Robotica (2022), **40**, pp. 2215–2230
doi:10.1017/S0263574721001612

CAMBRIDGE
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RESEARCH ARTICLE

Dynamic logarithmic state and control quantization for continuous-time linear systems

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Received: 1 February 2021; **Revised:** 19 July 2021; **Accepted:** 12 October 2021; **First published online:** 7 January 2022

Keywords: logarithmic quantizer, saturation, feedback stabilization, linear systems, dynamic sensitivity

Abstract
This paper addresses logarithmic quantizers with dynamic sensitivity design for continuous-time linear systems with a quantized feedback control law. The dynamics of state quantization and control quantization sensitivities during "zoom-in"/"zoom-out" stages are proposed. Dwell times of the dynamic sensitivities are co-designed. It is shown that with the proposed algorithm, a single-input continuous-time linear system can be stabilized by quantized feedback control via adopting sensitivity varying algorithm under certain assumptions. Also, the advantage of logarithmic quantization is sustained while achieving stability. Simulation results are provided to verify the theoretical analysis.

1. Introduction

With the development of computer technology and network technology, networked control system (NCS) [1] has a wide range of applications in smart transportation [2], smart cities [3], the internet of things (IoT) [4–6], and industrial automation. In NCSs, as the controller usually executes in digital form, signals on continuous sets need to be represented with limited accuracy to process digital information for a limited time, which necessitates the consideration of the effect of quantization during controller design stage.

Quantization process can take different algorithms, the most common of which are linear quantizers and logarithmic quantizers [7, 8]. Quantization algorithm has considerable impact on the stability and dynamic characteristics of the system as well as on the controller design especially by incurring quantization errors. Therefore, the study around quantization algorithm plays a very important role in the research of networked control systems. Examples of the most recent research papers about state estimation and control of networked systems with quantization effect considered are refs. [9] and [10].

As early as 1956, Kalman has described the quantization effects in sampled data systems and the performance of closed-loop systems in the case of control signals being quantized. Reference [11] pointed out that if the system is not excessively unstable, a quantized feedback control law can be designed so that the closed-loop system can be controlled to any position that is close to the origin at any time. In ref. [12], authors showed that the coarsest quantizer that can quadratically stabilize a single-input discrete-time invariant system is logarithmic. Authors of ref. [12] also proposed a quantized feedback mechanism which uses a so-called "sector bound" approach. Reference [13] proposed a new design concept, in which the algorithm is based on a quantizer whose sensitivity changes as the system dynamic evolves. The benefit of such adjustment is that the sensitivity can be increased in case of saturation so that the saturation value of the finite level quantizer also increases. Furthermore, the saturation threshold is increased at a speed higher than that of the system dynamic, in order to ensure the system is less saturated as the time elapses until saturation is prevented after a certain period of time. The manipulation is based on the facts that saturation is always undesirable in control systems as strong nonlinearity is incurred and that almost all quantizers in NCSs are of finite level for implementation considerations. It is worth noticing that although literatures such as refs. [14] and [15] deal with tracking control and fault tolerant control for systems containing nonlinearities successfully, to avoid introduction of nonlinear elements such as saturation into the system is still a smart approach to tackle control problems in complex systems. In ref. [13], when the saturation no longer happens, the sensitivity can be decreased to guarantee smaller quantization error and higher accuracy. This is an important innovation to the idea that the quantitative density is fixed, which has been universally adopted before. This achievement has been widely recognized [16], and its application is reflected in literature including [17–18]. Reference [19] investigates the robust stability of uncertain discrete-time linear systems subject to input and output quantization and packet loss. The quantization algorithm therein is also static and linear.

In the existing research, most performance analysis and controller design are based on linear quantizers. However, as mentioned above, for the same input, the logarithmic quantizer has the coarsest density, which brings to the algorithm significant advantages, for example shorter data packet to be transmitted. Also, it is realized that logarithmic quantizers may experience similar difficulties for example saturation, the solution to which is to adjust its sensitivity as the system evolves. In this paper, logarithmic quantizers with dynamic sensitivities are designed, with the dynamic of the sensitivities during "zoom-in"/"zoom-out" stages provided. Dynamic logarithmic quantization algorithm has the benefit of logarithmic quantizer, that is the coarsest density to reduce the transmission burden, as well as the benefit of dynamic sensitivity quantizer, that is the sensitivity can be increased in case of saturation so that the saturation threshold of the finite level quantizer also increases. Dynamic dwell time for the sensitivities are also proposed. Meanwhile, as an extension to ref. [19], two quantizers are designed for the state signal and the control signal respectively in this paper. Compared with one quantizer scheme adopted in ref. [19], such design is more reflective of the actual NCSs scenario. For the state quantizer, the quantization sensitivity is increased during "zoom-out" stage from an initial value until the quantization threshold is not violated. Then, it is time to enter the "zoom-in" stage during which the sensitivity is decreased. The main contribution of this paper is that the rules to adjust the sensitivity of control quantization and state quantization are both provided for a couple of scenarios considering the status of state saturation and input saturation. Furthermore, dwell time of the dynamic sensitivities is co-designed such that a comprehensive solution to the problem of when and how to adjust the sensitivities is provided. It is demonstrated that with those two dynamic logarithmic quantizers installed, the linear system considered can be stabilized if conditions on the dwell time and the laws following which the sensitivities of quantizers evolve are all satisfied. Then, the problem of stabilizing the quantized state feedback control system under a given control law and associated assumptions will be solved.

Notation: For a matrix M, $\lambda_{\min}(M)$ *and* $\lambda_{\max}(M)$ *express minimum eigenvalue and maximum eigenvalue, respectively. For an* nth *order vector* $v = [\, v_1 \;\, v_2 \;\, \ldots \;\, v_n \,]$, $\|v\|$ *represents Euclidean norm of v, that is* $\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$.

The paper is organized as follows. Section 2 presents the problem formulation. Main results of this paper are shown in Section 3. Section 4 is dedicated to the simulation results, and conclusions are drawn in Section 5.

2. Problem formulation

The diagram of a typical networked control system is shown in Fig. 1. It is observed that the measurement from the plant is processed by a quantization block before transmitted through a communication channel, which is normally wireless and lossy. The controller unit receives information from the channel and generates a control signal, which is then quantized and transmitted to the actuator through a communication channel with the same behavior.

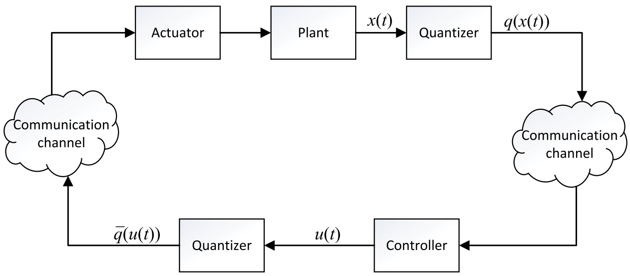

Figure 1. *Typical networked control system*

In terms of the plant model, we consider a single-input continuous-time linear system

$$\dot{x}(t) = Ax(t) + B\bar{q}\left(u(t)\right) \tag{1}$$

with a quantized state feedback control law

$$u(t) = -Kq\left(x(t)\right). \tag{2}$$

In (1), $x(t) \in R^n$ is the state vector with an arbitrary initial value $x(0)$, $q\left(x(t)\right)$ is the result of quantization of $x(t)$, $u \in R$ is the control input, $\bar{q}\left(u(t)\right)$ is the result of quantization of $u(t)$, A, B are known matrices or vectors of proper dimensions. B is assumed full rank and K is a feedback gain properly designed to ensure all eigenvalues of $A - BK$ have non-positive real parts that is according to Lyapunov stability theory, there exist positive definite symmetric matrices Q and D so that

$$(A - BK)^T Q + Q(A - BK) = -D. \tag{3}$$

A static logarithmic quantizer has the form [12]

$$V_s = \left\{\pm v_{si} : v_{si} = \frac{1}{\rho^i} v_{s0}, i = 0, 1, 2 \cdots M\right\} \cup \{0\}$$

$$q_s(a) = \begin{cases} v_{sM} & a > \dfrac{1}{1-\delta} v_{sM} \\ v_{si} & \dfrac{1}{1+\delta} v_{si} < a \leq \dfrac{1}{1-\delta} v_{si} \\ & i = 0, 1, \ldots, M \\ 0 & 0 < a \leq \dfrac{1}{1+\delta} v_{s0} \\ -q\left(-a\right) & a < 0 \end{cases} \tag{4}$$

$$\delta = \frac{1-\rho}{1+\rho}, 0 < \rho < 1, v_{s0} > 0$$

where $2M + 1$ is the order of quantizer determined by the characteristics of the communication channel, a is a scalar to be quantized, v_{s0} is an initialization parameter, V_{sM} and ρ are the range and density of the quantizer, respectively. A single parameter δ is related to ρ and also reflects the quantizer density.

Remark 1. For the state vector $x \in R^n$ in this paper, quantization is performed in an element-wise fashion with uniform quantization density for all elements.

In Fig. 2, the quantization of $x(t)$ is used as an example to demonstrate the mechanism of sensitivity varying quantization. It is assumed that the quantizer order $2M + 1$ is given. The sensitivity varying

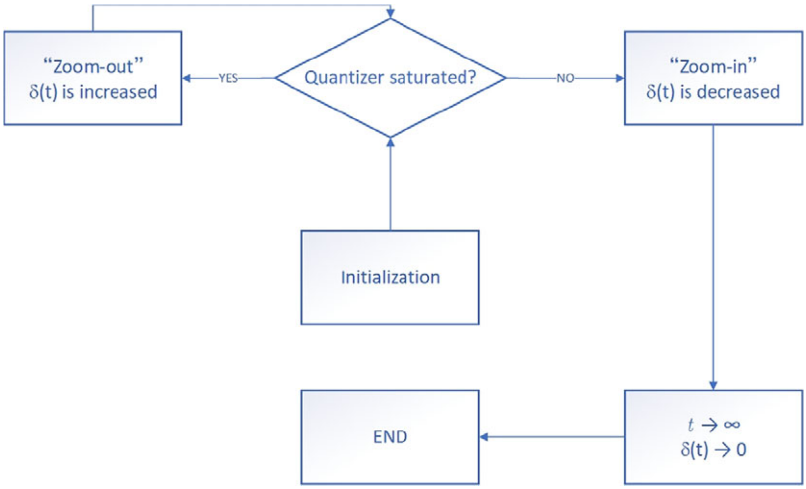

Figure 2. *Flowchart of sensitivity varying quantizing mechanism*

algorithm involves two stages that is "zoom-in" stage and "zoom-out" stage. When the initial input makes the quantizer saturated, $\delta(t)$ is increased at a speed faster than the dynamic of $x(t)$, and such process is called the "zoom-out" stage. When the system state once gets into the non-saturated area, $\delta(t)$ decreases following a certain law, which is called the "zoom-in" stage.

Therefore, with the dynamic logarithmic quantization scheme adopted in the following sections of this paper, the derivation of $q\,(x(t))$ in (2) follows the quantization algorithm as in (5), which is a significant extension from (4)

$$V(t) = \left\{ \pm v_i(t) : v_i(t) = \frac{1}{\rho^i(t)} v_0,\, i = 0, 1, \cdots M \right\} \cup \{0\},$$

$$q(x(t)) = \left[q(x_1(t))\; q(x_2(t))\; \dots\; q(x_n(t)) \right]^T,$$

$$q(x_j(t)) = \begin{cases} v_M(t) & x_j(t) > \dfrac{1}{1-\delta(t)} v_M(t) \\ v_i(t) & \dfrac{1}{1+\delta(t)} v_i(t) < x_j(t) \le \dfrac{1}{1-\delta(t)} v_i(t) \\ & i = 0, 1, \dots, M \\ 0 & 0 < x_j(t) \le \dfrac{1}{1+\delta(t)} v_0 \\ -q\left(-x_j(t)\right) & x_j(t) < 0 \end{cases} \tag{5}$$

$$\delta(t) = \frac{1-\rho(t)}{1+\rho(t)},\, 0 < \rho(t) < 1,\, v_0 > 0, \text{for all } 1 \le j \le n.$$

It is also easily derived that

$$\rho(t) = \frac{1-\delta(t)}{1+\delta(t)},\, 0 < \delta(t) < 1 \tag{6}$$

https://doi.org/10.1017/S0263574721001612 Published online by Cambridge University Press

And the derivation of $\bar{q}\,(u(t))$ in (1) follows the quantization algorithm as in (7),

$$\bar{V}(t)=\left\{\pm\bar{v}_i(t):\bar{v}_i(t)=\frac{1}{\bar{\rho}^i(t)}\bar{v}_0,\,i=0,1,2\cdots\bar{M}\right\}\cup\{0\}$$

$$\bar{q}(u(t))=\begin{cases}\bar{v}_{\bar{M}}(t) & u(t)>\dfrac{1}{1-\bar{\delta}(t)}\bar{v}_{\bar{M}}(t)\\ \bar{v}_i(t) & \dfrac{1}{1+\bar{\delta}(t)}\bar{v}_i(t)<u(t)\le\dfrac{1}{1-\bar{\delta}(t)}\bar{v}_i(t)\\ & i=0,1,...,\bar{M}\\ 0 & 0<u(t)\le\dfrac{1}{1+\bar{\delta}(t)}\bar{v}_0\\ -q\,(-u(t)) & u(t)<0\end{cases}\tag{7}$$

$$\bar{\delta}(t)=\frac{1-\bar{\rho}(t)}{1+\bar{\rho}(t)},\,0<\bar{\rho}(t)<1,\,\bar{v}_0>0$$

In (5) and (7), $2M+1$ and $2\bar{M}+1$ are the orders of quantizers, v_0 and $\bar{v}_0$ are initialization parameters, $V(t)$, $\bar{V}(t)$ and $\rho(t)$, $\bar{\rho}(t)$ are the range and density of quantizers respectively. $\delta(t)$, $\bar{\delta}(t)$ are associated with $\rho(t)$, $\bar{\rho}(t)$ respectively and reflect the quantizer densities.

3. Main results

For the considered continuous-time linear system with quantized state feedback controller (2), the closed-loop behavior is as follows

$$\begin{aligned}\dot{x}(t)&=Ax(t)+B\bar{q}\,(u(t))\\&=Ax(t)+B\,(-Kq(x(t))+\bar{e}(t))\\&=(A-BK)x(t)-BKe(t)+B\bar{e}(t)\end{aligned}\tag{8}$$

where $e(t)=q\,(x(t))-x(t)$, $\bar{e}(t)=\bar{q}\,(u(t))-u(t)$.

Let $x_{ia}(t)=\frac{1}{1+\delta(t)}v_i(t)$, $x_{ib}(t)=\frac{1}{1-\delta(t)}v_i(t)$, $i=1,2,\ldots,M$. Then, it is observed that $x_{Mb}(t)=\frac{1}{1-\delta(t)}v_M(t)$ with $v_M(t)=\frac{1}{\rho^M(t)}v_0$ is the saturation threshold of the finite level state quantizer. It can be also obtained:

$$x_{Mb}(t)=\frac{1}{\rho^M(t)\,(1-\delta(t))}v_0=\frac{(1+\delta(t))^M}{(1-\delta(t))^{M+1}}v_0\tag{9}$$

considering (6).

It can be derived that if the state quantizer is not saturated and $|x_i(t)|>\frac{1}{1+\delta(t)}v_0$,

$$e_i(t)=\delta_i(t)x_i(t),\text{ with }\,|\delta_i(t)|\le\delta(t),\forall i=1,2,...,n.$$

Also,

$$e(t)=\Delta(t)x(t)\text{ with }\Delta(t)=diag\{\delta_1(t),\delta_2(t),...,\delta_n(t)\}.$$

Then,

$$|e_i(t)|\le\delta(t)\,|x_i(t)|\,,\,\|e(\,\cdot\,)\|^2\le\delta^2(\,\cdot\,)\sum_{i=1}^{n}x_i^2\,(\cdot)=\delta^2(\,\cdot\,)\,\|x\,(\cdot)\|^2$$

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Therefore, $\|e(\cdot)\| \le \delta(\cdot)\|x(\cdot)\|$. In a similar way,

$$u_{\bar{M}b}(t) = \frac{(1+\bar{\delta}(t))^{\bar{M}+1}}{(1-\bar{\delta}(t))^{\bar{M}}}\bar{v}_0, \text{ with } \|\bar{e}(\cdot)\| \le \bar{\delta}(\cdot)\|u(\cdot)\| \tag{10}$$

in case no saturation happens to $\bar{q}(u(t))$.

Consider the scenario where the state quantizer is not saturated, and suppose $u(t)$ also falls into the non-saturation area of corresponding quantizer with an executable quantization result, then reminding ourselves of (3) and (8) gives us $\forall t_i < t \le t_{i+1}$

$$\begin{aligned}\frac{d}{dt}x^T(t)Qx(t) &= x^T(t)\,(A-BK)^T\,Qx(t) + \left(-e^T(t)\right)K^TB^TQx(t)\\ &\quad + \bar{e}^T(t)B^TQx(t) + x^T(t)Q\,(A-BK)\,x(t) + x^T(t)BK\,(-e(t)) + x^T(t)QB\bar{e}(t)\\ &\le -\|x(t)\|^2 D + 2\delta(t)\,\|QBK\|\,\|x(t)\|^2 + 2\bar{\delta}(t)\,\|QB\|\,\|x(t)\|\,\|K\|\,(\|x(t)\|\\ &\quad +\delta(t)\,\|x(t)\|)\\ &\le \|x(t)\|^2\,(-\lambda_{\min}(D) + 2\delta(t)\,\|QBK\| + 2\bar{\delta}(t)\,(1+\delta(t))\,\|QB\|\,\|K\|\,)\end{aligned} \tag{11}$$

where t_i is time sequence at which $\delta(t)$ and $\bar{\delta}(t)$ are renewed, t_0 is the time instant when the system gets into the non-saturated area, τ_i is the sequence of dwell time of $\delta(t)$ and $\bar{\delta}(t)$, that is $\tau_i = t_{i+1} - t_i$.

It can be seen from (11) that in order to achieve Lyapunov stability, $\delta(t)$ and $\bar{\delta}(t)$ have to be chosen small enough such that

$$2\delta(t)\,\|QBK\| + 2\bar{\delta}(t)(1+\delta(t))\,\|QB\|\,\|K\| < \lambda_{\min}(D) \tag{12}$$

For all t, let

$$2\delta(t)\,\|QBK\| + 2\bar{\delta}(t)\,(1+\delta(t))\,\|QB\|\,\|K\| = \sigma_{ns}(t)\lambda_{\min}(D) \tag{13}$$

then obviously one has to have $0 < \sigma_{ns}(t) < 1$.

However, it should be noticed that due to characteristics of the actuation component, an executable $\bar{q}\,(u(t))$ cannot be arbitrarily large. That means, the quantizer for $u(t)$ is saturated, if $|-Kq(x(t))| \ge u_{\max}$ with $u_{\max} > 0$ being the threshold enforced on $u(t)$ due to physical or practical limitations. Set $\bar{q}(u(t)) = \bar{q}(-Kq\,(x(t))\,) = sgn(-Kq(x(t)))u_{\max} = -\gamma(t)Kq\,(x(t))$ with $0 < \gamma(t) = \frac{u_{\max}}{|-Kq(x(t))|} < 1$. Then although the state quantizer is not saturated, instead of (8) the corresponding closed-loop behavior is as follows

$$\begin{aligned}\dot{x}(t) &= Ax(t) - \gamma(t)BKq\,(x(t))\\ &= (A-BK)\,x(t) + (1-\gamma(t))\,BKx(t) - \gamma(t)BKe(t)\end{aligned} \tag{14}$$

Assumption 1. As the ratio between $u_{\max}$ and $|-Kq(x(t))|$, $\gamma(t)$ is calculated every time, $|-Kq(x(t))|$ is updated. It is assumed in this paper that $\gamma(t)$ satisfies $\gamma(t) = \frac{u_{\max}}{|-Kq(x(t))|} > 1 - \frac{\lambda_{\min}(D)}{2\|QBD\|}\forall t$.

As a result, the differentiation of $x^T(t)Qx(t)$ is as follows

$$\begin{aligned}\frac{d}{dt}x^T(t)Qx(t) &= x^T(t)\,(A-BK)^T\,Qx(t) + (1-\gamma(t))\,x^T(t)K^TB^TQx(t)\\ &\quad -\gamma(t)e^T(t)K^TB^TQx(t) + x^T(t)Q\,(A-BK)\,x(t)\\ &\quad +(1-\gamma(t))\,x^T(t)QBKx(t) - \gamma(t)x^T(t)QBKe(t)\\ &\le -D\,\|x(t)\|^2 + 2\,(1-\gamma(t))\,\|QBK\|\,\|x(t)\|^2 + 2\gamma(t)\delta(t)\,\|QBK\|\,\|x(t)\|^2\\ &= \|x(t)\|^2\,(-\lambda_{\min}(D) + (2\,(1-\gamma(t)) + 2\gamma(t)\delta(t))\,\|QBK\|\,)\end{aligned} \tag{15}$$

It can be seen from (15) that in order to achieve Lyapunov stability, $\delta(t)$ has to chosen small enough such that

$$1-\gamma(t)+\gamma(t)\delta(t)<\frac{\lambda_{\min}(D)}{2\|QBK\|} \tag{16}$$

For all t, let

$$\sigma_s(t)=\frac{2\|QBK\|(1-\gamma(t)+\gamma(t)\delta(t))}{\lambda_{\min}(D)}, \tag{17}$$

then obviously one has to have $0<\sigma_s(t)<1$. Also, when $|\bar{q}(u(t))|\geq u_{\max}$, we leave $\bar{\delta}(t)$ static until the quantization of $u(t)$ is not saturated. This will happen in finite time as $x(t)$ enters "zoom-in" stage later on and $|-Kx(t)|$ shrinks accordingly.

Remark 2. It is given in Assumption 1 that $\gamma(t)>1-\frac{\lambda_{\min}(D)}{2\|QBD\|}\forall t$. This condition means in the system considered in this paper, the size of the quantized control signal $\bar{q}(u(t))$ should never exceed the threshold of the actuator $u_{\max}$ too much, that is $\frac{|-Kq(x)|}{u_{\max}}=\frac{1}{\gamma(t)}<\frac{1}{1-\frac{\lambda_{\min}(D)}{2\|QBD\|}}$ should hold for all t. This condition also guarantees that a $\delta(t)>0$ to satisfy (16) exists for all t. Furthermore, it is well understood that a system with input saturation and unstable state matrix can only be stabilized within a certain region. A limitation enforced on the extent to which the calculated quantized input signal may intrude the threshold of the actuator to ensure the system stays in such stabilization region is consistent with the common understanding.

Theorem 3.1. *For a continuous-time linear system as given in (1), assume K is a feedback gain ensuring the existence of positive definite symmetric matrices Q and D as in (3), and initialization parameter v_0 is chosen arbitrarily small. Then there exists a control policy as described in (2), such that the solutions of (1) with dynamic logarithmic quantizers for x(t) and u(t) as illustrated in (5) and (7) approach an arbitrarily small neighborhood of the origin as $t\to\infty$. Meanwhile, the evolution of quantizer sensitivities should follow the principles as below:*

1. *Case 1: If $\|x(t)\|>x_{Mb}(t)\sqrt{\frac{\lambda_{\min}(Q)}{\lambda_{\max}(Q)}}$, the state quantizer is in the "zoom-out" stage. During this stage, if the quantizer of $u(t)$ is saturated for some t, increase $\delta(t)$ fast enough to dominate the rate of growth of $x(t)$, that is $e^{m\|A-\varphi(t)(I_{n\times n}+\Delta(t))BK\|t}$, where $m>1$. If the quantization of $u(t)$ is not saturated at time t, increase $\delta(t)$ fast enough to dominate the rate of growth of $x(t)$, that is $e^{m\left\|A-\left(1+\bar{\delta}'(t)\right)(I_{n\times n}+\Delta(t))BK\right\|t}$, where $m>1$.*
2. *Case 2: Once $\|x(t)\|\leq x_{Mb}(t)\sqrt{\frac{\lambda_{\min}(Q)}{\lambda_{\max}(Q)}}$, the state quantizer is no longer saturated and comes into the "zoom-in" stage. The sensitivities of the quantizers, satisfy $\delta(t)=\delta(t_i)$, $\bar{\delta}(t)=\bar{\delta}(t_i)$, $\forall t_i\leq t<t_{i+1}$, with $t_0=0$, $t_{i+1}=t_i+\tau_i$, $0<\tau_i<\frac{\lambda_{\min}(Q)}{\eta(t_i)(1-\sigma(t_i))\lambda_{\min}(D)}$ where $\eta(t_i)=\frac{x^T(t_i)x(t_i)}{x_{Mb}^2(t_i)}$, $i\in Z^+\cup\{0\}$. The evolution of $\delta(t_i)$ is governed by a nonlinear function*

$$\delta(t_{i+1})=\mathcal{F}(\delta(t_i))=\frac{(1-h(t_i))+(1+h(t_i))\delta(t_i)}{(1+h(t_i))+(1-h(t_i))\delta(t_i)} \tag{18}$$

where $h(t_i)$ is the solution of $f(h(t_i))=2p(t_i)h^{M+1}(t_i)+(\delta(t_i)-1)h(t_i)-(\delta(t_i)+1)=0$, where

$$p(t_i)=\sqrt{\frac{\lambda_{\min}(Q)-\tau_i\eta(t_i)(1-\sigma(t_i))\lambda_{\min}(D)}{\lambda_{\min}(Q)}}$$

with Q,D as given in (3).

The evolution of $\bar{\delta}(t_i)$ is governed by the evolution of $u_{\bar{M}b}(t)$ and can be solved for according to (10) and (7). $u_{\bar{M}b}(t)$ can be easily calculated as $u_{\bar{M}b}(t)=u_{\bar{M}b}(t_i)$, $\forall t_i\leq t<t_{i+1}$, with

$$u_{\bar{M}b}(t_i)=\sum_{j=1}^{n}|k_j|\,x_{Mb}(t_i) \tag{19}$$

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Proof. Firstly we introduce the "zoom-out" stage of $x(t)$. During this stage, $\|x(t)\| > x_{Mb}(t)\sqrt{\frac{\lambda_{\min}(P_j)}{\lambda_{\max}(P_j)}}$, which corresponds to Case 1 in Theorem 1.

The quantizer of $u(t)$ is saturated if $|-Kq(x(t))| \geq u_{\max}$. Let $|-\varphi(t)Kq(x(t))| = u_{\max}$, $0 < \varphi(t) < 1$.

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu_{\max} sgn(-Kq(x(t)))\\ &= Ax(t) + B\,|-\varphi(t)Kq(x(t))|\, sgn(-Kq(x(t)))\\ &= Ax(t) + \varphi(t)B(-Kq(x(t)))\\ &= (A - \varphi(t)(I_{n\times n} + \Delta(t))BK)\, x(t)\end{aligned}$$

Define

$$\phi(t) = e^{\|A-\varphi(t)(I_{n\times n}+\Delta(t))BK\|t}.$$

It is worth noticing that as the ratio between $u_{\max}$ and $|-Kq(x(t))|$, $\varphi(t)$ is calculated every time $|-Kq(x(t))|$ is updated.

So during the "zoom-out" stage of $x(t)$, if $|-Kq(x(t))| \geq u_{\max}$ for some t, the quantization of $x(t)$ has to "zoom-out" faster than the rate of growth of $x(t)$ in order to catch the system state in finite time. That means in such cases, it is required that $\delta(t)$ is increased at a speed fast enough to dominate the rate of growth of $e^{m\|A-\varphi(t)(I_{n\times n}+\Delta(t))BK\|t}$, where $m > 1$. Obviously, the greater m is, the earlier x_{Mb} catches up with $x(t)$. But an extremely large m may deteriorate the system performance by incurring much too fast dynamic.

If the quantization of $u(t)$ is not saturated or some t during the "zoom-out" stage of the state quantizer, one has $|-Kq(x(t))| < u_{\max}$. Let $\bar{e}(t) = \bar{\delta}'(t)u(t) = -\bar{\delta}'(t)Kq(x(t))$, $\left|\bar{\delta}'(t)\right| < \bar{\delta}(t)$.

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B\bar{q}(-Kq(x(t)))\\ &= Ax(t) - BKq(x(t)) + B\bar{e}\,(u(t))\\ &= Ax(t) - BKq(x(t)) - \bar{\delta}'(t)BKq(x(t))\\ &= \left(A - \left(1 + \bar{\delta}'(t)\right)(I_{n\times n} + \Delta(t))BK\right) x(t)\end{aligned}$$

Define

$$\phi(t) = e^{\left\|A-\left(1+\bar{\delta}'(t)\right)(I_{n\times n}+\Delta(t))BK\right\|t}.$$

So during the "zoom-out" stage of $x(t)$, if $|-Kq(x(t))| \leq u_{\max}$ for some t, the quantization of $x(t)$ has to "zoom-out" faster than the rate of growth of $x(t)$ in order to catch the system state in finite time. That means in such cases, it is required that $\delta(t)$ is increased at a speed fast enough to dominate the rate of growth of $e^{m\left\|A-\left(1+\bar{\delta}'(t)\right)(I_{n\times n}+\Delta(t))BK\right\|t}$, where $m > 1$. The greater m is, the earlier x_{Mb} catches up with $x(t)$. But an extremely large m may deteriorate the system performance by incurring much too fast dynamic.

Then there exists a positive integer i_0 such that

$$\left\|x(t_{i_0})\right\| \leq x_{Mb}(t_{i_0})\sqrt{\frac{\lambda_{\min}(Q)}{\lambda_{\max}(Q)}} \tag{20}$$

Define

$$i_0 := \min\left\{r \geq 0\colon \|x(t_r)\| \leq x_{Mb}(t_r)\sqrt{\frac{\lambda_{\min}(Q)}{\lambda_{\max}(Q)}},\right\}$$

Then from (20), one has

$$\begin{aligned}x^T(t_{i_0})x(t_{i_0}) &\leq x_{Mb}^2(t_{i_0})\frac{\lambda_{\min}(Q)}{\lambda_{\max}(Q)}\\ x^T(t_{i_0})x(t_{i_0})\lambda_{\max}(Q) &\leq x_{Mb}^2(t_{i_0})\lambda_{\min}(Q).\end{aligned}$$

Considering the fact that $x^T(t_{i_0})Qx(t_{i_0}) \leq x^T(t_{i_0})x(t_{i_0})\lambda_{\max}(Q)$, one has

$$x^T(t_{i_0})Qx(t_{i_0}) \leq x_{Mb}^2(t_{i_0})\lambda_{\min}(Q)$$

Remark 3. It is not hard to see that with a higher rate of growth of $\delta(t)$, the rate of growth of $x_{Mb}(t)$ is also higher and i_0 becomes smaller.

Remark 4. Also, although the rule of tuning $\delta(t)$ continuously during "zoom-out" stage is provided in Theorem 1, the designer can still choose to update $\delta(t)$ in continuous time or only at certain discrete instants based on the application scenario. But the evolution of $\delta(t)$ should always satisfy the dominance rule, that is $\delta(t)$ increases with a rate dominating that of $x(t)$.

Next, we come to the "zoom-in" stage, which corresponds to Case 2 in Theorem 1. Considering (8) or (14) with (12) we have

$$\begin{aligned} x^T(t_{i_0}+1)Qx(t_{i_0}+1) &= x^T(t_{i_0})Qx(t_{i_0}) + \int_{t_{i_0}}^{t_{i_0}+\tau_{i_0}} \frac{d}{dt}x^T(t)Qx(t)dt \\ &\leq x^T(t_{i_0})Qx(t_{i_0}) + \int_{t_{i_0}}^{t_{i_0}+\tau_{i_0}} -(1-\sigma(t_{i_0}))\lambda_{\min}(D)x^T(t_{i_0})x(t_{i_0})dt \\ &\leq x_{Mb}^2(t_{i_0})\lambda_{\min}(Q) - \tau_{i_0}(1-\sigma(t_{i_0}))\lambda_{\min}(D)x^T(t_{i_0})x(t_{i_0}) \end{aligned} \tag{21}$$

where $\sigma(t_{i_0})$ is chosen either as $\sigma_{ns}(t_{i_0})$ in (13) or $\sigma_s(t_{i_0})$ in (17) depending on whether the actuation unit is saturated. As $x^T(t_i)x(t_i) = \eta(t_i)x_{Mb}^2(t_i)$ with $0 \leq \eta(t_i) < 1, \forall i$, we have

$$x^T(t_{i_0}+1)Qx(t_{i_0}+1) \leq x_{Mb}^2(t_{i_0})(\lambda_{\min}(Q) - \tau_{i_0}\eta(t_{i_0})(1-\sigma(t_{i_0}))\lambda_{\min}(D))$$

It is worth noticing that $\eta(t_i)$ is the ratio between $x^T(t_i)x(t_i)$ and x_{Mb}^2. Its value should be calculated every time $x(t_i)$ is updated. Then remind ourselves of (21) to derive

$$x^T(t_{i0+1})Qx(t_{i0+1}) \leq x_{Mb}^2(t_{i0+1})\lambda_{\min}(Q).$$

Remark 5. Based on Theorem 1, τ_i can be chosen as any value as long as $0 < \tau_i < \frac{\lambda_{\min}(Q)}{\eta(t_i)(1-\sigma(t_i))\lambda_{\min}(D)}$ is satisfied. But in real applications, trials may be carried out such that a τ_i which leads to better system performance is chosen.

By repeating the procedure above, we have $x^T(t_i)Qx(t_i) \leq x_{Mb}^2(t_i)\lambda_{\min}(Q)$. This also guarantees that the system will not go back to "zoom-out" stage.

Due to (19), we have

$$\begin{aligned} \left\|u\left(t_{i_0}\right)\right\| &\leq \left\|k_1 q\left(x_1\left(t_{i_0}\right)\right)\right\| + \left\|k_2 q\left(x_2\left(t_{i_0}\right)\right)\right\| + \cdots + \left\|k_n q\left(x_n\left(t_{i_0}\right)\right)\right\| \\ &\leq \|k_1\|\, x_{Mb}\left(t_{i_0}\right) + \|k_2\|\, x_{Mb}\left(t_{i_0}\right) + \cdots + \|k_n\|\, x_{Mb}\left(t_{i_0}\right) \\ &= u_{\bar{M}b}\left(t_{i_0}\right) \end{aligned}$$

Similarly, we have $\|u\,(t_i)\| \leq u_{\bar{M}b}\,(t_i)\,, \forall i \geq i_0$.

It is noticed that $f(0) = -\,(\delta(t_i)+1) < 0$ and $f(1) = 2\,(p(t_i)-1) < 0$. As $f(h(k))$ is continuously differentiable, the differential of $f(h(t_i))$ is shown as

$$f'(h(t_i)) = 2(M+1)p(t_i)h^M(t_i) + (\delta(t_i)-1)\,.$$

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Let $f'(\bar{h}(t_i)) = 0$, we have

$$\bar{h}(t_i) = \left| \sqrt[M]{\frac{1-\delta(t_i)}{2(M+1)p(t_i)}} \right|.$$

$$\begin{aligned} f(\bar{h}(t_i)) &= \frac{1-\delta(t_i)}{M+1} \left| \sqrt[M]{\frac{1-\delta(t_i)}{2(M+1)p(t_i)}} \right| - (1-\delta(t_i)) \left| \sqrt[M]{\frac{1-\delta(t_i)}{2(M+1)p(t_i)}} \right| - (\delta(t_i)+1) \\ &= -\frac{M}{M+1} (1-\delta(t_i)) \left| \sqrt[M]{\frac{1-\delta(t_i)}{2(M+1)p(t_i)}} \right| - (\delta(t_i)+1) \\ &< 0 \end{aligned}$$

Then, we have the following properties of $f(h(t_i))$:

1. $f(h(t_i))$ is monotonically decreasing between 0 and $\bar{h}(t_i)$ and monotonically increasing between $\bar{h}(t_i)$ and ∞.
2. $f(\infty) > 0$.
3. If $\bar{h}(t_i) > 1, f(h(t_i))$ has single root between $\bar{h}(t_i)$ and ∞. If $\bar{h}(t_i) \le 1, f(h(t_i))$ has single root between 1 and ∞.
4. Define the root of $f(h(t_i))$ as $h^*(t_i)$, such that $f(h^*(t_i)) = 0$ and $h^*(t_i) > 1$, which implies $\delta(t_i + 1) < \delta(t_i)$.
5. As i increases, $\delta(t_i)$ and $h(t_i)$ decrease, it can then be derived that $\lim_{i\to\infty} h(t_i) = 1$ and $\lim_{i\to\infty} \delta(t_i) = 0$.

As a result of (5), $x_{Mb}(t_i) = \frac{(1+\delta(t_i))^M}{(1-\delta(t_i))^{M+1}} v_0 \to v_0$ as $i \to \infty$ and

$$\begin{aligned} \lim_{i\to\infty} x^T(t_i)Qx(t_i) &\le \lim_{i\to\infty} x^2_{Mb}(t_i)\lambda_{\min}(Q) \\ &= v_0^2 \lambda_{\min}(Q) \end{aligned} \tag{22}$$

Obviously, (22) leads to $\|x(t_i)\| \le v_0$ as $i \to \infty$, which further guarantees $\|x(t)\| \le v_0$ as $t \to \infty$. Then, stability of system is guaranteed as v_0 is finite and can be chosen arbitrarily small. □

4. Numerical results

Example 1. Consider a third-order continuous-time linear system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.2 & 0.5 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} u(t)$$

with a quantized feedback control law designed as $K = \begin{bmatrix} 0.0024 & 0.0120 & 0.0800 \end{bmatrix}$. As a result, $eig(A - BK) = \begin{bmatrix} -0.1 & -0.2 & -0.3 \end{bmatrix}$.

Without loss of generality, set $M = \bar{M} = 10$, $v_0 = 0.1$, $\bar{v}_0 = 0.005$, $u_{\max} = 0.05$ for x_1, x_2, x_3. $x(0) = \begin{bmatrix} 1.6 & 0.9 & 1.8 \end{bmatrix}^T$.

The quantization of x_1, x_3 encounters saturation at $t = 0$, then during the "zoom-out" stage, choose $m = 3$, which results in $i_{01} = 13$ and $i_{03} = 1$. $\delta(t_{i0})$ and $\bar{\delta}(t_{i0})$ are both chosen as 0.13 which satisfy Eq. (12) or (16). Then during the "zoom-in" stage, it takes the states 17 time intervals to enter the

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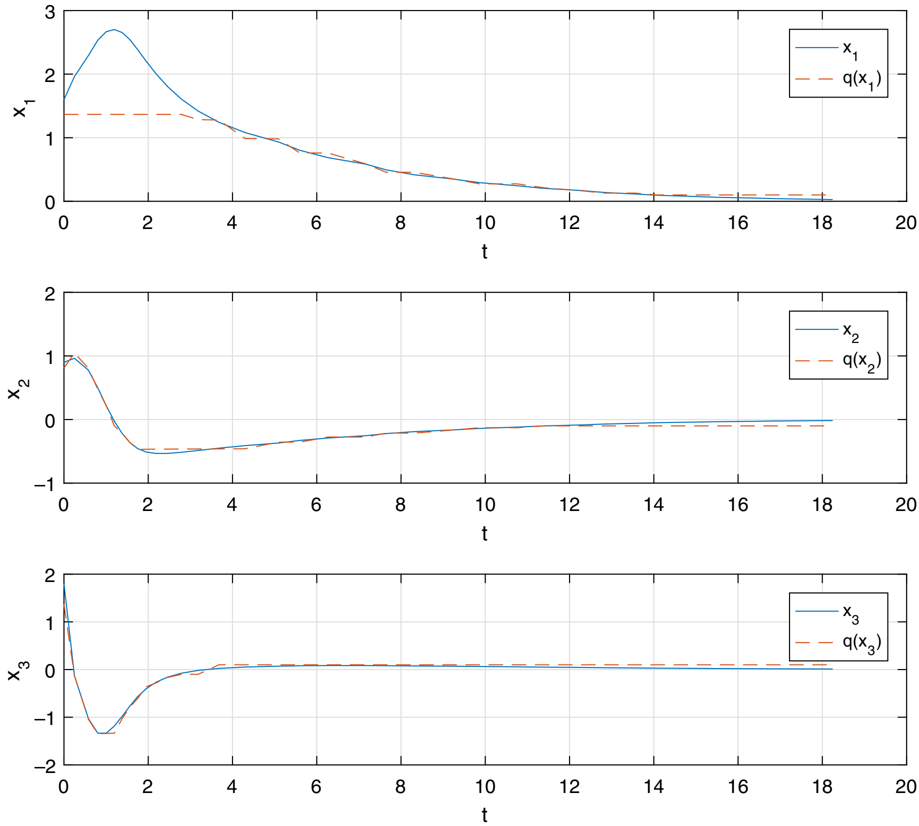

Figure 3. *Third-order system states using dynamic logarithmic quantizer*

neighborhood of the origin, that is $\|x(t)\| \leq v_0 = 0.1$. And, it is observed the states stay within the neighborhood ever after, that is $\|x(t)\| \leq v_0, \forall t \geq t_{30}$.

The state variables x_1, x_2, x_3 are shown as Figs. 3 and 4 gives a 3D view of system states. Figure 5 illustrates how $\|x(t)\|$ converges into a neighborhood of the origin the width of which is v_0.

The control signals u and $\bar{q}(u)$ are shown in Fig. 6. We could see that the control signal goes through four stages, that is saturated, unsaturated, saturated, then enters unsaturated stage and remains in this stage ever after.

It is observed that under quantized feedback control with proposed dynamic logarithmic quantizer, the system state converge to a spherical neighborhood of the origin whose radius is v_0. Figures 7 and 8 show the system states and control signal of the same system with linear state quantization algorithm as in ref. [13]. Compared with the dynamic logarithmic quantization algorithm proposed in this paper, in order to maintain the same saturation value and initial value of sensitivity, the order of dynamic linear quantizer must satisfies $M' \geq 13$ which is larger than the order of dynamic logarithmic quantizer $M = 10$. It is also noticed that the algorithm in ref. [13] did not consider the quantization of the control signal, which is equivalent to assuming the quantization of the control signal as ideal. However, with the algorithm proposed in this paper, quantizers are applied to both state and control signals which is closer to situation in the actual NCSs. Although more quantization error is inevitably brought into the system, the overall system performance is not deteriorated in comparison to that in ref. [13]. This is the benefit brought by suitable controller-quantizer co-design, which is another advantage of the proposed algorithm over that in ref. [13].

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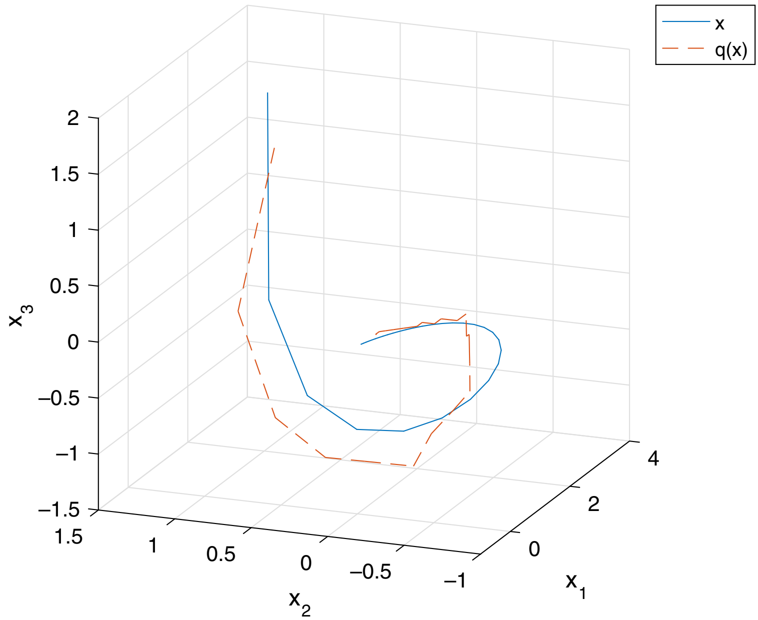

Figure 4. *3D view of system states using dynamic logarithmic quantizer*

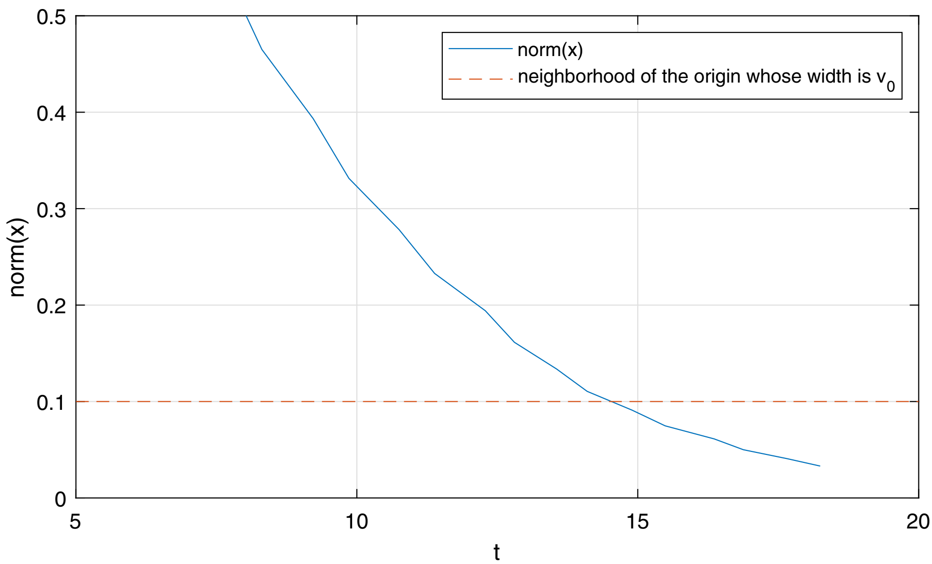

Figure 5. *Convergence of system state into a neighborhood of the origin*

Example 2. Consider the uninterruptible power system in ref. [20]

$$\dot{x}(t) = \begin{bmatrix} 0.9226 & 0.6330 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0.5 \\ 0 \\ 0.2 \end{bmatrix} u(t)$$

and choose a quantized feedback controller gain $K = \begin{bmatrix} 10.0150 & 24.2677 & 9.5755 \end{bmatrix}$. As a result, $eig(A - BK) = \begin{bmatrix} -1 & -2 & -3 \end{bmatrix}$. Set $M = \bar{M} = 10$, $v_0 = 0.1$, $\bar{v}_0 = 0.01$, $u_{\max} = 50$ for x_1, x_2, x_3. $x(0) = \begin{bmatrix} -0.7 & 0.6 & 0.3 \end{bmatrix}^T$.

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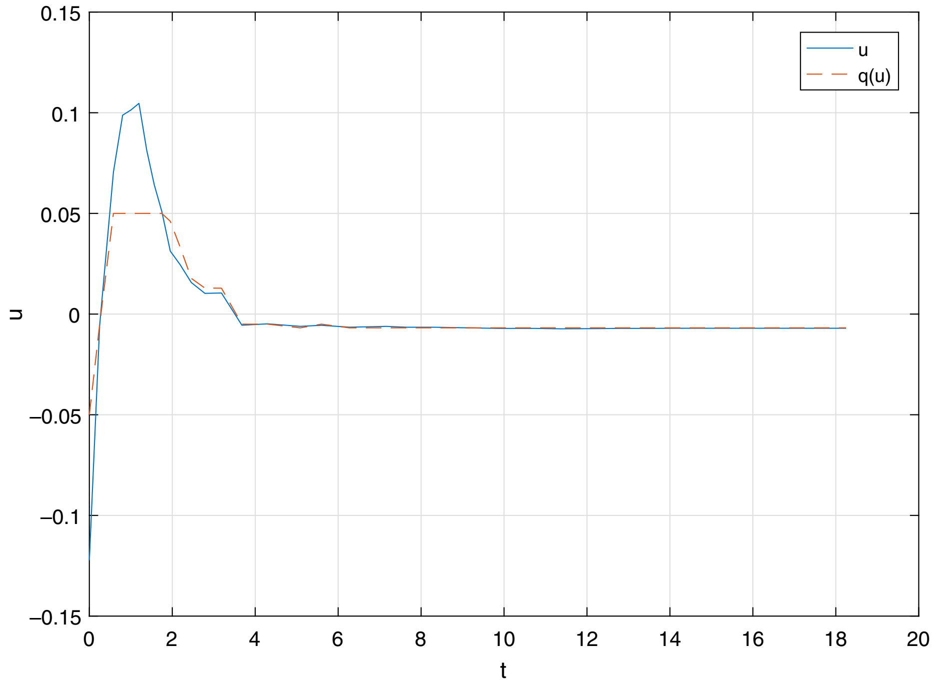

Figure 6. *Control signal using dynamic logarithmic quantizer*

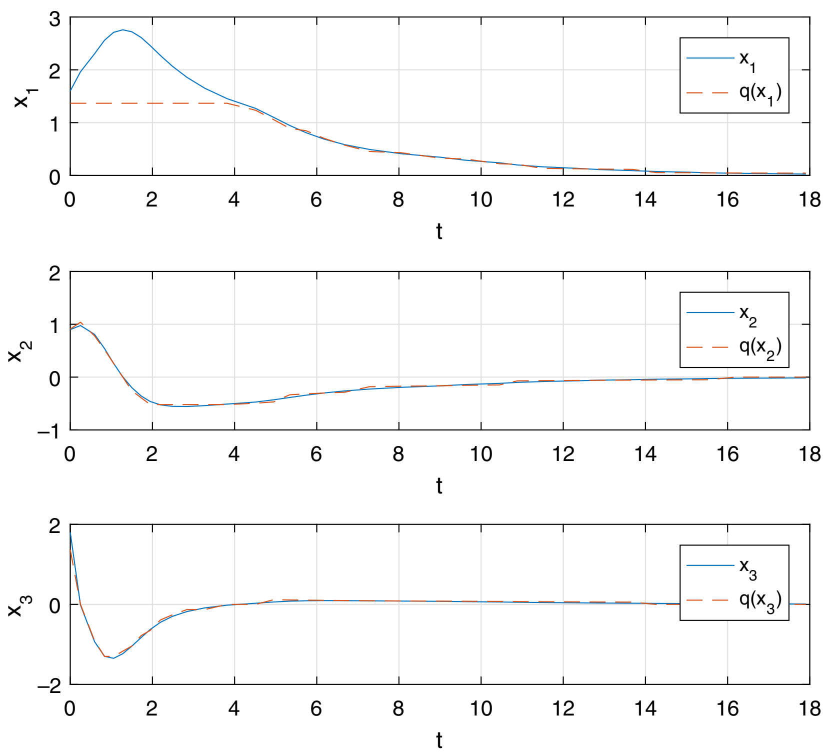

Figure 7. *System states using dynamic linear quantizer*

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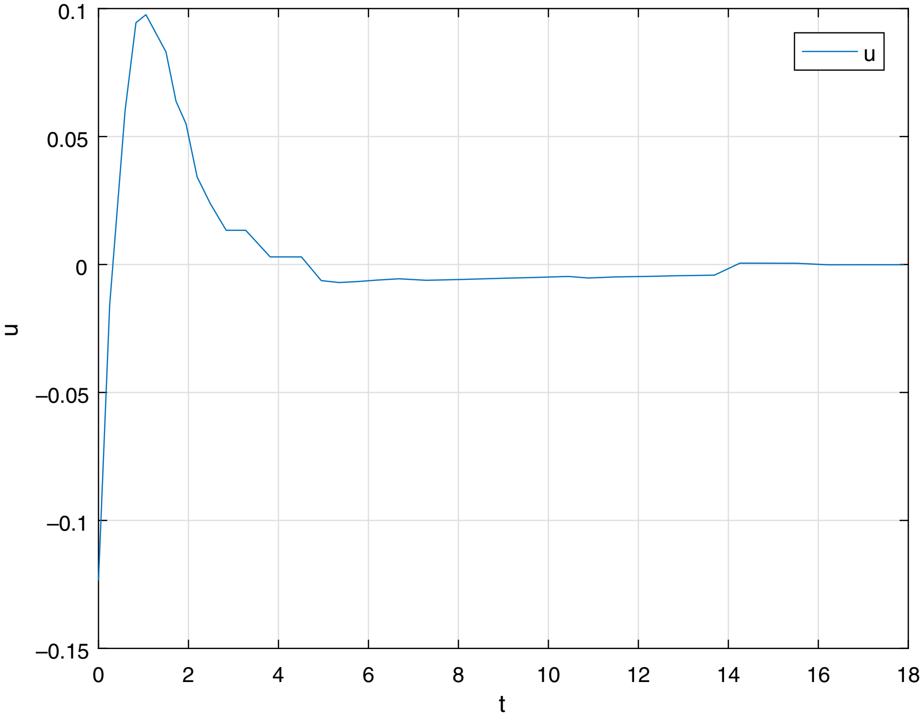

Figure 8. *Control signal using dynamic linear quantizer*

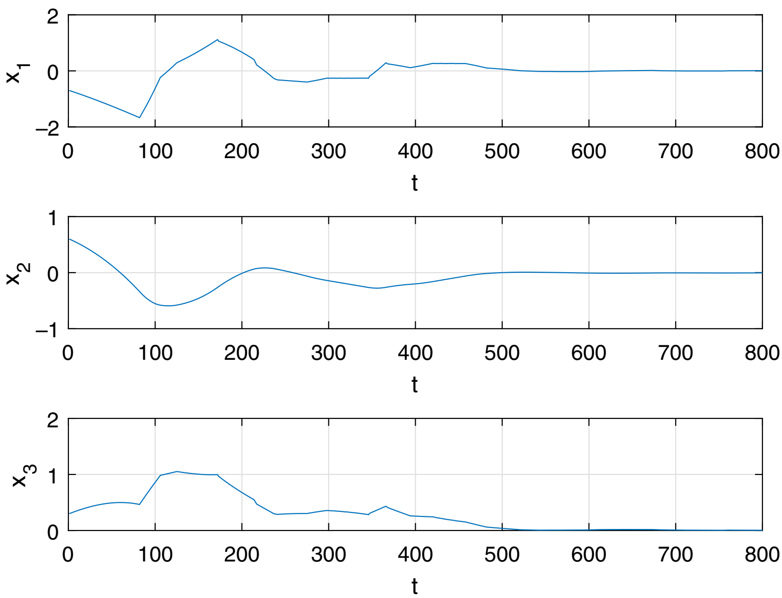

Figure 9. *System states of the uninterruptible power system*

During the "zoom-out" stage, choose $m = 1.5$, $\delta\,(t_{i0}) = 0.1$ and $\bar{\delta}\,(t_{i0}) = 0.1$. Then during the "zoom-in" stage, it takes the states 484 time intervals to enter the neighborhood of the origin, that is $\|x(t)\| \leq v_0 = 0.1$. And it is observed the states stay within the neighborhood ever after, that is $\|x(t)\| \leq v_0, \forall t \geq t_{484}$.

The state variables x_1, x_2, x_3 are shown as Figs. 9 and 10 gives a 3D view of system states.

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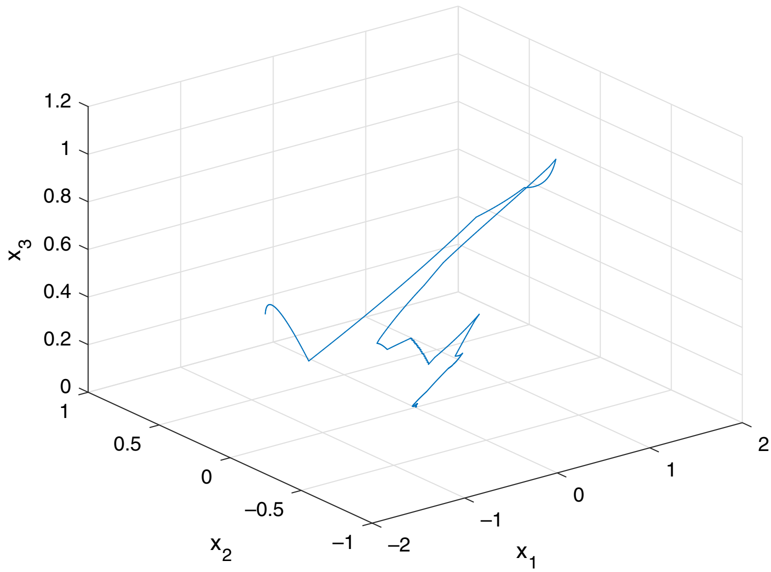

Figure 10. *3D view of system states*

5. Conclusion

In this paper, finite level quantization is performed to both the plant state and the control signal in continuous linear systems. In order to cope with the saturation phenomenon and guarantee coarseness of quantization, logarithmic quantizers with dynamic sensitivities are designed and dynamics of the sensitivities during "zoom-in" /"zoom-out" stages is proposed. With the proposed algorithm, closed-loop system trajectory approaches an arbitrarily small neighborhood of the origin as $t \to \infty$ with a properly designed quantized feedback control law. Numerical simulation results are given in the end. As the current results is restricted to single-input systems, the stability of multiple input continuous-time linear system with dynamic logarithmic quantizers will be looked into as the future research plan.

Acknowledgement. This work was supported by the National Natural and Science Foundation of China under Grant 61950410616.

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A. Appendix

As $h(t_i)$ is the solution of $f(h(t_i)) = 2p(t_i)h^{M+1}(t_i) + (\delta(t_i) - 1)\, h(t_i) - (\delta(t_i) + 1) = 0$, it is derived that

$$\frac{2h^{M+1}(t_i)}{(1 + h(t_i)) + (1 - h(t_i))\, \delta(t_i)} p(t_i) = 1 \tag{A1}$$

For $t = t_{i_0}$, rewrite (18) as

$$h(t_{i_0}) = \frac{1 + \delta(t_{i_0})}{1 + \delta(t_{i_0} + 1)} \cdot \frac{1 - \delta(t_{i_0} + 1)}{1 - \delta(t_{i_0})}. \tag{A2}$$

Combining (A2) with (A1) gives us

$$\left(\frac{1 + \delta(t_{i_0})}{1 + \delta(t_{i_0+1})}\right)^M \left(\frac{1 - \delta(t_{i_0+1})}{1 - \delta(t_{i_0})}\right)^{M+1} p(t_{i_0}) = 1$$

$$\frac{(1 + \delta(t_{i_0}))^M}{(1 - \delta(t_{i_0}))^{M+1}} p(t_{i_0}) = \frac{(1 + \delta(t_{i_0+1}))^M}{(1 - \delta(t_{i_0+1}))^{M+1}} \tag{A3}$$

As

$$0 < \tau_{i_0} < \frac{\lambda_{\min}(Q)}{\eta(t_{i_0})(1 - \sigma(t_{i_0}))\lambda_{\min}(D)},$$

one has

$$0 < p(t_{i_0}) = \sqrt{\frac{\lambda_{\min}(Q) - \tau_{i_0}\eta(t_{i_0})(1 - \sigma(t_{i_0}))\lambda_{\min}(D)}{\lambda_{\min}(Q)}} < 1 \tag{A4}$$

Combining (A3) with (A4) gives us

$$\frac{(1 + \delta(t_{i_0}))^M}{(1 - \delta(t_{i_0}))^{M+1}} \sqrt{\frac{\lambda_{\min}(Q) - \tau_{i_0}\eta(t_{i_0})(1 - \sigma(t_{i_0}))\lambda_{\min}(D)}{\lambda_{\min}(Q)}} = \frac{(1 + \delta(t_{i_0+1}))^M}{(1 - \delta(t_{i_0+1}))^{M+1}}$$

$$\left(\frac{(1 + \delta(t_{i_0}))^M}{(1 - \delta(t_{i_0}))^{M+1}}\right)^2 v_0^2[\lambda_{\min}(Q) - \tau_{i_0}\eta(t_{i_0})(1 - \sigma(t_{i_0}))\lambda_{\min}(D)] = \left(\frac{(1 + \delta(t_{i_0+1}))^M}{(1 - \delta(t_{i_0+1}))^{M+1}}\right)^2 v_0^2\lambda_{\min}(Q) \tag{A5}$$

Considering (9), (A5) implies

$$x_{Mb}^2(t_{i_0})(\lambda_{\min}(Q) - \tau_{i_0}\eta(t_{i_0})(1 - \sigma(t_{i_0}))\lambda_{\min}(D)) = x_{Mb}^2(t_{i_0+1})\lambda_{\min}(Q).$$

Cite this article: J. Wang, S. Pan and Z. Xi (2022). "Dynamic logarithmic state and control quantization for continuous-time linear systems", Robotica **40**, 2215–2230. https://doi.org/10.1017/S0263574721001612