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COMPUTABLE BOUNDS OF AN ℓ^2-SPECTRAL GAP FOR DISCRETE MARKOV CHAINS WITH BAND TRANSITION MATRICES

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Abstract

We analyse the $\ell^2(\pi)$-convergence rate of irreducible and aperiodic Markov chains with N-band transition probability matrix P and with invariant distribution π. This analysis is heavily based on two steps. First, the study of the essential spectral radius $r_{\mathrm{ess}}(P_{|\ell^2(\pi)})$ of $P_{|\ell^2(\pi)}$ derived from Hennion's quasi-compactness criteria. Second, the connection between the spectral gap property (SG_2) of P on $\ell^2(\pi)$ and the V-geometric ergodicity of P. Specifically, the (SG_2) is shown to hold under the condition $\alpha_0 := \sum_{m=-N}^{N} \limsup_{i\to+\infty}(P(i,i+m)P^*(i+m,i))^{1/2} < 1$. Moreover, $r_{\mathrm{ess}}(P_{|\ell^2(\pi)}) \le \alpha_0$. Effective bounds on the convergence rate can be provided from a truncation procedure.

Keywords: V-geometric ergodicity; essential spectral radius

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1. Introduction

Let $P := (P(i,j))_{(i,j)\in\mathbb{N}^2}$ be a Markov kernel on the countable state-space $\mathbb{N}$. Throughout the paper we assume that P is irreducible and aperiodic, that P has a unique invariant probability measure denoted by $\pi := (\pi(i))_{i\in\mathbb{N}}$, and, finally, that P satifies the following condition:

(AS1) there exist $i_0 \in \mathbb{N}$, $N \in \mathbb{N}^*$ such that, for all $i \ge i_0$, $|i-j| > N$ implies that $P(i,j) = 0$.

We denote by $(\ell^2(\pi), \|\cdot\|_2)$ the Hilbert space of sequences $(f(i))_{i\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ such that $\|f\|_2 := [\sum_{i\ge0}|f(i)|^2\pi(i)]^{1/2} < \infty$. Then P defines a linear contraction on $\ell^2(\pi)$, and its adjoint operator P^* on $\ell^2(\pi)$ is defined by $P^*(i,j) := \pi(j)P(j,i)/\pi(i)$. If $\pi(f) := \sum_{i\ge0} f(i)\pi(i)$ then the kernel P is said to have the spectral gap property on $\ell^2(\pi)$ if there exists $\rho \in (0,1)$ and $C \in (0,+\infty)$ such that the following holds:

(SG_2) for all $n \ge 1$, $f \in \ell^2(\pi)$, $\|P^n f - \Pi f\|_2 \le C\rho^n\|f\|_2$ with $\Pi f := \pi(f)\,\mathbf{1}_{\mathbb{N}}$,

where $\mathbf{1}$ denotes the indicator. A standard issue is to compute the value (or to find an upper bound) of

$$\varrho_2 := \inf\{\rho \in (0,1) \text{ such that } (\mathrm{SG}_2) \text{ holds}\}. \tag{1.1}$$

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In this work the quasi-compactness criteria of [3] is used to study (SG_2) and to estimate ϱ_2. In Section 2 it is proved that (SG_2) holds when the following condition holds:

(AS2) $\alpha_0 := \sum_{m=-N}^{N} \limsup_{i\to+\infty} (P(i, i+m)P^*(i+m, i))^{1/2} < 1.$

Moreover, $r_{\mathrm{ess}}(P_{|\ell^2(\pi)}) \leq \alpha_0$. We refer the reader to [3] for the definition of the essential spectral radius $r_{\mathrm{ess}}(T)$ and for quasi-compactness of a bounded linear operator T on a Banach space. Under the following assumptions:

(AS3) for all $m = -N, \dots, N$, $P(i, i+m) \to a_m \in [0, 1]$ as $i \to +\infty$;

(AS4) $\pi(i+1)/\pi(i) \to \tau \in [0, 1)$ as $i \to +\infty$;

(NERI) $\sum_{k=-N}^{N} k a_k < 0$;

property (AS2) holds (hence, (SG_2)) and α_0 can be explicitly computed as a function of τ and the a_m. Moreover, using $r_{\mathrm{ess}}(P_{|\ell^2(\pi)}) \leq \alpha_0$, (SG_2) is proved to be connected to the V-geometric ergodicity of P for $V := (\pi(n)^{-1/2})_{n\in\mathbb{N}}$. In particular, denoting the minimal V-geometrical ergodic rate by ϱ_V, it is proved that, either ϱ_2 and ϱ_V are both less than α_0, or $\varrho_2 = \varrho_V$. As a result, an accurate bound of ϱ_2 can be obtained for random walks (RW) with identically distributed bounded increments using the results of [5]. Actually, any estimation of ϱ_V, for instance that derived in Section 3 from the truncation procedure of [4], provides an estimation of ϱ_2. We point out that all the previous results hold without any reversibility properties.

The spectral gap property for Markov processes has been widely investigated in the discrete and continuous-time cases (see, e.g. [2] and [10]). There exist different definitions of the spectral gap property according to whether we are concerned with the discrete or continuous-time case (see, e.g. [8] and [14]). The focus of our paper is on the discrete-time case. In the reversible case, the equivalence between the geometrical ergodicity and (SG_2) was proved in [9] and $\varrho_2 \leq \varrho_V$ was obtained in [1, Theorem 6.1.]. This equivalence fails in the nonreversible case (see [7]). The link between ϱ_2 and ϱ_V stated in our Proposition 2.1 is obtained with no reversibility condition. Formulae for ϱ_2 are provided in [11] and [13] in terms of isoperimetric constants which are related to P in the reversible case and to P and P^* in the nonreversible case. However, to the best of the authors' knowledge, no explicit value (or upper bounds) of ϱ_2 can be derived from these formulae for discrete Markov chains with band transition matrices. Our explicit bound $r_{\mathrm{ess}}(P_{|\ell^2(\pi)}) \leq \alpha_0$ in Theorem 2.1 is the preliminary key result in this work. Recall that $r_{\mathrm{ess}}(P_{|\ell^2(\pi)})$ is a natural lower bound of ϱ_2 (apply [5, Proposition 2.1] with the Banach space $\ell^2(\pi)$). The essential spectral radius of Markov operators on an $\mathbb{L}^2$-type space was investigated for Markov chains with general state-space in [12], but no explicit bound for $r_{\mathrm{ess}}(P_{|\ell^2(\pi)})$ can be derived *a priori* from these theoretical results for Markov chains with band transition matrices, except in the reversible case [12, Theorem 5.5.].

2. Property (SG_2) and V-geometrical ergodicity

Theorem 2.1. *If* (AS2) *holds then P satisfies* (SG_2)*. Moreover, we have* $r_{\mathrm{ess}}(P_{|\ell^2(\pi)}) \leq \alpha_0$.

Proof. Let us introduce $\ell^1(\pi) := \{(f(i))_{i\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \|f\|_1 := \sum_{i\geq 0} |f(i)|\pi(i) < \infty\}$.

Lemma 2.1. *For any $\alpha > \alpha_0$, there exists a positive constant $L \equiv L(\alpha)$ such that*

$$\|Pf\|_2 \leq \alpha\|f\|_2 + L\|f\|_1 \quad \textit{for all } f \in \ell^2(\pi).$$

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Since the identity map is compact from $\ell^2(\pi)$ into $\ell^1(\pi)$ (from the Cantor diagonal procedure), it follows from Lemma 2.1 and from [3] that P is quasi-compact on $\ell^2(\pi)$ with $r_{\mathrm{ess}}(P_{|\ell^2(\pi)}) \le \alpha$. Since α can be chosen arbitrarily close to α_0, we obtain $r_{\mathrm{ess}}(P_{|\ell^2(\pi)}) \le \alpha_0$. Then ($SG_2$) is deduced from aperiodicity and irreducibility assumptions. □

Proof of Lemma 2.1. Under (AS1), define

$$\beta_m(i) := (P(i, i+m)P^*(i+m, i))^{1/2} \quad \text{for all } i \ge i_0,\ m = -N, \dots, N. \tag{2.1}$$

Let $\alpha > \alpha_0$, with α_0 given in (AS2). Fix $\ell \equiv \ell(\alpha) \ge i_0$ such that $\sum_{m=-N}^{N} \sup_{i\ge\ell} \beta_m(i) \le \alpha$. For $f \in \ell^2(\pi)$, we have, from Minkowski's inequality and the band structure of P, for $i \ge \ell$,

$$\begin{aligned}\|Pf\|_2 &\le \Big[\sum_{i<\ell} |(Pf)(i)|^2\pi(i)\Big]^{1/2} + \Big[\sum_{i\ge\ell}\Big|\sum_{m=-N}^{N} P(i,i+m)f(i+m)\Big|^2\pi(i)\Big]^{1/2} \\ &\le L\sum_{i<\ell}|(Pf)(i)|\pi(i) + \Big[\sum_{i\ge\ell}\Big|\sum_{m=-N}^{N} P(i,i+m)f(i+m)\Big|^2\pi(i)\Big]^{1/2},\end{aligned} \tag{2.2}$$

where $L \equiv L_\ell > 0$ comes from the equivalence of norms on $\mathbb{C}^\ell$. Moreover, we have

$$\sum_{i<\ell}|(Pf)(i)|\pi(i) \le \|Pf\|_1 \le \|f\|_1.$$

To control the second term in (2.2), define $F_m = (F_m(i))_{i\in\mathbb{N}} \in \ell^2(\pi)$ by

$$F_m(i) := P(i,i+m)f(i+m)(1 - 1_{\{0,\dots,\ell-1\}}(i)) \quad \text{for } -N \le m \le N.$$

Then

$$\Big[\sum_{i\ge\ell}\Big|\sum_{m=-N}^{N} P(i,i+m)f(i+m)\Big|^2\pi(i)\Big]^{1/2} = \Big\|\sum_{m=-N}^{N} F_m\Big\|_2 \le \sum_{m=-N}^{N}\|F_m\|_2.$$

and

$$\begin{aligned}\|F_m\|_2^2 &= \sum_{i\ge\ell} P(i,i+m)^2|f(i+m)|^2\pi(i) \\ &= \sum_{i\ge\ell} P(i,i+m)\frac{\pi(i)P(i,i+m)}{\pi(i+m)}|f(i+m)|^2\pi(i+m) \\ &\le \sup_{i\ge\ell}\beta_m(i)^2\|f\|_2^2\end{aligned}$$

from the definition of P^* and from (2.1). The statement in Lemma 2.1 can be deduced from the previous inequality and from (2.2). □

The core of our approach to estimate ϱ_2 is the relationship between (SG_2) and the V-geometric ergodicity. Indeed, specify Theorem 2.1 in terms of the V-geometric ergodicity with $V := (\pi(n)^{-1/2})_{n\in\mathbb{N}}$. Let $(\mathcal{B}_V, \|\cdot\|_V)$ denote the space of sequences $(g(n))_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ such that $\|g\|_V := \sup_{n\in\mathbb{N}} V(n)^{-1}|g(n)| < \infty$. Recall that P is said to be V-geometrically ergodic if P satisfies the spectral gap property on $\mathcal{B}_V$; namely, there exist $C \in (0, +\infty)$ and $\rho \in (0, 1)$ such that the following condition holds:

$$(\mathrm{SG}_V) \qquad \|P^n f - \Pi f\|_V \le C\rho^n \|f\|_V \quad \text{for all } n \ge 1,\ f \in \mathcal{B}_V.$$

When this property holds, we define

$$\varrho_V := \inf\{\rho \in (0, 1) \text{ such that } (\mathrm{SG}_V) \text{ holds}\}. \tag{2.3}$$

Remark 2.1. Under (AS3) and (AS4), we have

$$\alpha_0 := \sum_{m=-N}^{N} \limsup_{i\to+\infty} (P(i, i+m)P^*(i+m, i))^{1/2} = \begin{cases} \displaystyle\sum_{m=-N}^{N} a_m \tau^{-m/2} & \text{if } \tau \in (0, 1), \\ a_0 & \text{if } \tau = 0. \end{cases} \tag{2.4}$$

Indeed, if (AS4) holds with $\tau \in (0, 1)$, then the claimed formula follows from the definition of P^*. If $\tau = 0$ in (AS4), then $a_m = 0$ for every $m = 1, \dots, N$ from

$$\frac{\sum_{m=-N}^{N} P(i+m, i)\pi(i+m)}{\pi(i)} = 1.$$

Thus, $a_{-m} = 0$ when $m < 0$. Hence, $\alpha_0 = a_0$.

Proposition 2.1. *If P and π satisfy assumptions* (AS3), (AS4), *and* (NERI), *then P satisfies* (AS2) *(with $\alpha_0 < 1$ given in (2.4)). Moreover, when P satisfies both* (SG_2) *and* (SG_V) *with $V := (\pi(n)^{-1/2})_{n\in\mathbb{N}}$, we have $\max(r_{\mathrm{ess}}(P_{|\mathcal{B}_V}), r_{\mathrm{ess}}(P_{|\ell^2(\pi)})) \le \alpha_0$, and the following assertions hold:*

(i) *if $\varrho_V \le \alpha_0$ then $\varrho_2 \le \alpha_0$;*

(ii) *if $\varrho_V > \alpha_0$ then $\varrho_2 = \varrho_V$.*

Proof. If $\tau = 0$ in (AS4) then $\alpha_0 = a_0 < 1$ from (2.4) and (NERI). Now assume that (AS4) holds with $\tau \in (0, 1)$. Then $\alpha_0 = \sum_{m=-N}^{N} a_m \tau^{-m/2} = \psi(\sqrt{\tau})$, where, for all $t > 0$, $\psi(t) := \sum_{k=-N}^{N} a_k t^{-k}$. Moreover, it easily follows from the invariance of π that $\psi(\tau) = 1$. The inequality $\alpha_0 = \psi(\sqrt{\tau}) < 1$ is deduced from the following assertions. For all $t \in (\tau, 1)$, $\psi(t) < 1$, and for all $t \in (0, \tau) \cup (1, +\infty)$, we have $\psi(t) > 1$. To prove these properties, note that $\psi(\tau) = \psi(1) = 1$ and ψ is convex on $(0, +\infty)$. Moreover, we have $\lim_{t\to+\infty} \psi(t) = +\infty$ since $a_k > 0$ for some $k < 0$ (use $\psi(\tau) = \psi(1) = 1$ and $\tau \in (0, 1)$). Similarly, $\lim_{t\to 0^+} \psi(t) = +\infty$ since $a_k > 0$ for some $k > 0$. This gives the desired properties on ψ since $\psi'(1) > 0$ from (NERI).

Condition (SG_2) and $r_{\mathrm{ess}}(P_{|\ell^2(\pi)}) \le \alpha_0$ follow from Theorem 2.1. Next, (SG_V) is deduced from the well-known link between geometric ergodicity and the following drift inequality:

$$\text{for all } \alpha \in (\alpha_0, 1),\ \text{there exists } L \equiv L_\alpha > 0 \text{ such that } PV \le \alpha V + L\,\mathbf{1}_{\mathbb{N}}. \tag{2.5}$$

This inequality holds from the fact that $\lim_i (PV)(i)/V(i) = \alpha_0$.

Then (SG_V) is derived from (2.5) using aperiodicity and irreducibility. It also follows from (2.5) that $r_{\mathrm{ess}}(P_{|\mathcal{B}_V}) \le \alpha$ (see [5, Proposition 3.1]). Thus, $r_{\mathrm{ess}}(P_{|\mathcal{B}_V}) \le \alpha_0$.

Now we prove Propositions 2.1(i) and 2.1(ii) using the spectral properties of [5, Proposition 2.1] of both $P_{|\ell^2(\pi)}$ and $P_{|\mathcal{B}_V}$ (due to quasi-compactness, see [3]). We will also use the

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following obvious inclusion: $\ell^2(\pi) \subset \mathcal{B}_V$. In particular, every eigenvalue of $P_{|\ell^2(\pi)}$ is also an eigenvalue for $P_{|\mathcal{B}_V}$. First, assume that $\varrho_V \le \alpha_0$. Then there is no eigenvalue for $P_{|\mathcal{B}_V}$ in the annulus $\Gamma := \{\lambda \in \mathbb{C} \colon \alpha_0 < |\lambda| < 1\}$ since $r_{\mathrm{ess}}(P_{|\mathcal{B}_V}) \le \alpha_0$. From $\ell^2(\pi) \subset \mathcal{B}_V$, it follows that there is also no eigenvalue for $P_{|\ell^2(\pi)}$ in this annulus. Hence, $\varrho_2 \le \alpha_0$ since $r_{\mathrm{ess}}(P_{|\ell^2(\pi)}) \le \alpha_0$. Second, assume that $\varrho_V > \alpha_0$. Then $P_{|\mathcal{B}_V}$ admits an eigenvalue $\lambda \in \mathbb{C}$ such that $|\lambda| = \varrho_V$. Let $f \in \mathcal{B}_V$, $f \neq 0$, such that $Pf = \lambda f$. We know from [5, Proposition 2.2] that there exists some $\beta \equiv \beta_\lambda \in (0,1)$ such that $|f(n)| = \mathrm{O}(V(n)^\beta) = \mathrm{O}(\pi(n)^{-\beta/2})$, so that $|f(n)|^2\pi(n) = \mathrm{O}(\pi(n)^{(1-\beta)})$; thus, $f \in \ell^2(\pi)$ from (AS4). We have proved that $\varrho_2 \ge \varrho_V$. Finally, the converse inequality holds since every eigenvalue of $P_{|\ell^2(\pi)}$ is an eigenvalue for $P_{|\mathcal{B}_V}$. Thus, $\varrho_2 = \varrho_V$. □

From Proposition 2.1, any estimation of ϱ_V provides an estimation of ϱ_2. This is illustrated in Example 2.1 and Corollary 3.1. The Markov chains in Example 2.1 have been studied in detail in [5, Section 3]; we also mention that further technical details can be found in [6].

Example 2.1. (*RWs with identically distributed bounded increments.*) Let P be defined as follows. There exist some positive integers $c, g, d \in \mathbb{N}^*$ such that

$$\sum_{j=0}^{c} P(i,j) = 1 \quad \text{for all } i \in \{0,\dots,g-1\},$$

$$P(i,j) = \begin{cases} a_{j-i} & \text{if } i-g \le j \le i+d, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for all } i \ge g,\ j \in \mathbb{N},$$

and $(a_{-g},\dots,a_d) \in [0,1]^{g+d+1}$ such that $a_{-g} > 0$, $a_d > 0$, $\sum_{k=-g}^{d} a_k = 1$. Assume that P is aperiodic and irreducible, and satisfies (NERI). Then P has a unique invariant distribution π. It can be derived from standard results of a linear difference equation that $\pi(n) \sim c\tau^n$ when $n \to +\infty$, with $\tau \in (0,1)$ defined by $\psi(\tau) = 1$, where $\psi(t) := \sum_{k=-N}^{N} a_k t^{-k}$. Thus, if $\gamma := \tau^{-1/2}$ then $\mathcal{B}_V = \{(g(n))_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}, \sup_{n\in\mathbb{N}} \gamma^{-n}|g(n)| < \infty\}$. Then we know from [5, Proposition 3.2] that $r_{\mathrm{ess}}(P_{|\mathcal{B}_V}) = \alpha_0$ with α_0 given in (2.4), and that ϱ_V can be computed from an algebraic polynomial elimination. From this computation, Proposition 2.1 provides an accurate estimation of ϱ_2. Property (SG$_2$) was proved in [13, Theorem 2] under an extra weak reversibility assumption (with no explicit bound on ϱ_2). However, except in the $g = d = 1$ case, where reversibility is automatic, an RW with identically distributed bounded increments is not reversible or even weak reversible in general. No reversibility condition is required here.

3. Bound for ϱ_2 via truncation

Let P be any Markov kernel on $\mathbb{N}$, and let us consider the kth truncated (and augmented on the last column) matrix P_k associated with P as in [4]. If $\sigma(P_k)$ denotes the set of eigenvalues of P_k, define $\rho_k := \max\{|\lambda|, \lambda \in \sigma(P_k), |\lambda| < 1\}$. The weak perturbation method in [4] provides the following general result where (AS1) is not required and V is any unbounded increasing sequence.

Proposition 3.1. *Let P be an irreducible and aperiodic Markov kernel on $\mathbb{N}$ satisfying the following drift inequality for some unbounded increasing sequence $(V(n))_{n\in\mathbb{N}}$: there exist $\delta \in [0,1)$, $L > 0$, such that*

$$PV \le \delta V + L\,\mathbf{1}_{\mathbb{N}}. \tag{3.1}$$

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Let ϱ_V be defined in (2.3). Then, either $\varrho_V \leq \delta$ and $\limsup_k \rho_k \leq \delta$, or $\varrho_V > \delta$ and $\varrho_V = \lim_k \rho_k$.

Proof. Condition (3.1) ensures that the assumptions of [4, Lemma 6.1] are satisfied, so that $r_{\mathrm{ess}}(P_{|\mathcal{B}_V}) \leq \delta$. Then, using standard duality arguments, the spectral rank-stability property [4, Lemma 7.2] applies to $P_{|\mathcal{B}_V}$ and P_k. If $\varrho_V \leq \delta$ then, for each r such that $\delta < r < 1$, $\lambda = 1$ is the unique eigenvalue of $P_{|\mathcal{B}_V}$ in $C_r := \{\lambda \in \mathbb{C} \colon r < |\lambda| \leq 1\}$ (see [3]). From [4, Lemma 7.2] this property holds for P_k when k is large enough, so that $\limsup_k \rho_k \leq r$. Thus, $\limsup_k \rho_k \leq \delta$ since r is arbitrarily close to δ. Now assume that $\varrho_V > \delta$, and let r be such that $\delta < r < \varrho_V$. Then $P_{|\mathcal{B}_V}$ has a finite number of eigenvalues in C_r, say $\lambda_0, \lambda_1, \ldots, \lambda_N$, with $\lambda_0 = 1$, $|\lambda_1| = \varrho_V$, and $|\lambda_j| \leq \varrho_V$ for $j = 2, \ldots, N$ (see [3]). For $a \in \mathbb{C}$ and $\varepsilon > 0$, we define $D(a, \varepsilon) := \{z \in \mathbb{C} \colon |z - a| < \varepsilon\}$. Now consider any $\varepsilon > 0$ such that the disks $D(\lambda_j, \varepsilon)$ for $j = 0, \ldots, N$ are disjoint and are contained in C_r for $j \geq 1$. From [4, Lemma 7.2], for large enough k, 1 is the only eigenvalue of P_k in $D(1, \varepsilon)$, the others eigenvalues of P_k in C_r are contained in $\bigcup_{j=1}^{N} D(\lambda_j, \varepsilon)$, and, finally, each $D(\lambda_j, \varepsilon)$ contains at least one eigenvalue of P_k. Thus, each eigenvalue $\lambda \neq 1$ of P_k in C_r has modulus less than $\varrho_V + \varepsilon$, so that $\rho_k \leq \varrho_V + \varepsilon$. Moreover, the disk $D(\lambda_1, \varepsilon)$ contains at least an eigenvalue λ of P_k, so that $\rho_k \geq |\lambda| \geq \varrho_V - \varepsilon$. Thus, for large enough k, we have $\varrho_V - \varepsilon \leq \rho_k \leq \varrho_V + \varepsilon$. □

Under the assumptions of Proposition 2.1 we deduce the following result from Proposition 3.1.

Corollary 3.1. *If P satisfies the assumptions of Proposition 2.1, then the following properties hold with α_0 given in (2.4):*

(i) *$\varrho_2 \leq \alpha_0$ if and only if $\varrho_V \leq \alpha_0$, and in this case, we have $\limsup_k \rho_k \leq \alpha_0$;*

(ii) *$\varrho_2 > \alpha_0$ if and only if $\varrho_V > \alpha_0$, and in this case, we have $\varrho_2 = \varrho_V = \lim_k \rho_k$.*

As usual the reversible case is simpler. In particular, we can take $C = 1$ and $\rho = \varrho_2$ in (SG$_2$). Details and numerical illustrations for Metropolis–Hastings kernels can be found in [6].

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