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Formation behaviour of the kinetic Cucker–Smale model with non-compact support

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In this paper, we focus on the formation behaviour of the kinetic Cucker–Smale model for initial datum without compact support for the position variable. Comparing with the case of compact support, the attractive force between particles is weak. First, we obtain the existence and uniqueness of the classical solution to the kinetic Cucker–Smale model by standard approximation method. Second, by using the characteristic flow, we overcome the difficulty brought by the weak attractive force between particles through some estimates and establish the formation behaviour, i.e., consensus of velocity, of the classical solution to the kinetic Cucker–Smale model. Finally, for the measure-valued solution to the kinetic Cucker–Smale model, the formation behaviour is also established.

Keywords: Cucker–Smale model; formation behaviour; non-compact support; characteristic flow

2020 *Mathematics Subject Classification:* 35B40; 92D25

1. Introduction

Collective behaviour in many body systems is ubiquitous in real life, which can be interpreted as flocking, swarming, aggregation. The Cucker–Smale model is one of the well-known models to describe the emergent behaviour in flocks, introduced by Cucker and Smale in [8, 9]. The Cucker–Smale model of N particles is given by:

$$\begin{cases} \dfrac{\mathrm{d}x_i}{\mathrm{d}t} = v_i, \quad i = 1, 2, \dots, N, \\ \dfrac{\mathrm{d}v_i}{\mathrm{d}t} = \displaystyle\sum_{j=1}^{N} m_j H(|x_j - x_i|)(v_j - v_i), \end{cases} \tag{1.1}$$

where the communication rate H is

$$H(s) = \frac{1}{(1+s^2)^{\beta}}, \quad \beta \geqslant 0. \tag{1.2}$$

Here $(x_i(t), v_i(t)) \in \mathbb{R}^d \times \mathbb{R}^d$ represent the position and velocity of the ith particle, $1 \leqslant i \leqslant N$. This model has been extensively studied in the literature, from different aspects, for example, collision avoiding [1, 7], flocking with hierarchical leadership

[17, 24], rooted leadership [18, 19], bi-cluster flocking [11], discrete form flocking [21, 22] and so on. In order to describe the dynamics of large number of particles, Ha and Tadmor derived the following kinetic Cucker–Smale model in [13]:

$$\partial_t f + v\cdot \nabla_x f + \mathrm{div}_v(L[f]f) = 0, \quad f(0,x,v) = f_0(x,v), \tag{1.3}$$

where $L[f]$ is the alignment force given by

$$L[f](t,x,v) = -\int_{\mathbb{R}^{2d}} \frac{v-\omega}{(1+|x-y|^2)^\beta} f(t,y,\omega)\,\mathrm{d}y\,\mathrm{d}\omega, \quad \beta \geqslant 0. \tag{1.4}$$

The unknown function $f(t,\,x,\,v) \geqslant 0$ denotes a microscopic density of particle at time $t \geqslant 0$ and position $x \in \mathbb{R}^d$, moving with velocity $v \in \mathbb{R}^d$. Existence, uniqueness and stability results for model (1.3)–(1.4) have also been studied in [2–5, 10, 12, 15]. Besides, Karper, Mellet and Trivisa added a confinement potential to establish the global existence of the weak solutions in [14] for initial data in $L^1 \cap L^\infty$ with compact support in velocity variable. Moreover, the global existence of measure-valued solutions for (1.3)–(1.4) with a weak singular communication weight was established in [20].

The flocking behaviour of the kinetic Cucker–Smale model is proved in [4] when the initial datum f_0 is compactly supported in x and v, that is, the support in velocity shrinks towards its mean velocity exponentially fast while the support in position is bounded around the position of the centre of mass. More precisely, there are some positive constants C and α depending on $\mathrm{supp} f_0$ and β such that

$$\int_{\mathbb{R}^{2d}} |v-v_c|^2 f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \leqslant Ce^{-\alpha t}, \quad v_c = \|f_0\|_{L^1}^{-1}\int_{\mathbb{R}^{2d}} v f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v, \tag{1.5}$$

when $\beta \in [0,\,1/2]$. Besides, if the initial datum has compact velocity-position support, i.e., there exists a positive constant λ such that

$$\sup\left\{\left|v-\frac{x}{\lambda}\right| : (x,v)\in \mathrm{supp} f_0\right\} < \infty,$$

Chen and Yin established a new type of collective behaviour in [6] when $\beta > 1$, that is, some velocity-position moments decay:

$$\int_{\mathbb{R}^{2d}} \left|v - \frac{x}{t+\lambda}\right|^k f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \to 0, \quad \forall\ k \geqslant 2.$$

In the above results, they all require that the initial datum has a compact support of position in some sense. The target in this paper is to find some collective behaviours under more general condition for the support of position. We first observe that even if the support of position is unbounded while the velocity support is concentrate on one single point, the system (1.3)–(1.4) will still reach the equilibrium state (see example 1.2), which is very different from the previous results. Therefore, the following question is natural:

Question: Assume that the initial datum f_0 is not compactly supported in x, are there any other collective behaviours to the system (1.3)–(1.4)?

In this paper, we study the above question when $\beta \in [0,\,1/2]$ and show that the velocity support of the solution to (1.3)–(1.4) will be asymptotically concentrated on its mean velocity. We will call the above behaviour as formation behaviour defined as below:

DEFINITION 1.1. *The kinetic Cucker–Smale model* (1.3)–(1.4) *has a formation behaviour when the classical solution f to* (1.3)–(1.4) *satisfies the following result:*

$$\lim_{t\to\infty}\int_{\mathbb{R}^{2d}}|v-v_c|^2 f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v=0,\quad v_c=\|f_0\|_{L^1}^{-1}\int_{\mathbb{R}^{2d}} v f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v.$$

The formation behaviour is a relaxed concept from the original flocking behaviour, which only concerns the aggregation of velocities and do not care about the evolution of position support. Therefore, asymptotic formation behaviour even allows an unbounded position support, which is compatible with the non-compact framework of this paper. Next, we provide an example to illustrate the difference between the flocking behaviour and the formation behaviour. To this end, we introduce the following infinite-particle Cucker–Smale model:

$$\begin{cases} \dfrac{\mathrm{d}x_i}{\mathrm{d}t}=v_i, \\ \dfrac{\mathrm{d}v_i}{\mathrm{d}t}=\displaystyle\sum_{j=1}^{\infty} m_j H(|x_j-x_i|)(v_j-v_i), \quad t>0, \\ x_i(0)=x_{i0}, \quad v_i(0)=v_{i0}, \end{cases} \tag{1.6}$$

with total mass $M>0$,

$$\sum_{j=1}^{\infty} m_j = M, \quad m_j>0. \tag{1.7}$$

This model was first proposed in [**26**]. The measure curve given by

$$f(x,v,t)=\sum_{i=1}^{\infty} m_i\delta(x-x_i(t))\delta(v-v_i(t))$$

is a weak measure-valued solution to (1.3)–(1.4) for $\beta\leqslant\frac{1}{2}$ and initial data $\{x_{i0},\,v_{i0}\}_{i\in\mathbb{N}}$ belonging to $l^2_m(\mathbb{R}^d)\times l^\infty(\mathbb{R}^d)$, where $l^2_m(\mathbb{R}^d)$ is defined as:

$$l^2_m(\mathbb{R}^d)=\left\{x=(x_1,x_2,\ldots)\,\middle|\,\left(\sum_{i=1}^{\infty} m_i|x_i|^2\right)^{\frac{1}{2}}<\infty\right\}.$$

EXAMPLE 1.2. Suppose that the dimension $d=2$. We set the initial data as follows:

$$m_i=\frac{1}{(i)^4},\quad x_{i0}=(i,i),\quad v_{i0}=(1,1),\quad i=1,2,\ldots$$

All particles will move with the same velocity, but the position support of the corresponding measure-valued solution is always unbounded. This is a formation behaviour, but not a flocking behaviour.

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The main results of this paper are threefold. First, in subsection 2.1, we establish the existence and uniqueness of the classical solution to (1.3)–(1.4) without compact support by the standard approximation method; see theorem 2.5. The method to prove this theorem is described below. We first summarize the main results about the kinetic Cucker–Smale model for compactly supported initial data. Then, by using the characteristic flow, we show the non-expansion of velocity support (see lemma 2.4). Besides, we construct a sequence of approximate solutions and demonstrate the compactness of these approximate solutions. Moreover, we pass through the limit to obtain a global unique classical solution to (1.3)–(1.4) without compact support.

Second, in subsection 2.2, we obtain the formation behaviour of the classical solution to (1.3)–(1.4) by contradiction; see theorem 2.7. This theorem provides a rigorous proof of the emergence of asymptotic formation behaviour. By employing the boundedness of velocity support, we first provide some important differential equations (see lemma 2.6). Then, we establish the Grönwall's inequality on some positive time interval, which helps us to construct a decreasing sequence. By splitting the integral of the initial datum, we demonstrate that the velocity moment of order 2 of the solution after translation tends to zero asymptotically, i.e., the formation behaviour of the classical solution to (1.3)–(1.4).

Finally, in § 3, the formation behaviour of the measure-valued solution to (1.3)–(1.4) is presented. We first recall some related knowledge about the measure-valued solutions. Then, we show the stability of the classical solution to (1.3)–(1.4) under $p-$Wasserstein distance. Moreover, we regularize the initial datum to obtain a sequence of approximate solutions. By showing that the sequence of the approximate solutions is a Cauchy sequence, we get a measure-valued solution to (1.3)–(1.4). The proof of the formation behaviour of the measure-valued solution to (1.3)–(1.4) is similar to theorem 2.7.

The paper is organized as follows. In § 2, we establish the uniqueness, existence and formation behaviour of the classical solution to (1.3)–(1.4). In § 3, the corresponding results similar to the classical solution are obtained on the measure-valued solutions to the kinetic Cucker–Smale model. Finally, § 4 is devoted to the summary of our main results.

Notation.

$$\begin{aligned}
&C_b\left(\mathbb{R}^{2d}\right)=\left\{f\,\middle|\,f\in C\left(\mathbb{R}^{2d}\right)\ \text{ and bounded}\right\},\\
&C_c\left(\mathbb{R}^{2d}\right)=\left\{f\,\middle|\,f\in C\left(\mathbb{R}^{2d}\right)\ \text{ with compact support}\right\},\\
&C_0\left(\mathbb{R}^{2d}\right)=\left\{f\,\middle|\,f\in C\left(\mathbb{R}^{2d}\right)\ \text{ and vanishing at infinity}\right\},\\
&C_0^1\left(\mathbb{R}^{2d}\right)=\left\{f\,\middle|\,f\in C_0\left(\mathbb{R}^{2d}\right)\ \text{ and }\ \partial_{x^i}f,\ \partial_{v^i}f\in C_0\left(\mathbb{R}^{2d}\right),\ i=1,2,\ldots,d\right\},\\
&C_b^1\left(\mathbb{R}^{2d}\right)=\left\{f\,\middle|\,f\in C_b\left(\mathbb{R}^{2d}\right)\ \text{ and }\ \partial_{x^i}f,\ \partial_{v^i}f\in C_b\left(\mathbb{R}^{2d}\right),\ i=1,2,\ldots,d\right\}.
\end{aligned}$$

2. Classical solution

In this section, we establish the uniqueness, existence and formation behaviour of the classical solution to (1.3)–(1.4).

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2.1. Existence and uniqueness

To obtain the solution to (1.3)–(1.4) without compact support, we construct a sequence of solutions with compact support to approximate it. To this end, let us summarize the main findings of the kinetic Cucker–Smale model for compact support in the following lemma.

LEMMA 2.1 [**4**, **13**]. *Let $f_0 \in C_c^1(\mathbb{R}^{2d})$ be the nonnegative initial datum. Then, there exists a unique global classical solution $f \geqslant 0$ to (1.3)–(1.4) satisfying the following properties*:

(1) *The total mass and its average velocity are conserved*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}} f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = 0, \tag{2.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}} vf(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = 0. \tag{2.2}$$

(2) *The kinetic flocking behaviour*: *when $\beta \in [0,\, 1/2]$, there exist some positive C and α depending only on* $\mathrm{supp} f_0$ *and β such that*

$$\int_{\mathbb{R}^{2d}} |v-v_c|^2 f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \leqslant Ce^{-\alpha t}, \quad v_c = \|f_0\|_{L^1}^{-1}\int_{\mathbb{R}^{2d}} vf_0\,\mathrm{d}x\,\mathrm{d}v. \tag{2.3}$$

(3) *According to the fact that $L[f]$ is continuous in t and Lipschitz continuous in $(x,\, v)$, the corresponding characteristic flow $(X(t,\, 0,\, x,\, v),\, V(t,\, 0,\, x,\, v))$ associated to*

$$\begin{cases} \dot{X}(t,0,x,v) = V(t,0,x,v), \quad X(0,0,x,v) = x, \\ \dot{V}(t,0,x,v) = L[f](t,X(t,0,x,v),V(t,0,x,v)), \quad V(0,0,x,v) = v \end{cases}$$

is a well-defined homeomorphism for each fixed time t and a C^1-function of time t. Besides, the solution f to (1.3)–(1.4) is given by $f(t,\, x,\, v) = (X(t,\, 0,\, x,\, v),\, V(t,\, 0,\, x,\, v))\#f_0$ in the mass transportation notation, i.e., for all $\phi \in C_b^1(\mathbb{R}^{2d})$:

$$\int_{\mathbb{R}^{2d}} \phi(x,v)f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^{2d}} \phi(X(t,0,x,v),V(t,0,x,v))f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v.$$

REMARK 2.2.

(1) As mentioned in [**4**], if $f \in L^1([0,\, T) \times \mathbb{R}^{2d})$ is a classical solution to (1.3)–(1.4), then $\mu(t,\, x,\, v) = f(t,\, x,\, v)\,\mathrm{d}x\,\mathrm{d}v$ is a measure-valued solution to (1.3)–(1.4), where the definition of the measure-valued solution to (1.3)–(1.4) will be stated later in definition 3.1.

(2) The support of a continuous function f is the closure of the set $\{x : f(x) \neq 0\}$, and the support of a Borel measure μ on $\mathbb{R}^{2d}$ is the closure of the set $\{(x,\, v) \in \mathbb{R}^{2d} : \mu(B_r(x,\, v)) > 0,\, \forall\, r > 0\}$.

(3) Let μ be a Borel measure on $\mathbb{R}^n$, and let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a measurable map. The push-forward measure of μ by T is the measure $T\#\mu$ defined by $T\#\mu(O) = \mu(T^{-1}(O))$, for all Borel set $O \subset \mathbb{R}^n$.

Next, we show the non-expansion of velocity support of the classical solution to (1.3)–(1.4), which is important in the estimation for approximate solutions. We first recall a fundamental lemma below, which is obtained in [**23**].

LEMMA 2.3. *Let $g:[0, T] \to \mathbb{R}$ be a continuous function, we define:*

$$D^- g(t) = \overline{\lim_{h\to 0^+}} \frac{g(t)-g(t-h)}{h}.$$

If for any $t \in [0, T]$, we have $D^- g(t) \leqslant 0$, then $g(t)$ is non-increasing in $[0, T]$.

Then, we provide the following lemma to prepare for the existence and uniqueness of the classical solution to (1.3)–(1.4).

LEMMA 2.4. *Assume that f is a classical solution stated in lemma 2.1. Then f satisfies*

$$\sup_{(x,v)\in \mathrm{supp} f(t)} |v| \leqslant \sup_{(x,v)\in \mathrm{supp} f_0} |v| \quad \textit{for any} \quad t \geqslant 0. \tag{2.4}$$

Proof. First, the characteristic flow is well-defined:

$$\begin{cases} \dot{X}(t,0,x,v) = V(t,0,x,v), \quad X(0,0,x,v) = x, \\ \dot{V}(t,0,x,v) = L[f](t, X(t,0,x,v), V(t,0,x,v)), \quad V(0,0,x,v) = v. \end{cases} \tag{2.5}$$

Following from (1.3)–(1.4), we get that

$$\partial_t f + v\cdot \nabla_x f + L[f]\cdot \nabla_v f = -f \mathrm{div}_v L[f] = df \int_{\mathbb{R}^{2d}} \frac{f(t,y,\omega)}{(1+|x-y|^2)^\beta} \,\mathrm{d}y\,\mathrm{d}\omega.$$

Thus, we have

$$\begin{aligned} &f(t, X(t,0,x,v), V(t,0,x,v)) \\ &\quad = f_0(x,v)\cdot \exp\left\{ d\int_0^t \int_{\mathbb{R}^{2d}} \frac{f(s,y,\omega)}{(1+|X(s,0,x,v)-y|^2)^\beta} \,\mathrm{d}y\,\mathrm{d}\omega\,\mathrm{d}s \right\}, \end{aligned} \tag{2.6}$$

which means

$$(x,v) \in \mathrm{supp} f_0 \Longleftrightarrow (X(t,0,x,v), V(t,0,x,v)) \in \mathrm{supp} f(t).$$

Denote that

$$m(t) := \sup_{(x,v)\in \mathrm{supp} f(t)} |v| = \sup_{(x,v)\in \mathrm{supp} f_0} |V(t,0,x,v)|.$$

We claim that $m(t)$ is continuous with respect to t and

$$D^- m(t) := \overline{\lim_{h\to 0^+}} \frac{m(t)-m(t-h)}{h} \leqslant 0.$$

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(1) $m(t)$ is continuous with respect to t. If not, there exists some $\varepsilon_0 > 0$ and $t_n \to t$ such that

$$|m(t_n) - m(t)| \geqslant \varepsilon_0.$$

Choose $(x_0, v_0) \in \mathrm{supp} f_0$ such that

$$|V(t, 0, x_0, v_0)| = \sup_{(x,v)\in \mathrm{supp} f_0} |V(t, 0, x, v)|, \tag{2.7}$$

since $X(t, 0, x, v)$, $V(t, 0, x, v)$ are smooth on t, x, v and the support of f_0 is compact. We choose $(x_n, v_n) \in \mathrm{supp} f_0$ such that

$$|V(t_n, 0, x_0, v_0)| \leqslant |V(t_n, 0, x_n, v_n)| = m(t_n).$$

Since (t_n, x_n, v_n) are bounded, there exists a subsequence $(t_{n_k}, x_{n_k}, v_{n_k})$ converges to $(t, \bar{x}, \bar{v})$ such that

$$m(t) = \lim_{k\to\infty} |V(t_{n_k}, 0, x_0, v_0)| \leqslant \lim_{k\to\infty} m(t_{n_k}) = |V(t, 0, \bar{x}, \bar{v})| \leqslant m(t),$$

which contradicts with the selection of t_n.

(2) $D^- m(t) \leqslant 0$. First, using (2.5) we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}|V(t, 0, x_0, v_0)| \\
&\quad = -\frac{V(t, 0, x_0, v_0)}{|V(t, 0, x_0, v_0)|} \cdot L[f](t, X(t, 0, x_0, v_0), V(t, 0, x_0, v_0)) \\
&\quad = -\frac{V(t, 0, x_0, v_0)}{|V(t, 0, x_0, v_0))|} \cdot \left\{ \int_{\mathbb{R}^{2d}} \frac{V(t, 0, x_0, v_0) - \omega}{(1 + |X(t, 0, x_0, v_0) - y|^2)^\beta} f(t, y, \omega)\, \mathrm{d}y\, \mathrm{d}\omega \right\} \leqslant 0,
\end{aligned}$$

since $|\omega| \leqslant |V(t, 0, x_0, v_0)|$ for any $(y, \omega) \in \mathrm{supp} f(t)$. Then

$$\begin{aligned}
D^- m(t) &= \overline{\lim_{h\to 0^+}} \frac{m(t) - m(t-h)}{h} \\
&\leqslant \overline{\lim_{h\to 0^+}} \frac{|V(t, 0, x_0, v_0)| - |V(t-h, 0, x_0, v_0)|}{h} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}|V(t, 0, x_0, v_0)| \leqslant 0.
\end{aligned}$$

By lemma 2.3, we obtain

$$\sup_{(x,v)\in \mathrm{supp} f(t)} |v| \leqslant \sup_{(x,v)\in \mathrm{supp} f_0} |v|.$$

□

Now, we establish the existence and uniqueness of the classical solution to (1.3)–(1.4) when the initial datum f_0 is not compactly supported in x. The calculation in our proof for the following theorem is similar to [**6**, theorem 3.2], but the properties used are very different. For the convenience of the readers, we provide a complete proof below.

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Theorem 2.5. *Suppose that the initial datum $f_0 \in C_0^1(\mathbb{R}^{2d}) \cap L^1(\mathbb{R}^{2d})$ is nonnegative and satisfies*:

$$\rho := \sup_{(x,v)\in \operatorname{supp} f_0} |v| < \infty, \tag{2.8}$$

$$m_p := \int_{\mathbb{R}^{2d}} |x|^p f_0(x,v)\, \mathrm{d}x\, \mathrm{d}v < \infty, \tag{2.9}$$

for some constants $\rho > 0$ and $p \geqslant 1$. Then, there exists a unique global classical solution $f \in C^1([0,\infty) \times \mathbb{R}^{2d}) \cap L^\infty([0,\infty); L^1(\mathbb{R}^{2d}))$ to (1.3)–(1.4) *such that*

$$\sup_{(x,v)\in \operatorname{supp} f(t)} |v| \leqslant \rho. \tag{2.10}$$

Proof. We first use lemma 2.1 to construct a sequence of approximate solutions. Then, we prove the compactness of approximate solutions and pass to the limit to obtain a global solution. Furthermore, by the characteristic flow, the regularity of the solution is obtained. Finally, we demonstrate the uniqueness of the solution.

Step 1: Approximate solutions. Let $f_0^n = f_0(x,v)\cdot \chi_n$, where $\chi_n \in C_c^\infty(\mathbb{R}^{2d})$, $0 \leqslant \chi_n \leqslant 1$, $|\nabla \chi_n| \leqslant \frac{2}{n}$ and

$$\chi_n = \begin{cases} 1, & |x| \leqslant n, \\ 0, & |x| \geqslant 2n. \end{cases}$$

For any fixed n, there exists a unique global classical solution f_n to (1.3)–(1.4) with initial datum f_0^n by lemma 2.1. Combined (2.8) with lemma 2.4, the velocity support of f_n is uniformly bounded, i.e.,

$$\sup_{(x,v)\in \operatorname{supp} f_n(t)} |v| \leqslant \rho. \tag{2.11}$$

And (2.1) gives

$$\|f_n(t)\|_{L^1} = \|f_0^n\|_{L^1} \leqslant \|f_0\|_{L^1}. \tag{2.12}$$

Next, for any fixed n, we define the characteristic flow of f_n as follows:

$$\begin{cases} \dot X_n(s,t_0,x,v) = V_n(s,t_0,x,v), \quad X_n(t_0,t_0,x,v) = x, \\ \dot V_n(s,t_0,x,v) = L[f_n](s, X_n(s,t_0,x,v), V_n(s,t_0,x,v)), \ V_n(t_0,t_0,x,v) = v, \end{cases} \tag{2.13}$$

where $s \geqslant 0$, $t_0 \geqslant 0$. By the forward characteristic flow we have

$$\begin{aligned} &f_n(t, X_n(t,0,x,v), V_n(t,0,x,v)) \\ &\quad = f_0^n(x,v)\cdot \exp\left\{ d\int_0^t \int_{\mathbb{R}^{2d}} \frac{f_n(s,y,\omega)}{(1+|X_n(s,0,x,v)-y|^2)^\beta}\, \mathrm{d}y\, \mathrm{d}\omega\, \mathrm{d}s \right\}. \end{aligned} \tag{2.14}$$

For simplicity, the forward characteristic flow $(X_n(t,0,x,v), V_n(t,0,x,v))$ is denoted by $(X_n(t), V_n(t))$.

Step 2: Compactness. Fix $T > 0$, we show that the sequence $\{f_n\}$ is relatively compact in $C([0,T]; C_0(\mathbb{R}^{2d}))$.

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First, we claim that $\{f_n\}$ is uniformly bounded and equicontinuous with respect to $(x,\, v)$. It follows from (2.11) that for any $(x,\, v) \in \mathrm{supp} f_n(t)$,

$$|L[f_n]| \leqslant \int_{\mathbb{R}^{2d}} \frac{|v-\omega|}{(1+|x-y|^2)^\beta} f_n(t,y,\omega)\, \mathrm{d}y\, \mathrm{d}\omega \\ \leqslant 2\rho \|f_n(t)\|_{L^1} \leqslant C. \tag{2.15}$$

Moreover, we get that

$$\begin{aligned} |\partial_v L[f_n]| &= \left| \int_{\mathbb{R}^{2d}} \frac{1}{(1+|x-y|^2)^\beta} f_n(t,y,\omega)\, \mathrm{d}y\, \mathrm{d}\omega \right| \leqslant \|f_n(t)\|_{L^1} \leqslant C, \\ |\partial_x L[f_n]| &= 2\beta \left| \int_{\mathbb{R}^{2d}} \frac{v-\omega}{(1+|x-y|^2)^\beta} \cdot \frac{x-y}{(1+|x-y|^2)} f_n(t,y,\omega)\, \mathrm{d}y\, \mathrm{d}\omega \right| \\ &\leqslant 4\beta\rho \|f_n(t)\|_{L^1} \leqslant C, \\ |\mathrm{div}_v(\partial_{x^i} L[f_n])| &= 2\beta d \left| \int_{\mathbb{R}^{2d}} \frac{(x-y)\cdot(0,0,...,1,...,0,0)^i}{(1+|x-y|^2)^{\beta+1}} f_n(t,y,\omega)\, \mathrm{d}y\, \mathrm{d}\omega \right| \\ &\leqslant 2\beta d \|f_n(t)\|_{L^1} \leqslant C, \end{aligned} \tag{2.16}$$

where $(0,\, 0,\, \ldots,\, 1,\, \ldots,\, 0,\, 0)^i = (0,\, 0,\, \ldots,\, 0,\, \underbrace{1}_{i\text{-th}},\, 0,\, \ldots,\, 0,\, 0)$. Following from (2.15), (2.16) and the equalities

$$\begin{aligned} &\partial_t f_n + v\cdot\nabla_x f_n + L[f_n]\cdot\nabla_v f_n = -f_n \mathrm{div}_v(L[f_n]), \\ &\partial_t\partial_{x^i} f_n + v\cdot\nabla_x\partial_{x^i} f_n + L[f_n]\cdot\nabla_v\partial_{x^i} f_n = -(\partial_{x^i} f_n)\mathrm{div}_v(L[f_n]) - f_n \mathrm{div}_v(\partial_{x^i} L[f_n]) \\ &\quad - (\partial_{x^i} L[f_n])\cdot\nabla_v f_n, \\ &\partial_t\partial_{v^i} f_n + v\cdot\nabla_x\partial_{v^i} f_n + L[f_n]\cdot\nabla_v\partial_{v^i} f_n = -(\partial_{v^i} f_n)\mathrm{div}_v(L[f_n])) - \partial_{x^i} f_n \\ &\quad - (\partial_{v^i} L[f_n])\cdot\nabla_v f_n, \end{aligned}$$

there exists an increasing continuous function $R(t)$ independent of n such that

$$\|f_n(t)\|_{L^\infty} + \|\partial_x f_n(t)\|_{L^\infty} + \|\partial_v f_n(t)\|_{L^\infty} \leqslant R(t). \tag{2.17}$$

Therefore, $\{f_n\}$ is uniformly bounded and equicontinuous with respect to $(x,\, v)$.

Second, we demonstrate that $f_n \in C([0,\, T]; C_0(\mathbb{R}^{2d}))$. For any $\varepsilon > 0$, there exists some sufficiently large $r > 0$ such that

$$f_0(x,v) < \varepsilon \quad \text{if} \quad |v| > r,$$

since $f_0 \in C_0(\mathbb{R}^{2d})$. By the backward characteristic flow we can rewrite (2.14) as

$$\begin{aligned} f_n(t,x,v) &= f_0^n(X_n(0,t,x,v), V_n(0,t,x,v)) \\ &\quad \cdot \exp\left\{ d\int_0^t \int_{\mathbb{R}^{2d}} \frac{f_n(s,y,\omega)}{(1+|X_n(s,t,x,v)-y|^2)^\beta}\, \mathrm{d}y\, \mathrm{d}\omega\, \mathrm{d}s \right\} \\ &\leqslant f_0^n(X_n(0,t,x,v), V_n(0,t,x,v)) e^{Ct}. \end{aligned}$$

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Using (2.13), (2.15) and the definition of the characteristic flow,

$$|V_n(0,t,x,v)| \geqslant |v| - |V_n(0,t,x,v) - v| \geqslant |v| - Ct.$$

Combing with (2.18), we deduce

$$f_n(t,x,v) \leqslant \varepsilon e^{Ct} \quad \text{if} \quad |v| > r + Ct.$$

Similarly, for any $\varepsilon > 0$, there exists some sufficiently large $r > 0$ such that

$$f_0(x,v) < \varepsilon \quad \text{if} \quad |x| > r.$$

Then, using (2.11), we obtain

$$f_n(t,x,v) < \varepsilon e^{Ct} \quad \text{if} \quad |x| > r + Ct.$$

Combining above arguments, for any $\varepsilon > 0$, there exists some $r > 0$ such that

$$f_n(t,x,v) < \varepsilon \quad \text{if} \quad (t,x,v) \in [0,T] \times (B_r \times B_r)^c,$$

which yields that $f_n \in C([0,\, T]; C_0(\mathbb{R}^{2d}))$. And for any fixed $t \in [0,\, T]$, $\{f_n(t)\}$ is relatively compact in $C_0(\mathbb{R}^{2d})$ by above estimates.

Finally, we show that $\{f_n\}$ is equicontinuous with respect to t. For any $(t,\, x,\, v) \in [0,\, T] \times \mathbb{R}^d \times B_\rho$, from (2.15)–(2.17) we obtain

$$|\partial_t f_n(t,x,v)| \leqslant |v \cdot \nabla_x f_n| + |L[f_n] \cdot \nabla_v f_n| + |f_n \mathrm{div}_v(L[f_n])| \leqslant C.$$

Moreover, for any $(t,\, x,\, v) \in [0,\, T] \times \mathbb{R}^d \times B_\rho^c$, we have $f_n(t,\, x,\, v) \equiv 0$, which means $|\partial_t f_n(t,\, x,\, v)| = 0$. Thus, $\{f_n\}$ is equicontinuous with respect to t.

Combining above estimates, we know that f_n is relatively compact in $C([0,\, T]; C_0(\mathbb{R}^{2d}))$. Thus, there is a subsequence of f_n (still denoted as f_n) which uniformly converges to a continuous function $f \in C([0,\, T]; C_0(\mathbb{R}^{2d}))$. It follows from (2.11) and (2.12) we obtain that

$$\sup_{(x,v)\in \mathrm{supp} f(t)} |v| \leqslant \rho \tag{2.18}$$

and $f \in L^\infty([0,\, T]; L^1(\mathbb{R}^{2d}))$.

Step 3: Regularity. We define the characteristic flow of f as follows:

$$\begin{cases} \dot{X}(s,t_0,x,v) = V(s,t_0,x,v), \quad X(t_0,t_0,x,v) = x, \\ \dot{V}(s,t_0,x,v) = L[f](s,X(s,t_0,x,v),V(s,t_0,x,v)), \quad V(t_0,t_0,x,v) = v, \end{cases} \tag{2.19}$$

where $s \geqslant 0$, $t_0 \geqslant 0$. For simplicity, the forward characteristic flow $(X(t,\, 0,\, x,\, v),\, V(t,\, 0,\, x,\, v))$ is denoted by $(X(t),\, V(t))$. Then, for any $(x,\, v) \in \mathrm{supp} f_0$ we have

$$|X_n(t,0,x,v) - X(t,0,x,v)| \leqslant \int_0^t |V_n(s,0,x,v) - V(s,0,x,v)|\,\mathrm{d}s \tag{2.20}$$

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and

$$
\begin{aligned}
&|V_n(t,0,x,v) - V(t,0,x,v)| \\
&\leqslant \int_0^t |L[f_n](\tau, X_n(\tau), V_n(\tau)) - L[f](\tau, X(\tau), V(\tau))| \,\mathrm{d}\tau \\
&\leqslant \int_0^t \int_{\mathbb{R}^{2d}} \left| \frac{V_n(\tau)-\omega}{(1+|X_n(\tau)-y|^2)^\beta} f_n(\tau,y,\omega) - \frac{V(\tau)-\omega}{(1+|X(\tau)-y|^2)^\beta} f(\tau,y,\omega)\right| \mathrm{d}y\,\mathrm{d}\omega\,\mathrm{d}\tau.
\end{aligned} \tag{2.21}
$$

Now, recall that

$$
\sup_{(x,v)\in \operatorname{supp} f_0} |V_n(t)| \leqslant \rho, \qquad \sup_{(x,v)\in \operatorname{supp} f(t)} |v| \leqslant \rho.
$$

Then, for any $(x,\, v) \in \operatorname{supp} f_0$ we obtain

$$
\begin{aligned}
&\left| \frac{V_n(\tau)-\omega}{(1+|X_n(\tau)-y|^2)^\beta} f_n(\tau,y,\omega) - \frac{V(\tau)-\omega}{(1+|X(\tau)-y|^2)^\beta} f(\tau,y,\omega)\right| \\
&\leqslant \left| \frac{V_n(\tau)-\omega}{(1+|X_n(\tau)-y|^2)^\beta} f_n(\tau,y,\omega) - \frac{V_n(\tau)-\omega}{(1+|X_n(\tau)-y|^2)^\beta} f(\tau,y,\omega)\right| \\
&\quad + \left| \frac{V_n(\tau)-\omega}{(1+|X_n(\tau)-y|^2)^\beta} f(\tau,y,\omega) - \frac{V(\tau)-\omega}{(1+|X_n(\tau)-y|^2)^\beta} f(\tau,y,\omega)\right| \\
&\quad + \left| \frac{V(\tau)-\omega}{(1+|X_n(\tau)-y|^2)^\beta} f(\tau,y,\omega) - \frac{V(\tau)-\omega}{(1+|X(\tau)-y|^2)^\beta} f(\tau,y,\omega)\right| \\
&\leqslant C|f_n - f|(\tau,y,\omega) + C(|V_n(\tau)-V(\tau)| + |X_n(\tau)-X(\tau)|) f(\tau,y,\omega).
\end{aligned}
$$

Combining the above estimates, there exists some constant C depending only on T, β, ρ such that

$$
\begin{aligned}
&|V_n(t) - V(t)| + |X_n(t) - X(t)| \\
&\leqslant C \int_0^t \|f_n(\tau) - f(\tau)\|_{L^1} \,\mathrm{d}\tau + C\|f_0\|_{L^1} \int_0^t |V_n(\tau) - V(\tau)| + |X_n(\tau) - X(\tau)| \,\mathrm{d}\tau.
\end{aligned} \tag{2.22}
$$

Moreover, using (2.9), (2.11) and (2.13), there exists some C_T depending only on p, T, ρ, m_p, $\|f_0\|_{L^1}$ such that

$$
\begin{aligned}
\int_{\mathbb{R}^{2d}} (1+|x|+|v|)^p f_n(t,x,v)\,\mathrm{d}x\,\mathrm{d}v &= \int_{\mathbb{R}^{2d}} (1+|X_n(t)|+|V_n(t)|)^p f_0^n(x,v)\,\mathrm{d}x\,\mathrm{d}v \\
&\leqslant \int_{\mathbb{R}^{2d}} (1+|x|+|\rho t|+|\rho|)^p f_0^n(x,v)\,\mathrm{d}x\,\mathrm{d}v \\
&\leqslant C_T,
\end{aligned}
$$

which implies

$$
\int_{(B_r\times B_r)^c} (1+r)^p f_n(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \leqslant \int_{(B_r\times B_r)^c} (1+|x|+|v|)^p f_n(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \leqslant C_T.
$$

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Therefore, for any $\varepsilon > 0$, there exists some $r > 0$ such that

$$\int_{(B_r\times B_r)^c} f_n(t,x,v)\,\mathrm{d}x\,\mathrm{d}v < \varepsilon, \quad \forall t\in[0,T].$$

Combining above inequality with the fact that f_n uniformly converges to f, then

$$\lim_{n\to\infty}\|f_n - f\|_{L^\infty([0,T];L^1(\mathbb{R}^{2d}))} = 0. \tag{2.23}$$

Now, using (2.22) and (2.23), we prove that $(X_n(t,0,x,v), V_n(t,0,x,v))$ uniformly converges to $(X(t,0,x,v), V(t,0,x,v))$. Thus, we can pass to the limit in (2.14). Replacing variables (x, v) with $(X(0,t,x,v), V(0,t,x,v))$, we use the backward characteristic flow to get that

$$\begin{aligned} f(t,x,v) &= f_0(X(0,t,x,v), V(0,t,x,v)) \\ &\quad \cdot \exp\left\{ d\int_0^t \int_{\mathbb{R}^{2d}} \frac{f(s,y,\omega)}{(1+|X(s,t,x,v)-y|^2)^\beta}\,\mathrm{d}y\,\mathrm{d}\omega\,\mathrm{d}s \right\}. \end{aligned} \tag{2.24}$$

According to $L[f]\in C([0,T];C^1(\mathbb{R}^{2d}))$, we obtain that $X(s,t_0,x,v), V(s,t_0,x,v)\in C^1([0,T]\times[0,T]\times\mathbb{R}^{2d})$. Then, from (2.24) we get that $f\in C^1([0,T]\times\mathbb{R}^{2d})$ and it satisfies

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + \mathrm{div}_v(L[f]f) = 0, \\ f(0,x,v) = f_0(x,v). \end{cases} \tag{2.25}$$

Step 4: Uniqueness. Suppose that there exist two solutions, f and h, with the same initial datum f_0. The forward characteristic flows of f and h are denoted by $(X_f(t), V_f(t))$ and $(X_h(t), V_h(t))$. We will show that for any fixed $T > 0$,

$$E(t) := \int_{\mathbb{R}^{2d}} (|X_f(t)-X_h(t)| + |V_f(t)-V_h(t)|) f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v = 0, \quad \forall t\in[0,T].$$

Note that

$$f(s,y,\omega) = (X_f(s,0,y,\omega), V_f(s,0,y,\omega))\# f_0$$

and

$$h(s,y,\omega) = (X_h(s,0,y,\omega), V_h(s,0,y,\omega))\# f_0.$$

It is straightforward to check that

$$|X_f(t)-X_h(t)| \leqslant \int_0^t |V_f(s)-V_h(s)|\,\mathrm{d}s \tag{2.26}$$

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and

$$
\begin{aligned}
&|V_f(t) - V_h(t)| \\
&\quad \leqslant \int_0^t |L[f](s, X_f(s), V_f(s)) - L[h](s, X_h(s), V_h(s))| \,\mathrm{d}s \\
&\quad \leqslant \int_0^t \int_{\mathbb{R}^{2d}} \left| \frac{V_f(s) - V_f(s,0,y,\omega)}{(1 + |X_f(s) - X_f(s,0,y,\omega)|^2)^\beta} - \frac{V_h(s) - V_h(s,0,y,\omega)}{(1 + |X_h(s) - X_h(s,0,y,\omega)|^2)^\beta} \right| \\
&\qquad \cdot f_0(y,\omega) \,\mathrm{d}y \,\mathrm{d}\omega \,\mathrm{d}s.
\end{aligned}
$$

For simplicity, we denote

$$
\begin{aligned}
(Y_f(s), U_f(s)) &= (X_f(s,0,y,\omega), V_f(s,0,y,\omega)), \\
(Y_h(s), U_h(s)) &= (X_h(s,0,y,\omega), V_h(s,0,y,\omega)).
\end{aligned}
$$

Then, for any $(x,\, v) \in \operatorname{supp} f_0$ we have

$$
\begin{aligned}
&\int_{\mathbb{R}^{2d}} \left| \frac{V_f(s) - U_f(s)}{(1 + |X_f(s) - Y_f(s)|^2)^\beta} - \frac{V_h(s) - U_h(s)}{(1 + |X_h(s) - Y_h(s)|^2)^\beta} \right| f_0(y,\omega) \,\mathrm{d}y \,\mathrm{d}\omega \\
&\quad \leqslant \int_{\mathbb{R}^{2d}} \left| \frac{V_f(s) - U_f(s)}{(1 + |X_f(s) - Y_f(s)|^2)^\beta} - \frac{V_h(s) - U_h(s)}{(1 + |X_f(s) - Y_f(s)|^2)^\beta} \right| f_0(y,\omega) \,\mathrm{d}y \,\mathrm{d}\omega \\
&\qquad + \int_{\mathbb{R}^{2d}} \left| \frac{V_h(s) - U_h(s)}{(1 + |X_f(s) - Y_f(s)|^2)^\beta} - \frac{V_h(s) - U_h(s)}{(1 + |X_h(s) - Y_h(s)|^2)^\beta} \right| f_0(y,\omega) \,\mathrm{d}y \,\mathrm{d}\omega \\
&\quad \leqslant C(|X_f(s) - X_h(s)| + |V_f(s) - V_h(s)|) + \int_{\mathbb{R}^{2d}} |U_f(s) - U_h(s)| f_0(y,\omega) \,\mathrm{d}y \,\mathrm{d}\omega \\
&\qquad + 2\rho C_\beta \int_{\mathbb{R}^{2d}} |Y_f(s) - Y_h(s)| f_0(y,\omega) \,\mathrm{d}y \,\mathrm{d}\omega \\
&\quad \leqslant C(|X_f(s) - X_h(s)| + |V_f(s) - V_h(s)|) \\
&\qquad + C \int_{\mathbb{R}^{2d}} (|Y_f(s) - Y_h(s)| + |U_f(s) - U_h(s)|) f_0(y,\omega) \,\mathrm{d}y \,\mathrm{d}\omega. \qquad (2.27)
\end{aligned}
$$

Combing (2.26) with (2.27), for any $(x,\, v) \in \operatorname{supp} f_0$ we obtain

$$
\begin{aligned}
&|X_f(t) - X_h(t)| + |V_f(t) - V_h(t)| \\
&\quad \leqslant C \int_0^t (|X_f(s) - X_h(s)| + |V_f(s) - V_h(s)|) \,\mathrm{d}s \\
&\qquad + C \int_0^t \int_{\mathbb{R}^{2d}} (|Y_f(s) - Y_h(s)| + |U_f(s) - U_h(s)|) f_0(y,\omega) \,\mathrm{d}y \,\mathrm{d}\omega \,\mathrm{d}s \\
&\quad \leqslant C \int_0^t (|X_f(s) - X_h(s)| + |V_f(s) - V_h(s)|) \,\mathrm{d}s + C \int_0^t E(s) \,\mathrm{d}s,
\end{aligned}
$$

which implies that there exists a positive constant C depending only on T, β, ρ, $\|f_0\|_{L^1}$ such that

$$
E(t) \leqslant C \int_0^t E(s) \,\mathrm{d}s, \quad \forall t \in [0, T].
$$

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By the Grönwall's inequality, we get that $E = 0$ and then the uniqueness of solution. □

2.2. Formation behaviour of the classical solution

In order to establish the formation behaviour of the classical solution to the kinetic Cucker–Smale model, we provide the following lemma.

LEMMA 2.6. *If f is the classical solution as stated in theorem 2.5, then for any $t \geqslant 0$ we have*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}} f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = 0, \tag{2.28}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}} vf(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = 0, \tag{2.29}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}} |v|^2 f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = -\int_{\mathbb{R}^{4d}} \frac{|v-\omega|^2}{(1+|x-y|^2)^\beta} f(t,x,v)f(t,y,\omega)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\omega. \tag{2.30}$$

Proof. We first show that for any $\phi \in C_b^1(\mathbb{R}^{2d})$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}} \phi f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^{2d}} \{v\cdot\nabla_x\phi + \nabla_v\phi\cdot L[f]\}\, f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v. \tag{2.31}$$

Choose a smooth cut-off function

$$\begin{aligned}&\varphi_R(\cdot) = \varphi(\cdot/R) \in C_c^\infty(\mathbb{R}^d) \quad \text{such that} \quad 0 \leqslant \varphi_R \leqslant 1, \quad |\nabla\varphi_R| \leqslant 2/R,\\ &\varphi_R \equiv 1 \quad \text{on} \quad B_R(0) \quad \text{and} \quad \varphi_R \equiv 0 \quad \text{on} \quad \mathbb{R}^d \setminus B_{2R}(0).\end{aligned}$$

Then we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}} \varphi_R(v)\varphi_R(x)\phi f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v &= \int_{\mathbb{R}^{2d}} \varphi_R(v)\varphi_R(x)\phi\partial_t f\,\mathrm{d}x\,\mathrm{d}v\\ &= -\int_{\mathbb{R}^{2d}} \varphi_R(v)\varphi_R(x)\phi(v\cdot\nabla_x f + \mathrm{div}_v(L[f]f))\,\mathrm{d}x\,\mathrm{d}v.\end{aligned} \tag{2.32}$$

By direct calculation, we obtain

$$\begin{aligned}&-\int_{\mathbb{R}^{2d}} \varphi_R(v)\varphi_R(x)\phi(v\cdot\nabla_x f)\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^{2d}} \varphi_R(v) f v\cdot\nabla_x(\varphi_R(x)\phi)\,\mathrm{d}x\,\mathrm{d}v\\ &= \int_{\mathbb{R}^{2d}} \varphi_R(v) f\phi v\cdot\nabla_x(\varphi_R(x))\,\mathrm{d}x\,\mathrm{d}v + \int_{\mathbb{R}^{2d}} \varphi_R(v) f\varphi_R(x) v\cdot\nabla_x(\phi)\,\mathrm{d}x\,\mathrm{d}v\\ &\leqslant \int_{\mathbb{R}^{2d}} |\varphi_R(v) v f\phi||\nabla_x(\varphi_R(x))|\,\mathrm{d}x\,\mathrm{d}v + \int_{\mathbb{R}^{2d}} \varphi_R(v) f\varphi_R(x) v\cdot\nabla_x(\phi)\,\mathrm{d}x\,\mathrm{d}v\\ &\leqslant \frac{2}{R}\int_{\mathbb{R}^{2d}} |\varphi_R(v) v f\phi|\,\mathrm{d}x\,\mathrm{d}v + \int_{\mathbb{R}^{2d}} \varphi_R(v) f\varphi_R(x) v\cdot\nabla_x(\phi)\,\mathrm{d}x\,\mathrm{d}v.\end{aligned}$$

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This yields

$$\lim_{R\to\infty} -\int_{\mathbb{R}^{2d}} \varphi_R(v)\varphi_R(x)\phi(v\cdot\nabla_x f)\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^{2d}} v\cdot\nabla_x(\phi) f\,\mathrm{d}x\,\mathrm{d}v.$$

By repeating the above procedure, we have

$$\lim_{R\to\infty} -\int_{\mathbb{R}^{2d}} \varphi_R(v)\varphi_R(x)\phi\mathrm{div}_v(L[f]f)\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^{2d}} \nabla_v\phi\cdot L[f]f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v.$$

Thus, let $R\to\infty$, from (2.32) we can obtain (2.31).

Now, recall that

$$\sup_{(x,v)\in\mathrm{supp} f(t)} |v| \leqslant \rho.$$

Choose another smooth cut-off function $\chi_\rho \in C_c^1(\mathbb{R}^d)$ such that $0\leqslant\chi_\rho\leqslant 1$, $\chi_\rho\equiv 1$ on $B_{\rho+1}$, and $\chi_\rho\equiv 0$ on $\mathbb{R}^d\setminus B_{2\rho+2}$. The time derivative of velocity moments can be checked directly by taking $\phi = 1$, $v^i\chi_\rho$, $|v|^2\chi_\rho$ in (2.31), respectively.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}} v^i\chi_\rho f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v &= \int_{\mathbb{R}^{2d}} \nabla_v(v^i\chi_\rho)\cdot L[f]f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v\\
&= -\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} \frac{v^i-\omega^i}{(1+|x-y|^2)^\beta}\chi_\rho f(t,x,v)f(t,y,\omega)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\omega\\
&\quad -\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} \frac{v^i(v-\omega)}{(1+|x-y|^2)^\beta}\cdot\nabla_v(\chi_\rho) f(t,x,v)f(t,y,\omega)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\omega\\
&= -\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^d\times B_{\rho+1}} \frac{v^i-\omega^i}{(1+|x-y|^2)^\beta} f(t,x,v)f(t,y,\omega)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\omega\\
&= -\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} \frac{v^i-\omega^i}{(1+|x-y|^2)^\beta} f(t,x,v)f(t,y,\omega)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\omega\\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}} |v|^2\chi_\rho f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v &= \int_{\mathbb{R}^{2d}} \nabla_v(|v|^2\chi_\rho)\cdot L[f]f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v\\
&= -2\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} \frac{v\cdot(v-\omega)}{(1+|x-y|^2)^\beta}\chi_\rho f(t,x,v)f(t,y,\omega)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\omega\\
&\quad -\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} \frac{|v|^2(v-\omega)}{(1+|x-y|^2)^\beta}\cdot\nabla_v(\chi_\rho) f(t,x,v)f(t,y,\omega)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\omega\\
&= -2\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^d\times B_{\rho+1}} \frac{v\cdot(v-\omega)}{(1+|x-y|^2)^\beta} f(t,x,v)f(t,y,\omega)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\omega\\
&= -2\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} \frac{v\cdot(v-\omega)}{(1+|x-y|^2)^\beta} f(t,x,v)f(t,y,\omega)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\omega\\
&= -\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} \frac{|v-\omega|^2}{(1+|x-y|^2)^\beta} f(t,x,v)f(t,y,\omega)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\omega.
\end{aligned}$$

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Here we used the change of variable $(x,\, v) \leftrightarrow (y,\, \omega)$ and Fubini's theorem. □

With the above preparations, we establish the formation behaviour of the classical solution to (1.3)–(1.4) without compact support for position variable.

THEOREM 2.7. *Consider* (1.3)–(1.4) *with* $\beta \in [0,\, \frac{1}{2}]$. *Suppose that the initial datum* $f_0 \in C_0^1(\mathbb{R}^{2d}) \cap L^1(\mathbb{R}^{2d})$ *is nonnegative and satisfies*:

$$\rho := \sup_{(x,v)\in \operatorname{supp} f_0} |v| < \infty, \tag{2.33}$$

$$m_p := \int_{\mathbb{R}^{2d}} |x|^p f_0(x, v)\, \mathrm{d}x\, \mathrm{d}v < \infty, \tag{2.34}$$

for some constants $\rho > 0$ *and* $p \geqslant 1$. *Then, the following assertion holds*:

$$\lim_{t\to\infty} \int_{\mathbb{R}^{2d}} |v - v_c|^2 f(t, x, v)\, \mathrm{d}x\, \mathrm{d}v = 0, \quad v_c = \|f_0\|_{L^1}^{-1} \int_{\mathbb{R}^{2d}} v f_0\, \mathrm{d}x\, \mathrm{d}v.$$

First, the quadratic moment of the velocity will converge to some constant P. And we are devoted to prove $P = 0$ by contradiction. For this purpose, we construct some technical sequence $\{t_k\}_{k=1}^{\infty}$. If $P \neq 0$, we can derive some t_k are inconsistent with their own definitions. We elaborate our methodology in more detail in the following three steps:

• Step 1: Through the characteristic flow, some estimates are obtained to deduce the differential inequality (2.39), which will be further classified and discussed in step 2.

• Step 2: We begin with $t_1 = 0$ to find some R_{t_1} satisfying (2.40) and (2.41). Then, let t_2 be the supremum of time s, where s guarantees that the integral of the square of $V(t)$ in $B_{R_{t_1}}$ is greater than the integral outside $B_{R_{t_1}}$ during the time interval $[t_1,\, s)$($V(t)$ which is denoted by (2.38)). If $t_2 = \infty$, then we deduce $P = 0$ by (2.39). Otherwise, we can find some R_{t_2} and t_3, where R_{t_2} and t_3 are defined similarly as R_{t_1} and t_2 respectively. If all $t_k < \infty$, we obtain two sequences $\{t_k\}_{k=1}^{\infty}$ and $\{B_{R_{t_k}}\}_{k=1}^{\infty}$ by induction. Obviously, the integral of the square of $V(t)$ in $B_{R_{t_k}}$ is equal to the integral outside $B_{R_{t_k}}$ when $t = t_{k+1}$.

• Step 3: By splitting the integral of initial datum f_0, we construct some subsequence $\{t_{k_h}\}_{h=1}^{\infty}$ such that the equation (2.46) holds. If $P \neq 0$, we can derive some t_q such that the integral of the square of $V(t)$ in $B_{R_{t_q}}$ is greater than the integral outside $B_{R_{t_q}}$ when $t = t_{q+1}$.

Proof. Recall that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^{2d}} f(t, x, v)\, \mathrm{d}x\, \mathrm{d}v = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^{2d}} v f(t, x, v)\, \mathrm{d}x\, \mathrm{d}v = 0.$$

Hence

$$\int_{\mathbb{R}^{2d}} (v - v_c) f(t, x, v)\, \mathrm{d}x\, \mathrm{d}v = 0. \tag{2.35}$$

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For simplicity of notation, we use v instead of $v - v_c$ to calculation below and denote that

$$N_{v,2}(f(t)) := \int_{\mathbb{R}^{2d}} |v|^2 f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v.$$

Following from (2.30), we get that $N_{v,2}(f(t))$ is non-increasing in t. Thus, we have

$$\lim_{t\to\infty} N_{v,2}(f(t)) = P,$$

where $P \geqslant 0$. Next, we are devoted to prove that $P = 0$ by contradiction.

Step 1: A prior estimate. Let $(x,\, v)$, $(y,\, \omega)$ be the initial data of the forward characteristic flows of f, that is

$$\begin{cases} \dot{X}(t,0,x,v) = V(t,0,x,v), \quad X(0,0,x,v) = x, \\ \dot{V}(t,0,x,v) = L[f](t, X(t,0,x,v), V(t,0,x,v)), \quad V(0,0,x,v) = v \end{cases}$$

and

$$\begin{cases} \dot{Y}(t,0,y,\omega) = W(t,0,y,\omega), \quad Y(0,0,y,\omega) = y, \\ \dot{W}(t,0,y,\omega) = L[f](t, Y(t,0,y,\omega), W(t,0,y,\omega)), \quad W(0,0,y,\omega) = \omega. \end{cases}$$

We rewrite (2.30) as

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}} |V(t,0,x,v)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \\ &\quad = -\int_{\mathbb{R}^{4d}} \frac{|V(t,0,x,v) - W(t,0,y,\omega)|^2}{(1+|X(t,0,x,v)-Y(t,0,y,\omega)|^2)^{\beta}} f_0(x,v) f_0(y,\omega)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\omega. \end{aligned} \tag{2.36}$$

Note that

$$\sup_{(x,v)\in \mathrm{supp} f_0} |V(t,0,x,v)| \leqslant \rho, \qquad \sup_{(y,\omega)\in \mathrm{supp} f_0} |W(t,0,y,\omega)| \leqslant \rho.$$

Then, we have

$$|X(t,0,x,v) - Y(t,0,y,\omega)| \leqslant |x-y| + 2\rho t.$$

For any fixed $R > 0$, we denote that

$$H_R(t) := \inf_{x,y\in B_R} H(|X(t,0,x,v) - Y(t,0,y,\omega)|),$$

where H is defined by (1.2). It is obvious that

$$H_R(t) \geqslant H(2R + 2\rho t) =: \widetilde{H_R}(t).$$

For simplicity, we denote

$$(V(t), X(t)) = (V(t,0,x,v), X(t,0,x,v)), \tag{2.37}$$

$$(W(t), Y(t)) = (W(t,0,y,\omega), Y(t,0,y,\omega)). \tag{2.38}$$

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From (2.35) we have

$$\int_{\mathbb{R}^{2d}} V(t) f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^{2d}} W(t) f_0(y,\omega)\,\mathrm{d}y\,\mathrm{d}\omega = 0,$$

which means that for any $R > 0$

$$\int_{B_R\times\mathbb{R}^d} V(t) f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v = -\int_{B_R^c\times\mathbb{R}^d} V(t) f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v.$$

The above equality gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}} |V(t)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \\
&\quad= -\int_{\mathbb{R}^{4d}} \frac{|V(t)-W(t)|^2}{(1+|X(t)-Y(t)|^2)^{\beta}} f_0(x,v) f_0(y,\omega)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\omega \\
&\quad\leqslant -\int_{B_R\times\mathbb{R}^d\times B_R\times\mathbb{R}^d} \frac{|V(t)-W(t)|^2}{(1+|X(t)-Y(t)|^2)^{\beta}} f_0(x,v) f_0(y,\omega)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\omega \\
&\quad\leqslant -\widetilde{H_R}(t)\int_{B_R\times\mathbb{R}^d\times B_R\times\mathbb{R}^d} |V(t)-W(t)|^2 f_0(x,v) f_0(y,\omega)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\omega \\
&\quad= -2\widetilde{H_R}(t)\int_{B_R\times\mathbb{R}^d} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \cdot \int_{B_R\times\mathbb{R}^d} |V(t)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \\
&\qquad+ 2\widetilde{H_R}(t)\int_{B_R^c\times\mathbb{R}^d} V(t) f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \cdot \int_{B_R^c\times\mathbb{R}^d} W(t) f_0(y,\omega)\,\mathrm{d}y\,\mathrm{d}\omega \\
&\quad\leqslant -2\widetilde{H_R}(t)\int_{B_R\times\mathbb{R}^d} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \cdot \int_{B_R\times\mathbb{R}^d} |V(t)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \\
&\qquad+ 2\widetilde{H_R}(t)\int_{B_R^c\times\mathbb{R}^d} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \cdot \int_{B_R^c\times\mathbb{R}^d} |V(t)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v. \qquad (2.39)
\end{aligned}$$

The last inequality of (2.39) holds by

$$\begin{aligned}
&\left|\int_{B_R^c\times\mathbb{R}^d} V(t) f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \cdot \int_{B_R^c\times\mathbb{R}^d} W(t) f_0(y,\omega)\,\mathrm{d}y\,\mathrm{d}\omega\right| \\
&\quad\leqslant \left\{\int_{B_R^c\times\mathbb{R}^d} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v\right\}^{\frac{1}{2}} \left\{\int_{B_R^c\times\mathbb{R}^d} |V(t)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v\right\}^{\frac{1}{2}} \\
&\qquad\cdot \left\{\int_{B_R^c\times\mathbb{R}^d} f_0(y,\omega)\,\mathrm{d}y\,\mathrm{d}\omega\right\}^{\frac{1}{2}} \left\{\int_{B_R^c\times\mathbb{R}^d} |W(t)|^2 f_0(y,\omega)\,\mathrm{d}y\,\mathrm{d}\omega\right\}^{\frac{1}{2}} \\
&\quad= \int_{B_R^c\times\mathbb{R}^d} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \cdot \int_{B_R^c\times\mathbb{R}^d} |V(t)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v.
\end{aligned}$$

Besides, for any fixed $t \geqslant 0$, there exists some $R_t > 0$ such that

$$\int_{B_{R_t}\times\mathbb{R}^d} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \geqslant \frac{5}{6}\|f_0\|_{L^1} \qquad (2.40)$$

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and

$$\int_{B_{R_t}\times\mathbb{R}^d} |V(t)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \geqslant \frac{5}{6}\int_{\mathbb{R}^{2d}} |V(t)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v. \tag{2.41}$$

Step 2: The decreasing sequence. Now, we construct two sequences $\{t_k\}_{k=1}^{\infty}$ and $\{N_{v,2}(f(t_k))\}_{k=1}^{\infty}$. We begin with $t_1 = 0$ and there exists some R_{t_1} such that (2.40) and (2.41) hold for t_1. Set

$$T_1 = \left\{ s\in\mathbb{R} \,\middle|\, \text{for all } \tau\in[t_1,s), \int_{B_{R_{t_1}}\times\mathbb{R}^d} |V(\tau)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \right.$$
$$\left. > \int_{B^c_{R_{t_1}}\times\mathbb{R}^d} |V(\tau)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \right\}.$$

By continuity of $V(t)$ and (2.41), there exists some $\delta > 0$ such that $[t_1,\,\delta)\subset T_1$. Define

$$t_2 = \sup T_1.$$

Then, for all $s\in[0,\,t_2)$, we have

$$\begin{aligned}
&\int_{B_{R_{t_1}}\times\mathbb{R}^d} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \cdot \int_{B_{R_{t_1}}\times\mathbb{R}^d} |V(s)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \\
&\quad - \int_{B^c_{R_{t_1}}\times\mathbb{R}^d} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \cdot \int_{B^c_{R_{t_1}}\times\mathbb{R}^d} |V(s)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \\
&\geqslant \left\{\frac{5}{6}\|f_0\|_{L^1} - \frac{1}{6}\|f_0\|_{L^1}\right\}\frac{1}{2}\int_{\mathbb{R}^{2d}} |V(s)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \\
&\geqslant \frac{1}{3}\|f_0\|_{L^1}\int_{\mathbb{R}^{2d}} |V(s)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v.
\end{aligned} \tag{2.42}$$

We divide into two situations to discuss.

(i) $t_2 = \infty$. Combing (2.39) with (2.42) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}} |V(t)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \leqslant -\frac{2}{3}\widetilde{H_{R_{t_1}}}(t)\|f_0\|_{L^1}\int_{\mathbb{R}^{2d}} |V(t)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v.$$

Thus we have

$$\int_{\mathbb{R}^{2d}} |V(t)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \leqslant e^{\|f_0\|_{L^1}\int_0^t -\frac{2}{3}\widetilde{H_{R_{t_1}}}(s)ds}\int_{\mathbb{R}^{2d}} |v|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v$$

by the Grönwall's inequality. For $\beta\leqslant\frac{1}{2}$, the function $\frac{2}{3}\widetilde{H_{R_{t_1}}}$ is not integrable at ∞, which yields

$$\lim_{t\to\infty} N_{v,2}(f(t)) = 0.$$

(ii) $t_2 < \infty$. By the definition of t_2 we obtain

$$\int_{B_{R_{t_1}}\times\mathbb{R}^d} |V(t_2)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v = \int_{B^c_{R_{t_1}}\times\mathbb{R}^d} |V(t_2)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v.$$

Then, using the fact that $N_{v,2}(f(t))$ is non-increasing in t, we have

$$\begin{aligned}
&\int_{B_{R_{t_1}}\times\mathbb{R}^d} |V(t_1)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v - \int_{B_{R_{t_1}}\times\mathbb{R}^d} |V(t_2)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v\\
&\geqslant \frac{5}{6}\int_{\mathbb{R}^{2d}} |V(t_1)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v - \frac{1}{2}\int_{\mathbb{R}^{2d}} |V(t_2)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v\\
&\geqslant \frac{5}{6}\int_{\mathbb{R}^{2d}} |V(t_1)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v - \frac{1}{2}\int_{\mathbb{R}^{2d}} |V(t_1)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v\\
&\geqslant \frac{1}{3}\int_{\mathbb{R}^{2d}} |V(t_1)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v\\
&\geqslant \frac{P}{3}.
\end{aligned}$$

And for all $t \geqslant 0$

$$\begin{aligned}
\left|\frac{\mathrm{d}}{\mathrm{d}t}|V(t)|^2\right| &\leqslant 2\int_{\mathbb{R}^{2d}} \frac{|V(t)-\omega||V(t)|}{(1+|X(t)-y|^2)^\beta} f(t,y,\omega)\,\mathrm{d}y\,\mathrm{d}\omega\\
&\leqslant 4\rho^2\|f_0\|_{L^1}.
\end{aligned}$$

Hence

$$\begin{aligned}
4\rho^2\|f_0\|_{L^1}^2(t_2-t_1) &\geqslant \int_{B_{R_{t_1}}\times\mathbb{R}^d} 4\rho^2\|f_0\|_{L^1}(t_2-t_1) f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v\\
&\geqslant \int_{B_{R_{t_1}}\times\mathbb{R}^d} \int_{t_1}^{t_2} \left|\frac{\mathrm{d}}{\mathrm{d}t}|V(t)|^2\right| f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v\\
&\geqslant \int_{B_{R_{t_1}}\times\mathbb{R}^d} |V(t_1)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v\\
&\quad - \int_{B_{R_{t_1}}\times\mathbb{R}^d} |V(t_2)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v\\
&\geqslant \frac{P}{3}.
\end{aligned}$$

Thus, we get

$$t_2 - t_1 \geqslant \frac{P}{12\|f_0\|_{L^1}^2\rho^2} \tag{2.43}$$

and

$$N_{v,2}(f(t_2)) \leqslant N_{v,2}(f(t_1)). \tag{2.44}$$

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Next, according to theorem 2.5, we set $f(t_2, x, v)$ as initial datum. Then, there exists some R_{t_2} such that (2.40) and (2.41) hold for t_2. Similarly, we denote

$$T_2 = \left\{ s \in \mathbb{R} \,\middle|\, \text{for all } \tau \in [t_2, s), \int_{B_{R_{t_2}} \times \mathbb{R}^d} |V(\tau)|^2 f_0(x, v)\, \mathrm{d}x\, \mathrm{d}v \right. \\ \left. > \int_{B^c_{R_{t_2}} \times \mathbb{R}^d} |V(\tau)|^2 f_0(x, v)\, \mathrm{d}x\, \mathrm{d}v \right\}.$$

By continuity of $V(t)$ and (2.41), there exists some $\delta > 0$ such that $[t_2, \delta) \subset T_2$ and we define that

$$t_3 = \sup T_2.$$

We also divide into two situations to discuss.

(i) $t_3 = \infty$. The proof is similar to the case $t_2 = \infty$.

(ii) $t_3 < \infty$. We first use the definition of t_3 to obtain that

$$\int_{B_{R_{t_2}} \times \mathbb{R}^d} |V(t_3)|^2 f_0(x, v)\, \mathrm{d}x\, \mathrm{d}v = \int_{B^c_{R_{t_2}} \times \mathbb{R}^d} |V(t_3)|^2 f_0(x, v)\, \mathrm{d}x\, \mathrm{d}v.$$

Then, similar to the case $t_2 < \infty$ we have

$$t_3 - t_2 \geqslant \frac{P}{12\|f_0\|^2_{L^1} \rho^2}$$

and

$$N_{v,2}(f(t_3)) \leqslant N_{v,2}(f(t_2)).$$

By repeating the above procedure, if all $t_k < \infty$, there are two sequences $\{t_k\}_{k=1}^{\infty}$ and $\{N_{v,2}(f(t_k))\}_{k=1}^{\infty}$. If $P \neq 0$, we obtain

$$\sum_{k=1}^{\infty} \{t_{k+1} - t_k\} = \infty$$

from

$$t_{k+1} - t_k \geqslant \frac{P}{12\|f_0\|^2_{L^1} \rho^2}.$$

Step 3: Split the integral of initial datum. We split $\mathbb{R}^{2d} = \cup_{l=1}^{\infty} E_l$, where $E_i \cap E_j = \emptyset$ for $i \neq j$, E_l is bounded and

$$0 < m_l := \int_{E_l} f_0(x, v)\, \mathrm{d}x\, \mathrm{d}v < \|f_0\|_{L^1}.$$

Moreover, we denote

$$|v_l(t_k)|^2 := \frac{\int_{E_l} |V(t_k)|^2 f_0(x, v)\, \mathrm{d}x\, \mathrm{d}v}{\int_{E_l} f_0(x, v)\, \mathrm{d}x\, \mathrm{d}v} = \frac{\int_{E_l} |V(t_k)|^2 f_0(x, v)\, \mathrm{d}x\, \mathrm{d}v}{m_l},$$

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which yields

$$\int_{\mathbb{R}^{2d}} |V(t_k)|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v = \sum_{l=1}^{\infty} m_l |v_l(t_k)|^2. \tag{2.45}$$

Now, since $\{v_1(t_k)\}_{k=1}^{\infty}$ are bounded, there exists a convergence subsequence $\{v_1(t_{k_h})\}_{h=1}^{\infty}$ such that

$$\lim_{h\to\infty} |v_1(t_{k_h})| = |v_1^\star|.$$

Via the classical diagonal method, there exists some subsequence $\{k_h\}_{h=1}^{\infty}$ such that

$$\lim_{h\to\infty} |v_l(t_{k_h})| = |v_l^\star| \quad \text{for all} \quad l \in \mathbb{N}.$$

Then, using (2.45) we obtain

$$\lim_{h\to\infty} \sum_{l=1}^{\infty} m_l |v_l(t_{k_h})|^2 = \sum_{l=1}^{\infty} m_l |v_l^\star|^2 = P. \tag{2.46}$$

Here we used the dominated convergence theorem of discrete form. And there exists some N_1 such that

$$\sum_{l=1}^{N_1} m_l |v_l^\star|^2 \geqslant \frac{5}{6} \sum_{l=1}^{\infty} m_l |v_l^\star|^2.$$

For N_1, for any $\varepsilon_1 > 0$, there exists some $t_{k_{N_1}}$ such that for all $t_{k_h} > t_{k_{N_1}}$,

$$|v_l(t_{k_h})|^2 > |v_l^\star|^2 - \frac{\varepsilon_1}{2^l} \quad \text{for all} \quad l \leqslant N_1.$$

Thus, we can choose ε_1 sufficiently small such that

$$\begin{aligned} \sum_{l=1}^{N_1} m_l |v_l(t_{k_h})|^2 &\geqslant \sum_{l=1}^{N_1} m_l |v_l^\star|^2 - \|f_0\|_{L^1} \cdot \varepsilon_1 \\ &\geqslant \frac{5}{6} \sum_{l=1}^{\infty} m_l |v_l^\star|^2 - \|f_0\|_{L^1} \cdot \varepsilon_1 \\ &> \frac{2}{3} \sum_{l=1}^{\infty} m_l |v_l^\star|^2. \end{aligned} \tag{2.47}$$

Next, for $\varepsilon_2 = \frac{P}{3}$, there exists some $t_{k_{\varepsilon_2}}$ such that

$$N_{v,2}(f(t_k)) < P + \varepsilon_2 = \frac{4}{3} P \quad \text{for all} \quad t_k \geqslant t_{k_{\varepsilon_2}}. \tag{2.48}$$

It is obvious that the index R_{t_k} is unbounded, since

$$P = \lim_{k\to\infty} N_{v,2}(f(t_k)) \leqslant \lim_{k\to\infty} e^{\|f_0\|_{L^1} \sum_{q=1}^{k} \int_{t_q}^{t_{q+1}} -\frac{2}{3}\widetilde{H_{R_{t_q}}}(s) ds} N_{v,2}(f(0)).$$

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Above inequality gives $P = 0$ if the index R_{t_k} is bounded. Then, combing (2.47) with (2.48), there exists some $R_{t_q} > 0$ and $t_{q+1} > \max\left\{t_{k_{N_1}}, t_{k_{\varepsilon_2}}\right\}$ such that

$$\cup_{l=1}^{N_1} E_l \subset B_{R_{t_q}} \times \mathbb{R}^d$$

and

$$\begin{aligned}
\int_{B_{R_{t_q}} \times \mathbb{R}^d} |V(t_{q+1})|^2 f_0(x, v)\, \mathrm{d}x\, \mathrm{d}v &\geqslant \int_{\cup_{l=1}^{N_1} E_l} |V(t_{q+1})|^2 f_0(x, v)\, \mathrm{d}x\, \mathrm{d}v \\
&= \sum_{l=1}^{N_1} m_l |v_l(t_{q+1})|^2 > \frac{2}{3} \sum_{l=1}^{\infty} m_l |v_l^\star|^2 = \frac{2}{3} P > \frac{1}{2} N_{v,2}(f(t_{q+1})) \qquad (2.49) \\
&= \int_{B^c_{R_{t_q}} \times \mathbb{R}^d} |V(t_{q+1})|^2 f_0(x, v)\, \mathrm{d}x\, \mathrm{d}v,
\end{aligned}$$

which contradicts with the definition of t_{q+1}. Therefore, $P = 0$. □

REMARK 2.8. Similar to the proof of the classical solution to the model (1.3)–(1.4) in theorem 2.7, we can establish the corresponding formation behaviour of the solutions to the infinite-particle Cucker–Smale model (1.6)–(1.7) in $l_m^2(\mathbb{R}^d) \times l_m^2(\mathbb{R}^d)$.

3. Measure-valued solution

In this section, we study the existence, uniqueness and formation behaviour of the measure-valued solution to (1.3)–(1.4). Let us review the notion of the measure-valued solution to (1.3)–(1.4).

DEFINITION 3.1. *Let $\mathcal{P}(\mathbb{R}^d)$ be the set of probability measure on $\mathbb{R}^d$. For $T \in [0, \infty)$, $\mu \in L^\infty([0, T); \mathcal{P}(\mathbb{R}^{2d}))$ is a measure-valued solution to (1.3)–(1.4) with initial datum $\mu_0 \in \mathcal{P}(\mathbb{R}^{2d})$ if the following two assertions hold:*
(1) μ is weakly continuous in t:

$$\int_{\mathbb{R}^{2d}} \phi(x, v) \mu(t, \mathrm{d}x, \mathrm{d}v) \quad \textit{is continuous in } t \textit{ for any} \quad \phi \in C_0^1(\mathbb{R}^{2d}).$$

(2) μ satisfies (1.3)–(1.4) in the following weak sense:

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} \psi \mu(t, \mathrm{d}x, \mathrm{d}v) - \int_{\mathbb{R}^{2d}} \psi \mu(0, \mathrm{d}x, \mathrm{d}v) \\
&\quad = \int_0^t \int_{\mathbb{R}^{2d}} (\partial_s \psi + v \cdot \nabla_x \psi + \nabla_v \psi \cdot L[\mu]) \mu(s, \mathrm{d}x, \mathrm{d}v)\, \mathrm{d}s
\end{aligned}$$

for any $\psi \in C_0^1([0, T) \times \mathbb{R}^{2d})$.

The proof of the formation behaviour of the measure-valued solution to (1.3)–(1.4) is divided into the following four steps. First, we obtain the stability of the classical solution, which is formulated in terms of Wasserstein distance.

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Second, we regularize the initial datum and use the theorem 2.5 to obtain a sequence of approximate solutions, which is a Cauchy sequence in $(\mathcal{P}_p(\mathbb{R}^{2d}), W_p)$. Moreover, by passing the limit of the approximate solutions, we get a measure-valued solution. Finally, similar to the classical solution, the formation behaviour of the measure-valued solution to (1.3)–(1.4) is established.

For Wasserstein distance, we briefly recall the definition and some basic properties below.

DEFINITION 3.2. *Let $\mathcal{P}(\mathbb{R}^d)$ be the set of probability measure on $\mathbb{R}^d$, and let $p \in [1, \infty)$. For any two μ, $\nu \in \mathcal{P}(\mathbb{R}^d)$, the Wasserstein distance of order p between μ and ν is defined by the formula*

$$W_p(\mu,\nu) := \inf\left\{\left(\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^p\pi(\mathrm{d}x,\mathrm{d}y)\right)^{\frac{1}{p}} : \pi\in\Pi(\mu,\nu)\right\},$$

where $\Pi(\mu, \nu)$ is the set of probability measure on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals μ and ν, respectively. For all integrable (resp. nonnegative) measurable functions ψ, ϕ on $\mathbb{R}^d$,

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}(\psi(x)+\phi(y))\pi(\mathrm{d}x,\mathrm{d}y) = \int_{\mathbb{R}^d}\psi(x)\mu(\mathrm{d}x) + \int_{\mathbb{R}^d}\phi(y)\nu(\mathrm{d}y). \tag{3.1}$$

In order to avoid the trouble that W_p may take the value $+\infty$, we consider W_p on $\mathcal{P}_p(\mathbb{R}^d)$:

$$\mathcal{P}_p(\mathbb{R}^d) := \left\{\mu\in\mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d}|z|^p\mu(\mathrm{d}z)<\infty\right\}.$$

And list the characterization of the convergence in W_p below as a lemma.

LEMMA 3.3 [**25**]. *$\mathcal{P}_p(\mathbb{R}^d)$ endowed with the p-Wasserstein distance is a complete metric space. Let $(\mu_k)_{k\in\mathbb{N}}$ be a sequence of probability measures in $\mathcal{P}_p(\mathbb{R}^d)$ and let μ be another element of $\mathcal{P}_p(\mathbb{R}^d)$. We say that $(\mu_k)_{k\in\mathbb{N}}$ has uniformly integrable p-moments if for some $x_0 \in \mathbb{R}^d$:*

$$\lim_{r\to\infty}\int_{\mathbb{R}^d\setminus B_r(x_0)}|x-x_0|^p\,\mathrm{d}\mu_k(x) = 0 \quad \textit{uniformly with respect to } k\in\mathbb{N}.$$

Moreover, we say that μ_k converge weakly to μ if

$$\lim_{k\to\infty}\int_{\mathbb{R}^d}\phi(x)\mu_k(\mathrm{d}x) = \int_{\mathbb{R}^d}\phi(x)\mu(\mathrm{d}x) \quad \textit{for all} \quad \phi\in C_b(\mathbb{R}^d). \tag{3.2}$$

In particular, we have

$$\lim_{k\to\infty}W_p(\mu_k,\mu)=0 \Leftrightarrow \begin{cases}\mu_k & \textit{converge weakly to} \quad \mu,\\ (\mu_k)_{k\in\mathbb{N}} & \textit{has uniformly integrable } p\textit{-moments.}\end{cases}$$

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REMARK 3.4. From the fact that μ_k converge weakly to μ we can deduce

$$\mathrm{supp}\mu \subset \overline{\bigcup_k \mathrm{supp}\mu_k}.$$

Now, we first provide the stability result of the classical solution to (1.3)–(1.4).

LEMMA 3.5. *Assume that the initial probability densities f_0, h_0 satisfy the assumptions of theorem 2.7. For any $T > 0$, there exists some positive constant C_T depending only upon ρ, β, p, T such that*

$$W_p(f(t), h(t)) \leqslant C_T W_p(f_0, h_0), \quad \forall t \in [0, T],$$

where f, h are solutions to (1.3)–(1.4) with initial data f_0, h_0, respectively.

Proof. For any $\pi_0 \in \Pi(f_0, h_0)$, we define the forward characteristic flows of f and h with the initial data (x, v), $(y, \omega) \in \mathbb{R}^{2d}$ as follows:

$$\begin{cases} \dot{X}(t,0,x,v) = V(t,0,x,v), \quad X(0,0,x,v) = x, \\ \dot{V}(t,0,x,v) = L[f](t, X(t,0,x,v), V(t,0,x,v)), \quad V(0,0,x,v) = v \end{cases}$$

and

$$\begin{cases} \dot{Y}(t,0,y,\omega) = W(t,0,y,\omega), \quad Y(0,0,y,\omega) = y, \\ \dot{W}(t,0,y,\omega) = L[h](t, Y(t,0,y,\omega), W(t,0,y,\omega)), \quad W(0,0,y,\omega) = \omega. \end{cases}$$

The forward characteristic flows of f and h are defined by $(X(t), V(t))$ and $(Y(t), W(t))$. Following from the definition of the characteristic flows and (3.1), we get

$$|X(t) - Y(t)| \leqslant |x - y| + \int_0^t |V(s) - W(s)| \, \mathrm{d}s \tag{3.3}$$

and

$$\begin{aligned} &|V(t) - W(t)| \\ &\quad \leqslant |v - \omega| + \int_0^t \bigg| \int_{\mathbb{R}^{2d}} \frac{V(s) - u_f}{(1 + |X(s) - r_f|^2)^\beta} f(s, r_f, u_f) \, \mathrm{d}r_f \, \mathrm{d}u_f \\ &\qquad - \int_{\mathbb{R}^{2d}} \frac{W(s) - u_h}{(1 + |Y(s) - r_h|^2)^\beta} h(s, r_h, u_h) \, \mathrm{d}r_h \, \mathrm{d}u_h \bigg| \, \mathrm{d}s \\ &\quad = |v - \omega| + \int_0^t \bigg| \int_{\mathbb{R}^{2d}} \frac{V(s) - U_f(s,0,r_f,u_f)}{(1 + |X(s) - R_f(s,0,r_f,u_f)|^2)^\beta} f_0(r_f, u_f) \, \mathrm{d}r_f \, \mathrm{d}u_f \\ &\qquad - \int_{\mathbb{R}^{2d}} \frac{W(s) - U_h(s,0,r_h,u_h)}{(1 + |Y(s) - R_h(s,0,r_h,u_h)|^2)^\beta} h_0(r_h, u_h) \, \mathrm{d}r_h \, \mathrm{d}u_h \bigg| \, \mathrm{d}s \\ &\quad \leqslant |v - \omega| + \int_0^t \int_{\mathbb{R}^{4d}} \bigg| \frac{V(s) - U_f(s)}{(1 + |X(s) - R_f(s)|^2)^\beta} \\ &\qquad - \frac{W(s) - U_h(s)}{(1 + |Y(s) - R_h(s)|^2)^\beta} \bigg| \pi_0(\mathrm{d}\Omega_{r,u}) \, \mathrm{d}s, \end{aligned}$$

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where $\mathrm{d}\Omega_{r,u} = \mathrm{d}r_f\,\mathrm{d}u_f\,\mathrm{d}r_h\,\mathrm{d}u_h$ and we denote the forward characteristic flows with initial data $(r_f,\ u_f)$ and $(r_h,\ u_h)$ by $(R_f(s),\ U_f(s))$ and $(R_h(s),\ U_h(s))$, respectively. Similar to (2.27), for any $t \in [0,\ T]$ we get

$$\begin{aligned} |V(t)-W(t)| \leqslant |v-\omega| + C\int_0^t |V(s)-W(s)| + |X(s)-Y(s)|\,\mathrm{d}s \\ + C\int_0^t \int_{\mathbb{R}^{4d}} |U_f(s)-U_h(s)| + |R_f(s)-R_h(s)|\pi_0(\mathrm{d}\Omega_{r,u})\,\mathrm{d}s. \end{aligned} \tag{3.4}$$

Then, from (3.3) and (3.4) we have

$$\begin{aligned} &|V(t)-W(t)|^p + |X(t)-Y(t)|^p \\ &\leqslant C(|v-\omega|^p + |x-y|^p) + C\int_0^t |V(s)-W(s)|^p + |X(s)-Y(s)|^p\,\mathrm{d}s \\ &\quad + C\int_0^t \int_{\mathbb{R}^{4d}} |U_f(s)-U_h(s)|^p + |R_f(s)-R_h(s)|^p\pi_0(\mathrm{d}\Omega_{r,u})\,\mathrm{d}s \\ &= C(|v-\omega|^p + |x-y|^p) + C\int_0^t |V(s)-W(s)|^p + |X(s)-Y(s)|^p\,\mathrm{d}s \\ &\quad + C\int_0^t \int_{\mathbb{R}^{4d}} |X(s)-Y(s)|^p + |V(s)-W(s)|^p\pi_0(\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\omega)\,\mathrm{d}s, \end{aligned} \tag{3.5}$$

where we used the Hölder inequality. By denoting $\mathrm{d}\Omega_{x,v,y,\omega} = \mathrm{d}x\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\omega$ we obtain

$$\begin{aligned} &\int_{\mathbb{R}^{4d}} |V(t)-W(t)|^p + |X(t)-Y(t)|^p\pi_0(\mathrm{d}\Omega_{x,v,y,\omega}) \\ &\quad\leqslant C\int_{\mathbb{R}^{4d}} (|v-\omega|^p + |x-y|^p)\pi_0(\mathrm{d}\Omega_{x,v,y,\omega}) \\ &\qquad + C\int_0^t \int_{\mathbb{R}^{4d}} |X(s)-Y(s)|^p + |V(s)-W(s)|^p\pi_0(\mathrm{d}\Omega_{x,v,y,\omega})\,\mathrm{d}s. \end{aligned}$$

By the Grönwall's inequality, there exists some positive constant C_T depending only on $T,\ \beta,\ p,\ \rho$ such that

$$\begin{aligned} &\left[\int_{\mathbb{R}^{4d}} (|X(t)-Y(t)|^p + |V(t)-W(t)|^p)\pi_0(\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\omega)\right]^{\frac{1}{p}} \\ &\quad\leqslant C_T\left[\int_{\mathbb{R}^{4d}} (|x-y|^p + |v-\omega|^p)\pi_0(\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\omega)\right]^{\frac{1}{p}}. \end{aligned} \tag{3.6}$$

Note that

$$\pi(t, \mathrm{d}x\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\omega) := (X(t), V(t), Y(t), W(t))\#\pi_0(\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\omega) \in \Pi(f(t), h(t)).$$

It follows from (3.6) that

$$W_p(f(t),h(t)) \leqslant \left[\int_{\mathbb{R}^{4d}} (|x-y|^p + |v-\omega|^p)\pi(t, \mathrm{d}x\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\omega)\right]^{\frac{1}{p}}$$
$$\leqslant C_T \left[\int_{\mathbb{R}^{4d}} (|x-y|^p + |v-\omega|^p)\pi_0(\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\omega)\right]^{\frac{1}{p}},$$

and then by the definition of W_p we complete the proof. □

Next, we provide the following lemma to estimate the difference between the initial datum and the regularized one.

LEMMA 3.6 [**16**]. *Let $\zeta_\varepsilon \in C_c^\infty(\mathbb{R}^d)$ be the mollifier. Assume that $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$. Then, for $p \in [1, \infty)$ we have*

$$W_p(\mu * \zeta_\varepsilon, \mu) \leqslant \varepsilon \tag{3.7}$$

and

$$W_p(\mu * \zeta_\varepsilon, \nu * \zeta_\varepsilon) \leqslant W_p(\mu, \nu). \tag{3.8}$$

And then we show the global existence and uniqueness of measure-valued solution to (1.3)–(1.4).

LEMMA 3.7. *Let the initial datum $\mu_0 \in \mathcal{P}_p(\mathbb{R}^{2d})$. If there exists a positive constant ρ such that*

$$\rho := \sup_{(x,v)\in \mathrm{supp}\mu_0} |v| < \infty. \tag{3.9}$$

Then, there exists a global unique measure-valued solution $\mu \in L^\infty([0, T); \mathcal{P}_p(\mathbb{R}^{2d}))$ to (1.3)–(1.4) and

$$\sup_{(x,v)\in \mathrm{supp}\mu(t)} |v| \leqslant \rho. \tag{3.10}$$

Proof. Let $f_0^n = \mu_0 * \zeta_{\frac{1}{n}}$, it is easy to check that $f_0^n \in C_0^1(\mathbb{R}^{2d}) \cap \mathcal{P}_p(\mathbb{R}^{2d})$ and

$$\mathrm{supp} f_0^n \subset \overline{\mathrm{supp}\mu_0 + \mathrm{supp}\zeta_{\frac{1}{n}}},$$

which yields that

$$\sup_{(x,v)\in \mathrm{supp} f_0^n} |v| \leqslant \rho + 1.$$

Then, from theorem 2.5 we obtain a global unique classical solution f_n and

$$\sup_{(x,v)\in \mathrm{supp} f_n(t)} |v| \leqslant \rho + 1. \tag{3.11}$$

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Combined the lemma 3.5 with (3.7), there exists some integer N such that for any $n,\, m \geqslant N$,

$$\begin{aligned} W_p(f_n(t), f_m(t)) &\leqslant C_T W_p(f_0^n, f_0^m) \\ &\leqslant C_T W_p(f_0^n, \mu_0) + C_T W_p(\mu_0, f_0^m) \leqslant \frac{C_T}{N}, \quad t \in [0, T], \end{aligned} \tag{3.12}$$

which yields that $f_n(t)$ is a Cauchy sequence in $(\mathcal{P}_p(\mathbb{R}^{2d}), W_p)$. Thus, by lemma 3.3, there exists a probability measure $\mu(t)$ such that f_n converges weakly to $\mu(t)$ in $\mathcal{P}_p(\mathbb{R}^{2d})$ and

$$\sup_{(x,v)\in \mathrm{supp}\mu(t)} |v| \leqslant \rho + 1. \tag{3.13}$$

Depending on the remark 3.8, we can replace $\rho+1$ at the right side of the above inequality with ρ. Now, we claim that $\mu(t)$ is a measure-valued solution to (1.3)–(1.4). Note that

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} \psi f_n(t)\,\mathrm{d}x\,\mathrm{d}v - \int_{\mathbb{R}^{2d}} \psi f_0^n\,\mathrm{d}x\,\mathrm{d}v \\ &\quad = \int_0^t \int_{\mathbb{R}^{2d}} (\partial_s \psi + v\cdot\nabla_x\psi + \nabla_v\psi\cdot L[f_n]) f_n\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s \end{aligned} \tag{3.14}$$

for any $\psi \in C_0^1([0,\, T) \times \mathbb{R}^{2d})$, since f_n is the classical solution with initial datum f_0^n. Using (3.2), (3.11) and (3.13) we get

$$\begin{aligned} &\lim_{n\to\infty} \int_0^t \int_{\mathbb{R}^{2d}} v\cdot\nabla_x\psi f_n\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s = \lim_{n\to\infty} \int_0^t \int_{\mathbb{R}^d\times B_{\rho+2}} v\cdot\nabla_x\psi f_n\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s \\ &= \int_0^t \int_{\mathbb{R}^d\times B_{\rho+2}} v\cdot\nabla_x\psi\mu(s,\mathrm{d}x,\mathrm{d}v)\,\mathrm{d}s = \int_0^t \int_{\mathbb{R}^{2d}} v\cdot\nabla_x\psi\mu(s,\mathrm{d}x,\mathrm{d}v)\,\mathrm{d}s. \end{aligned} \tag{3.15}$$

Next, we denote $f_n(t)^{\otimes 2}$ the product measure $f_n(t)\otimes f_n(t)$. And recall the following inequality:

$$W_p(f_n(t)^{\otimes 2}, \mu(t)^{\otimes 2}) \leqslant 2^p W_p(f_n(t), \mu(t)), \tag{3.16}$$

which can be easily obtained from the definition of Wasserstein distance W_p. Combing with the fact that $f_n(t)$ converges weakly to $\mu(t)$ in $\mathcal{P}_p(\mathbb{R}^{2d})$, we deduce that $f_n(t)^{\otimes 2}$ converges weakly to $\mu(t)^{\otimes 2}$ in $\mathcal{P}_p(\mathbb{R}^{4d})$ from (3.16). Similar to (3.15), we have

$$\begin{aligned} &\lim_{n\to\infty} \int_0^t \int_{\mathbb{R}^{2d}} \nabla_v\psi\cdot L[f_n] f_n\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s \\ &= \lim_{n\to\infty} -\int_0^t \int_{\mathbb{R}^{4d}} \nabla_v\psi\cdot \frac{v-\omega}{(1+|x-y|^2)^\beta} f_n(s,x,v) f_n(s,y,\omega)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\omega\,\mathrm{d}s \\ &= -\int_0^t \int_{\mathbb{R}^{4d}} \nabla_v\psi\cdot \frac{v-\omega}{(1+|x-y|^2)^\beta} \mu(s,\mathrm{d}x,\mathrm{d}v)\mu(s,\mathrm{d}y,\mathrm{d}\omega)\,\mathrm{d}s \\ &= \int_0^t \int_{\mathbb{R}^{2d}} \nabla_v\psi\cdot L[\mu]\mu(s,\mathrm{d}x,\mathrm{d}v)\,\mathrm{d}s. \end{aligned} \tag{3.17}$$

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Then, let $n \to \infty$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^{2d}} \psi\mu(t, \mathrm{d}x, \mathrm{d}v) - \int_{\mathbb{R}^{2d}} \psi\mu(0, \mathrm{d}x, \mathrm{d}v) \\ = \int_0^t \int_{\mathbb{R}^{2d}} (\partial_s\psi + v\cdot\nabla_x\psi + \nabla_v\psi\cdot L[\mu])\mu(s, \mathrm{d}x, \mathrm{d}v)\,\mathrm{d}s\end{aligned} \tag{3.18}$$

from (3.14). Now, we show that μ is weakly continuous in t. For any $\phi \in C_0^1(\mathbb{R}^{2d})$, combing (3.13) with (3.18), there exists some constant C depending only on ρ such that

$$\begin{aligned}&\left|\int_{\mathbb{R}^{2d}} \phi\mu(t, \mathrm{d}x, \mathrm{d}v) - \int_{\mathbb{R}^{2d}} \phi\mu(s, \mathrm{d}x, \mathrm{d}v)\right| \\ &\quad = \left|\int_s^t \int_{\mathbb{R}^{2d}} (v\cdot\nabla_x\phi + \nabla_v\phi\cdot L[\mu])\mu(\tau, \mathrm{d}x, \mathrm{d}v)\,\mathrm{d}\tau\right| \leqslant C(t-s),\end{aligned}$$

which yields that μ is weakly continuous in t. At last, similar to the proof of the step 4 in theorem 2.5, μ is the global unique measure-valued solution to (1.3)–(1.4). □

Remark 3.8. If we fix the mollifier differently we can show

$$\sup_{(x,v)\in \mathrm{supp}\mu(t)} |v| \leqslant \rho + \varepsilon$$

for arbitrary small $\varepsilon > 0$.

Except for the regularity of the initial datum, our conditions for the initial datum of the measure-valued solution to (1.3)–(1.4) are almost the same as those for the classical solution to (1.3)–(1.4). With the above preparations, we establish the formation behaviour of the measure-valued solution to (1.3)–(1.4) as below:

Theorem 3.9. *Let the initial datum μ_0 satisfy the assumptions of lemma 3.7 and μ be the corresponding measure-valued solution to (1.3)–(1.4). When $\beta \in [0, \frac{1}{2}]$, we have*:

$$\lim_{t\to\infty} \int_{\mathbb{R}^{2d}} |v - v_c|^2\mu(t, \mathrm{d}x, \mathrm{d}v) = 0, \quad v_c = \|\mu_0\|_{L^1}^{-1} \int_{\mathbb{R}^{2d}} v\mu_0(\mathrm{d}x, \mathrm{d}v).$$

Proof. Since the overall proof of theorem 3.9 is almost the same as that of theorem 2.7, we will only sketch the proof.

Step 1: Some differential equalities. Similar to lemma 2.6, we can establish some differential equalities:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}} \mu(t, \mathrm{d}x, \mathrm{d}v) = 0, \tag{3.19}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}} v\mu(t, \mathrm{d}x, \mathrm{d}v) = 0, \tag{3.20}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}} |v|^2\mu(t, \mathrm{d}x, \mathrm{d}v) = -\int_{\mathbb{R}^{4d}} \frac{|v-\omega|^2}{(1+|x-y|^2)^\beta}\mu(t, \mathrm{d}x, \mathrm{d}v)\mu(t, \mathrm{d}y, \mathrm{d}\omega). \tag{3.21}$$

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For simplicity of notation, we also use v instead of $v - v_c$ to calculation below and denote that

$$M_{v,2}(\mu(t)) := \int_{\mathbb{R}^{2d}} |v|^2 \mu(t, \mathrm{d}x, \mathrm{d}v).$$

And (3.21) gives

$$\lim_{t\to\infty} M_{v,2}(\mu(t)) = P, \tag{3.22}$$

where $P \geqslant 0$. Next, we are devoted to prove that $P = 0$.

Step 2: The decreasing sequence. Let (x, v), (y, ω) be the initial data of the forward characteristic flows of μ, that is

$$\begin{cases} \dot{X}_\mu(t,0,x,v) = V_\mu(t,0,x,v), & X_\mu(0,0,x,v) = x, \\ \dot{V}_\mu(t,0,x,v) = L[\mu](t, X_\mu(t,0,x,v), V_\mu(t,0,x,v)), & V_\mu(0,0,x,v) = v \end{cases}$$

and

$$\begin{cases} \dot{Y}_\mu(t,0,y,\omega) = W_\mu(t,0,y,\omega), & Y_\mu(0,0,y,\omega) = y, \\ \dot{W}_\mu(t,0,y,\omega) = L[\mu](t, Y_\mu(t,0,y,\omega),\ W_\mu(t,0,y,\omega)), & W_\mu(0,0,y,\omega) = \omega. \end{cases}$$

For simplicity, we denote

$$(V_\mu(t), X_\mu(t)) = (V_\mu(t,0,x,v), X_\mu(t,0,x,v)),$$
$$(W_\mu(t), Y_\mu(t)) = (W_\mu(t,0,y,\omega), Y_\mu(t,0,y,\omega)).$$

And rewrite (3.21) as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^{2d}} |V_\mu(t)|^2 \mu_0(\mathrm{d}x, \mathrm{d}v) = -\int_{\mathbb{R}^{4d}} \frac{|V_\mu(t) - W_\mu(t)|^2}{(1 + |X_\mu(t) - Y_\mu(t)|^2)^\beta} \mu_0(\mathrm{d}x, \mathrm{d}v)\mu_0(\mathrm{d}y, \mathrm{d}\omega). \tag{3.23}$$

Note that

$$\sup_{(y,\omega)\in \mathrm{supp}\mu_0} |W_\mu(t)| \leqslant \rho, \qquad \sup_{(x,v)\in \mathrm{supp}\mu_0} |V_\mu(t)| \leqslant \rho.$$

Then

$$|X_\mu(t) - Y_\mu(t)| \leqslant |x - y| + 2\rho t.$$

For any fixed R, we denote

$$H_R(t) := \inf_{x,y\in B_R} H(|X_\mu(t) - Y_\mu(t)|).$$

It is obvious that

$$H_R(t) \geqslant \inf_{x,y\in B_R} H(2R + 2\rho t) =: \widetilde{H_R}(t).$$

Therefore, similar to (2.39) we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^{2d}} |V_\mu(t)|^2 \mu_0(\mathrm{d}x, \mathrm{d}v) \leqslant & -2\widetilde{H_R}(t) \int_{B_R\times\mathbb{R}^d\times B_R\times\mathbb{R}^d} |V_\mu(t)|^2 \mu_0(\mathrm{d}x, \mathrm{d}v)\mu_0(\mathrm{d}y, \mathrm{d}\omega) \\ & + 2\widetilde{H_R}(t) \int_{B_R^c\times\mathbb{R}^d\times B_R^c\times\mathbb{R}^d} |V_\mu(t)|^2 \mu_0(\mathrm{d}x, \mathrm{d}v)\mu_0(\mathrm{d}y, \mathrm{d}\omega). \end{aligned}$$

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Moreover, using the argument used in the proof of the step 2 in theorem 2.7, there are two sequences $\{t_k\}_{k=1}^{\infty}$ and $\{M_{v,2}(\mu(t_k))\}_{k=1}^{\infty}$.

Step 3: Split the integral of initial datum. We first split $\mathbb{R}^{2d} = \cup_{l=1}^{\infty} E_l$, where $E_i \cap E_j = \emptyset$ for $i \neq j$, E_l is bounded and

$$0 < m_l := \int_{E_l} \mu_0(\mathrm{d}x, \mathrm{d}v) < \|\mu_0\|_{L^1}.$$

Thus, we have

$$|v_l(t_k)|^2 := \frac{\int_{E_l} |V_\mu(t_k)|^2 \mu_0(\mathrm{d}x, \mathrm{d}v)}{\int_{E_l} \mu_0(\mathrm{d}x, \mathrm{d}v)} = \frac{\int_{E_l} |V_\mu(t_k)|^2 \mu_0(\mathrm{d}x, \mathrm{d}v)}{m_l}$$

and

$$\int_{\mathbb{R}^{2d}} |V_\mu(t_k)|^2 \mu_0(\mathrm{d}x, \mathrm{d}v) = \sum_{l=1}^{\infty} m_l |v_l(t_k)|^2. \tag{3.24}$$

Similar to the same argument used in the step 3 in theorem 2.7, we obtain that $P = 0$ by contradiction. □

4. Conclusion

In this paper, we obtained the formation behaviour of the kinetic Cucker–Smale model for classical solution as well as measure-valued solution, whose initial datum is not compactly supported in x. Even if the compactness condition of support has been relaxed in this paper, the velocity support has to be compact in any case. In terms of the classical solution, we first used the characteristic flow to establish the non-expansion of the velocity support. Then we established the existence and uniqueness of the classical solution to the kinetic Cucker–Smale model. Futhermore, we provided a rigorous proof of the emergence of asymptotic formation behaviour. Finally, for the measure-valued solution to the kinetic Cucker–Smale model, the formation behaviour is also established.

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