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Local intertwining relation for metaplectic groups

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Abstract

In an earlier paper of Wee Teck Gan and Gordan Savin, the local Langlands correspondence for metaplectic groups over a nonarchimedean local field of characteristic zero was established. In this paper, we formulate and prove a local intertwining relation for metaplectic groups assuming the local intertwining relation for non-quasi-split odd special orthogonal groups.

1. Introduction

In his long-awaited book [Art13], Arthur obtained a classification of irreducible representations of quasi-split symplectic and special orthogonal groups over local fields of characteristic zero (the local Langlands correspondence, which we shall refer to as LLC for short). Recall the basic form of the correspondence over nonarchimedean local fields of characteristic zero. Let F be a p-adic field, i.e. a finite extension of $\mathbb{Q}_p$, for some prime number p. Let Γ_F and W_F be the absolute Galois group and the absolute Weil group of F, respectively. We shall write WD_F for the Weil–Deligne group $W_F \times \mathrm{SL}_2(\mathbb{C})$.

Let G be a connected reductive algebraic group defined over F. The LLC proposes a classification of irreducible tempered admissible representations of $G(F)$ in terms of tempered admissible L-parameters for G. Let $\hat{G}$ be the connected complex Langlands dual group of G. We write $\Pi_{\mathrm{temp}}(G)$ for the set of equivalence classes of irreducible tempered admissible representations of $G(F)$, and $\Phi_{\mathrm{temp}}(G)$ for the set of equivalence classes of tempered admissible L-parameters $\phi : WD_F \to \hat{G} \rtimes W_F$. The basic form of the LLC is the following.

Conjecture 1.1.

(1) There exists a canonical map

$$LL : \Pi_{\mathrm{temp}}(G) \longrightarrow \Phi_{\mathrm{temp}}(G)$$

with some important properties.

(2) For each $\phi \in \Phi_{\mathrm{temp}}(G)$, the fiber $\Pi_\phi = \Pi_\phi(G) = LL^{-1}(\phi)$ is a finite set, called a packet.

There are further expected properties. We refer the reader to [Bor79], [Art89b], or [Kal16] for details.

As mentioned above, Arthur [Art13] established the LLC for quasi-split SO_{2n}, SO_{2n+1}, and Sp_{2n}, i.e. the even special orthogonal, odd special orthogonal, and symplectic groups of rank n,

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respectively. Moreover, Mœglin and Renard [MR18] provided a classification of irreducible tempered representations of non-quasi-split odd special orthogonal groups over p-adic fields, hence the LLC of Vogan type. Recall the LLC of Vogan type [Vog93, Conjecture 4.15] over p-adic fields. Let G^* be a quasi-split connected reductive algebraic group over a p-adic field F. The LLC of Vogan type treats pure inner twists of G^* at the same time. For each $\phi \in \Phi_{\mathrm{temp}}(G)$, we let $S_\phi = S_\phi(G)$ denote the centralizer $\mathrm{Cent}(\mathrm{Im}\phi, \hat{G})$ and $\pi_0(S_\phi)$ its component group. Then the LLC of Vogan type proposes the following.

CONJECTURE 1.2.

(1) There exists a canonical map

$$LLV\colon \bigsqcup_{(\xi,z)} \Pi_{\mathrm{temp}}(G) \longrightarrow \Phi_{\mathrm{temp}}(G^*),$$

where (ξ, z) runs over the isomorphism classes of pure inner twists of G^*, i.e. $\xi : G^* \to G$ is an inner twist and $z \in Z^1(\Gamma_F, G^*)$ is a 1-cocycle such that $\xi^{-1} \circ \sigma \circ \xi \circ \sigma^{-1} = \mathrm{Ad}(z(\sigma))$ for all $\sigma \in \Gamma_F$. This map satisfies some important properties.

(2) For each $\phi \in \Phi_{\mathrm{temp}}(G^*)$, the fiber $\Pi_\phi = LLV^{-1}(\phi)$ is a finite set.

(3) For each $\phi \in \Phi_{\mathrm{temp}}(G^*)$, there exists a bijective map

$$\iota : \Pi_\phi \longrightarrow \mathrm{Irr}(\pi_0(S_\phi)),$$

where $\mathrm{Irr}(\pi_0(S_\phi))$ denotes the set of equivalence classes of irreducible representations of the finite group $\pi_0(S_\phi)$, and this bijection ι satisfies the endoscopic character relations and other nice properties. Moreover, once we fix a Whittaker datum of G^*, the map ι is uniquely determined.

In this paper we consider the metaplectic groups, which are possibly not algebraic groups but whose representation theory is similar to that of algebraic groups. The LLC for metaplectic groups, which we now introduce, was established by Gan and Savin [GS12]. The metaplectic group, denoted by $\mathrm{Mp}_{2n}(F)$, is a unique nonlinear two-fold cover of $\mathrm{Sp}_{2n}(F)$ with an exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Mp}_{2n}(F) \longrightarrow \mathrm{Sp}_{2n}(F) \longrightarrow 1.$$

Thus we identify $\mathrm{Mp}_{2n}(F)$ with $\mathrm{Sp}_{2n}(F) \times \{\pm 1\}$ as sets. We say that a representation π of $\mathrm{Mp}_{2n}(F)$ is genuine if $\pi((1, -1))$ is not trivial. Let $\Pi_{\mathrm{temp}}(\mathrm{Mp}_{2n})$ be the set of equivalence classes of irreducible genuine tempered admissible representations of $\mathrm{Mp}_{2n}(F)$, and let $\Phi_{\mathrm{temp}}(\mathrm{Mp}_{2n}) = \Phi_{\mathrm{temp}}(\mathrm{SO}_{2n+1})$. Fix a nontrivial additive character $\psi : F \to \mathbb{C}^1$. We have the following LLC for $\mathrm{Mp}_{2n}(F)$ depending on the choice of ψ, due to Gan and Savin [GS12].

THEOREM 1.3.

(1) *There exists a map*

$$LL_\psi : \Pi_{\mathrm{temp}}(\mathrm{Mp}_{2n}) \longrightarrow \Phi_{\mathrm{temp}}(\mathrm{Mp}_{2n})$$

with some important properties.

(2) *For each $\phi \in \Phi_{\mathrm{temp}}(\mathrm{Mp}_{2n})$, the fiber $\Pi_{\phi,\psi} = LL_\psi^{-1}(\phi)$ is a finite set.*

(3) *For each* $\phi \in \Phi_{\text{temp}}(\text{Mp}_{2n})$*, there exists a unique bijective map*

$$\iota_\psi : \Pi_{\phi,\psi} \longrightarrow \text{Irr}(\pi_0(S_\phi))$$

which depends on the choice of ψ*, and this map satisfies some nice properties.*

Although in general the map LL may not be bijective, there is a formula that describes how the bijection ι classifies elements in the same packet in terms of intertwining operators. Specifically, this formula can distinguish the elements of each packet Π_ϕ more precisely by means of the eigenvalues of intertwining operators. We call this formula the local intertwining relation. This of course is closely related to the endoscopic character relations. Also, it is related to global theories such as the trace formula: global intertwining operators appear in the main terms of the trace formula, and local intertwining operators are their local factors.

In [Art13], Arthur proved the local intertwining relation for quasi-split special orthogonal and symplectic groups [Art13, Theorem 2.4.1]. Mok [Mok15] and Kaletha *et al.* [KMSW14] proved it for inner forms of unitary groups. Our aim in this paper is to formulate and prove a local intertwining relation for $\text{Mp}_{2n}(F)$ under the assumption that the local intertwining relation for the non-quasi-split odd special orthogonal groups holds.

Now we explain the local intertwining relation and our result in more detail. Let G be a classical group defined over F, and let P be a proper parabolic subgroup of G with a Levi subgroup M defined over F. We then have a canonical inclusion $\hat{M} \subset \hat{G}$. Composing this inclusion and an L-parameter for M gives an inclusion $\Phi_{\text{temp}}(M) \subset \Phi_{\text{temp}}(G)$. Let $\phi \in \Phi_{\text{temp}}(M)$ be an L-parameter for M, and also regard it as an L-parameter for G. Then the LLC and LLC of Vogan type conjecture that the packet $\Pi_\phi(G)$ consists of the irreducible constituents of the representations that are parabolically induced from the elements of $\Pi_\phi(M)$. For simplicity, we shall consider only the Vogan-type conjecture. The local intertwining relation can distinguish these constituents π of $\text{Ind}_P^G(\pi_M)$ in terms of the eigenvalues of certain maps for each $\pi_M \in \Pi_\phi(M)$. The relation asserts that for any $x \in \pi_0(S_\phi)$, one can construct an endomorphism $R_P(x, \pi_M)$ of $\text{Ind}_P^G(\pi_M)$ explicitly such that $R_P(x, \pi_M)$ acts on π by a scalar multiplication by $\iota(\pi)(x)$. In other words, we expect that for any $x \in \pi_0(S_\phi)$, the concretely defined endomorphism

$$R_P(x, \pi_M) \in \text{End}_G(\text{Ind}_P^G(\pi_M))$$

satisfies

$$R_P(x, \pi_M)\big|_\pi = \iota(\pi)(x)$$

for $\pi \subset \text{Ind}_P^G(\pi_M)$. This endomorphism is called the normalized self-intertwining operator.

In general, not only the proof of the local intertwining relation but also the definition of the normalized self-intertwining operator are not trivial. This is because we have to consider some constant factors, such as the ε-factors, Kottwitz sign, and Langlands constants (λ-factors), to define the normalizing factors. In particular, ε-factors depend on the representation π_M, so they are particularly important. See [Art89b] or [Art13] for details.

In this paper we treat the case where G is a metaplectic group Mp_{2n}. We shall define normalized intertwining operators $\mathcal{R}_P(x, \pi_M)$ for Mp_{2n} in § 7.3 by

$$\mathcal{R}_P(x, \pi_M) = \gamma_F(\psi)^{d(x,\pi_M)} \gamma(\tfrac{1}{2}, \phi_x, \psi)^{-1} \gamma(0, \rho^\vee \circ \phi, \psi) \mathcal{M}(x, \pi_M), \tag{1.1}$$

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where $d(x, \pi_M)$ is a certain nonnegative integer, ϕ_x and $\rho^\vee \circ \phi$ are certain L-parameters, and $\mathcal{M}(x, \pi_M)$ is an unnormalized intertwining operator. Our definition of the normalized intertwining operators resembles that of classical linear algebraic groups, but there are three subtle and important differences. First, unlike the case of linear algebraic groups, we can have the Weil index $\gamma_F(\psi)$ appearing in the normalizing factors; this is a constant that depends only on the additive character ψ. Second, the gamma factor $\gamma(\frac{1}{2}, \phi_x, \psi)^{-1}$ at $\frac{1}{2}$ appears. Third, the choice of the Haar measure on the unipotent radical of a parabolic subgroup of $\mathrm{Mp}_{2n}(F)$ is slightly different from that in the case of linear algebraic groups. These issues will be dealt with in §§ 7.2 and 7.3.

Then we define the normalized self-intertwining operator $R_P(x, \pi_M)$ in § 7.3 by using the normalized intertwining operator (1.1). The main theorem (Theorem 4.2) is the following.

THEOREM 1.4. *Assume the local intertwining relation for the odd special orthogonal groups (Hypothesis 5.2 below). Let $\phi \in \Phi_{\mathrm{temp}}(M)$ be an L-parameter for a Levi subgroup M of a parabolic subgroup P of $\mathrm{Mp}_{2n}(F)$, and let $\pi_M \in \Pi_{\phi,\psi}(M)$. Then for any $x \in \pi_0(S_\phi(\mathrm{Mp}_{2n}))$, the normalized self-intertwining operator*

$$R_P(x, \pi_M) \in \mathrm{End}_{\mathrm{Mp}_{2n}(F)}(\mathrm{Ind}_P^{\mathrm{Mp}_{2n}(F)}(\pi_M))$$

satisfies

$$R_P(x, \pi_M)\big|_\pi = \iota_\psi(\pi)(x)$$

for $\pi \subset \mathrm{Ind}_P^{\mathrm{Mp}_{2n}(F)}(\pi_M)$.

Notation. Let F be a p-adic field and $|-|_F$ the normalized absolute value on F. We shall write W_F and $WD_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ for the Weil group and Weil–Deligne group of F, respectively. We also write Γ_F for the Galois group of F. Let $(-,-)_F$ denote the quadratic Hilbert symbol of F. The Hilbert symbol defines a non-degenerate bilinear form on $F^\times/F^{\times 2}$. Fix a nontrivial additive character $\psi : F \to \mathbb{C}^1 = \{ z \in \mathbb{C} \mid z\overline{z} = 1 \}$. For any $c \in F$, we define an additive character ψ_c of F by

$$\psi_c(x) = \psi(cx).$$

For a non-degenerate quadratic form q on a finite-dimensional vector space over F, we write $\gamma_F(\psi \circ q)$ for the unnormalized Weil index of $\psi \circ q$, a character of second degree. See [Ran93, Appendix] for the definition of the Weil index. Note that if a quadratic form q is an orthogonal direct sum $q_1 \dotplus q_2$ of two non-degenerate quadratic forms q_1 and q_2, then

$$\gamma_F(\psi \circ q) = \gamma_F(\psi \circ q_1)\gamma_F(\psi \circ q_2).$$

Let us write $\gamma_F(\psi)$ for the unnormalized Weil index of $[x \mapsto \psi(x^2)]$ and $\gamma_F(a, \psi)$ for the normalized Weil index, which is defined by $\gamma_F(a, \psi) = \gamma_F(\psi_a)/\gamma_F(\psi)$ for $a \in F^\times$. For a totally disconnected locally compact group G, let $\mathrm{Irr}(G)$ denote the set of equivalence classes of irreducible smooth admissible representations of G. In this paper, we deal only with smooth admissible representations over $\mathbb{C}$, except for representations of WD_F. For simplicity, by representations of G we mean such representations of G. If G is a linear algebraic group (respectively a metaplectic group) over F, we write $\Pi_{\mathrm{temp}}(G)$ for the set of equivalence classes of irreducible tempered representations (respectively irreducible genuine tempered representations) of $G(F)$, and we may write $G = G(F)$ with an abuse of notation. For an algebraic group H, we define the component group of H by $\pi_0(H) = H/H^\circ$, where H° is the identity component of H. The connected complex Langlands dual group of a connected reductive linear algebraic group G is denoted by $\hat{G}$. For any finite-dimensional vector space X over F, we write $\mathcal{S}(X)$ for the space of compactly supported locally constant $\mathbb{C}$-valued functions on X. For any representation ρ, we write $\rho^\vee$ for its contragredient.

2. Metaplectic and orthogonal groups

Let us begin with a brief review of the metaplectic and orthogonal groups. In this section, we fix some notation for the groups of interest in this paper.

2.1 Symplectic group

First we introduce some notation for symplectic groups. Let $(W, \langle -, - \rangle_W)$ be a symplectic vector space of dimension $2n$ over F, with associated symplectic group

$$\mathrm{Sp}(W) = \left\{ g \in \mathrm{GL}(W) \;\middle|\; \langle gw, gw' \rangle_W = \langle w, w' \rangle_W \;\forall\, w, w' \in W \right\}.$$

Choose a symplectic basis $\{ y_1, \ldots, y_n, y_1^*, \ldots, y_n^* \}$ of W, and define

$$Y_k = \mathrm{span}_F(y_1, \ldots, y_k), \quad Y_k^* = \mathrm{span}_F(y_1^*, \ldots, y_k^*)$$

for $k = 1, \ldots, n$, so that we have a standard complete polarization $W = Y_n \oplus Y_n^*$. We also let

$$W_{n-k} = \mathrm{span}_F(y_{k+1}, \ldots, y_n, y_{k+1}^*, \ldots y_n^*)$$

so that

$$W = Y_k \oplus W_{n-k} \oplus Y_k^*.$$

If $n = 0$, then $W = \{0\}$, $\mathrm{Sp}(W) = \{1\}$, and the basis is the empty set.

We now describe the parabolic subgroups of $\mathrm{Sp}(W)$ up to conjugacy. Let $\mathbf{k} = (k_1, \ldots, k_m)$ be a sequence of positive integers such that $k_1 + \cdots + k_m \leq n$, and put $k_0 = 0$ and $n_0 = n - (k_1 + \cdots + k_m)$. Consider a flag of isotropic subspaces

$$Y_{k_1} \subset Y_{k_1+k_2} \subset \cdots \subset Y_{k_1+\cdots+k_m}$$

in Y_n. The stabilizer of such a flag is a parabolic subgroup $\overline{P_{\mathbf{k}}}$ whose Levi subgroup $\overline{M_{\mathbf{k}}}$ is given by

$$\overline{M_{\mathbf{k}}} \cong \mathrm{GL}_{k_1} \times \cdots \times \mathrm{GL}_{k_m} \times \mathrm{Sp}(W_{n_0}),$$

where GL_{k_i} is identified with the general linear group of a k_i-dimensional space

$$\mathrm{span}_F(y_{k_0+\cdots+k_{i-1}+1}, \ldots, y_{k_0+\cdots+k_{i-1}+k_i}).$$

The reason we use the overlines for $\overline{M_{\mathbf{k}}}$ and $\overline{P_{\mathbf{k}}}$ will become clear in the next subsection. We shall write $N_{\mathbf{k}}$ for the unipotent radical of $\overline{P_{\mathbf{k}}}$. Parabolic subgroups of this form are standard with respect to the splitting $\mathbf{spl}_{\mathrm{Sp}(W)}$ defined in § 7.1. Any parabolic subgroup of $\mathrm{Sp}(W)$ is conjugate to a parabolic subgroup of this form. If $m = 1$ and $\mathbf{k} = (k)$, we shall write $\overline{P_k}$, $\overline{M_k}$, and N_k instead of $\overline{P_{\mathbf{k}}}$, $\overline{M_{\mathbf{k}}}$, and $N_{\mathbf{k}}$, respectively, for simplicity.

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2.2 Metaplectic group

Next we come to metaplectic groups. If $n = 0$, we put $\mathrm{Mp}(W) = \{\pm 1\}$. If $n \geq 1$, then the symplectic group $\mathrm{Sp}(W)$ has a unique nonlinear two-fold central extension $\mathrm{Mp}(W)$, which is called the metaplectic group:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Mp}(W) \longrightarrow \mathrm{Sp}(W) \longrightarrow 1. \tag{2.1}$$

As a set, we may write

$$\mathrm{Mp}(W) = \mathrm{Sp}(W) \times \{\pm 1\}$$

with group law given by

$$(g, \epsilon) \cdot (g', \epsilon') = (gg', \epsilon\epsilon' c(g, g')),$$

where c is Ranga Rao's normalized cocycle, which is a 2-cocycle on $\mathrm{Sp}(W)$ valued in $\{\pm 1\}$. See [Ran93, § 5] or [Szp07, § 2] for details. For any subset $A \subset \mathrm{Sp}(W)$, we write $\widetilde{A}$ for its preimage under the covering map $\mathrm{Mp}(W) \to \mathrm{Sp}(W)$. Also, for any subset $B \subset \mathrm{Mp}(W)$, we write $\overline{B}$ for its image under the covering map.

By the parabolic subgroups of $\mathrm{Mp}(W)$ and their Levi subgroups we mean the preimages of the parabolic subgroups of $\mathrm{Sp}(W)$ and their Levi subgroups, respectively. Not only the metaplectic group $\mathrm{Mp}(W)$ but also its parabolic subgroups and Levi subgroups are in general nonlinear.

Let us describe the parabolic subgroup $P_{\mathbf{k}} = \widetilde{\overline{P_{\mathbf{k}}}}$ of $\mathrm{Mp}(W)$, which we shall call a standard parabolic subgroup (with respect to the splitting $\mathbf{spl}_{\mathrm{Sp}(W)}$). The covering (2.1) splits over the unipotent radical $N_{\mathbf{k}}$ of $\overline{P_{\mathbf{k}}}$ by $n \mapsto (n, 1)$, so we may canonically regard $N_{\mathbf{k}}$ as a subgroup of $\mathrm{Mp}(W)$, and we have a Levi decomposition

$$P_{\mathbf{k}} = M_{\mathbf{k}} \ltimes N_{\mathbf{k}}$$

where $M_{\mathbf{k}} = \widetilde{\overline{M_{\mathbf{k}}}}$ is a Levi subgroup. The covering $M_{\mathbf{k}}$ over $\overline{M_{\mathbf{k}}} \cong \mathrm{GL}_{k_1} \times \mathrm{GL}_{k_2} \times \cdots \times \mathrm{GL}_{k_m} \times \mathrm{Sp}(W_{n_0})$ is given by

$$M_{\mathbf{k}} \cong \widetilde{\mathrm{GL}}_{k_1} \times_{\mu_2} \cdots \times_{\mu_2} \widetilde{\mathrm{GL}}_{k_m} \times_{\mu_2} \mathrm{Mp}(W_{n_0}).$$

Here, the restriction of the covering to $\mathrm{Sp}(W_{n_0})$ is nothing but the metaplectic cover $\mathrm{Mp}(W_{n_0})$ of $\mathrm{Sp}(W_{n_0})$, and the covering over GL_{k_i} is

$$\widetilde{\mathrm{GL}}_{k_i} = \mathrm{GL}_{k_i} \times \{\pm 1\}$$

with group law

$$(g, \epsilon) \cdot (g', \epsilon') = (gg', \epsilon\epsilon'(\det g, \det g')_F).$$

Let k be a positive integer. The (genuine) representation theory of $\widetilde{\mathrm{GL}}_k$ can be easily related to the representation theory of GL_k. Indeed, to any irreducible representation τ of GL_k we can attach an irreducible genuine representation $\widetilde{\tau}$ of $\widetilde{\mathrm{GL}}_k$ as in [GS12, § 2.4], and this attachment $\tau \mapsto \widetilde{\tau}$ gives a bijection between $\mathrm{Irr}(\mathrm{GL}_k)$ and $\mathrm{Irr}(\widetilde{\mathrm{GL}}_k)$, where $\mathrm{Irr}(\widetilde{\mathrm{GL}}_k)$ is the set of equivalence classes of irreducible genuine representations of $\widetilde{\mathrm{GL}}_k$. We stress that this bijection depends on the choice of the additive character ψ because $\widetilde{\tau}$ is the twist $\tau \otimes \chi_\psi$ by a genuine character χ_ψ, which is defined using ψ, as in [GS12, § 2.4].

2.3 Orthogonal group

Now we come to the orthogonal groups. Let V be a $(2n+1)$-dimensional vector space over F equipped with a non-degenerate quadratic form $q = q_V$ of discriminant 1. Then we define a

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symmetric bilinear form b_q associated to q by

$$b_q(v, v') = q(v + v') - q(v) - q(v').$$

If $n \geq 1$, then up to isomorphism there are precisely two such quadratic spaces V. One of them, denoted by V^+, has maximal isotropic subspaces of dimension n, whereas the other, denoted by V^-, has maximal isotropic subspaces of dimension $n - 1$. As such, we call the former the split quadratic space and the latter the non-split quadratic space. We shall write

$$\epsilon(V) = \begin{cases} +1, & V = V^+, \\ -1, & V = V^-. \end{cases}$$

If $n = 0$, we have only one such V up to isomorphism, and we put $V^+ = V$ and $\epsilon(V) = +1$.

Let

$$\mathrm{O}(V) = \{ h \in \mathrm{GL}(V) \mid q(hv) = q(v) \ \forall v \in V \}$$

be the associated orthogonal group. Then observe that $\mathrm{O}(V) = \mathrm{SO}(V) \times \{\pm 1\}$, where

$$\mathrm{SO}(V) = \mathrm{O}(V) \cap \mathrm{SL}(V)$$

is the special orthogonal group. The group $\mathrm{SO}(V)$ is split (respectively non-quasi-split) if V is the split (respectively non-split) quadratic space. If $n \geq 1$, then up to isomorphism there are precisely two pure inner twists of $\mathrm{SO}(V^+)$, namely $\mathrm{SO}(V^+)$ and $\mathrm{SO}(V^-)$. Note that the Kottwitz sign [Kot83] of $\mathrm{SO}(V)$ is equal to $\epsilon(V)$.

Let r be the dimension of a maximal isotropic subspace of V, so that $r = n - (1 - \epsilon(V))/2$. Choose a basis $\{ x_1, \ldots, x_n, x_0, x_1^*, \ldots, x_n^* \}$ of V such that

$$b_q(x_i, x_j) = b_q(x_i^*, x_j^*) = 0, \quad b_q(x_i, x_j^*) = \delta_{i,j},$$
$$b_q(x_0, x_i) = b_q(x_0, x_i^*) = 0, \quad q(x_0) = 1$$

for $1 \leq i, j \leq r$ and, if $r = n - 1$,

$$b_q(x_n, x_i) = b_q(x_n, x_i^*) = b_q(x_n^*, x_i) = b_q(x_n^*, x_i^*) = 0$$

for any $1 \leq i \leq r$. For each $1 \leq k \leq r$, put

$$X_k = \mathrm{span}_F(x_1, \ldots, x_k), \quad X_k^* = \mathrm{span}_F(x_1^*, \ldots, x_k^*),$$
$$V_{n-k} = \mathrm{span}_F(x_{k+1}, \ldots, x_n, x_0, x_{k+1}^*, \ldots x_n^*),$$

so that

$$V = X_k \oplus V_{n-k} \oplus X_k^*.$$

We now describe the parabolic subgroups of $\mathrm{SO}(V)$ up to conjugacy. Let $\mathbf{k} = (k_1, \ldots, k_m)$ be a sequence of positive integers such that $k_1 + \cdots + k_m \leq r$. Put $k_0 = 0$ and $n_0 = n - (k_1 + \cdots + k_m)$. Consider a flag of isotropic subspaces

$$X_{k_1} \subset X_{k_1 + k_2} \subset \cdots \subset X_{k_1 + \cdots + k_m}$$

in X_r. The stabilizer of such a flag is a parabolic subgroup $Q_{\mathbf{k}}$ whose Levi subgroup $L_{\mathbf{k}}$ is given by

$$L_{\mathbf{k}} \cong \mathrm{GL}_{k_1} \times \cdots \times \mathrm{GL}_{k_m} \times \mathrm{SO}(V_{n_0}),$$

where GL_{k_i} is identified with the general linear group of a k_i-dimensional space

$$\mathrm{span}_F(x_{k_0+\cdots+k_{i-1}+1}, \ldots, x_{k_0+\cdots+k_{i-1}+k_i}).$$

We shall write $U_{\mathbf{k}}$ for the unipotent radical of $Q_{\mathbf{k}}$. Parabolic subgroups of this form are standard with respect to the splitting $\mathbf{spl}_{\mathrm{SO}(V^+)}$ defined in § 7.1 if $V = V^+$. Any parabolic subgroup of $\mathrm{SO}(V)$ is conjugate to a parabolic subgroup of this form.

3. Tempered L-parameters for $\mathrm{Mp}(W)$ and $\mathrm{SO}(V)$

In this section, we recall the notion of L-parameters for $\mathrm{Mp}(W)$ and $\mathrm{SO}(V)$. See [GGP12] for details.

3.1 Symplectic representations of WD_F and their component groups

We say that a homomorphism $\phi : WD_F \to \mathrm{GL}_d(\mathbb{C})$ is a representation of WD_F if:

- $\phi(\mathrm{Frob})$ is semi-simple, where $\mathrm{Frob} \in W_F$ is a geometric Frobenius;
- the restriction of ϕ to $\mathrm{SL}_2(\mathbb{C})$ is algebraic;
- the restriction of ϕ to W_F is smooth.

We say that ϕ is tempered if the image of W_F is bounded, and we say that ϕ is symplectic if there exists a non-degenerate anti-symmetric bilinear form $B : \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}$ such that $B(\phi(w)x, \phi(w)y) = B(x, y)$ for any $x, y \in \mathbb{C}^d$ and $w \in WD_F$; in this case, ϕ is self-dual.

Let $\phi : WD_F \to \mathrm{GL}_d(\mathbb{C})$ be a tempered symplectic representation. By changing bases if necessary, we may assume that $\phi : WD_F \to \mathrm{Sp}_d(\mathbb{C})$. Then, by [GGP12, § 4], we can write

$$\phi = \bigoplus_{i \in I_\phi} \ell_i \phi_i \oplus (\varphi \oplus \varphi^\vee),$$

where the ℓ_i are positive integers, I_ϕ is an indexing set for mutually non-equivalent irreducible symplectic representations ϕ_i of WD_F, and φ is a representation of WD_F such that all irreducible summands are non-symplectic. Let $S_\phi = \mathrm{Cent}(\mathrm{Im}\phi, \mathrm{Sp}_d(\mathbb{C}))$ be the centralizer of the image $\mathrm{Im}(\phi)$ in $\mathrm{Sp}_d(\mathbb{C})$. Then, by [GGP12, § 4], its component group $\pi_0(S_\phi)$ is canonically identified with a free $\mathbb{Z}/2\mathbb{Z}$-module of rank $\# I_\phi$:

$$\pi_0(S_\phi) \cong \bigoplus_{i \in I_\phi} (\mathbb{Z}/2\mathbb{Z}) a_i,$$

where $\{a_i\}$ is a formal basis associated to $\{\phi_i\}$. In the rest of this paper, we identify $\pi_0(S_\phi)$ with $\bigoplus_i (\mathbb{Z}/2\mathbb{Z}) a_i$. We shall write z_ϕ for the image of $-1 \in S_\phi$ in $\pi_0(S_\phi)$.

3.2 Tempered L-parameters for $\mathrm{Mp}(W)$ and $\mathrm{SO}(V)$

Let $\Phi_{\mathrm{temp}}(\mathrm{GL}_k)$ be the set of equivalence classes of tempered L-parameters for GL_k. Recall that it can be identified with the set of equivalence classes of tempered representations $\phi : WD_F \to \mathrm{GL}_k(\mathbb{C})$ of dimension k. Now let $\Phi_{\mathrm{temp}}(\mathrm{Mp}_{2n})$ and $\Phi_{\mathrm{temp}}(\mathrm{SO}_{2n+1})$ be the sets of equivalence classes of tempered L-parameters for $\mathrm{Mp}(W)$ and $\mathrm{SO}(V)$, respectively. Then, by [GGP12, §§ 11 and 8], we can identify $\Phi_{\mathrm{temp}}(\mathrm{Mp}_{2n})$ and $\Phi_{\mathrm{temp}}(\mathrm{SO}_{2n+1})$ with the set of equivalence classes of tempered symplectic representations $\phi : WD_F \to \mathrm{Sp}_{2n}(\mathbb{C})$ of dimension $2n$.

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Let $\mathbf{k} = (k_1, \ldots, k_m)$ and let n_0 be as in the previous section. For $(G, P, M) = (\mathrm{Mp}(W), P_{\mathbf{k}}, M_{\mathbf{k}})$ or $(\mathrm{SO}(V), Q_{\mathbf{k}}, L_{\mathbf{k}})$, put

$$\hat{G} = \mathrm{Sp}_{2n}(\mathbb{C}), \quad \hat{M} = \mathrm{GL}_{k_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{k_m}(\mathbb{C}) \times \mathrm{Sp}_{2n_0}(\mathbb{C}),$$

with a standard embedding $\hat{M} \hookrightarrow \hat{G}$ as a Levi subgroup of a standard parabolic subgroup $\hat{P}$ of $\hat{G}$. Let ϕ be a tempered L-parameter for G with the image $\mathrm{Im}(\phi)$ in $\hat{M}$. This is of the form

$$\phi = \phi_1 \oplus \cdots \oplus \phi_m \oplus \phi_0 \oplus \phi_m^\vee \oplus \cdots \oplus \phi_1^\vee, \tag{3.1}$$

where $\phi_i \in \Phi_{\mathrm{temp}}(\mathrm{GL}_{k_i})$ for $i = 1, \ldots, m$ and $\phi_0 \in \Phi_{\mathrm{temp}}(\mathrm{Mp}_{2n_0}) = \Phi_{\mathrm{temp}}(\mathrm{SO}_{2n_0+1})$. Let $A_{\hat{M}}$ be the maximal central torus of $\hat{M}$. Put

$$\begin{aligned} \mathfrak{N}_\phi(M, G) &= \mathrm{Norm}(A_{\hat{M}}, S_\phi)/\mathrm{Cent}(A_{\hat{M}}, S_\phi^\circ), \\ \mathrm{W}_\phi(M, G) &= \mathrm{Norm}(A_{\hat{M}}, S_\phi)/\mathrm{Cent}(A_{\hat{M}}, S_\phi), \\ S_\phi^\natural(M, G) &= \mathrm{Norm}(A_{\hat{M}}, S_\phi)/\mathrm{Norm}(A_{\hat{M}}, S_\phi^\circ). \end{aligned}$$

We have a natural surjection

$$\mathfrak{N}_\phi(M, G) \twoheadrightarrow S_\phi^\natural(M, G), \tag{3.2}$$

natural inclusions

$$\mathrm{W}_\phi(M, G) \subset \mathrm{W}(\hat{M}, \hat{G}), \quad S_\phi^\natural(M, G) \subset \pi_0(S_\phi),$$

and a natural short exact sequence

$$1 \longrightarrow \pi_0(S_{\phi_0}) \longrightarrow \mathfrak{N}_\phi(M, G) \longrightarrow \mathrm{W}_\phi(M, G) \longrightarrow 1.$$

By applying [Art13, p. 104] or [KMSW14, p. 103, after (2.4.1)] to $\mathrm{SO}(V)$, the injection $\pi_0(S_{\phi_0}) \to \mathfrak{N}_\phi(M, G)$ admits a canonical splitting

$$\mathfrak{N}_\phi(M, G) = \pi_0(S_{\phi_0}) \times \mathrm{W}_\phi(M, G).$$

4. Local Langlands correspondence for Mp(W) and the main theorem

In this section, we summarize some properties of the LLC for metaplectic groups and state the main theorem (Theorem 4.2). The correspondence is defined by combining the local Shimura correspondence with the LLC for odd special orthogonal groups, which we shall summarize in §§ 5 and 6 below.

The local Langlands correspondence for metaplectic groups was established by Gan and Savin; see [GS12, Corollary 1.2 and Theorem 1.3], and [Han19] for the last assertion of the theorem.

Theorem 4.1.

(1) *There exists a surjection (depending on ψ)*

$$LL_\psi : \Pi_{\mathrm{temp}}(\mathrm{Mp}(W)) \longrightarrow \Phi_{\mathrm{temp}}(\mathrm{Mp}_{2n}),$$

with finite fibers $\Pi_{\phi,\psi} = \Pi_{\phi,\psi}(\mathrm{Mp}(W)) = LL_\psi^{-1}(\phi)$.

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(2) *For each* $\phi \in \Phi_{\mathrm{temp}}(\mathrm{Mp}_{2n})$, *there exists a unique bijection (depending on* ψ*)*

$$\iota_\psi : \Pi_{\phi,\psi} \longrightarrow \mathrm{Irr}(\pi_0(S_\phi)).$$

(3) *Let* $\mathbf{k} = (k_1, \dots, k_m)$ *and* $n_0 = n - (k_1 + \cdots + k_m) \geq 0$, *and let* $\phi \in \Phi_{\mathrm{temp}}(\mathrm{Mp}_{2n})$ *be of the form (3.1). Then we have*

$$\Pi_{\phi,\psi} = \left\{ \pi \;\middle|\; \pi \subset \mathrm{Ind}_{P_{\mathbf{k}}}^{\mathrm{Mp}(W)}(\widetilde{\tau_1} \otimes \cdots \otimes \widetilde{\tau_m} \otimes \pi_0),\ \textit{irreducible constituent},\ \pi_0 \in \Pi_{\phi_0,\psi} \right\},$$

where τ_i *is the representation of* GL_{k_i} *which corresponds to* ϕ_i, *for* $i = 1, \dots, m$. *Moreover, for any* $\pi_0 \in \Pi_{\phi_0,\psi}$ *we have*

$$\mathrm{Ind}_{P_{\mathbf{k}}}^{\mathrm{Mp}(W)}(\widetilde{\tau_1} \otimes \cdots \otimes \widetilde{\tau_m} \otimes \pi_0) = \bigoplus_{\substack{\pi \in \Pi_{\phi,\psi} \\ \iota_\psi(\pi)|_{\pi_0(S_{\phi_0})} = \iota_\psi(\pi_0)}} \pi.$$

In the setting of Theorem 4.1(3), for $w \in \mathrm{W}_\phi(M_{\mathbf{k}}, \mathrm{Mp}(W))$ let

$$R_{P_{\mathbf{k}}}(w, \widetilde{\tau_1} \otimes \cdots \otimes \widetilde{\tau_m} \otimes \pi_0) \in \mathrm{End}_{\mathrm{Mp}(W)}(\mathrm{Ind}_{P_{\mathbf{k}}}^{\mathrm{Mp}(W)}(\widetilde{\tau_1} \otimes \cdots \otimes \widetilde{\tau_m} \otimes \pi_0))$$

be the normalized self-intertwining operator defined in § 7.3 below. Then we can state the main theorem as follows.

Theorem 4.2. *Assume the local intertwining relation for the odd special orthogonal groups (Hypothesis 5.2 below). Let* $x_w \in S_\phi^\natural(M_{\mathbf{k}}, \mathrm{Mp}(W))$ *be the image of* $w \in \mathrm{W}_\phi(M_{\mathbf{k}}, \mathrm{Mp}(W))$ *under the natural surjection (3.2). Then the restriction of* $R_{P_{\mathbf{k}}}(w, \widetilde{\tau_1} \otimes \cdots \otimes \widetilde{\tau_m} \otimes \pi_0)$ *to* $\pi \subset \mathrm{Ind}_{P_{\mathbf{k}}}^{\mathrm{Mp}(W)}(\widetilde{\tau_1} \otimes \cdots \otimes \widetilde{\tau_m} \otimes \pi_0)$ *is the scalar multiplication by* $\iota_\psi(\pi)(x_w)$.

We will reduce the main theorem to Proposition 7.3 in § 7.4, and we will complete a proof of the proposition in § 9.3.

5. Local Langlands correspondence and the local intertwining relation for SO(V)

The LLC for odd special orthogonal groups was established by Arthur [Art13] and by Mœglin and Renard [MR18]. In this section, we summarize some properties of the correspondence and the local intertwining relation.

Arthur [Art13] studied representations of $\mathrm{SO}(V^+)$ and Mœglin and Renard [MR18] representations of $\mathrm{SO}(V^-)$. Their results imply the LLC of Vogan type for $\mathrm{SO}(V)$, stated as follows.

Theorem 5.1.

(1) *There exists a surjection*

$$LLV : \Pi_{\mathrm{temp}}(\mathrm{SO}(V^+)) \sqcup \Pi_{\mathrm{temp}}(\mathrm{SO}(V^-)) \longrightarrow \Phi_{\mathrm{temp}}(\mathrm{SO}_{2n+1}),$$

with finite fibers $\Pi_\phi = \Pi_\phi(\mathrm{SO}(V^+)) \sqcup \Pi_\phi(\mathrm{SO}(V^-)) = LLV^{-1}(\phi)$.

(2) *For each* $\phi \in \Phi_{\mathrm{temp}}(\mathrm{SO}_{2n+1})$, *there exists a unique bijective map*

$$\iota : \Pi_\phi \longrightarrow \mathrm{Irr}(\pi_0(S_\phi))$$

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such that

$$\Pi_\phi(\mathrm{SO}(V^\pm)) = \{\, \sigma \in \Pi_\phi \mid \iota(\sigma)(z_\phi) = \pm 1 \,\}.$$

(3) *Let $V = V^+$ or V^-. Let $\mathbf{k} = (k_1, \ldots, k_m)$ be a sequence of positive integers such that $k_1 + \cdots + k_m \leq r$, and put $n_0 = n - (k_1 + \cdots + k_m)$. Let $\phi \in \Phi_{\mathrm{temp}}(\mathrm{SO}_{2n+1})$ be of the form (3.1). Then we have*

$$\Pi_\phi = \left\{ \sigma \;\middle|\; \sigma \subset \mathrm{Ind}_{Q_{\mathbf{k}}}^{\mathrm{SO}(V)}(\tau_1 \otimes \cdots \otimes \tau_m \otimes \sigma_0),\ \textit{irreducible constituent},\ \sigma_0 \in \Pi_{\phi_0} \right\},$$

where τ_i is the representation of GL_{k_i} which corresponds to ϕ_i, for $i = 1, \ldots, m$. Moreover, for any $\sigma_0 \in \Pi_{\phi_0}$, we have

$$\mathrm{Ind}_{Q_{\mathbf{k}}}^{\mathrm{SO}(V)}(\tau_1 \otimes \cdots \otimes \tau_m \otimes \sigma_0) = \bigoplus_{\substack{\sigma \in \Pi_\phi \\ \iota(\sigma)|_{\pi_0(S_{\phi_0})} = \iota(\sigma_0)}} \sigma.$$

In the setting of Theorem 5.1(3), for $w \in \mathrm{W}_\phi(L_{\mathbf{k}}, \mathrm{SO}(V))$ let

$$R_{Q_{\mathbf{k}}}(w, \tau_1 \otimes \cdots \otimes \tau_m \otimes \sigma_0) \in \mathrm{End}_{\mathrm{SO}(V)}(\mathrm{Ind}_{Q_{\mathbf{k}}}^{\mathrm{SO}(V)}(\tau_1 \otimes \cdots \otimes \tau_m \otimes \sigma_0))$$

be the normalized self-intertwining operator defined in § 7.3 below. The next hypothesis is the local intertwining relation for $\mathrm{SO}(V)$, and it has already been proven in the $V = V^+$ case by Arthur [Art13, § 2.4].

Hypothesis 5.2. Let $x_w \in S_\phi^\natural(L_{\mathbf{k}}, \mathrm{SO}(V))$ be the image of $w \in \mathrm{W}_\phi(L_{\mathbf{k}}, \mathrm{SO}(V))$ under the natural surjection (3.2). Then the restriction of $R_{Q_{\mathbf{k}}}(w, \tau_1 \otimes \cdots \otimes \tau_m \otimes \sigma_0)$ to $\sigma \subset \mathrm{Ind}_{Q_{\mathbf{k}}}^{\mathrm{SO}(V)}(\tau_1 \otimes \cdots \otimes \tau_m \otimes \sigma_0)$ is the scalar multiplication by $\iota(\sigma)(x_w)$.

6. Local Shimura correspondence

Gan and Savin [GS12] showed the local Shimura correspondence, which is the natural bijection between the set of isomorphism classes of irreducible genuine representations of $\mathrm{Mp}(W)$ and the set of isomorphism classes of irreducible representations of $\mathrm{SO}(V^+)$ and $\mathrm{SO}(V^-)$. This is given by the local theta correspondence, and we can construct the LLC for $\mathrm{Mp}(W)$ (Theorem 4.1) by combining the local Shimura correspondence with the LLC of Vogan type for $\mathrm{SO}(V)$ (Theorem 5.1). In this section we review their results. First we recall the Weil representation for $\mathrm{Mp}(W) \times \mathrm{O}(V)$ and the notion of local theta correspondence.

6.1 Weil representation

The group $\mathrm{Mp}(W) \times \mathrm{O}(V)$ has a natural representation $\omega_{V,W,\psi}$ depending on ψ, given as follows. The tensor product $\mathbb{W} = V \otimes_F W$ has a natural symplectic form $\langle -, - \rangle$ defined by

$$\langle v \otimes w, v' \otimes w' \rangle = b_q(v, v') \cdot \langle w, w' \rangle_W.$$

Then there is a natural map

$$\mathrm{Sp}(W) \times \mathrm{O}(V) \longrightarrow \mathrm{Sp}(\mathbb{W}). \tag{6.1}$$

One has the metaplectic $\mathbb{C}^1$-cover $\mathcal{M}p(\mathbb{W})$ of $\mathrm{Sp}(\mathbb{W})$, and the additive character ψ determines the Weil representation ω_ψ of $\mathcal{M}p(\mathbb{W})$. Kudla [Kud94] gives a splitting of the metaplectic cover

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over $\mathrm{Mp}(W)\times \mathrm{O}(V)$, and hence there exists a commutative diagram

$$\begin{array}{ccc} \mathrm{Mp}(W)\times \mathrm{O}(V) & \longrightarrow & \mathcal{M}p(\mathbb{W}) \\ \downarrow & & \downarrow \\ \mathrm{Sp}(W)\times \mathrm{O}(V) & \longrightarrow & \mathrm{Sp}(\mathbb{W}), \end{array}$$

where the right vertical map is given by the metaplectic $\mathbb{C}^1$-covering map, the left vertical map is given by the two-fold cover (2.1), and the lower horizontal map is (6.1). Thus, we have a Weil representation $\omega_{V,W,\psi}$ of $\mathrm{Mp}(W)\times \mathrm{O}(V)$. Later, in § 8.6, we will give some realizations of the Weil representation $\omega_{V,W,\psi}$ to show the main theorem. Here, a splitting over $\mathrm{Mp}(W)\times \mathrm{O}(V)$ is not unique, and we choose one following [Kud94].

6.2 Local theta correspondence

In this subsection, we summarize the Gan–Savin result [GS12]. First, note that the theorems in [GS12] had been verified only for the case of odd residue characteristic, as the Howe duality for even residue characteristic was conjecture at the time. However, the Howe duality for even residue characteristic was then verified by Gan and Takeda [GT16], so now we have the results of [GS12] for arbitrary residue characteristic.

Given an irreducible representation σ of $\mathrm{O}(V)$, the maximal σ-isotypic quotient of $\omega_{V,W,\psi}$ is of the form

$$\sigma \boxtimes \Theta_{V,W,\psi}(\sigma)$$

for some representation $\Theta_{V,W,\psi}(\sigma)$ of $\mathrm{Mp}(W)$ (called the big theta lift of σ). Then $\Theta_{V,W,\psi}(\sigma)$ either is zero or has finite length. The maximal semi-simple quotient of $\Theta_{V,W,\psi}(\sigma)$ is denoted by $\theta_{V,W,\psi}(\sigma)$ (called the small theta lift of σ).

Similarly, if π is an irreducible genuine representation of $\mathrm{Mp}(W)$, then one has its big theta lift $\Theta_{W,V,\psi}(\pi)$ and its small theta lift $\theta_{W,V,\psi}(\pi)$, which are representations of $\mathrm{O}(V)$.

By the Howe duality, each small theta lift is irreducible or zero [Wal90; GT16]. Gan and Savin [GS12, § 6] showed that:

(i) for $\pi \in \mathrm{Irr}(\mathrm{Mp}(W))$, exactly one of $\theta_{W,V^+,\psi}(\pi)$ or $\theta_{W,V^-,\psi}(\pi)$ is nonzero;
(ii) given $\sigma \in \mathrm{Irr}(\mathrm{SO}(V))$, with the extensions σ^+ and σ^- to $\mathrm{O}(V)$, exactly one of $\Theta_{V,W,\psi}(\sigma^+)$ or $\Theta_{V,W,\psi}(\sigma^-)$ is nonzero.

Here $\mathrm{Irr}(\mathrm{Mp}(W))$ is the set of equivalence classes of irreducible genuine representations of $\mathrm{Mp}(W)$, and $\sigma^\pm$ denote the extensions such that $-1 \in \mathrm{O}(V)$ acts as ± 1, respectively. Then Gan and Savin derived the following theorems [GS12, Theorems 1.1 and 1.3].

Theorem 6.1. *There is a bijection*

$$\Theta_\psi : \mathrm{Irr}(\mathrm{Mp}(W)) \longleftrightarrow \mathrm{Irr}(\mathrm{SO}(V^+)) \sqcup \mathrm{Irr}(\mathrm{SO}(V^-)),$$

given by the theta correspondence with respect to ψ.

Theorem 6.2. *Suppose that $\sigma \in \mathrm{Irr}(\mathrm{SO}(V))$ and $\pi \in \mathrm{Irr}(\mathrm{Mp}(W))$ correspond under Θ_ψ. Then the following hold.*

(1) *σ is a discrete series representation if and only if π is a discrete series representation.*

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(2) *σ is tempered if and only if π is tempered. Moreover, suppose that*

$$\sigma \subset \mathrm{Ind}_{Q_{\mathbf{k}}}^{\mathrm{SO}(V)}(\tau_1 \otimes \cdots \otimes \tau_m \otimes \sigma_0),$$

where $\mathbf{k} = (k_1, \ldots, k_m)$ is a sequence such that $k_1 + \cdots + k_m \leq r$, the τ_i are tempered representations of GL_{k_i}, σ_0 is a tempered representation of $\mathrm{SO}(V_{n_0})$, and $n_0 = n - (k_1 + \cdots + k_m)$. Then

$$\pi \subset \mathrm{Ind}_{P_{\mathbf{k}}}^{\mathrm{Mp}(W)}(\widetilde{\tau_1} \otimes \cdots \otimes \widetilde{\tau_m} \otimes \pi_0),$$

where $\pi_0 = \Theta_\psi(\sigma_0)$. In particular, Θ_ψ gives a bijection between the (isomorphism classes of) irreducible constituents of $\mathrm{Ind}_{Q_{\mathbf{k}}}^{\mathrm{SO}(V)}(\tau_1 \otimes \cdots \otimes \tau_m \otimes \sigma_0)$ and those of $\mathrm{Ind}_{P_{\mathbf{k}}}^{\mathrm{Mp}(W)}(\widetilde{\tau_1} \otimes \cdots \otimes \widetilde{\tau_m} \otimes \pi_0)$.

(3) *If σ is an irreducible representation of $\mathrm{SO}(V)$ and ρ is an irreducible representation of GL_l, then there is a Plancherel measure $\mu(s, \sigma \times \rho, \psi)$ associated to the parabolically induced representation $\mathrm{Ind}_{Q_l}^{\mathrm{SO}(V)}(\rho_s \otimes \sigma)$, where $\rho_s = \rho \otimes |\det|_F^s$. If $\pi = \Theta_\psi(\sigma)$, then one has*

$$\mu(s, \sigma \times \rho, \psi) = \mu(s, \pi \times \rho, \psi).$$

See [GI14, Appendix B] and [GI16, Appendix A.7] for details on Plancherel measures.

By combining Theorem 6.2 with Theorem 5.1 and [Han19], we obtain Theorem 4.1.

7. Intertwining operators

In this section, we define the normalized self-intertwining operators of $\mathrm{SO}(V)$ (following [Art13, § 2.3]) and those of $\mathrm{Mp}(W)$, which are used in Theorem 4.2 and Hypothesis 5.2 above. The definition of the normalized self-intertwining operators is very subtle because one has to choose the following data appropriately:

- representatives of a Weyl group element w;
- Haar measures on the unipotent radicals to define the unnormalized intertwining operators;
- normalizing factors $r_P(w, \pi_{M,\mathbf{s}})$ and $r_Q(w, \sigma_{L,\mathbf{s}})$;
- an intertwining isomorphism $\mathcal{A}_w$.

Let $\mathbf{k} = (k_1, \ldots, k_m)$ be a sequence of positive integers such that $k_1 + \cdots + k_m \leq r$. Put $k_0 = 0$, $k = k_1 + \cdots + k_m$, $n_0 = n - k$, and

$$P = P_{\mathbf{k}}, \quad M = M_{\mathbf{k}}, \quad N = N_{\mathbf{k}}, \quad Q = Q_{\mathbf{k}}, \quad L = L_{\mathbf{k}}, \quad U = U_{\mathbf{k}}.$$

7.1 Representatives of a Weyl group element

Let $w \in \mathrm{W}(\hat{M}, \mathrm{Sp}_{2n}(\mathbb{C}))$ be a Weyl group element. We shall identify w with elements in $\mathrm{W}(M, \mathrm{Mp}(W))$ and $\mathrm{W}(L, \mathrm{SO}(V))$ in a standard way, and take representatives $\widetilde{w}_P \in \mathrm{Mp}(W)$ and $\widetilde{w}_Q \in \mathrm{SO}(V)$ following the work of Langlands and Shelstad [LS87] and Gan and Li [GL18]. In this subsection we review the procedure; see [LS87, § 2.1], [Art13, § 2.3], or [GL18, Definition 4.1] for details.

First, we realize the relative Weyl group $\mathrm{W}(\hat{M}, \mathrm{Sp}_{2n}(\mathbb{C}))$ in $\mathfrak{S}_m \ltimes (\mathbb{Z}/2\mathbb{Z})^m$ and identify it with relative Weyl groups $\mathrm{W}(\overline{M}, \mathrm{Sp}(W))$ and $\mathrm{W}(L, \mathrm{SO}(V))$. We can do this in a canonical way, because the Levi subgroups $\hat{M}$, $\overline{M}$, and L are of the form

$$\mathrm{GL}_{k_1} \times \cdots \times \mathrm{GL}_{k_m} \times G_-$$

over $\mathbb{C}$ or F, where G_- is a semi-simple algebraic group.

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Next, we take standard splittings $\mathbf{spl}_{\mathrm{Sp}(W)}$ of $\mathrm{Sp}(W)$ and $\mathbf{spl}_{\mathrm{SO}(V^+)}$ of $\mathrm{SO}(V^+)$ by

$$\mathbf{spl}_{\mathrm{Sp}(W)} = (B_W, T_W, \{X_{\alpha_i}\}_{i=1,\dots,n}), \quad \mathbf{spl}_{\mathrm{SO}(V^+)} = (B_V, T_V, \{X_{\beta_i}\}_{i=1,\dots,n}),$$

where:

– B_W and B_V are, respectively, the Borel subgroups stabilizing the F-flags

$$Fy_1 \subset Fy_1 + Fy_2 \subset \cdots \subset Fy_1 + \cdots + Fy_n, \quad Fx_1 \subset Fx_1 + Fx_2 \subset \cdots \subset Fx_1 + \cdots + Fx_n;$$

– T_W and T_V are, respectively, their maximal tori which are diagonalized by the bases

$$\{\, y_1, \dots, y_n, y_1^*, \dots, y_n^* \,\}, \quad \{\, x_1, \dots, x_n, x_0, x_1^*, \dots, x_n^* \,\};$$

– X_{α_i} and X_{β_i} are simple root vectors given by

$$X_{\alpha_i} : y_j \mapsto \begin{cases} y_i & j = i+1, \\ 0 & \text{otherwise}, \end{cases} \qquad y_j^* \mapsto \begin{cases} -y_{i+1}^* & j = i, \\ 0 & \text{otherwise}, \end{cases} \qquad \text{for } 1 \le i \le n-1,$$

$$X_{\alpha_n} : y_j \mapsto 0, \qquad y_j^* \mapsto \begin{cases} y_n & j = n, \\ 0 & \text{otherwise}, \end{cases}$$

$$X_{\beta_i} : x_j \mapsto \begin{cases} x_i & j = i+1, \\ 0 & \text{otherwise}, \end{cases} \qquad x_j^* \mapsto \begin{cases} -x_{i+1}^* & j = i, \\ 0 & \text{otherwise}, \end{cases} \qquad \text{for } 1 \le i \le n-1,$$

$$X_{\beta_n} : x_j \mapsto \begin{cases} 2x_n & j = 0, \\ 0 & \text{otherwise}, \end{cases} \qquad x_j^* \mapsto \begin{cases} -x_0 & j = n, \\ 0 & \text{otherwise}. \end{cases}$$

Then M and P (respectively L and Q) are standard, in the sense that they contain $\widetilde{T_W}$ and $\widetilde{B_W}$ (respectively T_V and B_V), respectively. Let $\Phi(T_W, \mathrm{Sp}(W))$ and $\Delta(B_W) = \{\alpha_1, \dots, \alpha_n\}$ denote the set of roots and the set of simple positive roots with the indices relative to the basis. Then we can see that the simple root vectors X_{α_i} do indeed correspond to α_i. Let $X_{-\alpha_i}$ be the root vector for $-\alpha_i$ such that the Lie bracket $[X_{\alpha_i}, X_{-\alpha_i}]$ is the coroot for α_i. Let us take $\Phi(T_V, \mathrm{SO}(V^+))$, $\Delta(B_V) = \{\beta_1, \dots, \beta_n\}$, and $X_{-\beta_i}$ similarly.

Now let us take representatives $\widetilde{w}_P$ and $\widetilde{w}_Q$ of w. First assume that $V = V^+$. Let w_T and w'_T denote the representatives of w in the Weyl groups $\mathrm{W}(T_W, \mathrm{Sp}(W))$ and $\mathrm{W}(T_V, \mathrm{SO}(V^+))$ that stabilize the simple positive roots inside $\overline{M}$ and L, respectively. We shall write w_λ for the reflection corresponding to a root λ. We then have Langlands–Shelstad representatives

$$\widetilde{w}_P = \widetilde{w}_{\alpha_{(1)}} \cdots \widetilde{w}_{\alpha_{(\ell)}}, \quad \widetilde{w}_Q = \widetilde{w}_{\beta_{(1)}} \cdots \widetilde{w}_{\beta_{(\ell)}},$$

where $w_T = w_{\alpha_{(1)}} \cdots w_{\alpha_{(\ell)}}$ and $w'_T = w_{\beta_{(1)}} \cdots w_{\beta_{(\ell)}}$ are reduced decompositions of w_T and w'_T in $\mathrm{W}(T_W, \mathrm{Sp}(W))$ and $\mathrm{W}(T_V, \mathrm{SO}(V^+))$, respectively, and

$$\widetilde{w}_\alpha = (\exp(X_\alpha) \exp(-X_{-\alpha}) \exp(X_\alpha), 1), \quad \widetilde{w}_\beta = \exp(X_\beta) \exp(-X_{-\beta}) \exp(X_\beta)$$

for any $\alpha \in \Delta(B_W)$ and $\beta \in \Delta(B_V)$.

In the case of $V = V^-$, the representative $\widetilde{w}_Q \in \mathrm{SO}(V^-)$ is defined to be the corresponding element via the canonical pure inner twist $(\xi, z) : \mathrm{SO}(V^+) \to \mathrm{SO}(V^-)$. The following lemma is obvious but important.

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Lemma 7.1. *Let $w = w_1 \cdots w_l$ be a reduced decomposition of w as an element of the relative Weyl group* $\mathrm{W}(\hat{M}, \mathrm{Sp}_{2n}(\mathbb{C}))$. *Then we have*

$$\widetilde{w}_P = \widetilde{w_1}_P \cdots \widetilde{w_l}_P \quad \text{and} \quad \widetilde{w}_Q = \widetilde{w_1}_Q \cdots \widetilde{w_l}_Q.$$

The representatives can be given more explicitly in the $m = 1$ case. See Proposition 8.1 below.

7.2 Haar measures on the unipotent radicals

In this subsection we shall choose Haar measures on N and U. We first define Haar measures du on U for $V = V^+$ and $d'n$ on N with respect to the splittings $\mathbf{spl}_{\mathrm{SO}(V^+)}$ and $\mathbf{spl}_{\mathrm{Sp}(W)}$, following [Art13, § 2.3] or [KMSW14, § 2.2]. Here, $d'n$ is a Haar measure when we regard N as the unipotent radical of a parabolic subgroup $\overline{P}$ of $\mathrm{Sp}(W)$. On the unipotent radical N of the parabolic subgroup P of $\mathrm{Mp}(W)$, we take a Haar measure $dn = |2|_F^{-k/2} d'n$.

Since the splittings are given explicitly, one can describe these measures explicitly. We will give an explicit definition of the measures in the case of $m = 1$, i.e. $\mathbf{k} = (k)$, in § 8.5 below. The measures for $m = 1$ give us the following descriptions of du and dn.

For $1 \le i \le m$, put $n_i = n - (k_0 + \cdots + k_{i-1})$. As in §§ 2.1 and 2.3, let $N^{(i)}$ and $U^{(i)}$ be the unipotent radicals of the maximal parabolic subgroups of $\mathrm{Sp}(W_{n_i})$ and $\mathrm{SO}(V_{n_i})$ stabilizing

$$\mathrm{span}_F(y_{k_0+\cdots+k_{i-1}+1}, \ldots, y_{k_0+\cdots+k_i}), \quad \mathrm{span}_F(x_{k_0+\cdots+k_{i-1}+1}, \ldots, x_{k_0+\cdots+k_i}),$$

respectively. If we take Haar measures on each $N^{(i)}$ and $U^{(i)}$ as we will do on N_k and U_k in § 8.5 below, then the Haar measures dn on N and du on U are the measures defined via the homeomorphisms

$$N^{(1)} \times \cdots \times N^{(m)} \longrightarrow N, \quad (n_1, \ldots, n_m) \mapsto n_1 \cdots n_m,$$
$$U^{(1)} \times \cdots \times U^{(m)} \longrightarrow U, \quad (u_1, \ldots, u_m) \mapsto u_1 \cdots u_m.$$

In the case of $V = V^-$, we define the Haar measure du on U by using the above homeomorphism.

7.3 Intertwining operators

Now we define intertwining operators. Let τ_i be irreducible tempered representations of GL_{k_i} on a vector space $\mathscr{V}_{\tau_i}$, for $i = 1, \ldots, m$. For any $\mathbf{s} = (s_1, \ldots, s_m) \in \mathbb{C}^m$, we realize the representation $\tau_{i,s_i} = \tau_i \otimes |\det|_F^{s_i}$ on $\mathscr{V}_{\tau_i}$ by setting $\tau_{i,s_i}(a)v = |\det a|_F^{s_i} \tau_i(a)v$ for $v \in \mathscr{V}_{\tau_i}$ and $a \in \mathrm{GL}_{k_i}$. Let π_0 be an irreducible genuine tempered representation of $\mathrm{Mp}(W_{n_0})$ on $\mathscr{V}_{\pi_0}$ and σ_0 an irreducible tempered representation of $\mathrm{SO}(V_{n_0})$ on $\mathscr{V}_{\sigma_0}$ such that π_0 and σ_0 correspond under the bijection Θ_ψ. Put $\pi_{M,\mathbf{s}} = \widetilde{\tau}_{1,s_1} \otimes \cdots \otimes \widetilde{\tau}_{m,s_m} \otimes \pi_0$ and $\sigma_{L,\mathbf{s}} = \tau_{1,s_1} \otimes \cdots \otimes \tau_{m,s_m} \otimes \sigma_0$. In particular, we shall write $\pi_M = \pi_{M,0}$ and $\sigma_L = \sigma_{L,0}$.

The normalized parabolically induced representation $\mathrm{Ind}_P^{\mathrm{Mp}(W)}(\pi_{M,\mathbf{s}})$ is realized on the space of $(\mathscr{V}_{\tau_1} \otimes \cdots \otimes \mathscr{V}_{\tau_m} \otimes \mathscr{V}_{\pi_0})$ -valued smooth functions $\mathscr{F}_{\mathbf{s}}$ on $\mathrm{Mp}(W)$ such that

$$\mathscr{F}_{\mathbf{s}}(mng) = \delta_P(m)^{1/2} \pi_{M,\mathbf{s}}(m) \mathscr{F}_{\mathbf{s}}(g)$$

for any $m \in M$, $n \in N$, and $g \in \mathrm{Mp}(W)$, where δ_P is the modulus function. For $w \in \mathrm{W}(\hat{M}, \mathrm{Sp}_{2n}(\mathbb{C}))$ we define the unnormalized intertwining operator

$$\mathcal{M}(\widetilde{w}_P, \pi_{M,\mathbf{s}}) : \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\pi_{M,\mathbf{s}}) \to \mathrm{Ind}_P^{\mathrm{Mp}(W)}(w\pi_{M,\mathbf{s}})$$

by the meromorphic continuation of the integral

$$\mathcal{M}(\widetilde{w}_P, \pi_{M,\mathbf{s}})\mathscr{F}_{\mathbf{s}}(g) = \int_{({}^wN \cap N)\backslash N} \mathscr{F}_{\mathbf{s}}((\widetilde{w}_P)^{-1}ng)\, dn,$$

where ${}^wN = \widetilde{w}_P N \widetilde{w}_P^{-1}$ and $w\pi_{M,\mathbf{s}}$ is the representation of M on $\mathscr{V}_{\tau_1} \otimes \cdots \otimes \mathscr{V}_{\tau_m} \otimes \mathscr{V}_{\pi_0}$ given by

$$w\pi_{M,\mathbf{s}}(m) = \pi_{M,\mathbf{s}}((\widetilde{w}_P)^{-1} m \widetilde{w}_P)$$

for $m \in M$. The integral above converges absolutely on some open set of $\mathbb{C}^m$ in $\mathbf{s}$ and has a meromorphic continuation to $\mathbf{s} \in \mathbb{C}^m$. The operator is well-defined for $\mathbf{s} \in \mathbb{C}^m$ except at finite poles modulo $(2\pi i/\log q_F)\mathbb{Z}^m$. Similarly, we can define the unnormalized intertwining operator $\mathcal{M}(\widetilde{w}_Q, \sigma_{L,\mathbf{s}})$ from $\mathrm{Ind}_Q^{\mathrm{SO}(V)}(\sigma_{L,\mathbf{s}})$ to $\mathrm{Ind}_Q^{\mathrm{SO}(V)}(w\sigma_{L,\mathbf{s}})$.

Before stating the definition of the normalized self-intertwining operators, we need to normalize the operators $\mathcal{M}(\widetilde{w}_P, \pi_{M,\mathbf{s}})$ and $\mathcal{M}(\widetilde{w}_Q, \sigma_{L,\mathbf{s}})$ so that they are holomorphic at $\mathbf{s} = 0$. Put $P^w = (\widetilde{w}_P)^{-1} P \widetilde{w}_P$ and $Q^w = (\widetilde{w}_Q)^{-1} Q \widetilde{w}_Q$, and let $\phi_1, \ldots, \phi_m, \phi_0$ be the L-parameters corresponding to $\tau_1, \ldots, \tau_m, \sigma_0$ via the LLC. Since $\sigma_0 = \Theta_\psi(\pi_0)$, the L-parameter for π_0 is also ϕ_0. As in (3.1), put

$$\phi = \phi_1 \oplus \cdots \oplus \phi_m \oplus \phi_0 \oplus \phi_m^\vee \oplus \cdots \oplus \phi_1^\vee$$

so that $\phi \in \Phi_{\mathrm{temp}}(\mathrm{SO}_{2n+1}) = \Phi_{\mathrm{temp}}(\mathrm{Mp}_{2n})$ and $\mathrm{Im}(\phi) \subset \hat{M}$. We define the twist $\phi_{\mathbf{s}}$ of ϕ by $\mathbf{s}$ as follows:

$$\phi_{\mathbf{s}} = (\phi_1 \otimes |-|^{s_1}) \oplus \cdots \oplus (\phi_m \otimes |-|^{s_m}) \oplus \phi_0 \oplus (\phi_m^\vee \otimes |-|^{-s_m}) \oplus \cdots \oplus (\phi_1^\vee \otimes |-|^{-s_1}).$$

Then we define a normalizing factor

$$r_{P^w|P}(\mathbf{s}, \phi, \psi) = \gamma(0, \rho_{P^w|P}^\vee \circ \phi_{\mathbf{s}}, \psi)^{-1},$$

where $\rho_{P^w|P}$ denotes the representation of $\hat{M}$ defined in [Art13, pp. 80–81]. Similarly, define a normalizing factor $r_{Q^w|Q}(\mathbf{s}, \phi, \psi) = \gamma(0, \rho_{Q^w|Q}^\vee \circ \phi_{\mathbf{s}}, \psi)^{-1}$. The realization of $\mathrm{W}(\hat{M}, \mathrm{Sp}_{2n}(\mathbb{C}))$ as a subgroup of $\mathfrak{S}_m \ltimes (\mathbb{Z}/2\mathbb{Z})^m$ gives an expression

$$w = \sigma_w \ltimes (d_i)_{i=1}^m.$$

Let $y : \mathbb{Z}/2\mathbb{Z} \to \{0, 1\}$ be a map such that $y(2\mathbb{Z}) = 0$ and $y(1 + 2\mathbb{Z}) = 1$. We then define a representation $y(w, \phi_{\mathbf{s}})$ of WD_F and a complex number $y(w, \mathbf{s})$ by

$$y(w, \phi_{\mathbf{s}}) = \bigoplus_{i=1}^m y(d_i)\phi_i \otimes |-|^{s_i}, \quad y(w, \mathbf{s}) = \sum_{i=1}^m y(d_i) s_i.$$

Let us define normalized intertwining operators

$$\mathcal{R}_P(w, \pi_{M,\mathbf{s}}, \psi) = \gamma_F(\psi)^{\dim y(w,\phi)} |2|_F^{2y(w,\mathbf{s})} \gamma(\tfrac{1}{2}, y(w, \phi_{\mathbf{s}}), \psi)^{-1} r_{P^w|P}(\mathbf{s}, \phi, \psi)^{-1} \mathcal{M}(\widetilde{w}_P, \pi_{M,\mathbf{s}}),$$
$$\mathcal{R}_Q(w, \sigma_{L,\mathbf{s}}) = \epsilon(V)^{\dim y(w,\phi)} r_{Q^w|Q}(\mathbf{s}, \phi, \psi)^{-1} \mathcal{M}(\widetilde{w}_Q, \sigma_{L,\mathbf{s}}).$$

It is known that $\mathcal{R}_Q(w, \sigma_{L,\mathbf{s}})$ is independent of the choice of the additive character ψ (see [Art13, p. 83]). We can see that the intertwining operators can be defined at $\mathbf{s} = 0$.

Lemma 7.2. *The normalized intertwining operators $\mathcal{R}_P(w, \pi_{M,\mathbf{s}}, \psi)$ and $\mathcal{R}_Q(w, \sigma_{L,\mathbf{s}})$ are holomorphic at $\mathbf{s} = 0$.*

Proof. By [Art13, Proposition 2.3.1], $\mathcal{R}_Q(w, \sigma_{L,\mathbf{s}})$ is holomorphic at $\mathbf{s} = 0$ if $V = V^+$. Next, let us consider the metaplectic case. By the definition of the representative $\widetilde{w}_P$, we can decompose the operator into the product of the operators for simple reflections in $\mathrm{W}(\hat{M}, \mathrm{Sp}_{2n}(\mathbb{C}))$. There are two cases to consider: $w \in \mathfrak{S}_m$ and $w \in (\mathbb{Z}/2\mathbb{Z})^m$. If $w \in \mathfrak{S}_m$, the assertion is reduced to the case of GL_k and follows from [Sha81, Proposition 3.1.4 and (3.2.1)]. If $w \in (\mathbb{Z}/2\mathbb{Z})^m$, the assertion is reduced to the case of $m = 1$, i.e.$P = P_{\mathbf{k}}$ is a maximal parabolic subgroup P_k. In this case, since the explicit formula of Plancherel measures for the metaplectic group [GI16, Appendix A.7] is known, the assertion can be proven as in [Art89a, Theorem 2.1]. In the $V = V^-$ case, a similar argument goes. $\square$

We shall put

$$\mathcal{R}_P(w, \pi_M, \psi) = \mathcal{R}_P(w, \pi_{M,0}, \psi), \quad \mathcal{R}_Q(w, \sigma_L) = \mathcal{R}_Q(w, \sigma_{L,0}).$$

Let us define the normalized self-intertwining operators. Assume that

$$w \in \mathrm{W}_\phi(M, \mathrm{Mp}(W)) = \mathrm{W}_\phi(L, \mathrm{SO}(V)),$$

which is equivalent to $w\pi_M \cong \pi_M$ and $w\sigma_L \cong \sigma_L$. We take the unique Whittaker normalized isomorphism

$$\mathcal{A}_w : \mathscr{V}_{\tau_1} \otimes \cdots \otimes \mathscr{V}_{\tau_m} \longrightarrow \mathscr{V}_{\tau_1} \otimes \cdots \otimes \mathscr{V}_{\tau_m}$$

and define the normalized self-intertwining operators

$$R_P(w, \pi_M) : \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\pi_M) \longrightarrow \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\pi_M),$$
$$R_Q(w, \sigma_L) : \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\sigma_L) \longrightarrow \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\sigma_L)$$

as in [GI16, p. 756].

7.4 Reduction

Since the LLC for $\mathrm{Mp}(W)$ is defined by using the theta correspondence, to prove Theorem 4.2 it suffices to consider the relation between the theta correspondence and the intertwining operators. The following proposition will be proved later.

Proposition 7.3. *Put* $\breve{\sigma}_L = \tau_1 \otimes \cdots \otimes \tau_m \otimes \sigma_0^\vee$. *There exists a nonzero* $\mathrm{SO}(V) \times \mathrm{Mp}(W)$-*equivariant map*

$$\mathcal{T} : \omega_{V,W,\psi} \otimes \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\breve{\sigma}_L) \longrightarrow \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\pi_M)$$

such that:

(a) *for any irreducible constituent* σ *of* $\mathrm{Ind}_Q^{\mathrm{SO}(V)}(\breve{\sigma}_L)$, *the restriction of* $\mathcal{T}$ *to* $\omega_{V,W,\psi} \otimes \sigma$ *is nonzero;*

(b) *the diagram*

$$\begin{array}{ccc} \omega_{V,W,\psi} \otimes \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\breve{\sigma}_L) & \xrightarrow{\mathcal{T}} & \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\pi_M) \\ \Big\downarrow {\scriptstyle 1_{\omega_{V,W,\psi}} \otimes R_Q(w, \breve{\sigma}_L)} & & \Big\downarrow {\scriptstyle R_P(w, \pi_M)} \\ \omega_{V,W,\psi} \otimes \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\breve{\sigma}_L) & \xrightarrow{\mathcal{T}} & \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\pi_M) \end{array}$$

commutes.

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Once the above proposition is proven, we have Theorem 4.2.

Proposition 7.4. *Proposition 7.3 implies Theorem 4.2.*

Proof. Suppose that Proposition 7.3 holds. Because any irreducible representation of an odd special orthogonal group is self-dual [MVW87, Ch. 4, § II.1], we have $\sigma_0^\vee \cong \sigma_0$. Now fix an isomorphism $\sigma_L \cong \breve{\sigma}_L$ and identify $\breve{\sigma}_L$ with σ_L.

Let $\pi \subset \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\pi_M)$ be an irreducible tempered representation and put $\sigma = \Theta_\psi(\pi) \subset \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\sigma_L)$. Then, by Proposition 7.3 and the identification $\sigma_L \cong \breve{\sigma}_L$, there exists a nonzero $\mathrm{SO}(V) \times \mathrm{Mp}(W)$-equivariant map

$$\mathcal{T} : \omega_{V,W,\psi} \otimes \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\sigma_L) \longrightarrow \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\pi_M)$$

such that its restriction to $\omega_{V,W,\psi} \otimes \sigma$ is nonzero and it satisfies the following commutative diagram:

$$\begin{array}{ccc} \omega_{V,W,\psi} \otimes \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\sigma_L) & \xrightarrow{\mathcal{T}} & \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\pi_M) \\ {\scriptstyle 1_{\omega_{V,W,\psi}} \otimes R_Q(w,\sigma_L)}\downarrow & & \downarrow{\scriptstyle R_P(w,\pi_M)} \\ \omega_{V,W,\psi} \otimes \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\sigma_L) & \xrightarrow{\mathcal{T}} & \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\pi_M). \end{array}$$

By the Howe duality and the fact that $\sigma^\vee \cong \sigma$, $\mathcal{T}$ sends $\omega_{V,W,\psi} \otimes \sigma$ to π. Therefore, $\mathcal{T}$ gives a nonzero $\mathrm{SO}(V) \times \mathrm{Mp}(W)$-equivariant map

$$\mathcal{T}_{\sigma,\pi} : \omega_{V,W,\psi} \otimes \sigma \longrightarrow \pi$$

such that

$$R_P(w,\pi_M)\big|_\pi \circ \mathcal{T}_{\sigma,\pi} = \mathcal{T}_{\sigma,\pi} \circ \big(1_{\omega_{V,W,\psi}} \otimes R_Q(w,\sigma_L)\big|_\sigma\big). \tag{7.1}$$

Suppose that Hypothesis 5.2 holds. Then we have $R_Q(w,\sigma_L)|_\sigma = \eta(x_w)$ where $\eta = \iota(\sigma)$, and $x_w \in S_\phi^\natural(L, \mathrm{SO}(V)) = S_\phi^\natural(M, \mathrm{Mp}(W))$ is the image of w under the natural map (3.2). Now the relation (7.1) shows that

$$R_P(w,\pi_M)\big|_\pi \circ \mathcal{T}_{\sigma,\pi} = \eta(x_w)\mathcal{T}_{\sigma,\pi}.$$

Since $\mathcal{T}_{\sigma,\pi} \neq 0$ and π is irreducible, we have $R_P(w,\pi_M)|_\pi = \eta(x_w)$. We also have $\eta = \iota_\psi(\pi)$ by the definition of the LLC for $\mathrm{Mp}(W)$. This completes the proof. □

8. Preparations for the proof of Proposition 7.3

In the next section we shall give a proof of Proposition 7.3. For this, we introduce some more notation, following Gan and Ichino [GI16, §§ 7 and 8], in this section.

8.1 Maximal parabolic subgroups

We have described the parabolic subgroups of $\mathrm{Sp}(W)$, $\mathrm{Mp}(W)$, and $\mathrm{SO}(V)$ in §§ 2.1–2.3. Referring to [Ato18], we can describe their maximal parabolic subgroups more explicitl

Let k be a positive integer, and put $n_0 = n - k$. Put $Y = Y_k$ and $Y^* = Y_k^*$. We shall write an element in the symplectic group $\mathrm{Sp}(W)$ as a block matrix relative to the decomposition $W = Y \oplus W_{n_0} \oplus Y^*$. Following § 2.1 or 2.2, put $P = P_k$, $M = M_k$, and $N = N_k$ so that $\overline{P} = \overline{P_k}$

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and $\overline{M} = \overline{M_k}$. Then we have

$$\overline{M} = \{m(a)g_0 \mid a \in \mathrm{GL}(Y),\ g_0 \in \mathrm{Sp}(W_{n_0})\},$$
$$N = \{n^{\mathrm{b}}(b)n^{\mathrm{c}}(c) \mid b \in \mathrm{Hom}(W_{n_0}, Y),\ c \in \mathrm{Sym}(Y^*, Y)\},$$

where $m(a)$, $n^{\mathrm{b}}(b)$, $n^{\mathrm{c}}(c)$, and $\mathrm{Sym}(Y^*, Y)$ are defined as in [Ato18, § 2.4]. Recall that P and M are the double covers of $\overline{P}$ and $\overline{M}$, respectively. Note that the natural inclusion $\mathrm{Sp}(W_{n_0}) \subset \mathrm{Sp}(W)$ induces an inclusion $\mathrm{Mp}(W_{n_0}) \subset \mathrm{Mp}(W)$, $(g_0, \epsilon) \mapsto (g_0, \epsilon)$. Put

$$\rho_P = \frac{2n - k + 1}{2}.$$

Assume that $k \leq r$. Put $X = X_k$ and $X^* = X_k^*$, and write an element in the special orthogonal group $\mathrm{SO}(V)$ as a block matrix relative to the decomposition $V = X \oplus V_{n_0} \oplus X^*$, as above. Put $Q = Q_k$, $L = L_k$, and $U = U_k$, following § 2.3. Then we have

$$L = \{l(a)h_0 \mid a \in \mathrm{GL}(X),\ h_0 \in \mathrm{SO}(V_{n_0})\},$$
$$U = \{u = u^{\mathrm{b}}(b)u^{\mathrm{c}}(c) \mid b \in \mathrm{Hom}(V_{n_0}, X),\ c \in \mathrm{Alt}(X^*, X)\},$$

where $l(a)$, $u^{\mathrm{b}}(b)$, $u^{\mathrm{c}}(c)$, and $\mathrm{Alt}(X^*, X)$ are given in a similar way to [Ato18, § 2.4]. Put

$$\rho_Q = \frac{2n - k}{2}.$$

8.2 Representatives of w_M and w_L

Let w_M (respectively w_L) be the nontrivial element of the relative Weyl group $\mathrm{W}(\overline{M}, \mathrm{Sp}(W))$ (respectively $\mathrm{W}(L, \mathrm{SO}(V))$). Note that $\mathrm{W}(\overline{M}, \mathrm{Sp}(W)) \cong \mathrm{W}(L, \mathrm{SO}(V)) \cong \mathbb{Z}/2\mathbb{Z}$. In this subsection, we shall take representatives of w_M and w_L, following Langlands and Shelstad (see § 7.1), and calculate them explicitly.

First, let us define $I_X \in \mathrm{Hom}(X^*, X)$ and $I_Y \in \mathrm{Hom}(Y^*, Y)$ by $I_X x_i^* = x_i$ and $I_Y y_i^* = y_i$. With respect to the bases, I_X and I_Y correspond to the identity matrix. Put

$$J = \begin{pmatrix} & & & (-1)^{n+1} \\ & & (-1)^{n+2} & \\ & \cdot^{\cdot^{\cdot}} & & \\ (-1)^{n+k} & & & \end{pmatrix} \in \mathrm{GL}_k.$$

Using the bases, we can identify $\mathrm{GL}(X)$ and $\mathrm{GL}(Y)$ with GL_k and regard J as an element of $\mathrm{GL}(X)$ or $\mathrm{GL}(Y)$. Let us define elements $w_Y \in \mathrm{Sp}(W)$ and $w_X \in \mathrm{SO}(V)$ by

$$w_Y = \begin{pmatrix} & & I_Y \\ & (-1)^k 1_{W_{n_0}} & \\ -I_Y^{-1} & & \end{pmatrix}, \quad w_X = \begin{pmatrix} & & -I_X \\ & (-1)^k 1_{V_{n_0}} & \\ -I_X^{-1} & & \end{pmatrix}.$$

We take the representatives $w'_M \in \mathrm{Mp}(W)$ and $w'_L \in \mathrm{SO}(V)$ of w_M and w_L defined by

$$w'_M = \left((-1)^k \begin{pmatrix} & & -JI_Y \\ & 1_{W_{n_0}} & \\ JI_Y^{-1} & & \end{pmatrix}, \epsilon^{\mathrm{LS}} \right), \quad w'_L = (-1)^k \begin{pmatrix} & & JI_X \\ & 1_{V_{n_0}} & \\ JI_X^{-1} & & \end{pmatrix},$$

respectively, where $\epsilon^{\mathrm{LS}} = (-1, -1)_F^{k(k-1)/2}$.

Let $\widetilde{w}_P$ and $\widetilde{w}_Q$ denote Langlands and Shelstad's representatives (see [LS87, § 2.1] and [GL18, Definition 4.1]) of w_M and w_L with respect to the F-splittings $\mathbf{spl}_{\mathrm{Sp}(W)}$ and $\mathbf{spl}_{\mathrm{SO}(V^+)}$. Then we have the following proposition.

Proposition 8.1. *We have*

$$\widetilde{w}_P = w'_M,$$
$$\widetilde{w}_Q = w'_L.$$

The proof is similar to that of [GI16, Lemma 7.2]. Also, as pointed out in [GI16, p. 755], in the $V = V^-$ case one can see that w'_L corresponds to $\widetilde{w}_Q \in \mathrm{SO}(V^+)$ via the canonical pure inner twist. However, we shall give a proof of the first assertion in § 8.4, since the calculation of $\widetilde{w}_P$ is too complicated because we have to consider Ranga Rao's 2-cocycle.

8.3 Ranga Rao's 2-cocycle

Before proving Proposition 8.1, we introduce some notation and review Ranga Rao's x-function and normalized cocycle [Ran93].

For three nonnegative integers $r, s, t \in \mathbb{Z}_{\geq 0}$, define $\iota^{\mathrm{GL}}_{r,s,t}$ to be an embedding of GL_s into GL_{r+s+t} by

$$A \mapsto \begin{pmatrix} 1_r & & \\ & A & \\ & & 1_t \end{pmatrix}.$$

For $a \in \mathrm{GL}(Y_n)$, we write $m_n(a)$ for an element $\left(\begin{smallmatrix} a & \\ & (a^*)^{-1} \end{smallmatrix}\right)$ of a Levi subgroup $\overline{M_n}$ of the Siegel parabolic subgroup $\overline{P_n}$. For any subset $S \subset \{1, \ldots, n\}$, define σ_S and a_S by

$$\sigma_S \cdot y_i = \begin{cases} y_i^* & \text{if } i \in S, \\ y_i & \text{if } i \notin S, \end{cases} \quad \sigma_S \cdot y_i^* = \begin{cases} -y_i & \text{if } i \in S, \\ y_i^* & \text{if } i \notin S \end{cases}$$

and

$$a_S \cdot y_i = \begin{cases} -y_i & \text{if } i \in S, \\ y_i & \text{if } i \notin S, \end{cases} \quad a_S \cdot y_i^* = \begin{cases} -y_i^* & \text{if } i \in S, \\ y_i^* & \text{if } i \notin S. \end{cases}$$

When S is a singleton $\{i\}$, we shall write $\sigma_i = \sigma_{\{i\}}$ for simplicity.

Next, we review the notions of Ranga Rao's x-function and normalized cocycle. We have $\mathrm{Sp}(W) = \cup_S \overline{P_n}\sigma_S\overline{P_n}$, where the disjoint union runs over all subsets $S \subset \{1, \ldots, n\}$, and Ranga Rao's x-function is defined by

$$x(p_1\sigma_S p_2) = \det(p_1p_2|_{Y_n})(\mathrm{mod}(F^\times)^2), \quad p_1, p_2 \in \overline{P_n},\ S \subset \{1, \ldots, n\}.$$

This is well-defined [Ran93, Lemma 5.1]. Then, let $c(-,-)$ denote Ranga Rao's normalized cocycle, which is a 2-cocycle on $\mathrm{Sp}(W)$ valued in $\{\pm 1\}$. The precise definition of $c(-,-)$ is omitted here, but we list several of its properties. See [Ran93, § 5] or [Szp07, § 2] for details.

Proposition 8.2. *Let $p, p' \in \overline{P_n}$, $S, S' \subset \{1, \ldots, n\}$, and $g, g' \in \mathrm{Sp}(W)$. Put $j = |S \cap S'|$. Then*

$$c(\sigma_S, \sigma_{S'}) = (-1,-1)_F^{j(j+1)/2},$$

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$$c(pg, g'p') = c(g, g')(x(g), x(p))_F(x(g'), x(p'))_F(x(p), x(p'))_F(x(gg'), x(pp'))_F,$$
$$c(g, p) = c(p, g) = (x(p), x(g))_F.$$

Moreover, if $gg' = g'g$, then

$$c(g, g') = c(g', g).$$

8.4 Proof of Proposition 8.1

Now we begin the proof of the first assertion of Proposition 8.1.

Proof. First, we need a certain representative of w_M in $\mathrm{W}(T_W, \mathrm{Sp}(W))$. Take the representative $w_T \in \mathrm{W}(T_W, \mathrm{Sp}(W))$ of w_M such that w_T maps the positive roots inside $\overline{M}$ into the positive roots inside $\overline{M}$ and the positive roots outside $\overline{M}$ into the negative roots (not necessarily outside $\overline{M}$). Let $s_i \in \mathrm{W}(T_W, \mathrm{Sp}(W))$ be the simple reflection corresponding to $\alpha_i \in \Delta(B_W)$, and put

$$q_i = s_{k-1}s_{k-2}\cdots s_{i+1}s_i \quad \text{for } 1 \leq i \leq k-1,$$
$$r_i = s_i s_{i+1}\cdots s_{n-1}s_n s_{n-1}\cdots s_{i+1}s_i \quad \text{for } 1 \leq i \leq n.$$

Then

$$w_T = r_k q_1 r_k q_2 r_k \cdots q_{k-2} r_k q_{k-1} r_k$$

gives a reduced decomposition of w_T.

Second, let us consider the representative $\widetilde{w}_{\overline{P}}$ of w_T in the symplectic group $\mathrm{Sp}(W)$, following Langlands and Shelstad [LS87]. Put

$$\omega_i = \exp(X_{\alpha_i})\exp(-X_{-\alpha_i})\exp(X_{\alpha_i}) \quad \text{for } 1 \leq i \leq n,$$
$$u_i = \omega_{k-1}\cdots\omega_{i+1}\omega_i \quad \text{for } 1 \leq i \leq k-1,$$
$$v_i = \omega_i\cdots\omega_{n-1}\omega_n\omega_{n-1}\cdots\omega_i \quad \text{for } 1 \leq i \leq n,$$

which are representatives of s_i, q_i, and r_i, respectively. Then, by [LS87, § 2.1], we have the representative $\widetilde{w}_{\overline{P}}$ in $\mathrm{Sp}(W)$:

$$\widetilde{w}_{\overline{P}} = v_k u_1 v_k u_2 \cdots v_k u_{k-1} v_k.$$

In addition, put

$$z_i = \omega_{k-1}^{-1}\cdots\omega_{i+1}^{-1}\omega_i^{-1} \quad \text{for } 1 \leq i \leq k-1.$$

Then we obtain that $v_k u_i = z_i v_i$ for $1 \leq i \leq k-1$. Moreover, one can calculate v_i and z_j by descending induction on $i = n, \ldots, 1$ and $j = k-1, \ldots, 1$ to obtain

$$v_i = a_{\{i+1,\ldots,n\}}\sigma_i^{2(n-i-1)+1},$$
$$z_j = m_n\big(\iota^{\mathrm{GL}}_{j-1,k-j+1,n-k}(\kappa_{k-j+1})\big),$$

where

$$\kappa_l = \begin{pmatrix} 0 & -1 & & \\ & & \ddots & \\ & & & -1 \\ 1 & & & 0 \end{pmatrix} \in \mathrm{GL}_l.$$

A straightforward calculation then shows that $v_i z_j = z_j v_i$ for $1 \leq i < j \leq k-1$. This implies that $\widetilde{w}_{\overline{P}} = z_1\cdots z_{k-1}v_1\cdots v_k$.

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Finally, let us take their representatives in $\mathrm{Mp}(W)$ as follows. Put

$$\begin{aligned}\widetilde{\omega}_i &= \widetilde{w}_{\alpha_i} = (\omega_i, 1) \quad \text{for } 1 \le i \le n,\\ \widetilde{u}_i &= \widetilde{\omega}_{k-1} \cdots \widetilde{\omega}_{i+1}\widetilde{\omega}_i \quad \text{for } 1 \le i \le k-1,\\ \widetilde{v}_i &= \widetilde{\omega}_i \cdots \widetilde{\omega}_{n-1}\widetilde{\omega}_n\widetilde{\omega}_{n-1} \cdots \widetilde{\omega}_i \quad \text{for } 1 \le i \le n,\end{aligned}$$

and

$$\widetilde{z}_i = \widetilde{\omega}_{k-1}^{-1} \cdots \widetilde{\omega}_{i+1}^{-1}\widetilde{\omega}_i^{-1} \quad \text{for } 1 \le i \le k-1.$$

Then the required element $\widetilde{w}_P$ can be expressed as

$$\widetilde{w}_P = \widetilde{v}_k\widetilde{u}_1\widetilde{v}_k\widetilde{u}_2 \cdots \widetilde{v}_k\widetilde{u}_{k-1}\widetilde{v}_k.$$

We have $\widetilde{v}_k\widetilde{u}_i = \widetilde{z}_i\widetilde{v}_i$ for $1 \le i \le k-1$. Also, for $1 \le i < j \le k-1$ we have $\widetilde{v}_i\widetilde{z}_j = \widetilde{z}_j\widetilde{v}_i$ because $v_iz_j = z_jv_i$. Therefore,

$$\begin{aligned}\widetilde{w}_P &= \widetilde{z}_1\widetilde{v}_1\widetilde{z}_2\widetilde{v}_2 \cdots \widetilde{z}_{k-1}\widetilde{v}_{k-1}\widetilde{v}_k\\ &= \widetilde{z}_1\widetilde{z}_2 \cdots \widetilde{z}_{k-1}\widetilde{v}_1\widetilde{v}_2 \cdots \widetilde{v}_{k-1}\widetilde{v}_k.\end{aligned}$$

Since $\omega_1, \ldots, \omega_{n-1}$ are elements of the Siegel parabolic subgroup $\overline{P_n}$ and have determinant 1 on Y_n, one has

$$\begin{aligned}\widetilde{v}_i &= (v_i, 1) \quad \text{for } 1 \le i \le k,\\ \widetilde{z}_j &= (z_j, 1) \quad \text{for } 1 \le j \le k-1.\end{aligned}$$

Now let us compute $\widetilde{z}_1\widetilde{z}_2 \cdots \widetilde{z}_{k-1}$, $\widetilde{v}_1\widetilde{v}_2 \cdots \widetilde{v}_{k-1}\widetilde{v}_k$, and $\widetilde{w}_P$. First, we consider $\widetilde{z}_1\widetilde{z}_2 \cdots \widetilde{z}_{k-1}$. Since each z_j belongs to the Siegel parabolic subgroup and has determinant 1 on Y_n, we have $\widetilde{z}_1 \cdots \widetilde{z}_{k-1} = (z_1 \cdots z_{k-1}, 1)$. Additionally, by calculating its action on the basis $\{y_1, \ldots, y_n, y_1^*, \ldots, y_n^*\}$, we can compute the product $z_1 \cdots z_{k-1}$:

$$z_1 \cdots z_{k-1} = m_n\big(\iota^{\mathrm{GL}}_{0,k,n-k}((-1)^{n+k}J)\big).$$

Second, by descending induction, we can compute $\widetilde{v}_i \cdots \widetilde{v}_k$ for $i = k, \ldots, 1$. Note that Ranga Rao's normalized cocycle may not be trivial. By descending induction, one has

$$v_i \cdots v_k = \sigma_{\{i,\ldots,k\}}p_i',$$

where

$$p_i' = (a_{\{i,\ldots,k\}})^{n-i+1}(a_{\{k+1,\ldots,n\}})^{k-i+1}.$$

We also have

$$v_i = p_i\sigma_i,$$

where

$$p_i = (a_{\{i\}})^{n-i+1}a_{\{i+1,\ldots,n\}}.$$

Since p_i' and p_i are elements of the Siegel parabolic subgroup,

$$\begin{aligned} c(v_i, v_{i+1}\cdots v_k) &= c(p_i\sigma_i, \sigma_{\{i+1,\dots,k\}}p'_{i+1}) \\ &= (x(p_i), x(p'_{i+1}))_F \\ &= \big((-1)^{(n-i+1)+(n-i)}, (-1)^{(n-i)(k-i)+(k-i)(n-k)}\big)_F \\ &= (-1,-1)_F^{k+i}. \end{aligned}$$

Hence,

$$\begin{aligned} \widetilde{v}_1\cdots\widetilde{v}_k &= \Big(v_1\cdots v_k, \prod_{i=1}^{k-1} c(v_i, v_{i+1}\cdots v_k)\Big) \\ &= ((\sigma_{\{1,\dots,k\}})^{2n+1}(a_{\{k+1,\dots,n\}})^k, (-1,-1)_F^{k(k-1)/2}). \end{aligned}$$

Finally, since $z_1\cdots z_{k-1}$ belongs to the Siegel parabolic subgroup and has determinant 1 on Y_n, we have $\widetilde{w}_P = (\widetilde{w}_{\overline{P}}, \epsilon^{\mathrm{LS}})$ and $\widetilde{w}_{\overline{P}} = z_1\cdots z_{k-1}v_1\cdots v_k = (-1)^k m_n(\iota^{\mathrm{GL}}_{0,k,n-k}(J))\sigma_{\{1,\dots,k\}}$. This proves the first assertion of the proposition. □

8.5 Haar measures

In order to study the intertwining operators in more detail, or to describe some explicit formulas for the Weil representations, we need to take Haar measures appropriately and explicitly. Put

$$\begin{aligned} e &= x_1\otimes y_1^* + \cdots + x_k\otimes y_k^* \in X\otimes Y^*, \\ e^* &= x_1^*\otimes y_1 + \cdots + x_k^*\otimes y_k \in X^*\otimes Y, \\ e^{**} &= x_1^*\otimes y_1^* + \cdots + x_k^*\otimes y_k^* \in X^*\otimes Y^*. \end{aligned}$$

These vectors belong to the symplectic space $\mathbb{W} = V\otimes_F W$.

Let us define measures on each of the groups and vector spaces.

(i) Take the self-dual Haar measure $d_{\mathrm{M}_k}x$ on $\mathrm{M}_k(F)$ with respect to the pairing

$$\mathrm{M}_k(F)\times\mathrm{M}_k(F)\ni(x,y)\mapsto\psi(\mathrm{tr}(xy))\in\mathbb{C}^1.$$

In particular, write $d_\psi x$ when $k=1$.

(ii) Take the Haar measure dx on $\mathrm{GL}_k(F)$ defined by $dx = |\det x|_F^{-k} d_{\mathrm{M}_k}x$, and transfer it to $\mathrm{GL}(Y)$ and $\mathrm{GL}(X)$ via the identification.

(iii) Define the self-dual Haar measures on $V\otimes Y^*$, $X^*\otimes Y$, $X\otimes Y^*$, $V_{n_0}\otimes Y^*$, $X^*\otimes W_{n_0}$, $\mathrm{Hom}(V_{n_0}, X)$, $\mathrm{Hom}(W_{n_0}, Y)$, $\mathrm{Hom}(X,X)$, and $\mathrm{Hom}(Y,Y)$ in a similar way to [GI16, § 7.2].

(iv) Take the self-dual Haar measures on $\mathrm{Alt}(X^*, X)$ and $\mathrm{Sym}(Y^*, Y)$ with respect to the pairings

$$\begin{aligned} \mathrm{Alt}(X^*,X)\times\mathrm{Alt}(X^*,X)\ni(c,c')&\mapsto\psi(\langle I_Y ce^{**}, I_X^{-1}c'e^{**}\rangle)\in\mathbb{C}^1, \\ \mathrm{Sym}(Y^*,Y)\times\mathrm{Sym}(Y^*,Y)\ni(c,c')&\mapsto\psi(\langle I_X ce^{**}, I_Y^{-1}c'e^{**}\rangle)\in\mathbb{C}^1, \end{aligned}$$

respectively.

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(v) Take the Haar measures du on U for $u = u^{\mathrm{b}}(b)u^{\mathrm{c}}(c)$ and dn on N for $n = n^{\mathrm{b}}(b)n^{\mathrm{c}}(c)$ as follows:

$$du = |2|_F^{-k/2}\, db \cdot |2|_F^{-k(k-1)/4}\, dc, \quad b \in \mathrm{Hom}(V_{n_0}, X),\ c \in \mathrm{Alt}(X^*, X),$$
$$dn = |2|_F^{-k/2}\, db \cdot |2|_F^{-k(k-1)/4}\, dc, \quad b \in \mathrm{Hom}(W_{n_0}, Y),\ c \in \mathrm{Sym}(Y^*, Y).$$

(vi) Let us take measures on Q and $\overline{P}$. For $q = lu \in Q = LU$ and $p = mn \in \overline{P} = \overline{M}N$, define

$$dq = dl\, du, \quad dp = dm\, dn.$$

We have the modulus function $\delta_Q(l(a)h_0) = |\det a|_F^{2\rho_Q}$ for $a \in \mathrm{GL}(X)$ and $h_0 \in \mathrm{SO}(V_{n_0})$ and the modulus function $\delta_P(m(a)g_0) = |\det a|_F^{2\rho_P}$ for $a \in \mathrm{GL}(Y)$ and $g_0 \in \mathrm{Sp}(W_{n_0})$.

One can then check that the measures du on U and dn on N coincide with the Haar measures that we took in § 7.2 by using the splittings $\mathbf{spl}_{\mathrm{SO}(V^+)}$ and $\mathbf{spl}_{\mathrm{Sp}(W)}$, respectively (see [Ato18, § 6.3] for explicit calculations).

8.6 Big symplectic spaces and a mixed model

In this subsection we shall take a mixed model, which is a realization of the Weil representation, following Gan and Ichino [GI16, § 7.4].

Put $\mathbb{W}_0 = V \otimes W_{n_0} \subset \mathbb{W}$ and $\mathbb{W}_{00} = V_{n_0} \otimes W_{n_0} \subset \mathbb{W}_0 \subset \mathbb{W}$. These are symplectic subspaces of $\mathbb{W}$. Fix a polarization $W_{n_0} = W_{01} \oplus W_{02}$, where $W_{01} = \mathrm{span}_F(y_{k+1}, \ldots, y_n)$ and $W_{02} = \mathrm{span}_F(y_{k+1}^*, \ldots, y_n^*)$. We have the following natural complete polarizations of $\mathbb{W}$, $\mathbb{W}_0$, and $\mathbb{W}_{00}$:

$$\mathbb{W} = (V \otimes Y_n) \oplus (V \otimes Y_n^*),$$
$$\mathbb{W}_0 = (V \otimes W_{01}) \oplus (V \otimes W_{02}),$$
$$\mathbb{W}_{00} = (V_{n_0} \otimes W_{01}) \oplus (V_{n_0} \otimes W_{02}).$$

Let ω, ω_0, and ω_{00} be the realizations of the Weil representations $\omega_{V,W,\psi}$, $\omega_{V,W_{n_0},\psi}$, and $\omega_{V_{n_0},W_{n_0},\psi}$ of $\mathrm{O}(V) \times \mathrm{Mp}(W)$, $\mathrm{O}(V) \times \mathrm{Mp}(W_{n_0})$, and $\mathrm{O}(V_{n_0}) \times \mathrm{Mp}(W_{n_0})$, respectively, on a mixed Schrödinger model

$$\mathcal{S}_{00} = \mathcal{S}(V_{n_0} \otimes W_{02}),$$
$$\mathcal{S}_0 = \mathcal{S}(X^* \otimes W_0) \otimes \mathcal{S}_{00},$$
$$\mathcal{S} = \mathcal{S}(V \otimes Y^*) \otimes \mathcal{S}_0,$$

as in [GI16, § 7.4] or [Ato18, § 6.2]. We construct these models by using the following elements:

- the ordinary Schrödinger models

$$(\omega^{\mathrm{or}}, \mathcal{S}^{\mathrm{or}} = \mathcal{S}(V \otimes (Y^* \oplus W_{02}))), \quad (\omega_0^{\mathrm{or}}, \mathcal{S}_0^{\mathrm{or}} = \mathcal{S}(V \otimes W_{02})), \quad (\omega_{00}^{\mathrm{or}}, \mathcal{S}_{00}^{\mathrm{or}} = \mathcal{S}(V_{n_0} \otimes W_{02}))$$

  of $\omega_{V,W,\psi}$, $\omega_{V,W_{n_0},\psi}$, and $\omega_{V_{n_0},W_{n_0},\psi}$, respectively;
- canonical linear isomorphisms

$$\mathcal{S}(V \otimes (Y^* \oplus W_{02})) \cong \mathcal{S}(V \otimes Y^*) \otimes \mathcal{S}(V \otimes W_{02}),$$
$$\mathcal{S}(V \otimes W_{02}) \cong \mathcal{S}((X \oplus X^*) \otimes W_{02}) \otimes \mathcal{S}(V_{n_0} \otimes W_{02});$$

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– an isomorphism given by the partial inverse Fourier transform

$$\mathcal{S}((X \oplus X^*) \otimes W_{02}) \to \mathcal{S}(X^* \otimes W_0), \quad \varphi \mapsto \hat{\varphi}$$

defined by

$$\hat{\varphi}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \int_{y \in X \otimes W_{02}} \varphi \begin{pmatrix} y \\ x_2 \end{pmatrix} \psi(-\langle x_1, y\rangle)\, dy \quad \text{for } x_1 \in X^* \otimes W_{01},\ x_2 \in X^* \otimes W_{02},$$

where the Haar measure dy on $X \otimes W_{02}$ is defined by

$$dy = \prod_{\substack{1 \le i \le k \\ k+1 \le j \le n}} d_\psi c_{i,j} \quad \text{for } y = \sum_{\substack{1 \le i \le k \\ k+1 \le j \le n}} c_{i,j} x_i \otimes y_j^* \in X \otimes W_{02}.$$

Let $\mathscr{H}_0 = \mathbb{W}_0 \oplus F$ and $\mathscr{H}_{00} = \mathbb{W}_{00} \oplus F$ be the Heisenberg groups. Let ρ_0 and ρ_{00} be their Heisenberg representations associated with the Weil representations $(\omega_0, \mathcal{S}_0)$ and $(\omega_{00}, \mathcal{S}_{00})$, respectively. We regard $\mathrm{Sp}(W) = \mathrm{Sp}(W) \times \{1\} \subset \mathrm{Sp}(W) \times \{\pm 1\} = \mathrm{Mp}(W)$ as sets. Referring to [Ran93] or [Kud94, Theorem 3.1], we obtain some explicit formulas for the Weil representations.

For $\varphi \in \mathcal{S}$ and $x \in V \otimes Y^*$,

$$\begin{aligned}
[\omega(h)\varphi](x) &= \omega_0(h)\varphi(h^{-1}x), \quad h \in \mathrm{SO}(V),\\
[\omega(m(a))\varphi](x) &= \gamma_F(\det a, \psi)^{-1} |\det a|_F^{(2n+1)/2} \varphi(a^* x), \quad a \in \mathrm{GL}(Y),\\
[\omega(g_0)\varphi](x) &= \omega_0(g_0)\varphi(x), \quad g_0 \in \mathrm{Sp}(W_{n_0}),\\
[\omega(n^{\mathrm{b}}(b))\varphi](x) &= \rho_0((b^* x, 0))\varphi(x), \quad b \in \mathrm{Hom}(W_{n_0}, Y),\\
[\omega(n^{\mathrm{c}}(c))\varphi](x) &= \psi(\tfrac{1}{2}\langle n^{\mathrm{c}}(c)x, x\rangle)\varphi(x), \quad c \in \mathrm{Sym}(Y^*, Y),\\
[\omega(w_Y^{-1})\varphi](x) &= \gamma_F(\psi \circ q_V)^{-k} \omega_0((-1_{W_{n_0}})^k) \int_{Y^* \otimes V} \psi(\langle x', I_Y x\rangle)\varphi(x')\, dx'.
\end{aligned}$$

For $\varphi_0 \in \mathcal{S}_0 = \mathcal{S}(X^* \otimes W_{n_0}) \otimes \mathcal{S}_{00}$ and $y \in X^* \otimes W_{n_0}$,

$$\begin{aligned}
[\omega_0(g_0)\varphi_0](y) &= \omega_{00}(g_0)\varphi_0(g_0^{-1}y), \quad g_0 \in \mathrm{Sp}(W_{n_0}),\\
[\omega_0(l(a))\varphi_0](y) &= |\det a|_F^{n-k} \varphi_0(a^* y), \quad a \in \mathrm{GL}(X),\\
[\omega_0(h_0)\varphi_0](y) &= \omega_{00}(h_0)\varphi_0(y), \quad h_0 \in \mathrm{SO}(V_{n_0}),\\
[\omega_0(u^{\mathrm{b}}(b))\varphi_0](y) &= \rho_{00}((b^* y, 0))\varphi_0(y), \quad b \in \mathrm{Hom}(V_{n_0}, X),\\
[\omega_0(u^{\mathrm{c}}(c))\varphi_0](y) &= \psi(\tfrac{1}{2}\langle u^{\mathrm{c}}(c)y, y\rangle)\varphi_0(y), \quad c \in \mathrm{Alt}(X^*, X),\\
[\omega_0(w_X)\varphi_0](y) &= \omega_{00}((-1_{V_{n_0}})^k) \int_{X^* \otimes W_{n_0}} \psi(-\langle y', I_X y\rangle)\varphi_0(y')\, dy'.
\end{aligned}$$

For $\varphi_{00} \in \mathcal{S}_{00} = \mathcal{S}(V_{n_0} \otimes W_{02})$ and $x \in V_{n_0} \otimes W_{02}$,

$$\begin{aligned}
[\omega_{00}((-1_{W_{n_0}})^k)\varphi_{00}](x) &= \gamma_F((-1)^{k(n-k)}, \psi)^{-1} \varphi_{00}((-1)^k x),\\
[\omega_{00}((-1_{V_{n_0}})^k)\varphi_{00}](x) &= \varphi_{00}((-1)^k x).
\end{aligned}$$

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8.7 Gan and Ichino's equivariant maps

Next, we construct equivariant maps which realize the theta correspondence. Put

$$f_{\mathcal{S}}(\varphi)(gh) = [[\omega(gh)\varphi](e)](0),$$
$$\hat{f}_{\mathcal{S}}(\varphi)(gh) = \left[\int_{X\otimes Y^*} [\omega(gh)\varphi](x)\psi(-\langle e^*, x\rangle)\, dx\right](0)$$

for $\varphi \in \mathcal{S} = \mathcal{S}(V \otimes Y^*) \otimes \mathcal{S}(X^* \otimes W_{n_0}) \otimes \mathcal{S}_{00}$, $g \in \mathrm{Mp}(W)$, and $h \in \mathrm{O}(V)$. If $f = f_{\mathcal{S}}(\varphi)$ or $\hat{f}_{\mathcal{S}}(\varphi)$, then by the explicit formulas for the mixed Schrödinger model, we have

$$f(nugh) = f(gh), \quad n \in N,\ u \in U,$$
$$f(g_0h_0gh) = \omega_{00}(g_0h_0)f(gh), \quad g_0 \in \mathrm{Sp}(W_{n_0}),\ h_0 \in \mathrm{O}(V),$$
$$f(m(a)l(a)gh) = \gamma_F(\det a, \psi)^{-1}|\det a|_F^{\rho_Q+\rho_P} f(gh), \quad a \in \mathrm{GL}_k(F) \cong \mathrm{GL}(X) \cong \mathrm{GL}(Y), \tag{8.1}$$

for any $g \in \mathrm{Mp}(W)$ and $h \in \mathrm{O}(V)$. In the rest of this section we shall drop the subscript $\mathcal{S}$ for simplicity.

In this subsection, we shall write $\tau = \tau_1$ and assume that σ_0 and π_0 may be direct sums of irreducible tempered representations, whose summands have the same L-parameter ϕ_0 and correspond bijectively via Θ_ψ. For $\rho = \tau$, π_0, or σ_0, let $(\rho^\vee, \mathscr{V}_{\rho^\vee})$ be the contragredient representation of $(\rho, \mathscr{V}_\rho)$ and $\langle -, - \rangle$ the invariant non-degenerate bilinear form on $\mathscr{V}_\rho \times \mathscr{V}_{\rho^\vee}$. We fix a nonzero $\mathrm{Mp}(W_{n_0}) \times \mathrm{SO}(V_{n_0})$-equivariant map

$$\mathcal{T}_{00} : \omega_{00} \otimes \sigma_0^\vee \longrightarrow \pi_0. \tag{8.2}$$

For any $\varphi \in \mathcal{S}$, $\mathscr{F}_s \in \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\tau_s \otimes \sigma_0^\vee)$, $g \in \mathrm{Mp}(W)$, $\check{v} \in \mathscr{V}_{\tau^\vee}$, and $\check{v}_0 \in \mathscr{V}_{\pi_0^\vee}$, put

$$I(s, \varphi \otimes \mathscr{F}_s, \check{v} \otimes \check{v}_0, g) = \frac{1}{L(s+\frac{1}{2}, \tau)} \int_{U\mathrm{SO}(V_{n_0})\backslash \mathrm{SO}(V)} \langle \mathcal{T}_{00}(\hat{f}(\varphi)(gh) \otimes \langle \mathscr{F}_s(h), \check{v}\rangle), \check{v}_0\rangle\, dh$$

if the right-hand side converges absolutely. Here, $s \in \mathbb{C}$ is a complex variable.

Lemma 8.3. *The following hold.*

(1) *The integral $I(s, \varphi \otimes \mathscr{F}_s, \check{v} \otimes \check{v}_0, g)$ converges absolutely for $\mathrm{Re}(s) > -\frac{1}{2}$ and admits a holomorphic continuation to $s \in \mathbb{C}$.*
(2) *For $\mathrm{Re}(s) < \frac{1}{2}$, we have that $I(s, \varphi \otimes \mathscr{F}_s, \check{v} \otimes \check{v}_0, g)$ is equal to*

$$L\left(s+\frac{1}{2}, \tau\right)^{-1} \gamma\left(s+\frac{1}{2}, \tau, \psi\right)^{-1} \int_{U\mathrm{SO}(V_{n_0})\backslash \mathrm{SO}(V)} \langle \mathcal{T}_{00}(f(\varphi)(gh) \otimes \langle \mathscr{F}_s(h), \check{v}\rangle), \check{v}_0\rangle\, dh.$$

(3) *By virtue of (1.1), we define a vector $\mathcal{T}_s(\varphi \otimes \mathscr{F}_s)(g)$ of $\mathscr{V}_\tau \otimes \mathscr{V}_{\pi_0}$ by*

$$\langle \mathcal{T}_s(\varphi \otimes \mathscr{F}_s)(g), \check{v} \otimes \check{v}_0\rangle = I(s, \varphi \otimes \mathscr{F}_s, \check{v} \otimes \check{v}_0, g).$$

Then, for any $0 \neq \mathscr{F} \in \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\tau \otimes \sigma_0^\vee)$, there exists $\varphi \in \mathcal{S}$ such that

$$\mathcal{T}_{s=0}(\varphi \otimes \mathscr{F}) \neq 0.$$

Proof. The proof is similar to those of Lemmas 8.1, 8.2, and 8.3 in [GI16]. □

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When $s=0$ the assignment $\varphi\otimes\mathscr{F}\mapsto\mathcal{T}_0(\varphi\otimes\mathscr{F})$ gives an $\mathrm{Mp}(W)\times\mathrm{SO}(V)$-equivariant map $\omega\otimes\mathrm{Ind}_Q^{\mathrm{SO}(V)}(\tau\otimes\sigma_0^\vee)\to\mathrm{Ind}_P^{\mathrm{Mp}(W)}(\widetilde{\tau}\otimes\pi_0)$. We shall write $\mathcal{T}(k,\mathcal{T}_{00})$ for this map.

Now, we note the functorialities of the equivariant map $\mathcal{T}(k,\mathcal{T}_{00})$ here. We have the following two lemmas, which easily follow from the definition of $\mathcal{T}(k,\mathcal{T}_{00})$.

Lemma 8.4. *Let $(\tau',\mathscr{V}_{\tau'})$ be a representation of GL_k that is isomorphic to τ, and let $A:(\tau,\mathscr{V}_\tau)\to(\tau',\mathscr{V}_{\tau'})$ be an isomorphism of representations of GL_k. Then the diagram*

$$\begin{array}{ccc}
\omega\otimes\mathrm{Ind}_Q^{\mathrm{SO}(V)}(\tau\otimes\sigma_0^\vee) & \xrightarrow{\mathcal{T}(k,\mathcal{T}_{00})} & \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\widetilde{\tau}\otimes\pi_0)\\
{\scriptstyle 1\otimes\mathrm{Ind}(A)}\big\downarrow & & \big\downarrow{\scriptstyle\mathrm{Ind}(A)}\\
\omega\otimes\mathrm{Ind}_Q^{\mathrm{SO}(V)}(\tau'\otimes\sigma_0^\vee) & \xrightarrow[\mathcal{T}(k,\mathcal{T}_{00})]{} & \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\widetilde{\tau'}\otimes\pi_0)
\end{array}$$

commutes. Here, $\mathrm{Ind}(A)$ denotes an operator defined by $[\mathrm{Ind}(A)\mathscr{F}](x)=A(\mathscr{F}(x))$.

Lemma 8.5. *Let $(\sigma_0',\mathscr{V}_{\sigma_0'})$ (respectively $(\pi_0',\mathscr{V}_{\pi_0'})$) be a representation of $\mathrm{SO}(V_{n_0})$ (respectively $\mathrm{Mp}(W_{n_0})$) that is isomorphic to σ_0 (respectively π_0), and let $B:(\sigma_0^\vee,\mathscr{V}_{\sigma_0^\vee})\to({\sigma_0'}^\vee,\mathscr{V}_{{\sigma_0'}^\vee})$ (respectively $C:(\pi_0,\mathscr{V}_{\pi_0})\to(\pi_0',\mathscr{V}_{\pi_0'})$) be an isomorphism. Choose an $\mathrm{Mp}(W_{n_0})\times\mathrm{SO}(V_{n_0})$-equivariant map $\mathcal{T}_{00}':\omega_{00}\otimes{\sigma_0'}^\vee\to\pi_0$ such that the diagram*

$$\begin{array}{ccc}
\omega_{00}\otimes\sigma_0^\vee & \xrightarrow{\mathcal{T}_{00}} & \pi_0\\
{\scriptstyle 1\otimes B}\big\downarrow & & \big\downarrow{\scriptstyle C}\\
\omega_{00}\otimes{\sigma_0'}^\vee & \xrightarrow[\mathcal{T}_{00}']{} & \pi_0'
\end{array}$$

commutes. Then the diagram

$$\begin{array}{ccc}
\omega\otimes\mathrm{Ind}_Q^{\mathrm{SO}(V)}(\tau\otimes\sigma_0^\vee) & \xrightarrow{\mathcal{T}(k,\mathcal{T}_{00})} & \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\widetilde{\tau}\otimes\pi_0)\\
{\scriptstyle 1\otimes\mathrm{Ind}(B)}\big\downarrow & & \big\downarrow{\scriptstyle\mathrm{Ind}(C)}\\
\omega\otimes\mathrm{Ind}_Q^{\mathrm{SO}(V)}(\tau\otimes{\sigma_0'}^\vee) & \xrightarrow[\mathcal{T}(k,\mathcal{T}_{00}')]{} & \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\widetilde{\tau}\otimes\pi_0')
\end{array}$$

also commutes.

Finally, we note a key property of the assignment $\mathcal{T}_s$.

Proposition 8.6. *For $\varphi\in\mathcal{S}$ and $\mathscr{F}_s\in\mathrm{Ind}_Q^{\mathrm{SO}(V)}(\tau_s\otimes\sigma_0^\vee)$, we have*

$$\mathcal{R}_P(w_M,\widetilde{\tau}_s\otimes\pi_0)\mathcal{T}_s(\varphi\otimes\mathscr{F}_s)=\beta(s)\cdot\mathcal{T}_{-s}(\varphi\otimes\mathcal{R}_Q(w_L,\tau_s\otimes\sigma_0^\vee)\mathscr{F}_s),$$

where

$$\beta(s)=|2|_F^{2ks}\cdot\frac{L(-s+\frac{1}{2},\tau^\vee)}{L(s+\frac{1}{2},\tau)}\cdot\frac{\gamma(-s+\frac{1}{2},\tau^\vee,\psi)}{\gamma(s+\frac{1}{2},\tau,\psi)}.$$

Proof. Noting that $\phi_0^\vee\cong\phi_0$ and $\gamma_F(\psi\circ q_V)=\epsilon(V)\gamma_F(\psi)$, one can prove this proposition using a similar argument to the proof of [GI16, Corollary 8.5]. □

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9. Proof of Proposition 7.3

Now we can define the equivariant map $\mathcal{T}$ desired in Proposition 7.3 and give our proof of the proposition. We will define this map $\mathcal{T}$ to be the map $\mathcal{T}(k, \mathcal{T}_{00})$ constructed in § 8.7 when P is maximal, and by induction in stages when P is not maximal. We shall use the same notation as in § 7. and assume that $\sigma_0 = \Theta_\psi(\pi_0)$.

9.1 An equivariant map $\mathcal{T}$

In this subsection we define an $\mathrm{Mp}(W) \times \mathrm{SO}(V)$-equivariant map

$$\mathcal{T} : \omega_{V,W,\psi} \otimes \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\breve{\sigma}_L) \longrightarrow \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\pi_M)$$

that will satisfy Proposition 7.3. For a fixed $1 \le m' \le m$, we put $\mathbf{k}' = (k_1, \dots, k_{m'})$, $k' = k_1 + \cdots + k_{m'}$, $\mathbf{k}'' = (k_{m'+1}, \dots, k_m)$, $k'' = k_{m'+1} + \cdots + k_m$, and $n' = n - k'$. As in §§ 8.1 and 8.6, we take $X = X_k$, $X^* = X_k^*$, $Y = Y_k$, and $Y^* = Y_k^*$. Put $W_{02} = \mathrm{span}_F(y_{k+1}^*, \dots, y_n^*)$. Also, let us put

$$\begin{aligned} X' &= X_{k'} = \mathrm{span}_F(x_1, \dots, x_{k'}), & X'^* &= X_{k'}^* = \mathrm{span}_F(x_1^*, \dots, x_{k'}^*),\\ Y' &= Y_{k'} = \mathrm{span}_F(y_1, \dots, y_{k'}), & Y'^* &= Y_{k'}^* = \mathrm{span}_F(y_1^*, \dots, y_{k'}^*),\\ X'' &= \mathrm{span}_F(x_{k'+1}, \dots, x_k), & X''^* &= \mathrm{span}_F(x_{k'+1}^*, \dots, x_k^*),\\ Y'' &= \mathrm{span}_F(y_{k'+1}, \dots, y_k), & Y''^* &= \mathrm{span}_F(y_{k'+1}^*, \dots, y_k^*), \end{aligned}$$

$V' = V_{n'}$, and $W' = W_{n'}$, so that

$$\begin{aligned} V &= X' \oplus V' \oplus X'^*, & V' &= X'' \oplus V_{n_0} \oplus X''^*,\\ W &= Y' \oplus W' \oplus Y'^*, & W' &= Y'' \oplus W_{n_0} \oplus Y''^*, \end{aligned}$$

and we shall write $Q' = L' \ltimes U'$ and $P' = M' \ltimes N'$ for the maximal parabolic subgroups of $\mathrm{SO}(V')$ and $\mathrm{Mp}(W')$ stabilizing X'' and Y'', respectively.

Let $(\omega, \mathcal{S})$, $(\omega_0, \mathcal{S}_0)$, and $(\omega_{00}, \mathcal{S}_{00})$ be the models of the Weil representations constructed in § 8.6. Additionally, let ω'' be the realization of the Weil representation $\omega_{V',W',\psi}$ of $\mathrm{O}(V') \times \mathrm{Mp}(W')$ on a mixed model

$$\mathcal{S}'' = \mathcal{S}(V' \otimes Y''^*) \otimes \mathcal{S}(X''^* \otimes W_{n_0}) \otimes \mathcal{S}_{00},$$

and let $\omega' = \omega_{V,W,\psi}$ be the realization of the Weil representation of $\mathrm{O}(V) \times \mathrm{Mp}(W)$ on a mixed model

$$\mathcal{S}' = \mathcal{S}(V \otimes Y'^*) \otimes \mathcal{S}(X'^* \otimes W') \otimes \mathcal{S}''.$$

As in § 8.6, fix isomorphisms

$$(\omega, \mathcal{S}) \cong (\omega^{\mathrm{or}}, \mathcal{S}^{\mathrm{or}}) \cong (\omega', \mathcal{S}') \tag{9.1}$$

of the three realizations of $\omega_{V,W,\psi}$ and identify them.

Let $P_{\mathbf{k}}^{\mathrm{GL}}$ be the standard parabolic subgroup of $\mathrm{GL}_k \cong \mathrm{GL}(Y)$ stabilizing the flag

$$Y_{k_1} \subset Y_{k_1+k_2} \subset \cdots \subset Y_{k_1+\cdots+k_m}.$$

Similarly, we define the standard parabolic subgroups $P_{\mathbf{k}'}^{\mathrm{GL}}$ of $\mathrm{GL}_{k'}$ and $P_{\mathbf{k}''}^{\mathrm{GL}}$ of $\mathrm{GL}_{k''}$. Put $\tau = \mathrm{Ind}_{P_{\mathbf{k}}^{\mathrm{GL}}}^{\mathrm{GL}_k}(\tau_1 \otimes \cdots \otimes \tau_m)$, $\tau' = \mathrm{Ind}_{P_{\mathbf{k}'}^{\mathrm{GL}}}^{\mathrm{GL}_{k'}}(\tau_1 \otimes \cdots \otimes \tau_{m'})$, and $\tau'' = \mathrm{Ind}_{P_{\mathbf{k}''}^{\mathrm{GL}}}^{\mathrm{GL}_{k''}}(\tau_{m'+1} \otimes \cdots \otimes \tau_m)$.

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These representations are irreducible, since $\tau_1, \ldots, \tau_m$ are tempered. Define canonical isomorphisms

$$\Phi : \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\tau_1 \otimes \cdots \otimes \tau_m \otimes \sigma_0^\vee) \longrightarrow \mathrm{Ind}_{Q_k}^{\mathrm{SO}(V)}(\tau \otimes \sigma_0^\vee),$$
$$\Psi : \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\tau_1 \otimes \cdots \otimes \tau_m \otimes \sigma_0^\vee) \longrightarrow \mathrm{Ind}_{Q_{k'}}^{\mathrm{SO}(V)}\big(\tau' \otimes \mathrm{Ind}_{Q'}^{\mathrm{SO}(V')}(\tau' \otimes \sigma_0^\vee)\big)$$

by

$$\Phi\mathscr{F}(h)(x) = \delta_{Q_k}(l(x))^{-1/2}\mathscr{F}(l(x)h),$$
$$\Psi\mathscr{F}(h)(x', h') = \delta_{Q_{k'}}(l'(x'))^{-1/2}\mathscr{F}(l'(x')h'h),$$

where l and l' are the canonical embeddings $\mathrm{GL}_k \hookrightarrow L_k$ and $\mathrm{GL}_{k'} \hookrightarrow L_{k'}$, respectively, as in § 8.1.

Similarly, with an abuse of notation, we take canonical isomorphisms

$$\Phi : \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\widetilde{\tau_1} \otimes \cdots \otimes \widetilde{\tau_m} \otimes \pi_0) \longrightarrow \mathrm{Ind}_{P_k}^{\mathrm{Mp}(W)}(\widetilde{\tau} \otimes \pi_0),$$
$$\Psi : \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\widetilde{\tau_1} \otimes \cdots \otimes \widetilde{\tau_m} \otimes \pi_0) \longrightarrow \mathrm{Ind}_{P_{k'}}^{\mathrm{Mp}(W)}\big(\widetilde{\tau'} \otimes \mathrm{Ind}_{P'}^{\mathrm{Mp}(W')}(\widetilde{\tau''} \otimes \pi_0)\big)$$

and the canonical embeddings $m : \mathrm{GL}_k \hookrightarrow \overline{M_k}$ and $m' : \mathrm{GL}_{k'} \hookrightarrow \overline{M_{k'}}$.

Next, following § 8.7, we put $\mathcal{T}^a = \mathcal{T}(k, \mathcal{T}_{00})$ and $\mathcal{T}^r = \mathcal{T}(k', \mathcal{T}(k'', \mathcal{T}_{00}))$, which are $\mathrm{Mp}(W) \times \mathrm{SO}(V)$-equivariant maps

$$\omega \otimes \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\tau \otimes \sigma_0^\vee) \longrightarrow \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\widetilde{\tau} \otimes \pi_0)$$

and

$$\omega' \otimes \mathrm{Ind}_{Q_{k'}}^{\mathrm{SO}(V)}(\tau' \otimes \mathrm{Ind}_{Q'}^{\mathrm{SO}(V')}(\tau' \otimes \sigma_0^\vee)) \longrightarrow \mathrm{Ind}_{P_{k'}}^{\mathrm{Mp}(W)}(\widetilde{\tau'} \otimes \mathrm{Ind}_{P'}^{\mathrm{Mp}(W')}(\widetilde{\tau''} \otimes \pi_0)),$$

respectively. Here $\mathcal{T}_{00}$ is the fixed map (8.2).

Lemma 9.1. *The diagram*

$$\begin{array}{ccc}
\omega \otimes \mathrm{Ind}_{Q_k}^{\mathrm{SO}(V)}(\tau \otimes \sigma_0^\vee) & \xrightarrow{\mathcal{T}^a} & \mathrm{Ind}_{P_k}^{\mathrm{Mp}(W)}(\widetilde{\tau} \otimes \pi_0) \\
{\scriptstyle 1\otimes\Phi}\uparrow & & \uparrow{\scriptstyle \Phi} \\
\omega_{V,W,\psi} \otimes \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\tau_1 \otimes \cdots \otimes \tau_m \otimes \sigma_0^\vee) & & \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\widetilde{\tau_1} \otimes \cdots \otimes \widetilde{\tau_m} \otimes \pi_0) \\
{\scriptstyle 1\otimes\Psi}\downarrow & & \downarrow{\scriptstyle \Psi} \\
\omega' \otimes \mathrm{Ind}_{Q_{k'}}^{\mathrm{SO}(V)}(\tau' \otimes \mathrm{Ind}_{Q'}^{\mathrm{SO}(V')}(\tau' \otimes \sigma_0^\vee)) & \xrightarrow[\mathcal{T}^r]{} & \mathrm{Ind}_{P_{k'}}^{\mathrm{Mp}(W)}(\widetilde{\tau'} \otimes \mathrm{Ind}_{P'}^{\mathrm{Mp}(W')}(\widetilde{\tau''} \otimes \pi_0))
\end{array}$$

commutes.

Lemma 9.1 lets us define an $\mathrm{Mp}(W) \times \mathrm{SO}(V)$-equivariant map

$$\mathcal{T} : \omega_{V,W,\psi} \otimes \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\breve{\sigma}_L) \longrightarrow \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\pi_M),$$

so that the diagram will remain commutative if we insert $\mathcal{T}$ into the middle horizontal space. In other words,

$$\mathcal{T} = \Phi^{-1} \circ \mathcal{T}^a \circ (1 \otimes \Phi) = \Psi^{-1} \circ \mathcal{T}^r \circ (1 \otimes \Psi).$$

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9.2 Proof of Lemma 9.1

Let $\varphi \in \mathcal{S} \cong \mathcal{S}'$ and $\mathscr{F} \in \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\tau_1 \otimes \cdots \otimes \tau_m \otimes \sigma_0^\vee)$. It suffices to show that

$$\langle \mathcal{T}^r(\varphi \otimes \Psi\mathscr{F})(g)(1,1), \check{v}_1 \otimes \cdots \otimes \check{v}_m \otimes \check{v}_0\rangle = \langle \mathcal{T}^a(\varphi \otimes \Phi\mathscr{F})(g)(1), \check{v}_1 \otimes \cdots \otimes \check{v}_m \otimes \check{v}_0\rangle \tag{9.2}$$

for any $\check{v}_i \in \mathscr{V}_{\tau_i^\vee}$, $\check{v}_0 \in \mathscr{V}_{\pi_0^\vee}$, and $g \in \mathrm{Mp}(W)$.

Fix $\check{v}_i \in \mathscr{V}_{\tau_i^\vee}$, $\check{v}_0 \in \mathscr{V}_{\pi_0^\vee}$, and $g \in \mathrm{Mp}(W)$. Choose an element $\mathscr{K} = \mathcal{K} \otimes \mathscr{K}'$ of

$$\mathrm{Ind}_{P_{\mathbf{k}'}^{\mathrm{GL}}}^{\mathrm{GL}_{k'}}(\tau_1^\vee \otimes \cdots \otimes \tau_{m'}^\vee) \otimes \mathrm{Ind}_{P'}^{\mathrm{Mp}(W')}(\widetilde{\tau''}^\vee \otimes \pi_0^\vee) \cong \tau'^\vee \otimes \mathrm{Ind}_{P'}^{\mathrm{Mp}(W')}(\widetilde{\tau''} \otimes \pi_0)^\vee$$

such that

$$\mathrm{supp}(\mathscr{K}) \subset (P_{\mathbf{k}'}^{\mathrm{GL}} \times P') \cdot K',$$
$$\mathscr{K}(x) = \check{v}_1 \otimes \cdots \otimes \check{v}_m \otimes \check{v}_0$$

for any $x \in K'$, where $K' = K'_G \times K'_M \subset \mathrm{GL}_{k'} \times \mathrm{Mp}(W')$ is a compact open subgroup such that:

- $((\mathrm{GL}_{k_1} \times \cdots \times \mathrm{GL}_{k_{m''}}) \times \mathrm{Mp}(W')) \cap K'$ stabilizes $\check{v}_1, \ldots, \check{v}_m, \check{v}_0$;
- K' stabilizes $\omega'(g)\varphi$, i.e.$\omega'(m'(a')g'_0 g)\varphi = \omega'(g)\varphi$ for any $(a', g'_0) \in K'$.

Since K' stabilizes $\mathcal{T}^r(\varphi \otimes \Psi\mathscr{F})(g) = \mathcal{T}^r(\omega'(g)\varphi \otimes \Psi\mathscr{F})(1)$, we have

$$\begin{aligned}\langle \mathcal{T}^r(\varphi \otimes \Psi\mathscr{F})(g), \mathscr{K}\rangle &= \int_{(P_{\mathbf{k}'}^{\mathrm{GL}} \times P_{k''})\backslash(\mathrm{GL}_{k'} \times \mathrm{Mp}(W'))} \langle \mathcal{T}^r(\varphi \otimes \Psi\mathscr{F})(g)(x), \mathscr{K}(x)\rangle \, dx \\ &= \int_{(P_{\mathbf{k}'}^{\mathrm{GL}} \times P_{k''})\backslash(P_{\mathbf{k}'}^{\mathrm{GL}} \times P_{k''})K'} \langle \mathcal{T}^r(\varphi \otimes \Psi\mathscr{F})(g)(x), \mathscr{K}(x)\rangle \, dx \\ &= \mathrm{vol}(K')\langle \mathcal{T}^r(\varphi \otimes \Psi\mathscr{F})(g)(1,1), \check{v}_1 \otimes \cdots \otimes \check{v}_m \otimes \check{v}_0\rangle. \end{aligned} \tag{9.3}$$

On the other hand, by the definition of $\mathcal{T}^r$ and Lemma 8.3, we see that $\langle \mathcal{T}^r(\varphi \otimes \Psi\mathscr{F})(g), \mathscr{K}\rangle$ equals

$$L\left(\frac{1}{2}, \tau'\right)^{-1} \gamma\left(\frac{1}{2}, \tau', \psi\right)^{-1} \int_{U_{k'}\mathrm{SO}(V_{n'})\backslash \mathrm{SO}(V)} \langle \mathcal{T}'(f_{\mathcal{S}'}(\varphi)(gh) \otimes \langle \Psi\mathscr{F}(h), \mathcal{K}\rangle), \mathscr{K}'\rangle \, dh,$$

where

$$\mathcal{T}' = \mathcal{T}(k'', \mathcal{T}_{00}) : \omega' \otimes \mathrm{Ind}_{Q'}^{\mathrm{SO}(V')}(\tau' \otimes \sigma_0^\vee) \longrightarrow \mathrm{Ind}_{P'}^{\mathrm{Mp}(W')}(\widetilde{\tau''} \otimes \pi_0).$$

The last integral is equal to

$$\begin{aligned}&\int_{U_{k'}\mathrm{SO}(V_{n'})\backslash \mathrm{SO}(V)} \Big\langle \mathcal{T}'\Big(f_{\mathcal{S}'}(\varphi)(gh) \otimes \int_{P_{\mathbf{k}'}^{\mathrm{GL}}\backslash \mathrm{GL}_{k'}} \langle \Psi\mathscr{F}(h)(a', \bullet), \mathcal{K}(a')\rangle \, da'\Big), \mathscr{K}'\Big\rangle \, dh \\ &= \int_{U_{k'}\mathrm{SO}(V_{n'})\backslash \mathrm{SO}(V)} \Big\langle \mathcal{T}'\Big(f_{\mathcal{S}'}(\varphi)(gh) \otimes \int_{K'_G} \langle \Psi\mathscr{F}(l'(a')h)(1, \bullet), \check{v}_1 \otimes \cdots \otimes \check{v}_{m'}\rangle \, da'\Big), \mathscr{K}'\Big\rangle \, dh \\ &= \int_{K'_G} \int_{U_{k'}\mathrm{SO}(V_{n'})\backslash \mathrm{SO}(V)} \big\langle \mathcal{T}'\big(f_{\mathcal{S}'}(\varphi)(gl'(a')h) \otimes \langle \Psi\mathscr{F}(h)(1, \bullet), \check{v}_1 \otimes \cdots \otimes \check{v}_{m'}\rangle\big), \mathscr{K}'\big\rangle \, dh \, da'. \end{aligned}$$

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Thus we have that $\langle \mathcal{T}^r(\varphi \otimes \Psi \mathscr{F})(g), \mathscr{K} \rangle$ is equal to the product of $L(\frac{1}{2}, \tau')^{-1}\gamma(\frac{1}{2}, \tau', \psi)^{-1}$ and

$$\int_{K'_G} \int_{U_{k'}\mathrm{SO}(V_{n'})\backslash \mathrm{SO}(V)} \left\langle \mathcal{T}' \left(f_{\mathcal{S}'}(\varphi)(gl'(a')h) \otimes \langle \Psi \mathscr{F}(h)(1, \bullet), \check{v}_1 \otimes \cdots \otimes \check{v}_{m'} \rangle \right), \mathscr{K}' \right\rangle dh\, da'. \quad (9.4)$$

Moreover,

$$\begin{aligned} &\int_{K'_G} \left\langle \mathcal{T}' \left(f_{\mathcal{S}'}(\varphi)(gl'(a')h) \otimes \langle \Psi \mathscr{F}(h)(1, \bullet), \check{v}_1 \otimes \cdots \otimes \check{v}_{m'} \rangle \right), \mathscr{K}' \right\rangle da' \\ &\quad = \int_{K'_G} \int_{K'_M} \left\langle \mathcal{T}' \left(f_{\mathcal{S}'}(\varphi)(gl'(a')h) \otimes \langle \Psi \mathscr{F}(h)(1, \bullet), \check{v}_1 \otimes \cdots \otimes \check{v}_{m'} \rangle \right)(g'_0), \mathscr{K}'(g'_0) \right\rangle dg'_0\, da' \\ &\quad = \int_{K'} \left\langle \mathcal{T}' \left(f_{\mathcal{S}'}(\varphi)(g'_0 gl'(a')h) \otimes \langle \Psi \mathscr{F}(h)(1, \bullet), \check{v}_1 \otimes \cdots \otimes \check{v}_{m'} \rangle \right)(1), \check{v}_{m'+1} \otimes \cdots \otimes \check{v}_0 \right\rangle d(a', g'_0). \end{aligned} \quad (9.5)$$

Then (9.4) and (9.5) imply that $\langle \mathcal{T}^r(\varphi \otimes \Psi \mathscr{F})(g), \mathscr{K} \rangle$ is

$$\begin{aligned} &L\left(\frac{1}{2}, \tau'\right)^{-1} \gamma\left(\frac{1}{2}, \tau', \psi\right)^{-1} \int_{U_{k'}\mathrm{SO}(V_{n'})\backslash \mathrm{SO}(V)} \int_{K'} \\ &\quad \left\langle \mathcal{T}' \left(f_{\mathcal{S}'}(\varphi)(g'_0 gl'(a')h) \otimes \langle \Psi \mathscr{F}(h)(1, \bullet), \check{v}_1 \otimes \cdots \otimes \check{v}_{m'} \rangle \right)(1), \check{v}_{m'+1} \otimes \cdots \otimes \check{v}_0 \right\rangle d(a', g'_0)\, dh. \end{aligned} \quad (9.6)$$

Now, by the formula (8.1) and the choice of K', we have

$$f_{\mathcal{S}'}(\varphi)(g'_0 gl'(a')uh) = f_{\mathcal{S}'}(\varphi)(guh).$$

Therefore, (9.3) and (9.6) imply that

$$\begin{aligned} &\langle \mathcal{T}^r(\varphi \otimes \Psi \mathscr{F})(g)(1,1), \check{v}_1 \otimes \cdots \otimes \check{v}_m \otimes \check{v}_0 \rangle \\ &\quad = L\left(\frac{1}{2}, \tau'\right)^{-1} \gamma\left(\frac{1}{2}, \tau', \psi\right)^{-1} \int_{U_{k'}\mathrm{SO}(V_{n'})\backslash \mathrm{SO}(V)} \\ &\qquad \left\langle \mathcal{T}' \left(f_{\mathcal{S}'}(\varphi)(gh) \otimes \langle \Psi \mathscr{F}(h)(1, \bullet), \check{v}_1 \otimes \cdots \otimes \check{v}_{m'} \rangle \right)(1), \check{v}_{m'+1} \otimes \cdots \otimes \check{v}_0 \right\rangle dh. \end{aligned} \quad (9.7)$$

Now, the definition of $\mathcal{T}'$ gives that the last integral is equal to

$$\begin{aligned} &L\left(\frac{1}{2}, \tau''\right)^{-1} \gamma\left(\frac{1}{2}, \tau'', \psi\right)^{-1} \int_{U_{k'}\mathrm{SO}(V_{n'})\backslash \mathrm{SO}(V)} \int_{U'\mathrm{SO}(V_{n_0})\backslash \mathrm{SO}(V')} \\ &\quad \left\langle \mathcal{T}_{00} \left(f_{\mathcal{S}''} \left(f_{\mathcal{S}'}(\varphi)(gh) \right)(h') \otimes \langle \langle \Psi \mathscr{F}(h)(1, h'), \check{v}_1 \otimes \cdots \otimes \check{v}_{m'} \rangle, \check{v}_{m'+1} \otimes \cdots \otimes \check{v}_m \rangle \right), \check{v}_0 \right\rangle dh'\, dh \\ &= L\left(\frac{1}{2}, \tau''\right)^{-1} \gamma\left(\frac{1}{2}, \tau'', \psi\right)^{-1} \int_{U\mathrm{SO}(V_{n_0})\backslash \mathrm{SO}(V)} \int_{U_{k'}U'\backslash U} \\ &\quad \left\langle \mathcal{T}_{00} \left(f_{\mathcal{S}''} \left(f_{\mathcal{S}'}(\varphi)(guh) \right)(1) \otimes \langle \mathscr{F}(h), \check{v}_1 \otimes \cdots \otimes \check{v}_m \rangle \right), \check{v}_0 \right\rangle du\, dh. \end{aligned} \quad (9.8)$$

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By (9.7) and (9.8), we have

$$\begin{aligned} &\langle \mathcal{T}^r(\varphi \otimes \Psi\mathscr{F})(g)(1,1), \check{v}_1 \otimes \cdots \otimes \check{v}_m \otimes \check{v}_0\rangle \\ &\quad = L\left(\frac{1}{2},\tau\right)^{-1} \gamma\left(\frac{1}{2},\tau,\psi\right)^{-1} \int_{U\mathrm{SO}(V_{n_0})\backslash \mathrm{SO}(V)} \left\langle \mathcal{T}_{00}\left(f''(\varphi)(gh)\otimes \mathscr{F}_0(h)\right), \check{v}_0\right\rangle dh, \qquad (9.9) \end{aligned}$$

where

$$f''(\varphi)(gh) = \int_{U_{k'}U'\backslash U} f_{\mathcal{S}''}\left(f_{\mathcal{S}'}(\varphi)(ugh)\right)(1)\,du,$$
$$\mathscr{F}_0(h) = \langle \mathscr{F}(h), \check{v}_1 \otimes \cdots \otimes \check{v}_m\rangle.$$

Similarly, we can obtain

$$\begin{aligned} &\langle \mathcal{T}^a(\varphi \otimes \Phi\mathscr{F})(g)(1), \check{v}_1 \otimes \cdots \otimes \check{v}_m \otimes \check{v}_0\rangle \\ &\quad = L\left(\frac{1}{2},\tau\right)^{-1} \gamma\left(\frac{1}{2},\tau,\psi\right)^{-1} \int_{U\mathrm{SO}(V_{n_0})\backslash \mathrm{SO}(V)} \left\langle \mathcal{T}_{00}(f'(\varphi)(gh)\otimes \mathscr{F}_0(h)), \check{v}_0\right\rangle dh, \qquad (9.10) \end{aligned}$$

where

$$f'(\varphi)(gh) = \int_{U_k\backslash U} f_{\mathcal{S}}(\varphi)(ugh)\,du.$$

Now (9.9) and (9.10) tell us that it suffices to show that $f''(\varphi) = f'(\varphi)$, which will follow from Lemma 9.2 below. □

LEMMA 9.2. *Under the identification (9.1), for any $\varphi \in \mathcal{S}^{\mathrm{or}}$ we have*

$$\int_{U_{k'}U'\backslash U} f_{\mathcal{S}''}\left(f_{\mathcal{S}'}(\varphi)(u)\right)(1)\,du = \int_{U_k\backslash U} f_{\mathcal{S}}(\varphi)(u)\,du.$$

Proof. Put

$$e' = x_1 \otimes y_1^* + \cdots + x_{k'} \otimes y_{k'}^* \in X' \otimes Y'^*,$$
$$e'' = x_{k'+1} \otimes y_{k'+1}^* + \cdots + x_k \otimes y_k^* \in X'' \otimes Y''^*,$$

and let $\varphi \in \mathcal{S}^{\mathrm{or}}$. Because

$$\begin{aligned} \mathcal{S}^{\mathrm{or}} \cong{}& \mathcal{S}(V \otimes Y'^*) \ \otimes\ \mathcal{S}((X' \oplus X'^*) \otimes (Y''^* \oplus W_{02})) \\ &\otimes\ \mathcal{S}(V' \otimes Y''^*) \ \otimes\ \mathcal{S}((X'' \oplus X''^*) \otimes W_{02}) \ \otimes\ \mathcal{S}_{00}, \end{aligned}$$

we shall write

$$\varphi\left[x, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, x', \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}\right] = \varphi\left[x, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right]\left[x', \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}\right]$$

for the evaluation of φ at $x \in V \otimes Y'^*$, $y_1 \in X' \otimes (Y''^* \oplus W_{02})$, $y_2 \in X'^* \otimes (Y''^* \oplus W_{02})$, $x' \in V' \otimes Y''^*$, $y_1' \in X'' \otimes W_{02}$, and $y_2' \in X''^* \otimes W_{02}$, which is an element of $\mathcal{S}_{00}$. Then

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we have

$$\begin{aligned} f_{\mathcal{S}''}\left(f_{\mathcal{S}'}(\varphi)(u)\right)(1) &= \int_{y' \in X'' \otimes W_{02}} [f_{\mathcal{S}'}(\varphi)(u)] \left[e'', \begin{pmatrix} y' \\ 0 \end{pmatrix}\right] dy' \\ &= \int_{y' \in X'' \otimes W_{02}} \int_{y' \in X' \otimes (Y''^* \oplus W_{02})} \omega^{\mathrm{or}}(u)\varphi \left[e', \begin{pmatrix} y \\ 0 \end{pmatrix}, e', \begin{pmatrix} y' \\ 0 \end{pmatrix}\right] dy' \\ &= \int_{y' \in X'' \otimes W_{02}} \int_{y' \in X' \otimes (Y''^* \oplus W_{02})} \varphi \left[u^{-1}e', \begin{pmatrix} y \\ 0 \end{pmatrix}, e'', \begin{pmatrix} y' \\ 0 \end{pmatrix}\right] dy' \end{aligned}$$

for any $u \in U_{k'}U' \backslash U$. Thus, if we regard φ as an element of

$$\mathcal{S}^{\mathrm{or}} \cong \mathcal{S}((V \otimes Y^*) \oplus ((X \oplus X^*) \otimes W_{02})) \otimes \mathcal{S}_{00},$$

then we have

$$\int_{U_{k'}U' \backslash U} f_{\mathcal{S}''}(f_{\mathcal{S}'}(\varphi)(u))(1)\, du = \int_{c=(c_{i,j}) \in C} \varphi\left(\sum_{i=1}^{k} x_i \otimes y_i^* + \sum_{i,j} c_{i,j} x_i \otimes y_j^*\right) \prod_{i,j} d_\psi c_{i,j}, \quad (9.11)$$

where the integration region C is a direct product $C = C_1 \times \cdots \times C_m$ of sets

$$C_l = \left\{ (c_{i,j}) \;\middle|\; c_{i,j} \in F, \begin{array}{l} i = k_0 + \cdots + k_{l-1} + 1, \ldots, k_0 + \cdots + k_l, \\ j = k_0 + \cdots + k_l + 1, \ldots, n \end{array} \right\}$$

of $k_l \times (k_{l+1} + \cdots + k_m)$ matrices with certain shifted indices. Similarly, we have

$$\int_{U_k \backslash U} f_{\mathcal{S}}(\varphi)(u)\, du = \int_{c=(c_{i,j}) \in C} \varphi\left(\sum_{i=1}^{k} x_i \otimes y_i^* + \sum_{i,j} c_{i,j} x_i \otimes y_j^*\right) \prod_{i,j} d_\psi c_{i,j}. \quad (9.12)$$

Now the lemma follows from (9.11) and (9.12). □

9.3 Proof of Proposition 7.3

Let us finish the proof of Proposition 7.3. This follows from the propositions above and induction in stages. Assume that $w \in \mathrm{W}_\phi(M, \mathrm{Mp}(W))$, and let

$$w = w_1 \cdots w_l$$

be a reduced decomposition of w in $\mathrm{W}(\hat{M}, \mathrm{Sp}_{2n}(\mathbb{C}))$. Then it can be seen that

$$\begin{aligned} \mathcal{R}_P(w, \pi_M, \psi) &= \mathcal{R}_P(w_1, \pi_M, \psi) \circ \cdots \circ \mathcal{R}_P(w_l, \pi_M, \psi), \\ \mathcal{R}_Q(w, \sigma_L) &= \mathcal{R}_Q(w_1, \sigma_L) \circ \cdots \circ \mathcal{R}_Q(w_l, \sigma_L). \end{aligned}$$

Thus, it suffices to show that the following diagram commutes for any simple reflection $w \in \mathrm{W}(\hat{M}, \mathrm{Sp}_{2n}(\mathbb{C}))$:

$$\begin{array}{ccc} \omega_{V,W,\psi} \otimes \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\breve{\sigma}_L) & \xrightarrow{\mathcal{T}} & \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\pi_M) \\ {\scriptstyle 1 \otimes \mathcal{R}_Q(w, \breve{\sigma}_L)} \downarrow & & \downarrow {\scriptstyle \mathcal{R}_P(w, \pi_M, \psi)} \\ \omega_{V,W,\psi} \otimes \mathrm{Ind}_Q^{\mathrm{SO}(V)}(w\breve{\sigma}_L) & \xrightarrow{\mathcal{T}} & \mathrm{Ind}_P^{\mathrm{Mp}(W)}(w\pi_M). \end{array}$$

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Recall the realization $\mathrm{W}(\hat{M}, \mathrm{Sp}_{2n}(\mathbb{C})) \hookrightarrow \mathfrak{S}_m \ltimes (\mathbb{Z}/2\mathbb{Z})^m$. The commutativity follows from Lemma 8.4 and the equation $\mathcal{T} = \Phi^{-1} \circ \mathcal{T}^a \circ (1 \otimes \Phi)$ when $w \in \mathfrak{S}_m$, and from Lemma 8.5, Proposition 8.6, and the equation $\mathcal{T} = \Psi^{-1} \circ \mathcal{T}^r \circ (1 \otimes \Psi)$ applied repeatedly when $w \in (\mathbb{Z}/2\mathbb{Z})^m$. Then we have completed the proof of Proposition 7.3.

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