0^q - \gamma_1^2 c_0 + 1 = 0$. Note that we have $c_0 \equiv -1 \pmod{0+}$. Further, we choose a $q$th root $c_0^{1/q}$ of $c_0$.

PROPOSITION 4.3. *The reduction of the space* $\mathbf{Y}_{2,1}$ *is isomorphic to the affine curve defined by* $x^q y - xy^q = 1$.

*Proof.* We change variables as

$$u = \varpi^{1/(q(q+1))}\tilde{u}, \qquad X_1 = \varpi^{(q^2+q-1)/(q(q^2-1))}x_1, \qquad X_2 = \varpi^{1/(q^2-1)}x_2$$
$$\text{and} \qquad X_3 = \varpi^{1/(q^2(q^2-1))}x_3.$$

By Lemma 1.1, we have

$$\tilde{u} \equiv -x_1^{-(q-1)} \quad \left(\mathrm{mod}\ \frac{q^2-1}{q^2}+\right), \tag{4.6}$$

$$x_1 \equiv \tilde{u}x_2^q + \gamma_1^2(x_2^{q^2} + x_2) \quad \left(\mathrm{mod}\ \frac{q^2-1}{q^2}+\right), \tag{4.7}$$

$$x_2 \equiv x_3^{q^2} + \tilde{u}x_3^q \quad \left(\mathrm{mod}\ \frac{q-1}{q^2}+\right). \tag{4.8}$$

By (4.6) and (4.8), we can see that $\tilde{u}$ is written as a function of $x_1$, and that $x_2$ is written as a function of $x_1$ and $x_3$. We define a parameter $t$ by

$$\frac{x_2}{x_1} = c_0 + \gamma_2^2 \frac{x_2^q}{t}. \tag{4.9}$$

We note that $v(t) = 0$. By considering $x_1^{-1}\times$ (4.7), we have

$$\left(\frac{x_2}{x_1}\right)^q + 1 - \gamma_1^2 \frac{x_2}{x_1} \equiv \gamma_1^2 \frac{x_2^{q^2}}{x_1} \quad \left(\mathrm{mod}\ \frac{q^2-1}{q^2}+\right). \tag{4.10}$$

By substituting (4.9) into the left-hand side of the congruence (4.10), and dividing it by $\gamma_1^2 x_2^{q^2}$, we acquire

$$x_1 \equiv t^q \left(1 - \gamma_2^2 \frac{t^{q-1}}{x_2^{q(q-1)}}\right)^{-1} \quad \left(\mathrm{mod}\ \frac{q-1}{q^2}+\right). \tag{4.11}$$

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

By this congruence, we can see that $x_1$ is written as a function of $t$ and $x_3$. By considering $x_1^{-1}\times$ (4.8), we acquire

$$c_0+\gamma_2^2\frac{x_2^q}{t}\equiv\frac{x_3^{q^2}}{x_1}-\left(\frac{x_3}{x_1}\right)^q \quad \left(\mathrm{mod}\ \frac{q-1}{q^2}+\right) \tag{4.12}$$

by (4.9). Substituting (4.11) into (4.12), we have

$$\left(c_0^{1/q}-\frac{x_3^q}{t}+\frac{x_3}{x_1}\right)^q\equiv-\gamma_2^2\frac{(x_2+x_3)^{q^2}}{tx_2^{q(q-1)}} \quad \left(\mathrm{mod}\ \frac{q-1}{q^2}+\right). \tag{4.13}$$

By (4.9) and $c_0\equiv -1\ (\mathrm{mod}\ 0+)$, we have $x_2\equiv -x_1\ (\mathrm{mod}\ 0+)$. Therefore, we acquire

$$(x_2+x_3)^q\equiv x_2^{q-1}x_3^{q^2} \quad (\mathrm{mod}\ 0+)$$

by (4.6) and (4.8). In particular, we obtain $v(x_2+x_3)=0$. We introduce a new parameter $t_1$ as

$$c_0^{1/q}-\frac{x_3^q}{t}+\frac{x_3}{x_1}=-\gamma_3^q\frac{(x_2+x_3)^q}{t_1x_2^{q-1}}. \tag{4.14}$$

Substituting this into the left-hand side of the congruence (4.13), and dividing it by $-\gamma_2^2x_2^{-q(q-1)}(x_2+x_3)^{q^2}$, we acquire $t\equiv t_1^q\ (\mathrm{mod}\ 0+)$. By this congruence, we can see that $t$ is written as a function of $t_1$ and $x_3$. By (4.14), we obtain

$$x_3\equiv t_1^{q^2}(1+x_3t_1^{-1})^q \quad (\mathrm{mod}\ 0+)$$

using $t\equiv t_1^q\ (\mathrm{mod}\ 0+)$ and $x_1\equiv t^q\ (\mathrm{mod}\ 0+)$. Hence, by setting $x=t_1^{-1}$ and $y=t_1^q(1+x_3t_1^{-1})$, we acquire $x^qy-yx^q\equiv 1\ (\mathrm{mod}\ 0+)$. □

### 4.3 Reduction of the affinoid space $\mathbf{Z}_{1,1}^0$

In this subsection, we calculate the reduction of the affinoid space $\mathbf{Z}_{1,1}^0$. We define $\mathcal{S}_1$ as in the introduction. The reduction $\overline{\mathbf{Z}}_{1,1}^0$ is isomorphic to the affine curve defined by $Z^q+x_3^{q^2-1}+x_3^{-(q^2-1)}=0$. This affine curve has genus 0 and singularities at $x_3\in\mathcal{S}_1$.

We put

$$\omega_i=\varpi^{1/(2q^i(q-1))}, \qquad \epsilon_i=\frac{1}{2q^i}$$

for $1\leqslant i\leqslant 4$. We change variables as $u=\omega_1^{q-1}\tilde{u}$, $X_1=\omega_1^{2q-1}x_1$, $X_2=\omega_1x_2$ and $X_3=\omega_3x_3$. By Lemma 1.1, we have

$$\tilde{u}\equiv -x_1^{-(q-1)} \quad \left(\mathrm{mod}\ \tfrac{1}{2}+\right), \tag{4.15}$$

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

$$x_1 \equiv \tilde{u}x_2^q + \gamma_1 x_2^{q^2} + \gamma_1^2 x_2 \pmod{\tfrac{1}{2}+}, \tag{4.16}$$

$$x_2 \equiv x_3^{q^2} + \gamma_2 \tilde{u} x_3^q \pmod{\epsilon_1+}. \tag{4.17}$$

Note that we have $v(\gamma_1^2) > 1/2$ if $q \neq 2$. By (4.15) and (4.17), we can see that $\tilde{u}$ is written as a function of $x_1$, and that $x_2$ is written as a function of $x_1$ and $x_3$. We define a parameter $t$ by

$$\frac{x_2}{x_1} = -1 + \gamma_2 \frac{x_2^q}{t}. \tag{4.18}$$

By considering $x_1^{-1}\times$ (4.16), we acquire

$$\left(\frac{x_2}{x_1} + 1\right)^q \equiv \gamma_1 \frac{x_2^{q^2}}{x_1}\left(1 + \frac{\gamma_1}{x_2^{q^2-1}}\right) \quad \left(\mathrm{mod}\ \frac{1}{2}+\right) \tag{4.19}$$

by (4.15). Substituting (4.18) into (4.19), and dividing it by $\gamma_1 x_2^{q^2}$, we obtain

$$x_1 \equiv t^q \left(1 + \frac{\gamma_1}{x_2^{q^2-1}}\right) \pmod{\epsilon_1+}. \tag{4.20}$$

Therefore, we have $v(t) = 0$. By considering $x_1^{-1}\times$ (4.17), we acquire

$$\left(1 + \frac{x_3^q}{t}\right)^q - \gamma_1 \frac{x_3^{q^2}}{t^q x_2^{q^2-1}} \equiv \gamma_2 \left(\frac{x_2^q}{t} + \left(\frac{x_3}{x_1}\right)^q\right) \pmod{\epsilon_1+} \tag{4.21}$$

by (4.15), (4.18) and (4.20). We define a parameter $Z_0$ by

$$1 + \frac{x_3^q}{t} = \gamma_3 Z_0. \tag{4.22}$$

We note that $v(Z_0) \geqslant 0$. Substituting this into (4.21), and dividing it by $\gamma_2$, we obtain

$$Z_0^q \equiv \frac{x_2^q}{t} + \left(\frac{x_3}{x_1}\right)^q + \gamma_2^{q-1} \frac{x_3^{q^2}}{t^q x_2^{q^2-1}} \pmod{\epsilon_2+}. \tag{4.23}$$

By (4.22) and (4.23), we acquire

$$\left(Z_0 + \frac{x_2}{x_3} - \frac{x_3}{x_1}\right)^q \equiv \gamma_3 \left(\frac{x_2}{x_3}\right)^q Z_0 + \gamma_2^{q-1} \frac{x_3^{q^2}}{t^q x_2^{q^2-1}} \pmod{\epsilon_2+}. \tag{4.24}$$

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

We introduce a new parameter $Z$ as

$$Z_0 + \frac{x_2}{x_3} - \frac{x_3}{x_1} = \gamma_4 \frac{x_2}{x_3} Z. \tag{4.25}$$

We note that $v(Z) \geqslant 0$. Substituting this into the left-hand side of the congruence (4.24), and dividing it by $\gamma_3(x_2/x_3)^q$, we acquire

$$Z^q \equiv Z_0 + \gamma_3^{q^2-q-1} \frac{x_3^{q(q+1)}}{t^q x_2^{q^2+q-1}} \pmod{\epsilon_3+}\,. \tag{4.26}$$

By substituting (4.25) into (4.26), we obtain

$$Z^q + x_3^{q^2-1}(1-\gamma_4 Z) + x_3^{-(q^2-1)} \equiv -\gamma_3^{q^2-q-1} x_3^{-q(q^2-1)(q+1)} \pmod{\epsilon_3+} \tag{4.27}$$

by (4.17), (4.20) and (4.22). Note that we have $v(\gamma_3^{q^2-q-1}) > \epsilon_3$, if $q \neq 2$.

PROPOSITION 4.4. *The reduction of the space* $\mathbf{Z}^0_{1,1}$ *is isomorphic to the affine curve defined by* $Z^q + x_3^{q^2-1} + x_3^{-(q^2-1)} = 0$. *This affine curve has genus* 0 *and singularities at* $x_3 \in \mathcal{S}_1$.

*Proof.* The required assertion follows from the congruence (4.27) modulo $0+$. □

DEFINITION 4.5.

(1) For any $\zeta \in \mathcal{S}_1$, we define a subspace

$$\mathcal{D}_\zeta \subset \mathbf{Z}^0_{1,1} \times_{\widehat{K}^{\mathrm{ur}}} \widehat{K}^{\mathrm{ur}}(\omega_3)$$

by $\bar{x}_3 = \zeta$. We call the space $\mathcal{D}_\zeta$ a singular residue class of $\mathbf{Z}^0_{1,1}$.

(2) We define a subspace

$$\mathbf{Z}_{1,1} \subset \mathbf{Z}^0_{1,1} \times_{\widehat{K}^{\mathrm{ur}}} \widehat{K}^{\mathrm{ur}}(\omega_3)$$

by the complement $\mathbf{Z}^0_{1,1} \times_{\widehat{K}^{\mathrm{ur}}} \widehat{K}^{\mathrm{ur}}(\omega_3) \setminus \bigcup_{\zeta \in \mathcal{S}_1} \mathcal{D}_\zeta$.

PROPOSITION 4.6. *The reduction of the space* $\mathbf{Z}_{1,1}$ *is isomorphic to the affine curve defined by* $Z^q + x_3^{q^2-1} + x_3^{-(q^2-1)} = 0$ *with* $x_3 \notin \mathcal{S}_1$.

*Proof.* This follows from Proposition 4.4. □

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

### 4.4 Analysis of the singular residue classes of $\mathbf{Z}^0_{1,1}$

In this subsection, we analyze the singular residue classes $\{\mathcal{D}_\zeta\}_{\zeta\in\mathcal{S}_1}$ of $\mathbf{Z}^0_{1,1}$. If $q$ is odd, the space $\mathcal{D}_\zeta$ is a basic wide open space with an underlying affinoid $\mathbf{X}_\zeta$, whose reduction $\overline{\mathbf{X}}_\zeta$ is isomorphic to the affine curve defined by $z^q - z = w^2$. On the other hand, if $q$ is even, the situation is slightly complicated, because the space $\mathcal{D}_\zeta$ is not basic wide open. Hence, we have to cover $\mathcal{D}_\zeta$ by smaller basic wide open spaces. As a result, in $\mathcal{D}_\zeta$, we find an affinoid $\mathbf{P}^0_\zeta$, whose reduction is isomorphic to the affine curve defined by $z^2_{f+1} = w_1(w_1^{q-1} - 1)^2$. This affine curve has $q-1$ singular points at $w_1 \in k^\times$. Then, by analyzing the tubular neighborhoods of these singular points, we find an affinoid $\mathbf{X}_{\zeta,\zeta'} \subset \mathbf{P}^0_\zeta$ for each $\zeta' \in k^\times$, whose reduction is isomorphic to the affine curve defined by $z^2 + z = w^3$.

#### 4.4.1 $q$ odd

We assume that $q$ is odd. For each $\zeta \in \mu_{2(q^2-1)}(k^{\mathrm{ac}})$, we define an affinoid $\mathbf{X}_\zeta \subset \mathcal{D}_\zeta$ and compute its reduction $\overline{\mathbf{X}}_\zeta$.

For $\iota \in \mu_2(k^{\mathrm{ac}})$, we choose an element $c'_{1,\iota} \in \mathcal{O}^\times_{K^{\mathrm{ac}}}$ such that $\bar{c}'_{1,\iota} = -2\iota$ and $c'^{2q}_{1,\iota} = 4(1 - \gamma_4 c'_{1,\iota})$. We take $\zeta \in \mu_{2(q^2-1)}(k^{\mathrm{ac}})$. We put $c_{1,\zeta} = c'_{1,\zeta^{q^2-1}}$, and define $c_{2,\zeta} \in \mathcal{O}^\times_{K^{\mathrm{ac}}}$ by $c^{q^2-1}_{2,\zeta} = -2c^{-q}_{1,\zeta}$ and $\bar{c}_{2,\zeta} = \zeta$. We put

$$a_\zeta = \omega_4^{q-1} c_{2,\zeta}^{q+1}, \qquad b_\zeta = -2\zeta^{q^2-1}\omega_3^{(q-1)/2} c_{1,\zeta}^{-q} c_{2,\zeta}^{(q+3)/2}.$$

Note that we have $v(a_\zeta) = 1/(2q^4)$ and $v(b_\zeta) = 1/(4q^3)$.

For an element $\zeta \in \mu_{2(q^2-1)}(k^{\mathrm{ac}})$, we define an affinoid $\mathbf{X}_\zeta$ by $v(x_3 - c_{2,\zeta}) \geqslant 1/(4q^3)$. We change variables as

$$Z = a_\zeta z + c_{1,\zeta}, \qquad x_3 = b_\zeta w + c_{2,\zeta}.$$

Then, we acquire

$$a^q_\zeta(z^q - z - w^2) \equiv 0 \pmod{\epsilon_3+}$$

by (4.27). Dividing this by $a^q_\zeta$, we have $z^q - z = w^2 \pmod{0+}$. Hence, the reduction of $\mathbf{X}_\zeta$ is isomorphic to the affine curve defined by $z^q - z = w^2$.

PROPOSITION 4.7. *For each $\zeta \in \mu_{2(q^2-1)}(k^{\mathrm{ac}})$, the reduction $\overline{\mathbf{X}}_\zeta$ is isomorphic to the affine curve defined by $z^q - z = w^2$, and the complement $\mathcal{D}_\zeta \setminus \mathbf{X}_\zeta$ is an open annulus.*

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

*Proof.* We have already proved the first assertion. We prove the second assertion. We change variables as

$$Z = z' + c_{1,\zeta}, \qquad x_3 = w' + c_{2,\zeta}$$

with $0 < v(w') < 1/(4q^3)$. Substituting them into (4.27), we obtain

$$z'^q \equiv w'^2 \pmod{2v(w')+}.$$

Note that we have $0 < v(z') < 1/(2q^4)$. By setting $w' = z''z'^{(q-1)/2}$, we acquire

$$z''^2 \equiv z' \pmod{v(z')+}.$$

Hence, we can see that $z'$ is written as a function of $z''$. Then, $w'$ is also written as a function of $z''$. Therefore, $(\mathcal{D}_\zeta \setminus \mathbf{X}_\zeta)(\mathbf{C})$ is identified with $\{z'' \in \mathbf{C} \mid 0 < v(z'') < 1/(4q^4)\}$. □

### 4.4.2 $q$ even

We assume that $q$ is even. We put

$$Z_1 = x_3^{q^2-1}.$$

Then, the congruence (4.27) has the following form:

$$Z^q + Z_1(1 - \gamma_4 Z) + Z_1^{-1} \equiv -\gamma_3^{q^2-q-1} Z_1^{-q(q+1)} \pmod{\epsilon_3+}. \tag{4.28}$$

**1. Projective lines** For each $\zeta \in k_2^\times$, we define a subaffinoid $\mathbf{P}^0_\zeta \subset \mathcal{D}_\zeta$ by $v(Z) \geqslant 1/(4q^4)$. We change variables as

$$Z = \varpi^{1/(4q^4)} w_1, \qquad Z_1 = 1 + \varpi^{1/(8q^3)} z_1.$$

Substituting these into (4.28) and dividing it by $\varpi^{1/(4q^3)}$, we acquire

$$\begin{aligned}(z_1 + w_1^{q/2})^2 + \varpi^{1/(8q^3)} z_1^3 + \varpi^{1/(4q^3)} z_1^4 + \varpi^{(q-1)/(4q^4)} w_1 \\ + \varpi^{(3q-2)/(8q^4)} z_1 w_1 \equiv \varpi^{(2q-3)/(4q^3)} \left(\mathrm{mod}\ \frac{1}{4q^3}+\right).\end{aligned} \tag{4.29}$$

We can check that $v(z_1) \geqslant 0$. We set $q = 2^f$ and put

$$l_i = \frac{(2^i - 1)q}{2^i}, \qquad m_i = \frac{1}{2^{i+2}q^3}$$

for $1 \leqslant i \leqslant f+1$. Furthermore, we define parameters $z_i$ for $2 \leqslant i \leqslant f+1$ by

$$z_i + w_1^{l_i} = \varpi^{m_{i+1}} z_{i+1} \quad \text{for } 1 \leqslant i \leqslant f. \tag{4.30}$$

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

LEMMA 4.8. *We assume that* $v(Z) \geqslant 1/(4q^4)$. *Then, we have*

$$z_{f+1}^2 + w_1^{2q-1} + w_1 + \varpi^{1/(8q^4)} z_{f+1} w_1^q \equiv (q/2)\varpi^{1/(4q^4)} \quad \left(\mathrm{mod}\ \frac{1}{4q^4}+\right). \tag{4.31}$$

*Proof.* If $q = 2$, we can check that

$$z_2^2 + w_1^3 + w_1 + \varpi^{1/128} z_2 w_1^2 \equiv \varpi^{1/64}(w_1 z_2^2 + z_1^4 + w_1^2 + 1) \quad \left(\mathrm{mod}\ \frac{1}{64}+\right) \tag{4.32}$$

by

$$z_1 = -w_1 + \varpi^{1/128} z_2.$$

We have $v(z_2^2 + w_1^3 + w_1) > 0$. Therefore, we obtain

$$w_1 z_2^2 + z_1^4 + w_1^2 \equiv w_1(z_2^2 + w_1^3 + w_1) \equiv 0 \quad (\mathrm{mod}\ 0+).$$

Hence, the required assertion in this case follows from (4.32). Assume that $f \geqslant 2$. For $1 \leqslant i \leqslant f+1$, we put

$$n_i = \frac{q - 2^{i-1}}{2^{i+1} q^4}.$$

We prove

$$(z_i + w_1^{l_i})^2 + \varpi^{m_i} z_i w_1^q + \varpi^{n_i} w_1 \equiv 0 \quad \left(\mathrm{mod}\ \frac{1}{2^{i+1} q^3}+\right) \tag{4.33}$$

for $2 \leqslant i \leqslant f+1$ by induction on $i$. Eliminating $z_1$ from (4.29) by (4.30) and dividing it by $\varpi^{1/(8q^4)}$, we obtain

$$(z_2 + w_1^{3q/4})^2 + \varpi^{1/(16q^3)} z_2 w_1^q + \varpi^{(q-2)/(8q^4)} w_1 + \varpi^{1/(8q^3)} w_1^{q/2} (z_2 + w_1^{3q/4})^2 \equiv 0 \quad \left(\mathrm{mod}\ \frac{1}{8q^3}+\right).$$

This shows

$$v\left(z_2 + w_1^{3q/4}\right) \geqslant \frac{1}{32q^3}.$$

Hence, we have (4.33) for $i = 2$. Assume (4.33) for $i$. Eliminating $z_i$ from (4.33) by (4.30) and dividing it by $\varpi^{m_i}$, we obtain (4.33) for $i+1$. Hence, we have (4.33) for $f+1$, which is equivalent to (4.31). □

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

PROPOSITION 4.9. *For each $\zeta \in k_2^\times$, the reduction $\overline{\mathbf{P}}^0_\zeta$ is isomorphic to the affine curve defined by $z_{f+1}^2 = w_1(w_1^{q-1}-1)^2$, which has genus 0 and singularities at $w_1 \in k^\times$, and the complement $\mathcal{D}_\zeta \setminus \mathbf{P}^0_\zeta$ is an open annulus.*

*Proof.* The claim on $\overline{\mathbf{P}}^0_\zeta$ follows from the congruence (4.31) modulo 0+. We prove the last assertion. We change a variable as $Z_1 = 1 + z_1'$ with $0 < v(z_1') < 1/(8q^3)$. Similarly to (4.30), we introduce parameters $\{z_i'\}_{2 \leqslant i \leqslant f+1}$ by $z_i' + Z^{l_i} = z_{i+1}'$ for $1 \leqslant i \leqslant f$. Then, by similar computations to those in the proof of Lemma 4.8, we obtain

$$z_{f+1}'^2 \equiv Z^{2q-1} \pmod{2v(z_{f+1}')+}.$$

By setting $z_{f+2}' = Z^q / z_{f+1}'$, we obtain

$$z_{f+2}'^2 \equiv Z \pmod{v(Z)+}.$$

Then, we can see that all parameters $z_i'$ for $1 \leqslant i \leqslant f+1$ and $Z$ are written as functions of $z_{f+2}'$. Hence, $(\mathcal{D}_\zeta \setminus \mathbf{P}^0_\zeta)(\mathbf{C})$ is identified with

$$\{z_{f+2}' \in \mathbf{C} \mid 0 < v(z_{f+2}') < 1/(8q^4)\}. \qquad \square$$

**2. Elliptic curves** For $\zeta' \in k^\times$, we choose $c_{2,\zeta'} \in \mathcal{O}_{\mathbf{C}}^\times$ such that $\bar{c}_{2,\zeta'} = \zeta'$ and

$$c_{2,\zeta'}^{4(q-1)} + 1 + \varpi^{1/(4q^4)} c_{2,\zeta'}^{4q-3} = 0,$$

and a square root $c_{2,\zeta'}^{1/2}$ of $c_{2,\zeta'}$. Further, we choose $c_{1,\zeta'}$ such that

$$c_{1,\zeta'}^2 + \varpi^{1/(8q^4)} c_{2,\zeta'}^q c_{1,\zeta'} + c_{2,\zeta'}(c_{2,\zeta'}^{2(q-1)} + 1) = \frac{q}{2}\varpi^{1/(4q^4)},$$

and $b_{2,\zeta'}$ such that $b_{2,\zeta'}^3 = \varpi^{1/(4q^4)} c_{2,\zeta'}^4$. We put

$$a_{1,\zeta'} = \varpi^{1/(8q^4)} c_{2,\zeta'}^q, \qquad b_{1,\zeta'} = c_{2,\zeta'}^{(2q-3)/2} b_{2,\zeta'}.$$

For each $\zeta' \in k^\times$, we define a subspace $\mathcal{D}_{\zeta,\zeta'} \subset \mathbf{P}^0_\zeta$ by $v(w_1 - c_{2,\zeta'}) > 0$. Furthermore, we define $\mathbf{X}_{\zeta,\zeta'} \subset \mathcal{D}_{\zeta,\zeta'}$ by $v(w_1 - c_{2,\zeta'}) \geqslant 1/(12q^4)$. We put

$$\mathbf{P}_\zeta = \mathbf{P}^0_\zeta \Big\backslash \bigcup_{\zeta' \in k^\times} \mathcal{D}_{\zeta,\zeta'}.$$

We take $(\zeta, \zeta') \in k_2^\times \times k^\times$ and compute the reduction of $\mathbf{X}_{\zeta,\zeta'}$. In the following, we omit the subscript $\zeta'$ of $a_{1,\zeta'}$, $b_{1,\zeta'}$, $b_{2,\zeta'}$ $c_{1,\zeta'}$ and $c_{2,\zeta'}$, if there

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

is no confusion. We change variables as

$$z_{f+1} = a_1 z + b_1 w + c_1, \qquad w_1 = b_2 w + c_2.$$

By substituting these into (4.31), we acquire

$$a_1^2(z^2 + z + w^3) \equiv 0 \quad \left(\mathrm{mod}\ \frac{1}{4q^4}+\right) \tag{4.34}$$

by the definition of $a_1$, $b_1$, $b_2$, $c_1$ and $c_2$.

PROPOSITION 4.10. *For each $(\zeta, \zeta') \in k_2^\times \times k^\times$, the reduction of $\mathbf{X}_{\zeta,\zeta'}$ is isomorphic to the affine curve defined by $z^2 + z = w^3$, and the complement $\mathcal{D}_{\zeta,\zeta'} \setminus \mathbf{X}_{\zeta,\zeta'}$ is an open annulus.*

*Proof.* The first assertion follows from (4.34). We prove the second assertion. We change variables as

$$z_{f+1} = z' + c_2^{(2q-3)/2} w' + c_1, \qquad w_1 = w' + c_2$$

with $0 < v(w') < 1/(12q^4)$. Substituting them into (4.31), we acquire

$$z'^2 \equiv c_2^{2(q-2)} w'^3 \quad \left(\mathrm{mod}\ 2v(z')+\right)$$

by the choice of $c_2$. Note that we have

$$v(z') = 3v(w')/2 < \frac{1}{8q^4}.$$

By setting $z'' = z'/(c_2^{q-2} w')$, we obtain

$$z''^2 \equiv w' \quad \left(\mathrm{mod}\ v(w')+\right).$$

Then, we can see that $z'$ and $w'$ are written as functions of $z''$. Hence, $(\mathcal{D}_{\zeta,\zeta'} \setminus \mathbf{X}_{\zeta,\zeta'})(\mathbf{C})$ is identified with $\{z'' \in \mathbf{C} \mid 0 < v(z'') < 1/(24q^4)\}$. □

### 4.5 Stable covering of $\mathbf{X}_1(\mathfrak{p}^3)$

In this subsection, we show the existence of the stable covering of $\mathbf{X}_1(\mathfrak{p}^3)$ over some finite extension of the base field $\widehat{K}^{\mathrm{ur}}$. See [CM, Section 2.3] for the notion of semistable coverings. A semistable covering is called stable, if the corresponding semistable model is stable.

PROPOSITION 4.11. *There exists a stable covering of $\mathbf{X}_1(\mathfrak{p}^3)$ over a finite extension of the base field.*

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

*Proof.* First, we show that, after taking a finite extension of the base field, $\mathbf{X}_1(\mathfrak{p}^3)$ is a wide open space. By [St1, Theorem 2.3.1(i)], $\mathbf{X}_1(\mathfrak{p}^3)$ is the Raynaud generic fiber of the formal completion of an affine scheme over $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}$ at a closed point on the special fiber. Then, we can apply [CM, Theorem 2.29] to the formal completion of the affine scheme along its special fiber, after shrinking the affine scheme. Hence, $\mathbf{X}_1(\mathfrak{p}^3)$ is a wide open space over some extension.

By [CM, Theorem 2.18], a wide open space can be embedded to a proper algebraic curve so that its complement is a disjoint union of closed disks. Therefore, $\mathbf{X}_1(\mathfrak{p}^3)$ has a semistable covering over some finite extension by [CM, Theorem 2.40]. Then, a simple modification gives a stable covering. □

In the following, we construct a candidate of a semistable covering of $\mathbf{X}_1(\mathfrak{p}^3)$ over some finite extension. We put

$$\mathbf{V}_1 = \mathbf{W}^+_{1,1'} \cup \bigcup_{2\leqslant i\leqslant 6} \mathbf{W}_{i,1'}, \qquad \mathbf{V}_2 = \mathbf{W}^-_{1,1'} \cup \bigcup_{2\leqslant i\leqslant 4} \mathbf{W}_{1,i'},$$
$$\mathbf{U} = \mathbf{W}_{1,1'} \setminus \bigcup_{\zeta\in\mathcal{S}_1} \mathbf{X}_\zeta.$$

We note that $\mathbf{V}_1 \supset \mathbf{Y}_{1,2}$, $\mathbf{V}_2 \supset \mathbf{Y}_{2,1}$, $\mathbf{U} \supset \mathbf{Z}_{1,1}$, $\mathbf{V}_1 \cap \mathbf{V}_2 = \emptyset$, $\mathbf{V}_1 \cap \mathbf{U} = \mathbf{W}^+_{1,1'}$ and $\mathbf{V}_2 \cap \mathbf{U} = \mathbf{W}^-_{1,1'}$.

We consider the case where $q$ is even in this paragraph. We set $\hat{\mathcal{D}}_\zeta = \mathcal{D}_\zeta \setminus \bigl(\bigcup_{\zeta'\in k^\times} \mathbf{X}_{\zeta,\zeta'}\bigr)$ for $\zeta \in k_2^\times$. Then, $\hat{\mathcal{D}}_\zeta$ contains $\mathbf{P}_\zeta$ as the underlying affinoid. On the other hand, for $(\zeta,\zeta') \in k_2^\times \times k^\times$, the space $\mathcal{D}_{\zeta,\zeta'}$ has the underlying affinoid $\mathbf{X}_{\zeta,\zeta'}$.

We put

$$\mathcal{S} = \begin{cases} \mathcal{S}_1 & \text{if } q \text{ is odd,} \\ k_2^\times \times k^\times & \text{if } q \text{ is even.} \end{cases}$$

Now, we define an admissible covering of $\mathbf{X}_1(\mathfrak{p}^3)$ as

$$\mathcal{C}_1(\mathfrak{p}^3) = \begin{cases} \{\mathbf{V}_1, \mathbf{V}_2, \mathbf{U}, \{\mathcal{D}_\zeta\}_{\zeta\in\mathcal{S}_1}\} & \text{if } q \text{ is odd,} \\ \{\mathbf{V}_1, \mathbf{V}_2, \mathbf{U}, \{\hat{\mathcal{D}}_\zeta\}_{\zeta\in k_2^\times}, \{\mathcal{D}_{\zeta,\zeta'}\}_{(\zeta,\zeta')\in\mathcal{S}}\} & \text{if } q \text{ is even.} \end{cases}$$

In Section 7.2, we show that $\mathcal{C}_1(\mathfrak{p}^3)$ is a semistable covering of $\mathbf{X}_1(\mathfrak{p}^3)$ over some finite extension.

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

## §5. Action of the division algebra on the reductions

In this section, we determine the action of $\mathcal{O}_D^\times$ on the reductions $\overline{\mathbf{Y}}_{1,2}$, $\overline{\mathbf{Y}}_{2,1}$ $\overline{\mathbf{Z}}_{1,1}$, $\{\overline{\mathbf{P}}_\zeta\}_{\zeta\in k_2^\times}$ and $\{\overline{\mathbf{X}}_\zeta\}_{\zeta\in\mathcal{S}}$ by using the description of $\mathcal{O}_D^\times$-action in (1.8). We take

$$d = d_1 + \varphi d_2 \in \mathcal{O}_D^\times,$$

where $d_1 \in \mathcal{O}_{K_2}^\times$ and $d_2 \in \mathcal{O}_{K_2}$. We put

$$\kappa_1(d) = \bar{d}_1, \qquad \kappa_2(d) = -\bar{d}_1^{-q}\bar{d}_2.$$

LEMMA 5.1. *The element $d$ induces the following morphisms:*

$$\begin{aligned} &\overline{\mathbf{Y}}_{1,2} \to \overline{\mathbf{Y}}_{1,2}; \qquad (x,y) \mapsto (\kappa_1(d)x, \kappa_1(d)^{-q}y),\\ &\overline{\mathbf{Y}}_{2,1} \to \overline{\mathbf{Y}}_{2,1}; \qquad (x,y) \mapsto (\kappa_1(d)^{-1}x, \kappa_1(d)^{q}y). \end{aligned}$$

*Proof.* We prove the assertion for $\overline{\mathbf{Y}}_{1,2}$. By (1.5), we have

$$d^*x_1 \equiv d_1x_1, \qquad d^*x_3 \equiv d_1x_3 \pmod{0+}.$$

Therefore, the required assertion follows from (4.5). The assertion for $\overline{\mathbf{Y}}_{1,2}$ is proved similarly. □

Now, let the notation be as in Section 4.3. We put

$$\begin{aligned} &x_i' = d^*x_i \quad \text{for } 1 \leqslant i \leqslant 3,\\ &t' = d^*t, \qquad Z_0' = d^*Z_0, \qquad Z' = d^*Z. \end{aligned}$$

We have

$$\begin{aligned} j^{-1}(x_1) &\equiv x_1 + d_1^{-q}d_2\varpi^{\epsilon_1}\tilde{u}x_1 \pmod{\epsilon_1+},\\ j^{-1}(x_2) &\equiv x_2 + d_1^{-q}d_2\varpi^{\epsilon_1}\tilde{u}x_2 \pmod{\epsilon_1+},\\ j^{-1}(x_3) &\equiv x_3 \pmod{\epsilon_2+} \end{aligned}$$

by (1.11). On the other hand, we have

$$\begin{aligned} \tilde{d}(x_1) &\equiv d_1x_1 \pmod{\epsilon_1+},\\ \tilde{d}(x_2) &\equiv d_1x_2 + d_2^q\varpi^{\epsilon_1}x_2^q \pmod{\epsilon_1+},\\ \tilde{d}(x_3) &\equiv d_1x_3 + d_2^q\varpi^{\epsilon_3}x_3^q \pmod{\epsilon_2+} \end{aligned}$$

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

by (1.5). Hence, we obtain

$$x_1' \equiv d_1 x_1 + d_1^{-(q-1)} d_2 \varpi^{\epsilon_1} \tilde{u} x_1 \pmod{\epsilon_1+}, \tag{5.1}$$

$$x_2' \equiv d_1 x_2 + d_1^{-(q-1)} d_2 \varpi^{\epsilon_1} \tilde{u} x_2 + d_2^q \varpi^{\epsilon_1} x_2^q \pmod{\epsilon_1+}, \tag{5.2}$$

$$x_3' \equiv d_1 x_3 + d_2^q \varpi^{\epsilon_3} x_3^q \pmod{\epsilon_2+}. \tag{5.3}$$

By the definition of $t$ and the equation $x_2'/x_1' = -1 + \gamma_2(x_2'^q/t')$, we acquire

$$t' \equiv d_1^q t - d_1^{q-1} d_2^q t^{2-q} \varpi^{\epsilon_2} \pmod{\epsilon_2+} \tag{5.4}$$

using (5.1) and (5.2). We put

$$G_0 = d_1^{-q} d_2 x_3^{q(q-1)} + d_1^{-1} d_2^q x_3^{-q(q-1)}.$$

By the definition of $Z_0$ and the equation $1 + (x_3'^q/t') = \gamma_3 Z_0'$, we obtain

$$Z_0' \equiv Z_0 - \varpi^{\epsilon_3} G_0 \pmod{\epsilon_3+} \tag{5.5}$$

using (5.3) and (5.4). We put

$$G = G_0 + d_1^{-1} d_2^q (x_2 x_3^{q-2} + x_1^{-1} x_3^q).$$

By the definition of $Z$ and the equation

$$Z_0' + (x_2'/x_3') - (x_3'/x_1') = \gamma_4(x_2'/x_3') Z',$$

we obtain

$$Z' \equiv Z - \frac{x_3}{x_2} \varpi^{\epsilon_4} G \pmod{\epsilon_4+} \tag{5.6}$$

using (5.1), (5.2), (5.3) and (5.5). We have

$$G \equiv d_1^{-q} d_2 x_3^{q(q-1)} + d_1^{-1} d_2^q x_3^{(q-1)(q+2)} \pmod{0+}$$

by $x_1 \equiv -x_3^{q^2}$, $x_2 \equiv x_3^{q^2} \pmod{0+}$. We put

$$\Delta = d_1^{-q} d_2 x_3^{-(q-1)} + d_1^{-1} d_2^q x_3^{q-1}.$$

Then, the congruence (5.6) has the following form:

$$Z' \equiv Z - \varpi^{\epsilon_4} \Delta \pmod{\epsilon_4+}. \tag{5.7}$$

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

PROPOSITION 5.2. *The element $d$ acts on $\overline{\mathbf{Z}}_{1,1}$ by*

$$(Z, x_3) \mapsto (Z, \kappa_1(d)x_3).$$

*Proof.* This follows from (5.3) and (5.7). □

PROPOSITION 5.3. *The element $d$ induces the morphism*

$$\overline{\mathbf{P}}_{\zeta} \to \overline{\mathbf{P}}_{\kappa_1(d)\zeta}; \qquad w_1 \mapsto w_1.$$

*Proof.* This follows from (5.7), Proposition 5.2 and $Z = \varpi^{1/(4q^4)}w_1$. □

PROPOSITION 5.4. *We take $\zeta \in \mathcal{S}_1$. Further, we take $\zeta' \in k^{\times}$, if $q$ is even. We set as follows:*

$$\eta = \begin{cases} \zeta & \text{if } q \text{ is odd,} \\ (\zeta, \zeta') & \text{if } q \text{ is even,} \end{cases}$$

$$d\eta = \begin{cases} \kappa_1(d)\zeta & \text{if } q \text{ is odd,} \\ (\kappa_1(d)\zeta, \zeta') & \text{if } q \text{ is even,} \end{cases}$$

$$f_d = \begin{cases} \mathrm{Tr}_{k_2/k}(\zeta^{-2q}\kappa_2(d)) & \text{if } q \text{ is odd,} \\ \mathrm{Tr}_{k_2/\mathbb{F}_2}(\zeta^{1-q}\zeta'^{-2}\kappa_2(d)) & \text{if } q \text{ is even,} \end{cases}$$

*where $\eta, d\eta \in \mathcal{S}$. Then, the element $d$ induces*

$$\overline{\mathbf{X}}_{\eta} \to \overline{\mathbf{X}}_{d\eta} : \begin{cases} (z, w) \mapsto (\kappa_1(d)^{-(q+1)}(z + f_d), \kappa_1(d)^{-(q+1)/2}w) & \text{if } q \text{ is odd,} \\ (z, w) \mapsto (z + f_d, w) & \text{if } q \text{ is even.} \end{cases}$$

*Proof.* First, we assume that $q$ is odd. Recall that $Z = a_{\zeta}z + c_{1,\zeta}$ and $x_3 = b_{\zeta}w + c_{2,\zeta}$. Similarly, we have $Z' = a_{\bar{d}_1\zeta}z' + c_{1,\bar{d}_1\zeta}$ and $x_3 = b_{\bar{d}_1\zeta}w' + c_{2,\bar{d}_1\zeta}$. Then, the claim follows from (5.7).

Next, we assume that $q$ is even. By (5.7) and $d^*x_3 \equiv d_1x_3 \pmod{(\epsilon_3/2)+}$, we acquire

$$d^*z_{f+1} - z_{f+1} \equiv \varpi^{\epsilon_4/4} \sum_{i=1}^{f} w_1^{q-2^i} \Delta^{2^{i-1}} \quad \left(\mathrm{mod}\ \frac{\epsilon_4}{4}+\right) \tag{5.8}$$

on the locus where $v(Z) \geqslant \epsilon_4/2$. By $z_{f+1} = a_{1,\zeta'}z + b_{1,\zeta'}w + c_{1,\zeta'}$ and $w_1 = b_{2,\zeta'}w + c_{2,\zeta'}$, we obtain

$$d^*z - z \equiv \sum_{i=1}^{f} c_{2,\zeta'}^{-2^i} \Delta^{2^{i-1}} \quad (\mathrm{mod}\ 0+)\,,$$

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

$$d^*w \equiv w \quad \left(\mathrm{mod}\ \frac{\epsilon_4}{3}+\right)$$

on $\mathbf{X}_{\zeta,\zeta'}$ by (5.7) and (5.8). On the other hand, we have

$$\sum_{i=1}^{f} \bar{c}_{2,\zeta'}^{-2^i} \overline{\Delta}^{2^{i-1}} = f_d,$$

because $\bar{x_3} = \zeta$ and $\bar{c}_{2,\zeta'} = \zeta'$. Hence, we have proved the claim. □

## §6. Action of the Weil group on the reductions

In this section, we compute the actions of the Weil group on the reductions $\overline{\mathbf{Y}}_{1,2}$, $\overline{\mathbf{Y}}_{2,1}$, $\overline{\mathbf{Z}}_{1,1}$, $\{\overline{\mathbf{P}}_\zeta\}_{\zeta\in k_2^\times}$ and $\{\overline{\mathbf{X}}_\eta\}_{\eta\in\mathcal{S}}$.

Let $\mathbf{X}$ be a reduced affinoid over $\mathbf{C}$ with an action of $W_K$. For $P \in \mathbf{X}(\mathbf{C})$, the image of $P$ under the natural reduction map $\mathbf{X}(\mathbf{C}) \to \overline{\mathbf{X}}(k^{\mathrm{ac}})$ is denoted by $\overline{P}$. The action of $W_K$ on $\overline{\mathbf{X}}$ is a homomorphism

$$w_{\mathbf{X}} : W_K \to \mathrm{Aut}(\overline{\mathbf{X}})$$

characterized by $\overline{\sigma(P)} = w_{\mathbf{X}}(\sigma)(\overline{P})$ for $\sigma \in W_K$ and $P \in \mathbf{X}(\mathbf{C})$. For $\sigma \in W_K$, we define $r_\sigma \in \mathbb{Z}$ so that $\sigma$ induces the $q^{-r_\sigma}$th power map on the residue field of $K^{\mathrm{ac}}$.

Remark 6.1. In the usual sense, $W_K$ does not act on $\mathbf{X}_1(\mathfrak{p}^3)$, because the action of $W_K$ does not preserve the connected components of $\mathrm{LT}_1(\mathfrak{p}^3)$. Precisely, $w_{\mathbf{X}}$ is the action of

$$\{(\sigma, \varphi^{-r_\sigma}) \in W_K \times D^\times\},$$

which preserves the connected components of $\mathrm{LT}_1(\mathfrak{p}^3)$.

### 6.1 Actions of the Weil group on $\overline{\mathbf{Y}}_{1,2}$, $\overline{\mathbf{Y}}_{2,1}$ and $\overline{\mathbf{Z}}_{1,1}$

For $\sigma \in W_K$, we put

$$\lambda(\sigma) = \overline{\sigma(\varpi^{1/(q^2-1)})/\varpi^{1/(q^2-1)}} \in k_2^\times.$$

We note that $\lambda$ is not a group homomorphism in general.

Lemma 6.2. *Let $\sigma \in W_K$. Then, the element $\sigma$ induces the automorphisms*

$$\overline{\mathbf{Y}}_{1,2} \to \overline{\mathbf{Y}}_{1,2}; \qquad (x,y) \mapsto (\lambda(\sigma)^q x^{q^{-r_\sigma}}, \lambda(\sigma)^{-1} y^{q^{-r_\sigma}}),$$
$$\overline{\mathbf{Y}}_{2,1} \to \overline{\mathbf{Y}}_{2,1}; \qquad (x,y) \mapsto (\lambda(\sigma)^{-1} x^{q^{-r_\sigma}}, \lambda(\sigma)^q y^{q^{-r_\sigma}})$$

*as schemes over $k$.*

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

*Proof.* We prove the claim for $\overline{\mathbf{Y}}_{1,2}$. We set

$$\sigma(\varpi^{1/(q^3(q^2-1))}) = \xi\varpi^{1/(q^3(q^2-1))}$$

with $\xi \in \mu_{q^3(q^2-1)}(K^{\mathrm{ac}})$. Let $P \in \mathbf{Y}_{1,2}(\mathbf{C})$. We have $X_3(\sigma(P)) = \sigma(X_3(P))$. By applying $\sigma$ to $X_3(P) = \varpi^{1/(q^3(q^2-1))}x_3(P)$, we obtain

$$x_3(\sigma(P)) = \xi\sigma(x_3(P)) \equiv \xi x_3(P)^{q^{-r_\sigma}} \pmod{0+}.$$

In the same way, we have

$$x_1(\sigma(P)) \equiv \xi^{q^4} x_1(P)^{q^{-r_\sigma}} \pmod{0+}.$$

Therefore, we acquire $x^\sigma = \bar{\xi}x^{q^{-r_\sigma}}$ and $y^\sigma = \bar{\xi}^{-q}y^{q^{-r_\sigma}}$ by (4.5). Hence, the claim follows from $\bar{\xi} = \lambda(\sigma)^q$. We can prove the claim for $\overline{\mathbf{Y}}_{2,1}$ similarly. □

For $\sigma \in W_K$, we put

$$\xi_\sigma = \frac{\sigma(\omega_3)}{\omega_3} \in \mu_{2q^3(q-1)}(K^{\mathrm{ac}}).$$

LEMMA 6.3. *Let $\sigma \in W_K$. Then, $\sigma$ acts on $\overline{\mathbf{Z}}_{1,1}$ by $(Z, x_3) \mapsto (Z^{q^{-r_\sigma}}, \bar{\xi}_\sigma x_3^{q^{-r_\sigma}})$.*

*Proof.* We use the notation in Section 4.3. Let $P \in \mathbf{Z}_{1,1}(\mathbf{C})$. Since we set $X_1 = \omega_1^{2q-1}x_1$, $X_2 = \omega_1 x_2$ and $X_3 = \omega_3 x_3$, we have

$$\begin{aligned} x_1(\sigma(P)) &= \xi_\sigma^{q^2(2q-1)}\sigma(x_1(P)), \\ x_2(\sigma(P)) &= \xi_\sigma^{q^2}\sigma(x_2(P)), \\ x_3(\sigma(P)) &= \xi_\sigma\sigma(x_3(P)). \end{aligned}$$

Hence, we obtain

$$\frac{x_2(\sigma(P))}{x_1(\sigma(P))} = \xi_\sigma^{-2q^2(q-1)}\sigma\left(\frac{x_2(P)}{x_1(P)}\right) \equiv \sigma\left(\frac{x_2(P)}{x_1(P)}\right) \pmod{\epsilon_1+}.$$

Since we set $x_2/x_1 = -1 + \gamma_2(x_2^q/t)$, we acquire

$$t(\sigma(P)) \equiv \xi_\sigma^{q^3}\sigma(t(P)) \pmod{\epsilon_2+}.$$

Therefore, we obtain

$$\frac{x_3(\sigma(P))^q}{t(\sigma(P))} = \xi_\sigma^{-q(q^2-1)}\sigma\left(\frac{x_3(P)^q}{t(P)}\right) \equiv \sigma\left(\frac{x_3(P)^q}{t(P)}\right) \pmod{\epsilon_2+}.$$

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

Since we set $1+(x_3^q/t)=\gamma_3 Z_0$, we obtain

$$Z_0(\sigma(P)) \equiv \sigma(Z_0(P)) \pmod{\epsilon_3+}\,.$$

Therefore, we acquire

$$Z(\sigma(P)) \equiv \sigma(Z(P)) \pmod{\epsilon_4+} \tag{6.1}$$

by $Z_0+(x_2/x_3)-(x_3/x_1)=\gamma_4(x_2/x_3)Z$.

The assertion follows from

$$x_3(\sigma(P)) = \xi_\sigma \sigma(x_3(P)) \equiv \xi_\sigma x_3(P)^{q^{-r_\sigma}} \pmod{0+}$$

and (6.1). □

### 6.2 Action of the Weil group on $\overline{\mathbf{X}}_\eta$

In this subsection, let $\zeta \in \mu_{2(q^2-1)}(k^{\mathrm{ac}})$. Until Lemma 6.8, let $\sigma \in W_K$.

#### 6.2.1 $q$ odd

We assume that $q$ is odd. We use the notation in Section 4.4.1. By (6.1) and $x_3(\sigma(P))=\xi_\sigma\sigma(x_3(P))$, we have

$$\begin{aligned} a_{\bar{\xi}_\sigma\zeta^{q-r_\sigma}} z(\sigma(P)) + c_{1,\bar{\xi}_\sigma\zeta^{q-r_\sigma}} &= Z(\sigma(P)) \equiv \sigma(Z(P)) \\ &= \sigma(a_\zeta)\sigma(z(P)) + \sigma(c_{1,\zeta}) \pmod{\epsilon_4+} \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} b_{\bar{\xi}_\sigma\zeta^{q-r_\sigma}} w(\sigma(P)) + c_{2,\bar{\xi}_\sigma\zeta^{q-r_\sigma}} &= x_3(\sigma(P)) = \xi_\sigma\sigma(x_3(P)) \\ &= \xi_\sigma\sigma(b_\zeta)\sigma(w(P)) + \xi_\sigma\sigma(c_{2,\zeta}) \end{aligned} \tag{6.3}$$

for $P \in \mathbf{X}_\zeta(\mathbf{C})$. Note that $c_{1,\bar{\xi}_\sigma\zeta^{q-r_\sigma}} = c_{1,\zeta}$ and $c_{2,\bar{\xi}_\sigma\zeta^{q-r_\sigma}} = \xi_\sigma^{q^4}\zeta^{q^{-r_\sigma}-1}c_{2,\zeta}$. We have

$$v(\sigma(c_{1,\zeta}) - c_{1,\zeta}) \geqslant \epsilon_4$$

by (6.2). We put

$$a_{\sigma,\zeta} = \frac{\sigma(a_\zeta)}{\zeta^{r_\sigma(q^2-1)}\xi_\sigma^{q+1}a_\zeta}, \qquad b_{\sigma,\zeta} = \frac{\sigma(c_{1,\zeta})-c_{1,\zeta}}{\zeta^{r_\sigma(q^2-1)}\xi_\sigma^{q+1}a_\zeta},$$

$$c_{\sigma,\zeta} = \frac{\sigma(b_\zeta)}{\zeta^{(q^{-r_\sigma}-1)(q+3)/2}\xi_\sigma^{(q+1)/2}b_\zeta}.$$

Then, we have $a_{\sigma,\zeta}, b_{\sigma,\zeta}, c_{\sigma,\zeta} \in \mathcal{O}_{K^{\mathrm{ac}}}$. In the following, we omit the subscript $\zeta$ of $a_{\sigma,\zeta}$, $b_{\sigma,\zeta}$ and $c_{\sigma,\zeta}$.

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

PROPOSITION 6.4. *We have* $\bar{a}_\sigma \in k^\times$, $\bar{b}_\sigma \in k$ *and* $\bar{a}_\sigma = \bar{c}_\sigma^2$. *Further,* $\sigma$ *induces the morphism*

$$\overline{\mathbf{X}}_\zeta \to \overline{\mathbf{X}}_{\bar{\xi}_\sigma \zeta^{q^{-r_\sigma}}}; \qquad (z, w) \mapsto (\bar{a}_\sigma z^{q^{-r_\sigma}} + \bar{b}_\sigma, \bar{c}_\sigma w^{q^{-r_\sigma}}).$$

*Proof.* We have

$$v(\xi_\sigma \sigma(c_{2,\zeta}) - \xi_\sigma^{q^4} \zeta^{q^{-r_\sigma}-1} c_{2,\zeta}) \geqslant \epsilon_3$$

by $v(\sigma(c_{1,\zeta}) - c_{1,\zeta}) \geqslant \epsilon_4$. Hence, we have the last assertion by (6.2) and (6.3). By the definition of $a_\zeta$, $b_\zeta$ and $c_{1,\zeta}$, we can check that

$$\bar{a}_\sigma^{q-1} = 1, \qquad \bar{b}_\sigma^q = \bar{b}_\sigma, \qquad \bar{a}_\sigma = \bar{c}_\sigma^2$$

using $c_{1,\zeta}^q \equiv -\iota(2 - \gamma_4 c_{1,\zeta}) \pmod{(q-1)/q^4}$. □

We put $L = K(\varpi^{1/2})$ and $L_2 = K_2(\varpi^{1/2})$ in $K^{\mathrm{ac}}$. Let $\mathrm{LT}_{L_2}$ be the formal $\mathcal{O}_{L_2}$-module over $\mathcal{O}_{L^{\mathrm{ur}}}$ of dimension 1 such that

$$[\varpi^{1/2}]_{\mathrm{LT}_{L_2}}(X) = \varpi^{1/2} X - X^{q^2},$$
$$[\zeta]_{\mathrm{LT}_{L_2}}(X) = \zeta X \quad \text{for } \zeta \in \mu_{q^2-1}(L_2) \cup \{0\}.$$

We put $\varpi_{1,L_2} = \varpi^{1/(2(q^2-1))}$ and take $\varpi_{2,L_2} \in \mathcal{O}_{K^{\mathrm{ac}}}$ such that $[\varpi^{1/2}]_{\mathrm{LT}_{L_2}}(\varpi_{2,L_2}) = \varpi_{1,L_2}$. Let $\mathrm{Art}_{L_2} : L_2^\times \xrightarrow{\sim} W_{L_2}^{\mathrm{ab}}$ be the Artin reciprocity map normalized so that the image by $\mathrm{Art}_{L_2}$ of a uniformizer is a lift of the geometric Frobenius. We consider the following homomorphism:

$$I_{L_2} \to k_2^\times \times k_2; \qquad \sigma \mapsto \big(\lambda_1(\sigma), \lambda_2(\sigma)\big)$$
$$= \left( \overline{\frac{\sigma(\varpi_{1,L_2})}{\varpi_{1,L_2}}}, \overline{\frac{\varpi_{1,L_2}\sigma(\varpi_{2,L_2}) - \sigma(\varpi_{1,L_2})\varpi_{2,L_2}}{\sigma(\varpi_{1,L_2})\varpi_{1,L_2}}} \right).$$

This map is equal to the composite

$$I_{L_2} \to \mathcal{O}_{L_2}^\times \to k_2^\times \times k_2,$$

where the first homomorphism is induced from the inverse of $\mathrm{Art}_{L_2}$, and the second homomorphism is given by $a + b\varpi^{1/2} \mapsto (\bar{a}, \bar{b}/\bar{a})$ for $a \in \mu_{q^2-1}(L_2)$ and $b \in \mathcal{O}_{L_2}$. Then, we rewrite Proposition 6.4 as follows.

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

COROLLARY 6.5. *Let $\sigma \in I_L$. We put*

$$g_0 = 2\zeta^{-(q+1)}(\lambda_2(\sigma)^q + \zeta^{q^2-1}\lambda_2(\sigma)) \in k.$$

*Then, $\sigma$ induces the morphism*

$$\overline{\mathbf{X}}_\zeta \to \overline{\mathbf{X}}_{\lambda_1(\sigma)^{q+1}\zeta}; \qquad (z, w) \mapsto (\lambda_1(\sigma)^{-2(q+1)}(z + g_0), \lambda_1(\sigma)^{-(q+1)}w).$$

*Proof.* We can check that $\bar{a}_\sigma = \lambda_1(\sigma)^{-2(q+1)}$ and $\bar{c}_\sigma = \lambda_1(\sigma)^{-(q+1)}$ easily. We prove that

$$\bar{b}_\sigma = \lambda_1(\sigma)^{-2(q+1)}g_0.$$

We simply write $\varpi_i$ for $\varpi_{i,L_2}$. We put $\iota = \zeta^{q^2-1}$ and

$$C = \varpi_1^{(q^2-1)/q}\left\{\left(\frac{\varpi_2}{\varpi_1}\right)^q + \iota\left(\frac{\varpi_2}{\varpi_1}\right)\right\}.$$

Then, we have

$$C^q - \iota\gamma_1 C \equiv -1 \quad \left(\mathrm{mod}\ \frac{1}{2}+\right)$$

by $\varpi_2^{q^2} - \varpi^{1/2}\varpi_2 = -\varpi_1$. We can easily check the equality

$$\sigma(C) - C \equiv \varpi^{\epsilon_1}(\lambda_2(\sigma)^q + \iota\lambda_2(\sigma)) \quad (\mathrm{mod}\ \epsilon_1+)\,.$$

On the other hand, we can check

$$c_{1,\zeta}^q \equiv -\iota(2 - \gamma_4 c_{1,\zeta}) \quad \left(\mathrm{mod}\ \frac{q-1}{2q^4}+\right)$$

by the definition of $c_{1,\zeta}$. Therefore, the elements $C$ and $c_{1,\zeta}^{q^3}/(2\iota)$ satisfy

$$x^q - \iota\gamma_1 x \equiv -1 \quad \left(\mathrm{mod}\ \frac{1}{2}+\right).$$

Hence, we obtain $C \equiv c_{1,\zeta}^{q^3}/(2\iota)\ (\mathrm{mod}\ \epsilon_1+)$. This implies that

$$(\sigma(c_{1,\zeta}) - c_{1,\zeta})^{q^3} \equiv 2\iota(\sigma(C) - C) \quad (\mathrm{mod}\ \epsilon_1+)\,.$$

Therefore, we obtain

$$\bar{b}_\sigma \equiv \bar{b}_\sigma^{q^3} \equiv \lambda_1(\sigma)^{-2(q+1)}g_0 \quad (\mathrm{mod}\ 0+)$$

by $\xi_\sigma = \lambda_1(\sigma)^{q+1}\ (\mathrm{mod}\ 0+)$. ∎

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

6.2.2 $q$ even

We assume that $q$ is even. We use the notation in Section 4.4.2. For $P \in \mathbf{P}^0(\mathbf{C})$, we have

$$w_1(\sigma(P)) \equiv \sigma(w_1(P)) \quad \left(\mathrm{mod}\ \frac{1}{4q^4}+\right) \tag{6.4}$$

by (6.1). We can see that

$$z_{f+1}(\sigma(P)) \equiv \sigma(z_{f+1}(P)) \quad \left(\mathrm{mod}\ \frac{1}{8q^4}+\right) \tag{6.5}$$

using (4.31) and (6.4).

LEMMA 6.6. *The element $\sigma$ induces the morphism*

$$\overline{\mathbf{P}}_\zeta \to \overline{\mathbf{P}}_{\bar{\xi}_\sigma \zeta^{q-r_\sigma}}; \qquad w_1 \mapsto w_1^{q^{-r_\sigma}}.$$

*Proof.* This follows from Lemma 6.3 and (6.4). □

We take $\zeta' \in k^\times$. By (6.4) and (6.5), we have

$$\begin{aligned} &a_{1,\zeta'} z(\sigma(P)) + b_{1,\zeta'} w(\sigma(P)) + c_{1,\zeta'} \\ &\equiv a_{1,\zeta'} \sigma(z(P)) + \sigma(b_{1,\zeta'}) \sigma(w(P)) + \sigma(c_{1,\zeta'}) \quad \left(\mathrm{mod}\ \frac{1}{8q^4}+\right) \end{aligned} \tag{6.6}$$

and

$$b_{2,\zeta'} w(\sigma(P)) + c_{2,\zeta'} \equiv \sigma(b_{2,\zeta'}) \sigma(w(P)) + \sigma(c_{2,\zeta'}) \quad \left(\mathrm{mod}\ \frac{1}{4q^3}+\right) \tag{6.7}$$

using $\sigma(a_{1,\zeta'}) \equiv a_{1,\zeta'} \pmod{1/(8q^4)+}$. We put

$$a_{\sigma,\zeta'} = \frac{\sigma(b_{2,\zeta'})}{b_{2,\zeta'}}, \qquad b_{\sigma,\zeta'} = \frac{\sigma(b_{1,\zeta'}) b_{2,\zeta'} - b_{1,\zeta'} \sigma(b_{2,\zeta'})}{a_{1,\zeta'} b_{2,\zeta'}},$$

$$b'_{\sigma,\zeta'} = \frac{\sigma(c_{2,\zeta'}) - c_{2,\zeta'}}{b_{2,\zeta'}}, \qquad c_{\sigma,\zeta'} = \frac{\sigma(c_{1,\zeta'}) - c_{1,\zeta'} - b_{1,\zeta'} b_{2,\zeta'}^{-1} (\sigma(c_{2,\zeta'}) - c_{2,\zeta'})}{a_{1,\zeta'}}.$$

In the following, we omit the subscript $\zeta'$ of $a_{\sigma,\zeta'}$, $b_{\sigma,\zeta'}$, $b'_{\sigma,\zeta'}$ and $c_{\sigma,\zeta'}$. We note that $v(a_\sigma) = 0$. We have $v(b'_\sigma) \geqslant 0$ by (6.7). This implies that $v(b_\sigma) \geqslant 0$. By (6.6) and (6.7), we obtain $v(c_\sigma) \geqslant 0$ using $v(b_\sigma) \geqslant 0$.

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

PROPOSITION 6.7. *The element $\sigma$ induces the morphism*

$$\overline{\mathbf{X}}_{\zeta,\zeta'} \to \overline{\mathbf{X}}_{\bar{\xi}_\sigma \zeta^{q^{-r_\sigma}},\zeta'}; \qquad (z,w) \mapsto (z^{q^{-r_\sigma}} + \bar{b}_\sigma w^{q^{-r_\sigma}} + \bar{c}_\sigma, \bar{a}_\sigma w^{q^{-r_\sigma}} + \bar{b}'_\sigma).$$

*Proof.* This follows from (6.6) and (6.7). □

In the following, we simplify the description of $\bar{a}_\sigma$, $\bar{b}_\sigma$, $\bar{b}'_\sigma$ and $\bar{c}_\sigma$. Let $\tilde{\zeta}' \in \mu_{q-1}(K)$ be the lift of $\zeta'$. We put

$$h_{\zeta'}(x) = x^4 - \varpi^{1/4}\tilde{\zeta}'^4 x - \tilde{\zeta}'^4.$$

LEMMA 6.8. *There is a root $\delta_1$ of $h_{\zeta'}(x) = 0$ such that*

$$\delta_1 \equiv c_{2,\zeta'}^{q^4} + \frac{q}{2}\varpi^{1/4}\tilde{\zeta}'^2 \quad \left(\mathrm{mod}\ \frac{1}{4}+\right).$$

*Proof.* We put

$$h(x) = x^{4(q-1)} + 1 + \varpi^{1/4}x^{4q-3}.$$

By the definition of $c_{2,\zeta'}$, we have $h(c_{2,\zeta'}^{q^4}) \equiv 0 \,(\mathrm{mod}\ 1)$. Hence, we have a root $c'_2$ of $h$ such that $c'_2 \equiv c_{2,\zeta'}^{q^4} \,(\mathrm{mod}\ 3/4)$ by Newton's method. We can check that

$$c'_2 \equiv \tilde{\zeta}' + \varpi^{1/16}\tilde{\zeta}'^{5/4} \quad \left(\mathrm{mod}\ \frac{1}{16}+\right).$$

We define a parameter $s$ with $v(s) \geqslant 1/16$ by $x = \tilde{\zeta}' + s$. Then, we have

$$\begin{aligned} h(\tilde{\zeta}' + s) &\equiv \tilde{\zeta}'^{-4}s^4 + \left(\frac{q}{2} - 1\right)\tilde{\zeta}'^{-8}s^8 + \varpi^{1/4}(\tilde{\zeta}' + s) \\ &\equiv \tilde{\zeta}'^{-4}h_{\zeta'}(x) + \left(\frac{q}{2} - 1\right)\tilde{\zeta}'^{-8}s^8 + \varpi^{1/4}s^4\tilde{\zeta}'^{-3} \quad \left(\mathrm{mod}\ \frac{1}{2}+\right). \end{aligned}$$

This implies that

$$h_{\zeta'}(c'_2) \equiv \frac{q}{2}\varpi^{1/2}\tilde{\zeta}'^6 \quad \left(\mathrm{mod}\ \frac{1}{2}+\right).$$

Therefore, we have a root $\delta_1$ of $h_{\zeta'}(x) = 0$ such that

$$\delta_1 \equiv c'_2 + \frac{q}{2}\varpi^{1/4}\tilde{\zeta}'^2 \quad \left(\mathrm{mod}\ \frac{1}{4}+\right)$$

by Newton's method. □

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

By the definition of $b_{2,\zeta'}$, we have

$$b_{2,\zeta'}^{3q^4}\varpi^{-1/4} \equiv \tilde{\zeta}'^4 \pmod{0+}.$$

Let $\zeta''$ be the element of $\mu_{3(q-1)}(K^{\mathrm{ur}})$ satisfying $\zeta'' \equiv b_{2,\zeta'}^{q^4}\varpi^{-1/12} \pmod{0+}$. Note that $\zeta''^3 = \tilde{\zeta}'^4$. We take $\delta_1$ as in Lemma 6.8 and put $\delta = \delta_1/(\zeta''\varpi^{1/12})$. Then, we have

$$\delta^4 - \delta = \frac{1}{\zeta''\varpi^{1/3}}.$$

Note that $v(\delta) = -1/12$. We take $\zeta_3 \in \mu_3(K^{\mathrm{ur}})$ such that $\zeta_3 \neq 1$, and put

$$h_{\delta_1}(x) = x^2 - (1+2\zeta_3)\varpi^{1/4}\delta_1^{2q}x - \varpi^{1/4}\delta_1^{4q-1}(1+2\varpi^{1/4}\delta_1).$$

LEMMA 6.9. *There is a root $\theta_1$ of $h_{\delta_1}(x) = 0$ such that*

$$\theta_1 \equiv c_{1,\zeta'}^{2q^4} \pmod{\frac{1}{4}+}.$$

*Proof.* By the definition of $c_{1,\zeta'}$ and $c_{2,\zeta'}$, we have $h_{\delta_1}(c_{1,\zeta'}^{2q^4}) \equiv 0 \pmod{1/2+}$. Hence, we can show the claim using Newton's method. □

We take $\theta_1$ as in Lemma 6.9 and put

$$\theta = \frac{\theta_1}{\varpi^{1/4}\delta_1^{2q}} - \zeta_3.$$

Then, we have $\theta^2 - \theta = \delta^3$. Note that $v(\theta) = -1/8$. Let $\sigma \in W_K$ in this paragraph. We put

$$\zeta_{3,\sigma} = \frac{\sigma(\zeta''\varpi^{1/3})}{\zeta''\varpi^{1/3}}.$$

We take $\nu_\sigma \in \mu_3(K^{\mathrm{ur}}) \cup \{0\}$ such that $\sigma(\delta) \equiv \zeta_{3,\sigma}^{-1}(\delta + \nu_\sigma) \pmod{5/6}$. Then, we have

$$\begin{aligned} \big(\sigma(\theta) - \theta + \nu_\sigma^2\delta\big)^2 &\equiv \sigma(\theta) - \theta + \nu_\sigma^2\delta + \nu_\sigma^3, \\ \big(\sigma(\theta) - \theta + \nu_\sigma^2\delta + \nu_\sigma^3\big)^2 &\equiv \sigma(\theta) - \theta + \nu_\sigma^2\delta \pmod{0+}. \end{aligned} \tag{6.8}$$

By these equations, we can take $\mu_\sigma \in \mu_3(K^{\mathrm{ur}}) \cup \{0\}$ such that

$$\mu_\sigma \equiv \sigma(\theta) - \theta + \nu_\sigma^2\delta + \nu_\sigma^3 + \sigma(\zeta_3) - \zeta_3 \pmod{0+}.$$

Then, we have $\mu_\sigma^2 + \mu_\sigma \equiv \nu_\sigma^3 \pmod 1$ by (6.8) and $\nu_\sigma, \mu_\sigma \in \mu_3(K^{\mathrm{ur}}) \cup \{0\}$.

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

LEMMA 6.10.

(1) *Let $\sigma \in W_K$. Then, we have*

$$a_\sigma \equiv \zeta_{3,\sigma}, \qquad b_\sigma \equiv \zeta_{3,\sigma}\nu_\sigma^2, \qquad b'_\sigma \equiv \nu_\sigma, \qquad c_\sigma \equiv \mu_\sigma \pmod{0+}.$$

(2) *Let $\sigma \in W_K$. Then we have $\bar{a}_\sigma \in \mathbb{F}_4^\times$ and $\bar{b}_\sigma, \bar{b}'_\sigma, \bar{c}_\sigma \in \mathbb{F}_4$. Further, $\bar{a}_\sigma \bar{b}_\sigma^2 = \bar{b}'_\sigma$ and $\bar{b}_\sigma^3 = \bar{c}_\sigma^2 + \bar{c}_\sigma$ hold.*

*Proof.* By the definition of $b_{2,\zeta'}$, we have

$$a_\sigma^{4q^4} \equiv \frac{\sigma(\zeta''^4 \varpi^{1/3})}{\zeta''^4 \varpi^{1/3}} \pmod{0+}.$$

Hence, we have $\bar{a}_\sigma^{4q^4} = \bar{\zeta}_{3,\sigma} \in \mathbb{F}_4^\times$. This implies that $\bar{a}_\sigma = \bar{\zeta}_{3,\sigma} \in \mathbb{F}_4^\times$.

By the definition of $a_{1,\zeta'}$ and $b_{1,\zeta'}$, we have

$$\begin{aligned} b_\sigma^{2q^4} &\equiv \frac{\sigma(b_{2,\zeta'}^{2q^4})\big(\sigma(c_{2,\zeta'}^{q^4(2q-3)}) - c_{2,\zeta'}^{q^4(2q-3)}\big)}{\varpi^{1/4} c_{2,\zeta'}^{2q^5}} \equiv \frac{a_\sigma^{2q^4} b_{2,\zeta'}^{2q^4}\big(\sigma(\delta_1^{2q-3}) - \delta_1^{2q-3}\big)}{\varpi^{1/4}\delta_1^{2q}} \\ &\equiv \frac{\zeta_{3,\sigma}^2 \zeta''^2\big(\sigma(\delta_1) - \delta_1\big)}{\varpi^{1/12}\delta_1^4} \equiv \zeta_{3,\sigma}^2 \left(\frac{\sigma(\zeta''\varpi^{1/12})}{\zeta''\varpi^{1/12}}\sigma(\delta) - \delta\right) \\ &\equiv \zeta_{3,\sigma}^2 \nu_\sigma \pmod{0+}, \end{aligned}$$

where we use Lemma 6.8 in the second congruence, $b_{2,\zeta'}^{q^4}/\varpi^{1/12} \equiv \zeta'' \pmod{0+}$ in the third congruence, $\delta_1^4 = \tilde{\zeta}'^4 \pmod{1/4}$ and $\zeta''^3 = \tilde{\zeta}'^4$ in the fourth congruence, and $\sigma(\zeta''\varpi^{1/12})/(\zeta''\varpi^{1/12}) \equiv \zeta_{3,\sigma} \pmod{0+}$ in the last congruence. Hence, we obtain $\bar{b}_\sigma = \bar{\zeta}_{3,\sigma}\bar{\nu}_\sigma^2 \in \mathbb{F}_4$.

By Lemma 6.8 and $b_{2,\zeta'}^{q^4}/\varpi^{1/12} \equiv \zeta'' \pmod{0+}$, we have

$$b_\sigma'^{q^4} \equiv \frac{\sigma(c_{2,\zeta'}^{q^4}) - c_{2,\zeta'}^{q^4}}{b_{2,\zeta'}^{q^4}} \equiv \frac{\sigma(\delta_1) - \delta_1}{\zeta''\varpi^{1/12}} = \frac{\sigma(\zeta''\varpi^{1/12})}{\zeta''\varpi^{1/12}}\sigma(\delta) - \delta \equiv \nu_\sigma \pmod{0+}.$$

Hence, we obtain $\bar{b}'_\sigma = \bar{\nu}_\sigma \in \mathbb{F}_4$.

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

By Lemmas 6.8 and 6.9 and the definition of $a_{1,\zeta'}$, we have

$$\begin{aligned} c_\sigma^{2q^4} &\equiv \frac{\sigma(\theta_1)-\theta_1-\delta_1^{2q-3}(\sigma(\delta_1^2)-\delta_1^2)}{\varpi^{1/4}\delta_1^{2q}} \\ &\equiv \frac{\delta_1^{-2q}\sigma\big(\delta_1^{2q}\varpi^{1/4}(\theta+\zeta_3)\big)-\varpi^{1/4}(\theta+\zeta_3)-\delta_1^{-3}(\sigma(\delta_1)-\delta_1)^2}{\varpi^{1/4}} \\ &\equiv \frac{\sigma\big(\varpi^{1/4}(\theta+\zeta_3)\big)-\varpi^{1/4}(\theta+\zeta_3)-\varpi^{1/12}\delta\big(\sigma(\varpi^{1/12}\delta)-\varpi^{1/12}\delta\big)^2}{\varpi^{1/4}} \\ &\equiv \sigma(\theta)-\theta+\nu_\sigma^2\delta+\sigma(\zeta_3)-\zeta_3 \pmod{0+}, \end{aligned}$$

where we use $\sigma(\delta_1)\equiv\delta_1 \pmod{1/4}$ in the second congruence, and $\delta_1^4=\tilde{\zeta}'^4 \pmod{1/4}$ in the third congruence. Then, we have $\bar{c}_\sigma^{2q^4}\in\mathbb{F}_4$ by (6.8). Hence, we have $\bar{c}_\sigma\in\mathbb{F}_4$ and $c_\sigma\equiv\mu_\sigma \pmod{0+}$, again by (6.8).

By the above calculations, we can easily check that $\bar{a}_\sigma\bar{b}_\sigma^2=\bar{b}'_\sigma$ and $\bar{b}_\sigma^3=\bar{c}_\sigma^2+\bar{c}_\sigma$. □

LEMMA 6.11. *The field $K(\zeta_3,\zeta''\varpi^{1/3},\theta)$ is a Galois extension over $K$.*

*Proof.* Let $\sigma\in W_K$. It suffices to show that $\sigma(\theta)\in K(\zeta_3,\zeta''\varpi^{1/3},\theta)$. We put

$$\theta_\sigma=\theta+\nu_\sigma^2\delta+\nu_\sigma^3+\mu_\sigma+\sigma(\zeta_3)-\zeta_3.$$

Then, we have $\theta_\sigma^2-\theta_\sigma\equiv\sigma(\delta)^3 \pmod{2/3}$. Hence, we can find $\theta'$ such that $\theta'^2-\theta'=\sigma(\delta)^3$ and $\theta'\equiv\theta_\sigma \pmod{2/3}$. By the choice of $\mu_\sigma$, we have $\theta'=\sigma(\theta) \pmod{0+}$. Hence, we obtain $\theta'=\sigma(\theta)$.

We take $\sigma'\in W_K$ such that $\sigma'(\theta)\neq\sigma(\theta)$. We can define $\theta_{\sigma'}$ as above, and have $\sigma'(\theta)\equiv\theta_{\sigma'} \pmod{2/3}$. If $\nu_\sigma=\nu_{\sigma'}$, then we have $\zeta_{3,\sigma}\sigma(\delta)\equiv\zeta_{3,\sigma'}\sigma'(\delta) \pmod{5/6}$, which implies that $\zeta_{3,\sigma}\sigma(\delta)=\zeta_{3,\sigma'}\sigma'(\delta)$ because both sides are roots of

$$x^4-x-\frac{1}{\zeta''\varpi^{1/3}}=0.$$

Hence, if $\sigma(\delta)^3\neq\sigma'(\delta)^3$, we have $\nu_\sigma\neq\nu_{\sigma'}$, which implies that

$$\sigma(\theta)\equiv\theta_\sigma\not\equiv\theta_{\sigma'}\equiv\sigma'(\theta) \pmod{0+}.$$

If $\sigma(\delta)^3=\sigma'(\delta)^3$, we have $\sigma(\theta)\not\equiv\sigma'(\theta) \pmod{0+}$. Therefore, we have

$$v(\sigma(\theta)-\theta_\sigma)>v(\sigma'(\theta)-\theta_\sigma).$$

Then, we obtain

$$\sigma(\theta)\in K(\theta_\sigma)\subset K(\zeta_3,\zeta''\varpi^{1/3},\theta)$$

by Krasner's lemma. □

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

Let $E$ be the elliptic curve over $k^{\mathrm{ac}}$ defined by $z^2+z=w^3$. We put

$$Q=\left\{ g(\alpha,\beta,\gamma)=\begin{pmatrix}\alpha & \beta & \gamma\\ & \alpha^2 & \beta^2\\ & & \alpha\end{pmatrix}\in \mathrm{GL}_3(\mathbb{F}_4)\;\middle|\;\alpha\gamma^2+\alpha^2\gamma=\beta^3\right\}.$$

We note that $|Q|=24$ and $Q$ is isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$ (cf. [Se, 8.5. Exercices 2]). Let $Q\rtimes\mathbb{Z}$ be a semidirect product, where $r\in\mathbb{Z}$ acts on $Q$ by $g(\alpha,\beta,\gamma)\mapsto g(\alpha^{q^r},\beta^{q^r},\gamma^{q^r})$. Then, $Q\rtimes\mathbb{Z}$ acts faithfully on $E$ as a scheme over $k$, where $(g(\alpha,\beta,\gamma),r)\in Q\rtimes\mathbb{Z}$ acts on $E$ by

$$(z,w)\mapsto\left(z^{q^{-r}}+\alpha^{-1}(\beta w^{q^{-r}}+\gamma),\alpha(w^{q^{-r}}+(\alpha^{-1}\beta)^2)\right)$$

for $k^{\mathrm{ac}}$-valued points.

Proposition 6.12. *The element $\sigma\in W_K$ sends $\overline{\mathbf{X}}_{\zeta,\zeta'}$ to $\overline{\mathbf{X}}_{\bar\xi_\sigma\zeta^{q^{-r_\sigma}},\zeta'}$. We identify $\overline{\mathbf{X}}_{\zeta,\zeta'}$ with $\overline{\mathbf{X}}_{\bar\xi_\sigma\zeta^{q^{-r_\sigma}},\zeta'}$ by $(z,w)\mapsto(z,w)$. Then, the action of $W_K$ gives a homomorphism*

$$\Theta_{\zeta'}:W_K\to Q\rtimes\mathbb{Z}\subset\mathrm{Aut}_k(\overline{\mathbf{X}}_{\zeta,\zeta'});\qquad \sigma\mapsto\left(g(\bar\zeta_{3,\sigma},\bar\zeta_{3,\sigma}^2\bar\nu_\sigma^2,\bar\zeta_{3,\sigma}\bar\mu_\sigma),r_\sigma\right).$$

*Proof.* This follows from Proposition 6.7 and Lemma 6.10. □

Proposition 6.13. *The homomorphism $\Theta_{\zeta'}$ factors through $W(K^{\mathrm{ur}}(\varpi^{1/3},\theta)/K)$ and gives an isomorphism $W(K^{\mathrm{ur}}(\varpi^{1/3},\theta)/K)\simeq Q\rtimes\mathbb{Z}$.*

*Proof.* By Lemma 6.10(1), the homomorphism $\Theta_{\zeta'}$ factors through $W(K^{\mathrm{ur}}(\varpi^{1/3},\theta)/K)$ and induces an injective homomorphism

$$W(K^{\mathrm{ur}}(\varpi^{1/3},\theta)/K)\to Q\rtimes\mathbb{Z}.$$

To prove the surjectivity, it suffices to show that $\Theta_{\zeta'}$ sends $I_K$ onto $Q$. Let $g=g(\alpha,\beta,\gamma)\in Q$. We take $\zeta_\alpha\in\mu_3(K^{\mathrm{ur}})$, $\nu_\beta,\mu_\gamma\in\mu_3(K^{\mathrm{ur}})\cup\{0\}$ such that $\bar\zeta_\alpha=\alpha$, $\bar\nu_\beta=\alpha^{-1}\beta$ and $\bar\mu_\gamma=\alpha^{-1}\gamma$. We put $\delta_g=\zeta_\alpha^{-1}(\delta+\nu_\beta)$ and $\theta_g=\theta+\nu_\beta^2\delta+\nu_\beta^3+\mu_\gamma$. Then, we have

$$\delta_g^4-\delta_g\equiv\frac{1}{\zeta_\alpha\zeta''\varpi^{1/3}}\quad\left(\mathrm{mod}\ \frac{5}{6}\right).$$

Hence, we can find $\delta_g'$ such that $\delta_g'^4-\delta_g'=1/(\zeta_\alpha\zeta''\varpi^{1/3})$ and $\delta_g'\equiv\delta_g$ (mod 5/6). Further, we have $\theta_g^2-\theta_g\equiv\delta_g'^3$ (mod 2/3). Hence, we can find $\theta_g'$ such that $\theta_g'^2-\theta_g'=\delta_g'^3$ and $\theta_g'\equiv\theta_g$ (mod 2/3). Then, $\varpi^{1/3}\mapsto\zeta_\alpha\varpi^{1/3}$, and $\theta\mapsto\theta_g'$ gives an element of $I_K$, whose image by $\Theta_{\zeta'}$ is $g$. □

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

## §7. Cohomology of $\mathbf{X}_1(\mathfrak{p}^3)$

In this section, we show that the covering $\mathcal{C}_1(\mathfrak{p}^3)$ is semistable, and study a structure of $\ell$-adic cohomology of $\mathbf{X}_1(\mathfrak{p}^3)$. In the following, for a projective smooth curve $X$ over $k$, we simply write $H^1(X, \overline{\mathbb{Q}}_\ell)$ for $H^1(X_{k^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell)$. For a finite abelian group $A$, the character group $\mathrm{Hom}_{\mathbb{Z}}(A, \overline{\mathbb{Q}}_\ell^\times)$ is denoted by $A^\vee$.

### 7.1 Cohomology of reductions

Let $X_{\mathrm{DL}}$ be the smooth compactification of the affine curve over $k$ defined by $X^q - X = Y^{q+1}$. The curve $X_{\mathrm{DL}}$ is also the smooth compactification of the Deligne–Lusztig curve $x^q y - xy^q = 1$ for $\mathrm{SL}_2(\mathbb{F}_q)$. Then, $a \in k$ acts on $X_{\mathrm{DL}}$ by

$$\alpha_a : (X, Y) \mapsto (X + a, Y).$$

On the other hand, $\zeta \in k_2^\times$ acts on $X_{\mathrm{DL}}$ by

$$\beta_\zeta : (X, Y) \mapsto (\zeta^{q+1} X, \zeta Y).$$

By these actions, we consider $H^1(X_{\mathrm{DL}}, \overline{\mathbb{Q}}_\ell)$ as a $\overline{\mathbb{Q}}_\ell[k \times k_2^\times]$-module.

LEMMA 7.1. *We have an isomorphism*

$$H^1(X_{\mathrm{DL}}, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\psi \in k^\vee \setminus \{1\}} \bigoplus_{\chi \in \mu_{q+1}(k_2)^\vee \setminus \{1\}} \psi \otimes \chi$$

*as* $\overline{\mathbb{Q}}_\ell[k \times \mu_{q+1}(k_2)]$*-modules.*

*Proof.* As $\overline{\mathbb{Q}}_\ell[k \times \mu_{q+1}(k_2)]$-modules, we have the short exact sequence

$$0 \to \bigoplus_{\psi \in k^\vee} \psi \to H_c^1(X_{\mathrm{DL}} \setminus X_{\mathrm{DL}}(k), \overline{\mathbb{Q}}_\ell) \to H^1(X_{\mathrm{DL}}, \overline{\mathbb{Q}}_\ell) \to 0. \tag{7.1}$$

Let $\mathcal{L}_\psi$ denote the Artin–Schreier $\overline{\mathbb{Q}}_\ell$-sheaf associated to $\psi \in k^\vee$. Let $\mathcal{K}_\chi$ denote the Kummer $\overline{\mathbb{Q}}_\ell$-sheaf associated to $\chi \in \mu_{q+1}(k_2)^\vee$. Since

$$X_{\mathrm{DL}} \setminus X_{\mathrm{DL}}(k) \to \mathbb{G}_m; \qquad (X, Y) \mapsto Y^{q+1}$$

is a finite etale Galois covering with a Galois group $k \times \mu_{q+1}(k_2)$, we have the isomorphism

$$H_c^1(X_{\mathrm{DL}} \setminus X_{\mathrm{DL}}(k), \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\psi \in k^\vee} \bigoplus_{\chi \in \mu_{q+1}(k_2)^\vee} H_c^1(\mathbb{G}_m, \mathcal{L}_\psi \otimes \mathcal{K}_\chi) \tag{7.2}$$

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

as $\overline{\mathbb{Q}}_\ell[k \times \mu_{q+1}(k_2)]$-modules. Note that we have

$$\dim H^1_c(\mathbb{G}_m, \mathcal{L}_\psi \otimes \mathcal{K}_\chi) = 1$$

if $\psi \neq 1$ by the Grothendieck–Ogg–Shafarevich formula (cf. [SGA5, Exposé X, Théorème 7.1]). Clearly, if $\chi \neq 1$, we have $H^1_c(\mathbb{G}_m, \mathcal{K}_\chi) = 0$ and $H^1_c(\mathbb{G}_m, \mathcal{L}_\psi) \simeq \psi$. Hence, we acquire the isomorphism

$$\begin{aligned} &\bigoplus_{\psi \in k^\vee} \bigoplus_{\chi \in \mu_{q+1}(k_2)^\vee} H^1_c(\mathbb{G}_m, \mathcal{L}_\psi \otimes \mathcal{K}_\chi) \\ &\simeq \bigoplus_{\psi \in k^\vee \setminus \{1\}} \bigoplus_{\chi \in \mu_{q+1}(k_2)^\vee \setminus \{1\}} H^1_c(\mathbb{G}_m, \mathcal{L}_\psi \otimes \mathcal{K}_\chi) \oplus \bigoplus_{\psi \in k^\vee} \psi \end{aligned} \tag{7.3}$$

as $\overline{\mathbb{Q}}_\ell[k \times \mu_{q+1}(k_2)]$-modules. By (7.1), (7.2) and (7.3), the required assertion follows. □

For a character $\psi \in k^\vee$ and an element $\zeta \in k^\times$, we denote by $\psi_\zeta$ the character $x \mapsto \psi(\zeta x)$. We consider a character group $(k^\times)^\vee$ as a subgroup of $(k_2^\times)^\vee$ by $\mathrm{Nr}^\vee_{k_2/k}$.

LEMMA 7.2. *We have an isomorphism*

$$H^1(X_{\mathrm{DL}}, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\tilde{\chi} \in (k_2^\times)^\vee \setminus (k^\times)^\vee} \tilde{\chi}$$

*as* $\overline{\mathbb{Q}}_\ell[k_2^\times]$*-modules.*

*Proof.* By Lemma 7.1, we take a basis

$$\{e_{\psi,\chi}\}_{\psi \in k^\vee \setminus \{1\},\, \chi \in \mu_{q+1}(k_2)^\vee \setminus \{1\}}$$

of $H^1(X_{\mathrm{DL}}, \overline{\mathbb{Q}}_\ell)$ over $\overline{\mathbb{Q}}_\ell$ such that $k \times \mu_{q+1}(k_2)$ acts on $e_{\psi,\chi}$ by $\psi \otimes \chi$. For $\zeta \in k_2^\times$ and $a \in k$, we have

$$\beta_\zeta \circ \alpha_a \circ \beta_\zeta^{-1} = \alpha_{\zeta^{q+1} a}$$

in $\mathrm{Aut}_{k_2}(X_{\mathrm{DL}})$. Hence, $\zeta \in k_2^\times$ acts on $H^1(X_{\mathrm{DL}}, \overline{\mathbb{Q}}_\ell)$ by

$$e_{\psi,\chi} \mapsto c_{\psi,\chi,\zeta} e_{\psi_{\zeta^{-(q+1)}},\chi}$$

with some constant $c_{\psi,\chi,\zeta} \in \overline{\mathbb{Q}}_\ell^\times$. Therefore, we acquire an isomorphism

$$H^1(X_{\mathrm{DL}}, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\chi \in \mu_{q+1}(k_2)^\vee \setminus \{1\}} \mathrm{Ind}_{\mu_{q+1}(k_2)}^{k_2^\times}(\chi)$$

as $\overline{\mathbb{Q}}_\ell[k_2^\times]$-modules. Hence, the required assertion follows. □

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

PROPOSITION 7.3. *We have isomorphisms*

$$H^1(\overline{\mathbf{Y}}^{\mathrm{c}}_{1,2}, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\tilde{\chi} \in (k_2^\times)^\vee \setminus (k^\times)^\vee} (\tilde{\chi} \circ \lambda) \otimes (\tilde{\chi}^q \circ \kappa_1),$$

$$H^1(\overline{\mathbf{Y}}^{\mathrm{c}}_{2,1}, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\tilde{\chi} \in (k_2^\times)^\vee \setminus (k^\times)^\vee} (\tilde{\chi} \circ \lambda) \otimes (\tilde{\chi} \circ \kappa_1)$$

*as* $(I_K \times \mathcal{O}_D^\times)$*-representations over* $\overline{\mathbb{Q}}_\ell$*.*

*Proof.* This follows from Lemmas 5.1, 6.2 and 7.2. □

Let $X_{\mathrm{AS}}$ be the smooth compactification of the affine curve $X'_{\mathrm{AS}}$ over $k$ defined by $z^q - z = w^2$. Let $a \in k$ act on $X_{\mathrm{AS}}$ by

$$\alpha_a : (z, w) \mapsto (z + a, w).$$

By this action, we consider $H^1(X_{\mathrm{AS}}, \overline{\mathbb{Q}}_\ell)$ as a $\overline{\mathbb{Q}}_\ell[k]$-module. On the other hand, let $b \in \mu_{2(q-1)}(k^{\mathrm{ac}})$ act on $X_{\mathrm{AS}}$ by

$$\beta_b : (z, w) \mapsto (b^2 z, bw).$$

LEMMA 7.4. *We assume that* $q$ *is odd. Let* $G$ *be the Galois group of the Galois extension* $F$ *over* $k((s))$ *defined by* $z^q - z = 1/s^2$*. Let* $G^r$ *be the upper numbering ramification filtration of* $G$*. Then,* $G^r = G$ *if* $r \leqslant 2$*, and* $G^r = 1$ *if* $r > 2$*.*

*Proof.* We take $a \in F$ such that $a^q - a = 1/s^2$. Then, $sa^{(q-1)/2}$ is a uniformizer of $F$. Let $v_F$ be the normalized valuation of $F$. For $\sigma \in G$ and an integer $i$, the condition

$$v_F\big(\sigma(sa^{(q-1)/2}) - sa^{(q-1)/2}\big) \geqslant i$$

is equivalent to the condition

$$v_F\big(\sigma(a) - a\big) \geqslant i - 3.$$

Hence, the claim follows. □

For a character $\psi \in k^\vee$ and $x \in k^\times$, we write $\psi_x \in k^\vee$ for the character $y \mapsto \psi(xy)$. We set

$$V = \bigoplus_{\psi \in k^\vee \setminus \{1\}} \psi$$

as $\overline{\mathbb{Q}}_\ell[k]$-modules. Let $\{e_\psi\}_{\psi \in k^\vee \setminus \{1\}}$ be the standard basis of $V$.

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

LEMMA 7.5. *We assume that $q$ is odd.*

(1) *Then, we have $H^1(X_{\mathrm{AS}}, \overline{\mathbb{Q}}_\ell) \simeq V$ as $\overline{\mathbb{Q}}_\ell[k]$-modules.*
(2) *For $b \in \mu_{2(q-1)}(k^{\mathrm{ac}})$, the automorphism $\beta_b$ of $X_{\mathrm{AS}}$ induces the action*

$$e_\psi \mapsto c_{\psi,b} e_{\psi_{b^{-2}}}$$

*on $H^1(X_{\mathrm{AS}}, \overline{\mathbb{Q}}_\ell) \simeq V$ with some constant $c_{\psi,b} \in \overline{\mathbb{Q}}_\ell^\times$. Furthermore, we have $c_{\psi,-1} = -1$.*

*Proof.* We have $H^1(X_{\mathrm{AS}}, \overline{\mathbb{Q}}_\ell) \simeq H^1_c(X'_{\mathrm{AS}}, \overline{\mathbb{Q}}_\ell)$, because the complement $X_{\mathrm{AS}} \backslash X'_{\mathrm{AS}}$ consists of one point. The curve $X'_{\mathrm{AS}}$ is a finite etale Galois covering of $\mathbb{A}^1$ with a Galois group $k$ by $(z, w) \mapsto w$. For $\psi \in k^\vee$, let $\mathcal{L}_{2,\psi}$ be the smooth $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{A}^1$ defined by the covering $X'_{\mathrm{AS}}$ and $\psi$. Then, we have

$$H^1_c(X'_{\mathrm{AS}}, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\psi \in k^\vee \backslash \{1\}} H^1_c(\mathbb{A}^1, \mathcal{L}_{2,\psi})$$

as $\overline{\mathbb{Q}}_\ell[k]$-modules. By Lemma 7.4 and the Grothendieck–Ogg–Shafarevich formula, we have

$$\dim H^1_c(\mathbb{A}^1, \mathcal{L}_{2,\psi}) = 1$$

and $H^1_c(\mathbb{A}^1, \mathcal{L}_{2,\psi}) \simeq \psi$ as $\overline{\mathbb{Q}}_\ell[k]$-modules for $\psi \in k^\vee \backslash \{1\}$. Hence, the first assertion follows.

The second assertion follows from the fact that $\beta_b \alpha_a \beta_b^{-1} = \alpha_{ab^2}$ for $a \in k$ and $b \in \mu_{2(q-1)}(k^{\mathrm{ac}})$. The assertion $c_{\psi,-1} = -1$ follows from the Lefschetz trace formula. □

We put

$$U_D = \{d \in \mathcal{O}_D^\times \mid \kappa_1(d) \in k^\times\}.$$

We take $\zeta_0 \in \mu_{2(q^2-1)}(k^{\mathrm{ac}}) \backslash k_2^\times$. Let $\Delta \in (k^\times)^\vee$ be the character defined by

$$x \mapsto x^{(q-1)/2} \in \{\pm 1\} \subset \overline{\mathbb{Q}}_\ell^\times$$

for $x \in k^\times$. If $q$ is odd, we put

$$\begin{aligned}
\tau_{\chi,\psi} &= \mathrm{Ind}_{I_L}^{I_K}\big((\chi \circ \lambda_1^{q+1}) \otimes (\psi^2 \circ \mathrm{Tr}_{k_2/k} \circ \lambda_2)\big), \\
\tau'_{\chi,\psi} &= \mathrm{Ind}_{I_L}^{I_K}\Big((\chi \circ \lambda_1^{q+1}) \otimes \big(\psi^2 \circ \mathrm{Tr}_{k_2/k} \circ (-\zeta_0^{-(q+1)} \lambda_2)\big)\Big), \\
\theta_{\chi,\psi} &= (\Delta\chi \circ \kappa_1) \otimes (\psi \circ \mathrm{Tr}_{k_2/k} \circ \kappa_2),
\end{aligned}$$

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

$$\theta'_{\chi,\psi} = (\Delta\chi \circ \kappa_1) \otimes \big(\psi \circ \mathrm{Tr}_{k_2/k} \circ (\zeta_0^{-2q}\kappa_2)\big),$$
$$\rho_{\chi,\psi} = \mathrm{Ind}_{U_D}^{\mathcal{O}_D^\times}\, \theta_{\chi,\psi},$$
$$\rho'_{\chi,\psi} = \mathrm{Ind}_{U_D}^{\mathcal{O}_D^\times}\, \theta'_{\chi,\psi}$$

for $\chi \in (k^\times)^\vee$ and $\psi \in k^\vee\backslash\{1\}$. We note that

$$\dim \rho_{\chi,\psi} = \dim \rho'_{\chi,\psi} = q+1.$$

For different $\psi, \psi' \in k^\vee\backslash\{1\}$, we can check that $\tau_{\chi,\psi} = \tau_{\chi,\psi'}$ if and only if $\psi' = \psi^{-1}$, and $\rho_{\chi,\psi} = \rho_{\chi,\psi'}$ if and only if $\psi' = \psi^{-1}$. Similar conditions hold also for $\tau'_{\chi,\psi}$ and $\rho'_{\chi,\psi}$. We define an equivalence relation $\sim$ on $k^\vee\backslash\{1\}$ by $\psi \sim \psi^{-1}$. We put

$$\Pi_{\chi,\psi} = \tau_{\chi,\psi} \otimes \rho_{\chi,\psi}, \qquad \Pi'_{\chi,\psi} = \tau'_{\chi,\psi} \otimes \rho'_{\chi,\psi}$$

for $\chi \in (k^\times)^\vee$ and $\psi \in k^\vee\backslash\{1\}$.

PROPOSITION 7.6. *We assume that $q$ is odd. Then, we have an isomorphism*

$$\bigoplus_{\zeta \in \mu_{2(q^2-1)}(k^{\mathrm{ac}})} H^1(\overline{\mathbf{X}}^{\mathrm{c}}_\zeta, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\chi \in (k^\times)^\vee} \bigoplus_{\psi \in (k^\vee\backslash\{1\})/\sim} \Pi_{\chi,\psi} \oplus \Pi'_{\chi,\psi}$$

*as representations of $I_K \times \mathcal{O}_D^\times$.*

*Proof.* The actions of $I_L$ and $U_D$ on $\bigoplus_{\zeta \in k^\times} H^1(\overline{\mathbf{X}}^{\mathrm{c}}_\zeta, \overline{\mathbb{Q}}_\ell)$ factor through $k^\times \times k$ by Proposition 5.4 and Corollary 6.5. On the other hand, the action of $k^\times \times k$ on $\bigoplus_{\zeta \in k^\times} H^1(\overline{\mathbf{X}}^{\mathrm{c}}_\zeta, \overline{\mathbb{Q}}_\ell)$ is induced from the action of $\{1\} \times k$ on $H^1(\overline{\mathbf{X}}^{\mathrm{c}}_1, \overline{\mathbb{Q}}_\ell)$. Hence, we have

$$\bigoplus_{\zeta \in k^\times} H^1(\overline{\mathbf{X}}^{\mathrm{c}}_\zeta, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\chi \in (k^\times)^\vee} \bigoplus_{\psi \in k^\vee\backslash\{1\}} \chi \otimes \psi$$

as representations of $k^\times \times k$ by Lemma 7.5.1. Therefore, we have an isomorphism

$$\bigoplus_{\zeta \in k^\times} H^1(\overline{\mathbf{X}}^{\mathrm{c}}_\zeta, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\chi \in (k^\times)^\vee} \bigoplus_{\psi \in k^\vee\backslash\{1\}} (\chi \circ \lambda_1^{q+1}) \otimes (\psi^2 \circ \mathrm{Tr}_{k_2/k} \circ \lambda_2) \otimes \theta_{\chi,\psi}$$

as representations of $I_L \times U_D$ by Proposition 5.4, Corollary 6.5 and Lemma 7.5.2. Inducing this representation from $U_D$ to $\mathcal{O}_D^\times$, we obtain an isomorphism

$$\bigoplus_{\zeta \in k_2^\times} H^1(\overline{\mathbf{X}}^{\mathrm{c}}_\zeta, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\chi \in (k^\times)^\vee} \bigoplus_{\psi \in k^\vee\backslash\{1\}} (\chi \circ \lambda_1^{q+1}) \otimes (\psi^2 \circ \mathrm{Tr}_{k_2/k} \circ \lambda_2) \otimes \rho_{\chi,\psi}$$

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

as representations of $I_L \times \mathcal{O}_D^\times$. On the left-hand side of this isomorphism, we have an action of $I_K$ that commutes with the action of $\mathcal{O}_D^\times$. Hence, we have

$$\bigoplus_{\zeta \in k_2^\times} H^1(\overline{\mathbf{X}}_\zeta^{\mathrm{c}}, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\chi \in (k^\times)^\vee} \bigoplus_{\psi \in (k^\vee \setminus \{1\})/\sim} \tau_{\chi,\psi} \otimes \rho_{\chi,\psi}$$

as representations of $I_K \times \mathcal{O}_D^\times$. By similar arguments, we have

$$\bigoplus_{\zeta \in \mu_{2(q^2-1)}(k^{\mathrm{ac}}) \setminus k_2^\times} H^1(\overline{\mathbf{X}}_\zeta^{\mathrm{c}}, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\chi \in (k^\times)^\vee} \bigoplus_{\psi \in (k^\vee \setminus \{1\})/\sim} \tau'_{\chi,\psi} \otimes \rho'_{\chi,\psi}$$

as representations of $I_K \times \mathcal{O}_D^\times$. Therefore, we have the isomorphism in the assertion. □

Let $E$ and $Q$ be as in Section 6.2.2. Let $Z \subset Q$ be the subgroup consisting of $g(1,0,\gamma)$ with $\gamma^2 + \gamma = 0$, and let $\phi$ be the unique nontrivial character of $Z$. By [BH, Lemma 22.2], there exists a unique irreducible two-dimensional representation $\tau$ of $Q$ such that

$$\tau|_Z \simeq \phi^{\oplus 2}, \qquad \operatorname{Tr} \tau(g(\alpha,0,0)) = -1 \tag{7.4}$$

for $\alpha \in \mathbb{F}_4^\times \setminus \{1\}$. Then, it is easily checked that the determinant character of $\tau$ is trivial. Note that every two-dimensional irreducible representation of $Q$ has a form $\tau \otimes \chi$ with $\chi \in (\mathbb{F}_4^\times)^\vee$, where we consider $\chi$ as a character of $Q$ by $g(\alpha,\beta,\gamma) \mapsto \chi(\alpha)$.

LEMMA 7.7. *The $Q$-representation $H^1(E, \overline{\mathbb{Q}}_\ell)$ is isomorphic to $\tau$.*

*Proof.* The $Q$-representation $H^1(E, \overline{\mathbb{Q}}_\ell)$ satisfies (7.4) by Lemma 7.1. Hence, the assertion follows. □

Let $\tau_{\zeta'}$ be the representation of $W_K$ induced from the $(Q \rtimes \mathbb{Z})$-representation $H^1(E, \overline{\mathbb{Q}}_\ell)$ by $\Theta_{\zeta'}$. Then, the restriction to $I_K$ of $\tau_{\zeta'}$ is isomorphic to the representation induced from $\tau$ by Lemma 7.7.

We say that a continuous two-dimensional irreducible representation $V$ of $W_K$ over $\overline{\mathbb{Q}}_\ell$ is primitive, if there is no pair of a quadratic extension $K'$ and a continuous character $\chi$ of $W_{K'}$ such that $V \simeq \operatorname{Ind}_{W_{K'}}^{W_K} \chi$.

LEMMA 7.8. *The representation $\tau_{\zeta'}$ is primitive of Artin conductor* 3.

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

*Proof.* We use the notations in the proof of Lemma 6.11. The element $1/(\varpi^{1/3}\theta^3)$ is a uniformizer of $K^{\mathrm{ur}}(\varpi^{1/3},\theta)$. For $\sigma\in I_K$, we can show that

$$v\left(\sigma\left(\frac{1}{\varpi^{1/3}\theta^3}\right)-\frac{1}{\varpi^{1/3}\theta^3}\right)=\begin{cases}\frac{1}{24} & \text{if } \zeta_{3,\sigma}\neq 1,\\ \frac{1}{12} & \text{if } \zeta_{3,\sigma}=1,\ \nu_\sigma\neq 0,\\ \frac{1}{6} & \text{if } \zeta_{3,\sigma}=1,\ \nu_\sigma=0,\ \mu_\sigma\neq 0,\end{cases}$$

using $\sigma(\theta)\equiv\theta_\sigma \pmod{2/3}$. The claim on the Artin conductor follows from this.

The unique index-2 subgroup of $Q\rtimes\mathbb{Z}$ is $Q\rtimes 2\mathbb{Z}$, because $Q$ has no index-2 subgroup. Hence, if $\tau_{\zeta'}$ is not primitive, it is induced from a character of $W_{K_2}$. However, this is impossible, because the restriction of $\tau_{\zeta'}$ to $W_{K_2}$ is irreducible. □

We define a character $\lambda_\xi : W_K\to k^\times$ by $\lambda_\xi(\sigma)=\bar{\xi}_\sigma$. We put

$$\begin{aligned}\tau_{\zeta',\chi}&=\tau_{\zeta'}\otimes(\chi\circ\lambda_\xi),\\ \theta_{\zeta',\chi}&=(\chi\circ\kappa_1)\otimes(\phi\circ\mathrm{Tr}_{k_2/\mathbb{F}_2}(\zeta'^{-2}\kappa_2)),\\ \rho_{\zeta',\chi}&=\mathrm{Ind}_{U_D}^{\mathcal{O}_D^\times}\theta_{\zeta',\chi},\\ \Pi_{\zeta',\chi}&=\tau_{\zeta',\chi}\otimes\rho_{\zeta',\chi}\end{aligned}$$

for $\zeta'\in k^\times$ and $\chi\in(k^\times)^\vee$. In the following, we consider $\tau_{\zeta',\chi}$ as a representation of $I_K$.

PROPOSITION 7.9. *We assume that $q$ is even. Let $\zeta'\in k^\times$. Then, we have an isomorphism*

$$\bigoplus_{\zeta\in k_2^\times} H^1(\overline{\mathbf{X}}^{\mathrm{c}}_{\zeta,\zeta'},\overline{\mathbb{Q}}_\ell)\simeq\bigoplus_{\chi\in(k^\times)^\vee}\Pi_{\zeta',\chi}$$

*as representations of $I_K\times\mathcal{O}_D^\times$.*

*Proof.* The actions of $I_K$ and $U_D$ on $\bigoplus_{\zeta\in k^\times}H^1(\overline{\mathbf{X}}^{\mathrm{c}}_{\zeta,\zeta'},\overline{\mathbb{Q}}_\ell)$ factor through $Q\times k^\times$ by Propositions 5.4 and 6.12. On the other hand, the action of $Q\times k^\times$ on $\bigoplus_{\zeta\in k^\times}H^1(\overline{\mathbf{X}}^{\mathrm{c}}_{\zeta,\zeta'},\overline{\mathbb{Q}}_\ell)$ is induced from the action of $Q$ on $H^1(\overline{\mathbf{X}}^{\mathrm{c}}_{1,\zeta'},\overline{\mathbb{Q}}_\ell)$. Hence, we have an isomorphism

$$\bigoplus_{\zeta\in k^\times} H^1(\overline{\mathbf{X}}^{\mathrm{c}}_{\zeta,\zeta'},\overline{\mathbb{Q}}_\ell)\simeq\bigoplus_{\chi\in(k^\times)^\vee}\tau\otimes\chi$$

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

as representations of $Q \times k^\times$. Therefore, we have an isomorphism

$$\bigoplus_{\zeta \in k^\times} H^1(\overline{\mathbf{X}}^{\mathrm{c}}_{\zeta,\zeta'}, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\chi \in (k^\times)^\vee} \tau_{\zeta',\chi} \otimes \theta_{\zeta',\chi}$$

as representations of $I_K \times U_D$ by Propositions 5.4 and 6.12. Inducing this representation from $U_D$ to $\mathcal{O}_D^\times$, we obtain the isomorphism in the assertion. □

### 7.2 Genus calculation

LEMMA 7.10. *We have* $\dim H^1_{\mathrm{c}}(\mathbf{X}_1(\mathfrak{p}^3)_{\mathbf{C}}, \overline{\mathbb{Q}}_\ell) = 2q^3 - 2q + 1$.

*Proof.* It suffices to show that

$$\dim H^1_{\mathrm{c}}\big((\mathrm{LT}_1(\mathfrak{p}^3)/\varpi^{\mathbb{Z}})_{\mathbf{C}}, \overline{\mathbb{Q}}_\ell\big) = 4q^3 - 4q + 2,$$

because we have

$$\dim H^1_{\mathrm{c}}\big((\mathrm{LT}_1(\mathfrak{p}^3)/\varpi^{\mathbb{Z}})_{\mathbf{C}}, \overline{\mathbb{Q}}_\ell\big) = 2 \dim H^1_{\mathrm{c}}(\mathbf{X}_1(\mathfrak{p}^3)_{\mathbf{C}}, \overline{\mathbb{Q}}_\ell).$$

For an irreducible smooth representation $\pi$ of $\mathrm{GL}_2(K)$, we write $c(\pi)$ for the conductor of $\pi$. By Proposition 2.1, we have

$$H^1_{\mathrm{c}}((\mathrm{LT}_1(\mathfrak{p}^3)/\varpi^{\mathbb{Z}})_{\mathbf{C}}, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\pi} (\pi^{K_1(\mathfrak{p}^3)})^{\oplus 2 \dim \mathrm{LJ}(\pi)} \oplus \bigoplus_{\chi} (\mathrm{St} \otimes \chi)^{K_1(\mathfrak{p}^3)},$$

where $\pi$ runs through irreducible cuspidal representations of $\mathrm{GL}_2(K)$ such that $c(\pi) \leqslant 3$ and $\omega_\pi(\varpi) = 1$, and $\chi$ runs through characters of $K^\times$ such that $c(\mathrm{St} \otimes \chi) \leqslant 3$ and $\chi(\varpi^2) = 1$. We have the following list of discrete series representations $\pi$ of $\mathrm{GL}_2(K)$ such that $c(\pi) \leqslant 3$ and $\omega_\pi(\varpi) = 1$.

(1) $\pi \simeq \mathrm{St} \otimes \chi$ for an unramified character $\chi : K^\times \to \overline{\mathbb{Q}}_\ell^\times$ such that $\chi(\varpi^2) = 1$. Then, $c(\pi) = 1$ and $\dim \mathrm{LJ}(\pi) = 1$. There are two such representations.

(2) $\pi \simeq \mathrm{St} \otimes \chi$ for a tamely ramified character $\chi : K^\times \to \overline{\mathbb{Q}}_\ell^\times$ that is not unramified and satisfies $\chi(\varpi^2) = 1$. Then, $c(\pi) = 2$ and $\dim \mathrm{LJ}(\pi) = 1$. There are $2(q-2)$ such representations.

(3) $\pi \simeq \pi_\chi$, in the notation of [BH, 19.1], for a character $\chi : K_2^\times \to \overline{\mathbb{Q}}_\ell^\times$ of level zero such that $\chi$ does not factor through $\mathrm{Nr}_{K_2/K}$ and $\chi(\varpi) = 1$. Then, $c(\pi) = 2$ and $\dim \mathrm{LJ}(\pi) = 2$. There are $q(q-1)/2$ such representations.

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

(4) The cuspidal representations $\pi$ of $\mathrm{GL}_2(K)$ such that $c(\pi)=3$ and $\omega_\pi(\varpi)=1$. Then, $\dim \mathrm{LJ}(\pi)=q+1$ by [Tu, Theorem 3.6]. There are $2(q-1)^2$ such representations by [Tu, Theorem 3.9].

We note that $\dim \pi^{K_1(\mathfrak{p}^3)}=4-c(\pi)$ if $\pi$ is a discrete series representation of $\mathrm{GL}_2(K)$ such that $c(\pi)\leqslant 3$. Then, we obtain the claim by taking a summation according to the above list. □

For an affinoid rigid space $X$, a Zariski subaffinoid of $X$ is the inverse image of a nonempty open subscheme of $\overline{X}$ under the reduction map $X\to\overline{X}$.

PROPOSITION 7.11. *Let $W$ be a wide open rigid curve over a finite extension of $\widehat{K}^{\mathrm{ur}}$ with a stable covering $\{(U_i,U_i^{\mathrm{u}})\}_{i\in I}$. Let $X$ be a subaffinoid space of $W$ such that $\overline{X}$ is a connected smooth curve with a positive genus. Then, there exists $i\in I$ such that $X$ is a Zariski subaffinoid of $U_i^{\mathrm{u}}$.*

*Proof.* Assume that $X\cap U_i^{\mathrm{u}}$ is contained in a finite union of residue classes of $X$ for any $i\in I$. Then, a Zariski subaffinoid of $X$ appears in an open annulus. This is a contradiction, because $\overline{X}$ has a positive genus. Hence, there exists $i'\in I$ such that $X\cap U_{i'}^{\mathrm{u}}$ is not contained in any finite union of residue class of $X$. We fix such $i'$ in the following.

Then, some open irreducible subscheme of the reduction of $X\cap U_{i'}^{\mathrm{u}}$ does not go to one point in $\overline{X}$ under the natural map $\overline{X\cap U_{i'}^{\mathrm{u}}}\to\overline{X}$. Let $Y$ be the inverse image of such an open subscheme under the reduction map $X\cap U_{i'}^{\mathrm{u}}\to\overline{X\cap U_{i'}^{\mathrm{u}}}$. Then, we see that $Y$ is a Zariski subaffinoid of $X$ by [CM, Lemma 2.24(i)]. Each connected component of $X\setminus Y$ is an open disk, and is included in $U_{i'}^{\mathrm{u}}$ or $U_i^{\mathrm{u}}$ for $i\neq i'$ or an open annulus outside the underlying affinoids. This can be checked by applying [CM, Corollary 2.39] to every closed disk in a connected component of $X\setminus Y$. Hence, $X\cap U_{i'}^{\mathrm{u}}$ is a Zariski subaffinoid of $X$. If $X\cap U_{i'}^{\mathrm{u}}\neq X$, then $U_{i'}^{\mathrm{u}}$ is connected to an open disk in $U_i^{\mathrm{u}}$ for $i\neq i'$ or in an open annulus outside the underlying affinoids. This is a contradiction. Therefore, we have $X\subset U_i^{\mathrm{u}}$. Then, we obtain the claim by [CM, Lemma 2.24(i)]. □

LEMMA 7.12. *Let $W$ be a wide open rigid curve over a finite extension of $\widehat{K}^{\mathrm{ur}}$ with a stable covering. Let $X$ be a subaffinoid space of $W$ such that $\overline{X}$ is a connected smooth curve with genus zero. Then, there is a basic wide open subspace of $W$ such that its underlying affinoid is $X$.*

*Proof.* We note that we have the claim if $X$ appears in an open subannulus of $W$. Let $\{(U_i,U_i^{\mathrm{u}})\}_{i\in I}$ be the stable covering of $W$.

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

First, we consider the case where $X \cap U_i^{\mathrm{u}}$ is contained in a finite union of residue classes of $X$ for any $i \in I$. Then, a Zariski subaffinoid of $X$ appears in an open annulus. Further, $X$ itself appears in the open annulus, because $X$ is connected. Hence, we have the claim in this case.

Therefore, we may assume that there exists $i' \in I$ such that $X \cap U_{i'}^{\mathrm{u}}$ is not contained in any finite union of residue class of $X$. We fix such $i'$. By the same argument as in the proof of Proposition 7.11, we have $X \subset U_i^{\mathrm{u}}$. If the image of the induced map $\overline{X} \to \overline{U}_i^{\mathrm{u}}$ is one point, we have the claim because $X$ appears in an open disk. Otherwise, $X$ is a Zariski subaffinoid of $U_i^{\mathrm{u}}$, and we have the claim. □

We consider the natural level-lowering map

$$\pi_f : \mathbf{X}_1(\mathfrak{p}^3) \to \mathbf{X}_1(\mathfrak{p}^2); \qquad (u, X_3) \mapsto (u, X_2).$$

LEMMA 7.13. *The connected components of* $\mathbf{W}_{1,2'}$, $\mathbf{W}_{1,3'}$, $\mathbf{W}_{2,1'}$ *and* $\mathbf{W}_{4,1'} \cup \mathbf{W}_{5,1'} \cup \mathbf{W}_{6,1'}$ *are not open balls.*

*Proof.* Let $\mathbf{W}_0'$ be a subannulus of $\mathbf{W}_0$ defined by $v(u) < 1/(q(q+1))$. Then, we have $\pi_f^{-1}(\mathbf{W}_{k^\times}) = \mathbf{W}_{2,1'}$, $\pi_f^{-1}(\mathbf{W}_\infty) = \mathbf{W}_{4,1'} \cup \mathbf{W}_{5,1'} \cup \mathbf{W}_{6,1'}$ and $\pi_f^{-1}(\mathbf{W}_0') = \mathbf{W}_{1,2'} \cup \mathbf{W}_{1,3'}$. Hence, we have the claim by Proposition 3.1 and [Co, Lemma 1.4]. □

The smooth projective curves $\overline{\mathbf{Y}}_{1,2}^{\mathrm{c}}$ and $\overline{\mathbf{Y}}_{2,1}^{\mathrm{c}}$ have defining equations $X^qY - XY^q = Z^{q+1}$ determined by the equation in Propositions 4.2 and 4.3. The infinity points of $\overline{\mathbf{Y}}_{1,2}$ in $\mathbb{P}_k^2$ consist of $P_a^+ = (a, 1, 0)$ for $a \in k$ and $P_\infty^+ = (1, 0, 0)$. The infinity points of $\overline{\mathbf{Y}}_{2,1}$ consist of $P_a^- = (a, 1, 0)$ for $a \in k$ and $P_\infty^- = (1, 0, 0)$.

For a wide open space $W$, let $e(W)$ be the number of ends of $W$, and let $g(W)$ be the genus of $W$ (cf. [CM, p. 369 and p. 380]). For a proper smooth curve $C$ over $k^{\mathrm{ac}}$, we write $g(C)$ for the genus of $C$.

THEOREM 7.14. *The covering* $\mathcal{C}_1(\mathfrak{p}^3)$ *is a semistable covering of* $\mathbf{X}_1(\mathfrak{p}^3)$ *over some finite extension.*

*Proof.* We consider the stable covering of $\mathbf{X}_1(\mathfrak{p}^3)_{\mathbf{C}}$ by Proposition 4.11. Then, $\overline{\mathbf{Y}}_{1,2}^{\mathrm{c}}$ and $\overline{\mathbf{Y}}_{2,1}^{\mathrm{c}}$ appear in the stable reduction of $\mathbf{X}_1(\mathfrak{p}^3)_{\mathbf{C}}$ as irreducible components by Proposition 7.11. The point $P_0^+$ is the unique infinity point of $\overline{\mathbf{Y}}_{1,2}$ whose tube is contained in $\mathbf{W}_{1,1'}^+$, because $v(X_3) > 1/(q^3(q^2-1))$ in $\mathbf{W}_{1,1'}^+$. Similarly, $P_0^-$ is the unique infinity point of $\overline{\mathbf{Y}}_{2,1}$ whose tube is contained in $\mathbf{W}_{1,1'}^-$. Hence, we have $e(\mathbf{X}_1(\mathfrak{p}^3)_{\mathbf{C}}) \geqslant 2q$ by Lemma 7.13.

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

Therefore, we have $g(\mathbf{X}_1(\mathfrak{p}^3)_{\mathbf{C}}) \leqslant q^3 - 2q + 1$ by Lemma 7.10. On the other hand, we have

$$g(\mathbf{X}_1(\mathfrak{p}^3)_{\mathbf{C}}) \geqslant g(\overline{\mathbf{Y}}^{\mathrm{c}}_{1,2}) + g(\overline{\mathbf{Y}}^{\mathrm{c}}_{2,1}) + \begin{cases} \displaystyle\sum_{\zeta \in \mu_{2(q^2-1)}(k^{\mathrm{ac}})} g(\overline{\mathbf{X}}^{\mathrm{c}}_{\zeta}) & \text{if } q \text{ is odd,} \\ \displaystyle\sum_{\zeta \in k_2^{\times},\, \zeta' \in k^{\times}} g(\overline{\mathbf{X}}^{\mathrm{c}}_{\zeta,\zeta'}) & \text{if } q \text{ is even,} \end{cases}$$

where the summation on the right-hand side is $q^3 - 2q + 1$ by Propositions 7.3, 7.6 and 7.9. Then, the affinoids $\mathbf{Y}_{1,2}$, $\mathbf{Y}_{2,1}$, $\mathbf{X}_{\zeta}$ for $\zeta \in \mu_{2(q^2-1)}(k^{\mathrm{ac}})$ and $\mathbf{X}_{\zeta,\zeta'}$ for $\zeta \in k_2^{\times}$ and $\zeta' \in k^{\times}$ are underlying affinoids of basic wide open spaces in the stable covering by Proposition 7.11 and Lemma 7.13. Therefore, by the above genus inequalities, we see that $e(\mathbf{X}_1(\mathfrak{p}^3)_{\mathbf{C}}) = 2q$, and the connected components of $\mathbf{W}_{1,2'}$, $\mathbf{W}_{1,3'}$, $\mathbf{W}_{2,1'}$ and $\mathbf{W}_{4,1'} \cup \mathbf{W}_{5,1'} \cup \mathbf{W}_{6,1'}$ are open annuli.

The connected components of $\mathbf{X}_1(\mathfrak{p}^3) \setminus \mathbf{Z}^0_{1,1}$ are two wide open spaces, because each connected component is connected to $\mathbf{Z}^0_{1,1}$ at an open subannulus by Lemma 7.12. Then, we see that these two wide open spaces are basic wide open spaces with underlying affinoids $\mathbf{Y}_{1,2}$ and $\mathbf{Y}_{2,1}$ by the above genus inequalities. Therefore, we have the claim by Propositions 4.7, 4.9 and 4.10. □

### 7.3 Structure of cohomology

In this subsection, we study the action of $I_K \times \mathcal{O}_D^{\times}$ on $\ell$-adic cohomology of $\mathbf{X}_1(\mathfrak{p}^3)$. We put

$$(W_K \times D^{\times})^0 = \{(\sigma, \varphi^{-r_\sigma}) \in W_K \times D^{\times}\}.$$

Although it is possible to study the action of $(W_K \times D^{\times})^0$ using the result of Section 6, here we study only the inertia action for simplicity. The result in this subsection is essentially used in [IT3].

Let $\mathcal{X}_1(\mathfrak{p}^3)$ be the semistable formal scheme constructed from $\mathcal{C}_1(\mathfrak{p}^3)$ by [IT2, Theorem 3.5]. The semistable reduction of $\mathcal{X}_1(\mathfrak{p}^3)$ means the underlying reduced scheme of $\mathcal{X}_1(\mathfrak{p}^3)$, which is denoted by $\mathcal{X}_1(\mathfrak{p}^3)_{k^{\mathrm{ac}}}$.

LEMMA 7.15. *The smooth projective curves* $\overline{\mathbf{Y}}^{\mathrm{c}}_{1,2}$ *and* $\overline{\mathbf{Y}}^{\mathrm{c}}_{2,1}$ *intersect with* $\overline{\mathbf{Z}}^{\mathrm{c}}_{1,1}$ *at* $P_0^+$ *and* $P_0^-$ *respectively in the stable reduction* $\mathcal{X}_1(\mathfrak{p}^3)_{k^{\mathrm{ac}}}$.

*Proof.* We see this from the proof of Theorem 7.14. □

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

Let $\Gamma$ be the graph defined by the following.

- The set of the vertexes of $\Gamma$ consists of $P_0$, $P_\infty$, $P_a^+$ and $P_a^-$ for $a \in \mathbb{P}^1(k) \setminus \{0\}$.
- The set of the edges of $\Gamma$ consists of $P_0P_a^+$, $P_0P_a^-$, $P_\infty P_a^+$ and $P_\infty P_a^-$ for $a \in \mathbb{P}^1(k) \setminus \{0\}$.

We note that $P_a^+$ and $P_a^-$ for $a \in \mathbb{P}^1(k) \setminus \{0\}$ are points of $\overline{\mathbf{Y}}^{\mathrm{c}}_{1,2}$ and $\overline{\mathbf{Y}}^{\mathrm{c}}_{2,1}$ that are not on $\overline{\mathbf{Z}}^{\mathrm{c}}_{1,1}$ by Lemma 7.15. Let $H^1(\Gamma, \overline{\mathbb{Q}}_\ell)$ be the cohomology group of $\Gamma$ with coefficients in $\overline{\mathbb{Q}}_\ell$ (cf. [IT2, Section 2]). The group $I_K \times \mathcal{O}_D^\times$ acts on $P_a^+$ and $P_a^-$ for $a \in \mathbb{P}^1(k) \setminus \{0\}$ via the action on $\overline{\mathbf{Y}}^{\mathrm{c}}_{1,2}$ and $\overline{\mathbf{Y}}^{\mathrm{c}}_{2,1}$. Let $I_K \times \mathcal{O}_D^\times$ act on $P_0$ and $P_\infty$ trivially. By this action, we consider $H^1(\Gamma, \overline{\mathbb{Q}}_\ell)$ as a $\overline{\mathbb{Q}}_\ell[I_K \times \mathcal{O}_D^\times]$-module.

THEOREM 7.16. *We have an exact sequence*

$$0 \longrightarrow H^1(\Gamma, \overline{\mathbb{Q}}_\ell) \longrightarrow H^1_{\mathrm{c}}(\mathbf{X}_1(\mathfrak{p}^3)_{\mathbf{C}}, \overline{\mathbb{Q}}_\ell) \longrightarrow H^1(\mathcal{X}_1(\mathfrak{p}^3)_{k^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell)^*(-1) \longrightarrow 0$$

*as representations of* $(W_K \times D^\times)^0$. *Further, as* $(I_K \times \mathcal{O}_D^\times)$*-representations,* $H^1(\mathcal{X}_1(\mathfrak{p}^3)_{k^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell)$ *is isomorphic to*

$$\bigoplus_{\tilde{\chi} \in (k_2^\times)^\vee \setminus (k^\times)^\vee} \Pi_{\tilde{\chi}} \oplus \begin{cases} \displaystyle\bigoplus_{\chi \in (k^\times)^\vee} \bigoplus_{\psi \in (k^\vee \setminus \{1\})/\sim} \Pi_{\chi,\psi} \oplus \Pi'_{\chi,\psi} & \text{if } q \text{ is odd,} \\ \displaystyle\bigoplus_{\zeta' \in k^\times} \bigoplus_{\chi \in (k^\times)^\vee} \Pi_{\zeta',\chi} & \text{if } q \text{ is even,} \end{cases}$$

*where we put* $\Pi_{\tilde{\chi}} = (\tilde{\chi} \circ \lambda) \otimes (\tilde{\chi} \circ \kappa_1 \oplus \tilde{\chi}^q \circ \kappa_1)$, *and* $H^1(\Gamma, \overline{\mathbb{Q}}_\ell)$ *is isomorphic to*

$$1 \oplus \bigoplus_{\chi \in (k^\times)^\vee} \left((\chi \circ \lambda^{q+1}) \otimes (\chi \circ \kappa_1^{q+1})\right)^{\oplus 2}.$$

*Proof.* The existence of the exact sequence follows from [IT2, Theorem 5.3] and Lemma 7.15 using Poincaré duality (cf. [Far1, Proposition 5.9.2]). We know the structure of $H^1(\mathcal{X}_1(\mathfrak{p}^3)_{k^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell)$ by Propositions 7.3, 7.6 and 7.9.

We study the structure of $H^1(\Gamma, \overline{\mathbb{Q}}_\ell)$. By Lemma 5.1 and Lemma 6.2, the action of $I_K \times \mathcal{O}_D^\times$ on $H^1(\Gamma, \overline{\mathbb{Q}}_\ell)$ factors through $k^\times$. We can check that

$$H^1(\Gamma, \overline{\mathbb{Q}}_\ell) \simeq 1 \oplus \bigoplus_{\chi \in (k^\times)^\vee} \chi^{\oplus 2}$$

as representations of $k^\times$. Hence, the claim follows from Lemmas 5.1 and 6.2. ∎

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press

**Acknowledgments.** The authors thank Seidai Yasuda for helpful discussion on the subject in Section 6.2.2. They are grateful to a referee for suggestions for improvements.

## References

[BH] C. J. Bushnell and G. Henniart, *The Local Langlands Conjecture for* GL(2), Grundlehren der Mathematischen Wissenschaften **335**, Springer, Berlin, 2006.

[Ca] H. Carayol, *Non-abelian Lubin–Tate theory, Automorphic forms, Shimura varieties, and L-functions, Vol. II (Ann Arbor, MI, 1988)*, Perspect. Math. **11**, Academic Press, Boston, MA, 1990, 15–39.

[Co] R. Coleman, *Stable maps of curves*, Doc. Math. (2003), 217–225; Kazuya Kato's Fiftieth Birthday.

[CM] R. Coleman and K. McMurdy, *Stable reduction of* $X_0(p^3)$, with an appendix by Everette W. Howe, Algebra Number Theory **4**(4) (2010), 357–431.

[Da] J.-F. Dat, *Théorie de Lubin–Tate non-abélienne et représentations elliptiques*, Invent. Math. **169**(1) (2007), 75–152.

[De] P. Deligne, *Formes modulaires et représentations de* GL(2), *Modular Functions of one Variable II, Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972*, Lecture Notes in Math. **349**, Springer, Berlin, 1973, 55–105.

[DR] P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques, Modular functions of one variable, II, Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972*, Lecture Notes in Math. **349**, Springer, Berlin, 1973, 143–316.

[Dr] V. G. Drinfel'd, *Elliptic modules*, Mat. Sb. (N.S.) **94**(136) (1974), 594–627.

[Fal] G. Faltings, *A relation between two moduli spaces studied by V. G. Drinfeld, Algebraic Number Theory and Algebraic Geometry*, Contemp. Math. **300**, Amer. Math. Soc., Providence, RI, 2002, 115–129.

[Far1] L. Fargues, *Cohomologie des espaces de modules de groupes p-divisibles et correspondances de Langlands locales*, Astérisque **291** (2004), 1–199.

[Far2] L. Fargues, *Dualité de Poincaré et involution de Zelevinsky dans la cohomologie équivariante des espaces rigides*, preprint.

[FGL] L. Fargues, A. Genestier and V. Lafforgue, *L'isomorphisme entre les tours de Lubin–Tate et de Drinfeld*, Progress in Mathematics **262**, Birkhäuser, Basel, 2008.

[GH] B. Gross and M. Hopkins, *Equivariant vector bundles on the Lubin–Tate moduli space, Topology and Representation Theory (Evanston, IL, 1992)*, Contemp. Math. **158**, Amer. Math. Soc., Providence, RI, 1994, 23–88.

[SGA5] A. Grothendieck, *Cohomologie l-adique et Fonctions L, Séminaire de Géométrie Algébrique du Bois-Marie 1965–1966*, Lecture Notes in Mathematics **589**, Springer, Berlin–New York, 1977.

[Ha] M. Hazewinkel, *Formal Groups and Applications*, Academic Press, New York-London, 1978.

[IT1] N. Imai and T. Tsushima, *Explicit construction of semi-stable models of Lubin–Tate curves with low level, Algebraic Number Theory and Related Topics 2012*, RIMS Kôkyûroku Bessatsu **B51**, Res. Inst. Math. Sci. (RIMS), Kyoto, 2014, 15–31.

[IT2] N. Imai and T. Tsushima, *Cohomology of rigid curves with semi-stable coverings*, Asian J. Math. **19**(4) (2015), 637–650.

[IT3] N. Imai and T. Tsushima, *Geometric realization of the local Langlands correspondence for representations of conductor three*, preprint, arXiv:1205.0734.

[IT4] N. Imai and T. Tsushima, *Affinoids in the Lubin–Tate perfectoid space and simple epipelagic representations I: tame case*, preprint, arXiv:1308.1276.
[IT5] N. Imai and T. Tsushima, *Affinoids in the Lubin–Tate perfectoid space and simple epipelagic representations II: wild case*, preprint, arXiv:1603.04693.
[IT6] N. Imai and T. Tsushima, *Affinoids in Lubin–Tate surfaces with exponential full level two*, to appear in Around Langlands Correspondences, Contemp. Math.
[Mi] Y. Mieda, *Geometric approach to the local Jacquet–Langlands correspondence*, Amer. J. Math. **136**(4) (2014), 1067–1091.
[Se] J.-P. Serre, *Représentations linéaires des groupes finis*, Hermann, Paris, 1967.
[St1] M. Strauch, *Deformation spaces of one-dimensional formal modules and their cohomology*, Adv. Math. **217**(3) (2008), 889–951.
[St2] M. Strauch, *Geometrically connected components of Lubin–Tate deformation spaces with level structures*, Pure Appl. Math. Q. **4**(4) (2008), 1215–1232; Special Issue: In honor of Jean-Pierre Serre. Part 1.
[Tu] J. B. Tunnell, *On the local Langlands conjecture for* GL(2), Invent. Math. **46**(2) (1978), 179–200.
[Weil] A. Weil, *Exercices dyadiques*, Invent. Math. **27** (1974), 1–22.
[Wein] J. Weinstein, *Semistable models for modular curves of arbitrary level*, Invent. Math. **205**(2) (2016), 459–526.

Naoki Imai
*Graduate School of Mathematical Sciences*
*The University of Tokyo*
*3-8-1 Komaba*
*Meguro-ku*
*Tokyo 153-8914*
*Japan*
naoki@ms.u-tokyo.ac.jp

Takahiro Tsushima
*Department of Mathematics and Informatics*
*Faculty of Science*
*Chiba University*
*1-33 Yayoi-cho*
*Inage*
*Chiba*
*263-8522*
*Japan*
tsushima@math.s.chiba-u.ac.jp

https://doi.org/10.1017/nmj.2016.36 Published online by Cambridge University Press