J. Fluid Mech. (2012), *vol.* 700, *pp.* 441–458.
doi:10.1017/jfm.2012.142

Instability of compressible drops and jets

Umpei Miyamoto†

Department of Physics, Rikkyo University, Tokyo 171-8501, Japan

(Received 30 May 2011; revised 14 December 2011; accepted 11 March 2012; first published online 30 April 2012)

We revisit the classic problem of the stability of drops and jets held by surface tension, while regarding the compressibility of bulk fluids and spatial dimensions as free parameters. By mode analysis, it is shown that there exists a critical compressibility above which the drops (and discs) become unstable for a spherical perturbation. For a given value of compressibility (and of the surface tension and the density at equilibrium), this instability criterion provides a minimal radius below which the drop cannot be in stable equilibrium. According to the existence of the above unstable mode of the drop, which corresponds to a homogeneous perturbation of a cylindrical jet, the dispersion relation of Rayleigh–Plateau instability for cylinders drastically changes. In particular, we identify another critical compressibility above which the homogeneous unstable mode is predominant. The analysis is carried out for non-relativistic and relativistic perfect fluids, the self-gravity of which is ignored.

Key words: drops, liquid bridges

1. Introduction

The free oscillations of liquid droplets were studied by Kelvin (1890) and Rayleigh (1894) more than a hundred years ago. Later, Lamb (1932), Chandrasekhar (1959), Reid (1960) and others generalized the analysis to take into account the effects of an outer fluid, viscosity and so on – see e.g. the introduction of Becker, Hiller & Kowalewski (1991) for a brief but nice review and a more complete list of references. In the simplest case of an inviscid droplet in vacuum (or approximately in air), the droplet with unperturbed radius r and constant density ρ oscillates with angular frequency given by

$$\Omega^2 = \frac{\sigma}{\rho r^3}(\ell - 1)\ell(\ell + 2), \tag{1.1}$$

where $\ell = 0, 1, 2, \ldots$ denotes the mode number and σ is the surface tension responsible for the oscillation.

Another important phenomenon associated with surface tension is the drop formation resulting from an instability of cylindrical jets. Theoretical studies date back to the early investigations by Plateau (1873) and Rayleigh (1879), and later by Chandrasekhar (1961) – see Eggers (1997) for a comprehensive review. For example,

† Email address for correspondence: umpei@rikkyo.ac.jp

for an inviscid cylinder with non-perturbed radius r, the sinusoidal perturbation with wavenumber k evolves in time, with the growth rate given by

$$\omega^2 = \frac{\sigma}{\rho r^3} \frac{kr(1-(kr)^2)\mathrm{I}_1(kr)}{\mathrm{I}_0(kr)}, \tag{1.2}$$

where I is the modified Bessel function of the first kind. This dispersion relation tells us that any cylinder that is longer than $2\pi r$ is unstable, and the most unstable mode, which roughly determines the size of droplets forming, appears at wavelength $\lambda \sim 9r$.

The above two phenomena, the oscillations of droplets and the instability of cylinders, are not only of fundamental importance in theoretical fluid mechanics but also important from industrial points of view. Therefore, they have been studied theoretically and experimentally in a considerable variety of physical situations. However, the effect of the *non-zero compressibility* or *finite sound velocity* of fluids has not been well studied, to the present author's knowledge. The reason is that the compressibility of liquids in many non-extreme situations is expected to be negligible and to give rise to no perceivable effects.

Notwithstanding the above general expectation, in this paper we reinvestigate the two classic problems, while allowing the bulk fluids to have non-zero compressibility or finite sound velocity. Our analysis reproduces the results (1.1) and (1.2) in the incompressible limit. The results are a little surprising and intriguing, at least, at the theoretical level. The stability structure of droplets and cylinders with finite compressibility is rather richer than expected. It will be shown that there exists a critical compressibility above which the droplet (or a disc in the two-dimensional case) becomes unstable for a spherically symmetric perturbation. According to the existence of such an instability for a disc, a cylinder (whose cross-section is a disc) becomes unstable above the critical compressibility. These instability criteria can be interpreted as follows. For given parameters of the fluid and surface, namely the sound velocity $c_s^2 = \mathrm{d}p/\mathrm{d}\rho$, surface tension and density at equilibrium, there exists a minimum radius of droplet and cylinder below which they cannot be in stable equilibrium. Such a minimum radius is identified to be

$$r_{min} = \frac{n}{n+1} \frac{\sigma}{\rho c_s^2}, \tag{1.3}$$

where $n = 1$ for cylinders and $n = 2$ for droplets. When the fluid is relativistic (e.g. when the pressure is comparable to the energy density ϵ),

$$r_{min} = \frac{n}{n+1} \frac{\sigma}{\epsilon c_s^2}(1-(n+1)c_s^2), \tag{1.4}$$

where $c_s^2 = \mathrm{d}p/\mathrm{d}\epsilon$ in this case.

Fortunately (or to the author's regret), in many non-extreme systems, such as water in air at room temperature, r_{min} is extremely small ($r_{min} \sim 10^{-11}$ m), and instability will not play a crucial role in the dynamics. (The author confesses that he cannot say for certain whether or not there are systems where r_{min} is macroscopic. See appendix A for a discussion on the values of r_{min}.) From the theoretical or mathematical point of view, however, the existence of a minimal radius is significant in proving that the Euler equation supplemented by the Young–Laplace stress balance relation at the surface, which governs the perfect fluid systems considered in this paper, is not well defined for arbitrary values of (σ, c_s, ρ), while these parameters are usually supposed to take any positive finite values.

Here, let us give some notes on the analyses and results in this paper. The first is this. If one is familiar with the theories of stellar structure, the result (the instability of droplets) may not be totally surprising. That is, a self-gravitating fluid ball is unstable if the heat capacity ratio (or adiabatic index), denoted by γ conventionally, exceeds $4/3$ in Newtonian gravitational theory (see e.g. Shapiro & Teukolsky 1983). This critical value is changed to a higher value in general relativity (see e.g. Shapiro & Teukolsky 1983). Furthermore, if γ is slightly larger than $4/3$, then the spherical fluid ball is unstable to radial perturbation provided that the radius is less than a critical radius, which is much larger than the Schwarzschild radius (Chandrasekhar 1964). The author thanks an anonymous referee for pointing out these points.

The second note is the following. We will consider axially symmetric fluids in arbitrary dimensions, by leaving n as a free parameter. The reasons for doing so are twofold. By leaving the spatial dimension as a free parameter, one can treat discs, drops and cylinders at the same time. The other reason is related to the so-called *fluid–gravity correspondence* (Bhattacharyya *et al.* 2008) – see also Rangamani (2009) for a review and a complete list of references – which was found in the context of string/M theories and relates the fluid mechanics in d dimensions to a gravitational theory in $(d+1)$ dimensions. In this context, the spatial dimension often has to be treated as a free parameter, and some phenomena such as the Rayleigh–Plateau instability and its subsequent dynamics have been known to depend crucially on the dimension (Caldarelli *et al.* 2009; Maeda & Miyamoto 2009). However, it should be added that, within the analysis in this paper, we could not find any qualitative difference originating from the difference of dimension.

The organization of this paper is as follows. In § 2 we review the Euler equation and Young–Laplace relation for non-relativistic fluids, and then we derive the dispersion relation for the perturbations of spherical and cylindrical equilibria. In § 3, we analyse the dispersion relation to obtain the stability criteria for droplets (§ 3.1) and cylinders (§ 3.2). We generalize the analysis to the relativistic fluids in §§ 4 and 5. Section 6 is devoted to a summary and discussion. The stability of droplets for non-spherical ($\ell \neq 0$) perturbations is shown in appendix B.

2. Non-relativistic perfect fluid with boundary

2.1. *Euler equation and Young–Laplace relation*

We consider compressible inviscid fluids in d-dimensional ($d \geqslant 3$) flat spacetime $\mathbb{R}^{1,d-1}$. Denoting the density, pressure and velocity field by ρ, p and v^I $(I, J = 1, 2, \ldots, d-1)$, respectively, the continuity equation and Euler equations (Landau & Lifshitz 1987) are

$$\partial_t \rho + \nabla_I(\rho v^I) = 0, \tag{2.1}$$

$$\partial_t(\rho v^I) + \nabla_J(\rho v^I v^J + p g^{IJ}) = 0, \tag{2.2}$$

where ∇_I is the covariant derivative compatible with a flat metric g_{IJ} (Wald 1984).

We assume that a lump of fluid is supported by a constant surface tension $\sigma > 0$. Then, the Young–Laplace relation (Landau & Lifshitz 1987), describing the normal-stress balance at the surface, is given by

$$p = \sigma \kappa|_{f=0}. \tag{2.3}$$

Here, f is a scalar function with which the surface is identified by $f = 0$, and κ is $(d-2)$ times the mean curvature of the surface, given as the divergence of the unit

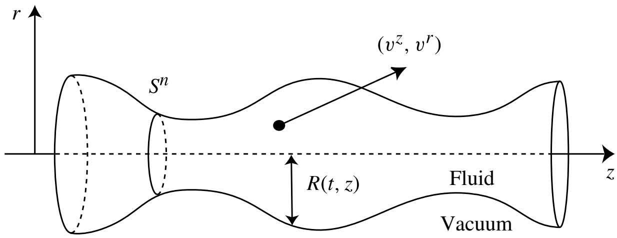

FIGURE 1. An axially symmetric fluid and the cylindrical coordinates in an $(n+3)$-dimensional spacetime $\mathbb{R}^{1,n+2}$. The fluid surface at $z=$ const. is an n-sphere S^n.

normal vector n^I (or as the trace of extrinsic curvature),

$$\kappa = \nabla_I n^I, \quad n_I = \frac{\nabla_I f}{(\nabla^J f \nabla_J f)^{1/2}}. \tag{2.4}$$

Furthermore, we assume that the surface is convected with the fluid, which is expressed as

$$(\partial_t + v^I \nabla_I) f = 0|_{f=0}. \tag{2.5}$$

Since we are interested in axially symmetric fluids, it is convenient to work in cylindrical coordinates. Writing the spacetime dimensions as $d = n+3$ $(n = 1, 2, \ldots)$, the line element is written as

$$g_{IJ}\, \mathrm{d}x^I\, \mathrm{d}x^J = \mathrm{d}z^2 + \mathrm{d}r^2 + r^2\, \mathrm{d}s_n^2 = \delta_{ab}\, \mathrm{d}x^a\, \mathrm{d}x^b + r^2 \gamma_{ij}(\theta)\, \mathrm{d}\theta^i\, \mathrm{d}\theta^j, \tag{2.6}$$

where $x^a := (z, r)$ and $\gamma_{ij}(\theta)\, \mathrm{d}\theta^i\, \mathrm{d}\theta^j$ $(i, j = 1, 2, \ldots, n)$ is the line element on the unit n-sphere. For $d = 3$ we just discard the z-coordinate, and the cylindrical coordinates reduce to polar coordinates. Now, we assume that the fluid and its surface have $SO(n+1)$ symmetry around the z-coordinate. Then, the coordinate dependences of fluid quantities are given by

$$p = p(t, z, r), \quad v^a = v^a(t, z, r), \quad v^i = 0, \quad f(t, z, r) = r - R(t, z), \tag{2.7}$$

where $R(t, z)$ is a function representing the local radius of the fluid surface (see figure 1).

With the above ansatz, the continuity equation (2.1) and the Euler equation (2.2) can be written as

$$(\partial_t + v_z \partial_z + v_r \partial_r)\rho + \rho\left(\partial_z v_z + \partial_r v_r + \frac{n}{r} v_r\right) = 0, \tag{2.8}$$

$$\rho(\partial_t + v_z \partial_z + v_r \partial_r) v_a = -\partial_a p. \tag{2.9}$$

The mean curvature of the surface, appearing in the Young–Laplace relation (2.3), is given by

$$\kappa = \frac{n}{R\,[1 + (\partial_z R)^2]^{1/2}} - \frac{\partial_z^2 R}{[1 + (\partial_z R)^2]^{3/2}}. \tag{2.10}$$

Finally, the kinematic boundary condition (2.5) is reduced to

$$\partial_t R + v_z \partial_z R = v_r|_{r=R}. \tag{2.11}$$

https://doi.org/10.1017/jfm.2012.142 Published online by Cambridge University Press

2.2. *Linear perturbation of cylinders*

The above system obviously allows the static cylinder as an equilibrium solution, where the constant pressure p_0 and the radius of the cylinder $R = r_0$ satisfy

$$p_0 = \sigma \frac{n}{r_0}. \tag{2.12}$$

Now, we perturb this equilibrium solution. We can assume that the perturbation results from a sinusoidal disturbance of the local radius given by

$$R(t, z) = r_0[1 + \varepsilon \mathrm{e}^{\omega t} \cos(kz)], \tag{2.13}$$

where $|\varepsilon| \ll 1$ is a small parameter. Such a disturbance leads to the disturbance of both pressure and velocity, where the disturbed pressure at $O(\varepsilon)$ may take form

$$p(t, z, r) = p_0[1 + \varepsilon \mathrm{e}^{\omega t} P(r) \cos(kz)]. \tag{2.14}$$

In general, from the perturbations of the continuity equation (2.1) and the Euler equation (2.2), the pressure perturbation has to satisfy the wave equation

$$(\partial_t^2 - c_s^2 \nabla^J \nabla_J)\delta p = 0, \tag{2.15}$$

where $c_s^2 := \mathrm{d}p_0/\mathrm{d}\rho_0$ (ρ_0 is the density at equilibrium and $0 < c_s < \infty$) is the squared sound velocity of the bulk fluid, and we denote the $O(\varepsilon)$ perturbation of any quantity X by δX hereafter. Plugging expression (2.14) into (2.15), we obtain

$$\frac{\mathrm{d}^2 P}{\mathrm{d}r^2} + \frac{n}{r}\frac{\mathrm{d}P}{\mathrm{d}r} - \left(k^2 + \frac{\omega^2}{c_s^2}\right) P = 0. \tag{2.16}$$

With the regularity at the axis ($r = 0$), this equation is solved by the modified Bessel function of the first kind

$$P(r) = C\frac{\mathrm{I}_{(n-1)/2}(Kr)}{r^{(n-1)/2}}, \quad K := \left(k^2 + \frac{\omega^2}{c_s^2}\right)^{1/2}, \tag{2.17}$$

where C is an integration constant. The perturbation of the Young–Laplace relation (2.3), $\delta p = \sigma \delta\kappa|_{r=R}$, fixes the integration constant as

$$C = -\frac{[n - (kr_0)^2] r_0^{(n-1)/2}}{n \mathrm{I}_{(n-1)/2}(Kr_0)}. \tag{2.18}$$

The perturbation of the Euler equation in the r-direction (2.9) is $\rho_0 \partial_t \delta v_r = -\partial_r \delta p$. On the other hand, the velocity in the r-direction at the surface is given by $\delta v_r = \partial_t \delta R|_{r=R}$ from the kinetic boundary condition (2.11). The combination of these two yields

$$\partial_r \delta p = -\rho_0 \partial_t^2 \delta R|_{r=R}. \tag{2.19}$$

Plugging equations (2.13), (2.14), (2.17) and (2.18) into (2.19), and eliminating the derivative of the modified Bessel function, we finally obtain the dispersion relation of perturbations for the compressible cylinder:

$$\omega^2 = \frac{\sigma}{\rho_0 r_0^3}[n - (kr_0)^2] K r_0 \frac{\mathrm{I}_{(n+1)/2}(Kr_0)}{\mathrm{I}_{(n-1)/2}(Kr_0)}. \tag{2.20}$$

In the incompressible limit ($c_s \to \infty$), this dispersion relation reduces to (1.2) for $n = 1$ and to the equation derived in Cardoso & Gualtieri (2006) for general n.

To simplify the dispersion relation, we introduce here the following dimensionless quantities:

$$\hat{k} := r_0 k, \quad \hat{\omega} := \left(\frac{\rho_0 r_0^3}{\sigma}\right)^{1/2} \omega, \quad \beta := \left(\frac{\sigma}{\rho_0 r_0}\right)^{1/2} c_s^{-1} > 0. \tag{2.21}$$

Here, β, being basically the reciprocal of the sound velocity, serves as the parameter representing the (adiabatic) compressibility of the bulk fluid. In terms of these dimensionless quantities, the dispersion relation (2.20) is equivalent to the relation between $\hat{k}$ and $\hat{\omega}$, for which the following function vanishes:

$$F(\hat{\omega}, \hat{k}) := \hat{\omega}^2 - (n - \hat{k}^2)\hat{K}\frac{\mathrm{I}_{(n+1)/2}(\hat{K})}{\mathrm{I}_{(n-1)/2}(\hat{K})}, \quad \hat{K} := (\hat{k}^2 + \beta^2\hat{\omega}^2)^{1/2}. \tag{2.22}$$

3. Analysis of dispersion relation for the non-relativistic fluid

We have derived dispersion relation (2.20) for the perturbation of cylinders in $\mathbb{R}^{1,n+2}$. However, since the cross-section of a cylinder is a disc, (2.20) with $k = 0$ provides the oscillation frequency (or the growth rate of a possible instability) of droplets in $\mathbb{R}^{1,n+1}$ (i.e. without the z-direction). Thus, we divide this section into two parts, §§ 3.1 and 3.2. In the former, we investigate the dispersion relation with $k = 0$, which corresponds to both a spherically symmetric perturbation of droplets and homogeneous (in the z-direction) perturbation of cylinders. Then, in the latter, we investigate the dispersion relation for general modes with $k \geqslant 0$.

3.1. *Instability of drops ($k = 0$ mode)*

Setting $k = 0$ in (2.20), we have

$$F(\hat{\omega}, 0) = \hat{\omega}^2 - n\beta\hat{\omega}\frac{\mathrm{I}_{(n+1)/2}(\beta\hat{\omega})}{\mathrm{I}_{(n-1)/2}(\beta\hat{\omega})}. \tag{3.1}$$

It is noted that, in the incompressible limit ($\beta \to 0$), the zero of $F(\hat{\omega}, 0)$ identically vanishes ($\hat{\omega} = 0$). This just says that the incompressible droplets cannot oscillate nor collapse (nor expand) while keeping the spherical symmetry.

Let us see the behaviour of $F(\hat{\omega}, 0)$ for small $\hat{\omega}$. Expanding (3.1) around $\hat{\omega} = 0$, one has

$$F(\hat{\omega}, 0) = \left(1 - \frac{n}{n+1}\beta^2\right)\hat{\omega}^2 + \frac{n}{(n+1)^2(n+3)}\beta^4\hat{\omega}^4 + O(\hat{\omega}^6). \tag{3.2}$$

From this, one can see that $F(\hat{\omega}, 0)|_{\hat{\omega}=0} = \partial_{\hat{\omega}} F(\hat{\omega}, 0)|_{\hat{\omega}=0} = 0$. On the other hand, it is easy to see that $\lim_{\hat{\omega}\to\infty} F(\hat{\omega}, 0) = +\infty$ from (3.1). Therefore, if $\partial_{\hat{\omega}}^2 F(\hat{\omega}, 0)|_{\hat{\omega}=0} < 0$ holds, then $F(\hat{\omega}, 0)$ must have at least one positive zero from continuity. From (3.2) it is clear that $\partial_{\hat{\omega}}^2 F(\hat{\omega}, 0)|_{\hat{\omega}=0} < 0$ holds if β is larger than a critical value,

$$\beta > \beta_{c,1} := \left(\frac{n+1}{n}\right)^{1/2}. \tag{3.3}$$

In this case, from (3.2) the behaviour of $\hat{\omega}$ near $\beta_{c,1}$ is

$$\hat{\omega} \simeq \left(\frac{4n^3\,(n+3)^2}{n+1}\right)^{1/4} (\beta - \beta_{c,1})^{1/2} + O((\beta - \beta_{c,1})^{3/2}). \tag{3.4}$$

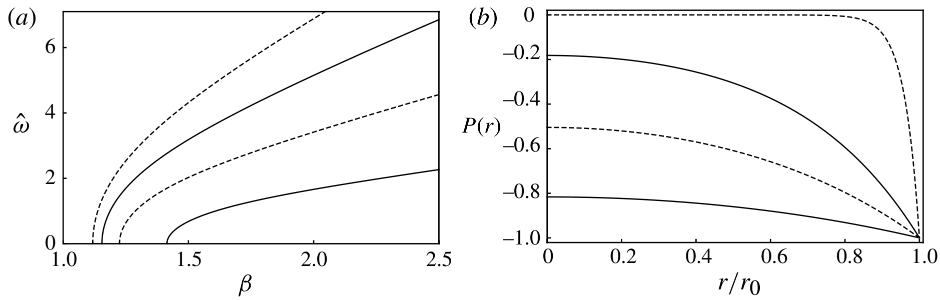

FIGURE 2. (*a*) Compressibility parameter β versus dimensionless growth rate $\hat{\omega}$ of the radial perturbation of non-relativistic droplets in $\mathbb{R}^{1,n+1}$ for several n: $n = 1,\ 2,\ 3,\ 4$ from bottom to top. (*b*) The radial eigenfunction $P(r)$ in the $n = 2$ case for several values of compressibility parameter: $\beta/\beta_{c,1} = 1.04,\ 1.13,\ 1.30,\ 3.00$ from bottom to top. Negative $P(r)$ corresponds to a decrease in pressure, resulting from an increase in the radius of the droplet.

The global β-dependence of $\hat{\omega}$ obtained from (3.1) is shown in figure 2(*a*). One can see that, in the large-β limit, $\hat{\omega}$ increases linearly with β. This asymptotic behaviour, that is in fact $\hat{\omega} \simeq n\beta$, can be derived from (3.1) by using a property of the modified Bessel function, $\lim_{\hat{\omega}\to\infty} \mathrm{I}_{(n+1)/2}(\beta\hat{\omega})/\mathrm{I}_{(n-1)/2}(\beta\hat{\omega}) = 1$. The radial function $P(r)$ in the $n = 2$ case for several values of β is shown in figure 2(*b*). One can observe the non-uniform (in the r-direction) decrease in the pressure, resulting from the increase in the droplet radius. The non-uniformity of the pressure perturbation is amplified as the compressibility increases, the interpretation of which is that the compressibility reduces the propagation speed of density fluctuations generated near the surface.

Using (2.12) and (2.21), the instability criterion (3.3) can be written in several forms, namely $c_s^2 < p_0/[(n+1)\rho_0]$ or equivalently

$$r_0 < r_{min} := \frac{n}{n+1}\frac{\sigma}{\rho_0 c_s^2}. \tag{3.5}$$

Inequality (3.5) is striking, as it means that there exists a minimum radius r_{min} only above which the droplet and cylinder can exist stably. The critical radius is determined by three parameters (ρ_0, σ, c_s), which do not restrict each other, at least from a macroscopic point of view, although they should be correlated microscopically.

The value of r_{min} for water in air at 25 °C is around 2.13×10^{-11} m (estimated with $n = 2$, $\sigma = 72.0 \times 10^{-3}$ J m^{-2}, $\rho_0 = 1.00 \times 10^3$ kg m^{-3}, and $c_s = 1.50 \times 10^3$ m s^{-1}; Weast 1978), where the fluid-mechanical description has already broken down. Thus, the instability found plays no central role for such a fluid. It would be interesting to look for a system in which r_{min} is larger than or comparable with the length scale where the fluid approximation breaks down. See also appendix A.

One could consider non-spherical perturbations of the drop by allowing a θ-dependence of R in (2.13). As in the incompressible case, however, the non-spherical modes turn out to be oscillatory for compressible fluids too, proving that the droplets are stable for such perturbations. See § B.1 for a proof.

3.2. *Rayleigh–Plateau instability ($k > 0$ modes)*

We proceed to the dispersion relation (2.20) for general values of $k \geqslant 0$. Since the homogeneous ($k = 0$) mode becomes unstable ($\omega > 0$) above a critical

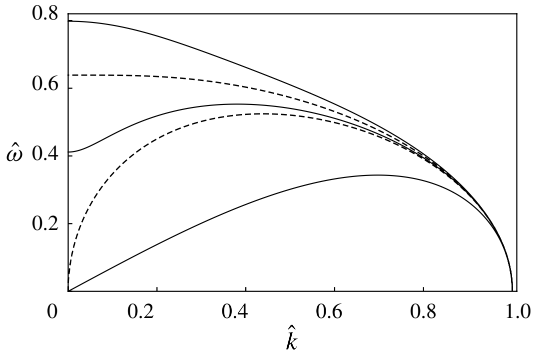

FIGURE 3. Dimensionless wavenumber $\hat{k}$ versus dimensionless growth rate $\hat{\omega}$ of the instability of a non-relativistic cylinder in $\mathbb{R}^{1,3}$ ($n = 1$) for several values of compressibility: $\beta = 0.00,\ 1.41\ (=\beta_{c,1}),\ 1.45,\ 1.49\ (=\beta_{c,2}),\ 1.53$ from bottom to top.

n	1	2	3	4	5	6	7	8	9	10
$\beta_{c,1}$	1.414	1.225	1.155	1.118	1.095	1.080	1.069	1.061	1.054	1.049
$\hat{k}_{*,1}$	0.4388	0.6252	0.7596	0.8672	0.9580	1.037	1.107	1.171	1.229	1.282
$\hat{\omega}_{*,1}$	0.5244	0.8504	1.115	1.345	1.552	1.742	1.919	2.086	2.244	2.394
$\beta_{c,2}$	1.489	1.293	1.215	1.172	1.144	1.124	1.110	1.098	1.089	1.081
$\hat{\omega}_{*,2}$	0.6388	1.039	1.358	1.632	1.875	2.097	2.301	2.492	2.671	2.840

TABLE 1. Critical values of the compressibility parameter, $\beta_{c,1}$ and $\beta_{c,2}$, and characteristic dimensionless wavenumber and growth rate: $\beta_{c,1}$ is the compressibility above which the cylinder is unstable for the homogeneous ($k = 0$) perturbation; $\hat{k}_{*,1}$ and $\hat{\omega}_{*,1}$ respectively are the wavenumber and growth rate of the most unstable mode when $\beta = \beta_{c,1}$; $\beta_{c,2}$ is the compressibility above which the homogeneous ($k = 0$) mode becomes the most unstable one; and $\hat{\omega}_{*,2}$ is the largest growth rate at $\beta = \beta_{c,2}$.

compressibility $\beta_{c,1}$, one can expect the behaviour of the dispersion relation for $k > 0$ to change at the critical compressibility.

The dispersion relation (2.20) for $n = 1$ is shown in figure 3 for several values of β. The qualitative behaviours are independent of n. For $0 < \beta \leqslant \beta_{c,1}$, the dispersion relation is qualitatively the same as the usual dispersion relation of the Rayleigh–Plateau instability (for an incompressible inviscid fluid), although the growth rate for all $\hat{k} \in (0, \sqrt{n})$ increases somewhat with β. The wavenumber and growth rate of the most unstable mode at $\beta = \beta_{c,1}$ obtained numerically are given in table 1. As β exceeds $\beta_{c,1}$, the homogeneous ($k = 0$) mode begins to have positive growth rate, as shown in § 3.1. Incidentally, the dispersion relation deviates from that of the usual Rayleigh–Plateau instability. Note that, even if β exceeds $\beta_{c,1}$ only slightly, the most unstable mode is still in $k > 0$. However, there exists another critical value of β (we call it $\beta_{c,2} > \beta_{c,1}$) above which the homogeneous mode becomes the most unstable one. The values of $\beta_{c,2}$ and the growth rate of the most unstable mode therein, which we denote by $\hat{\omega}_{*,2}$, are also given in table 1. They can be obtained by solving numerically the following coupled algebraic equations for β and $\hat{\omega}$:

$$F(\hat{\omega}, \hat{k})|_{\hat{k}=0} = 0, \quad \partial_{\hat{k}}^2 F(\hat{\omega}, \hat{k})|_{\hat{k}=0} = 0. \tag{3.6}$$

We note that $\beta_{c,2}$ is of course larger than $\beta_{c,1}$, but only slightly. Thus, we can say that, for generic values of β larger than $\beta_{c,1}$, the homogeneous unstable mode is predominant.

4. Relativistic perfect fluid with boundary

The argument in the preceding sections can be generalized to relativistic fluids. We consider relativistic fluids in the d-dimensional flat spacetime $\mathbb{R}^{1,d-1}$ ($d \geqslant 3$) with the spacetime coordinates $x^\mu = (ct, x^I)$ ($\mu, \nu = 0, 1, 2, \ldots, d-1$; $I, J = 1, 2, \ldots, d-1$), and denoting the flat metric by $g_{\mu\nu}$ (with the so-called almost-plus notation $(-, +, \ldots, +)$). The speed of light is set to unity ($c = 1$). Symbols appearing hereafter have the same meanings as in the non-relativistic case, unless otherwise noted.

4.1. *Euler equation and Young–Laplace relation*

The energy–momentum tensor of a relativistic perfect fluid held by surface tension (see e.g. Misner, Thorne & Wheeler 1974; Lahiri & Minwalla 2008) is given by

$$T^{\mu\nu} = (\epsilon u^\mu u^\nu + pP^{\mu\nu})\Theta(-f) - \sigma|\nabla f| h^{\mu\nu}\delta(f). \tag{4.1}$$

Here, ϵ is the energy density, u^μ is the normalized d-velocity field ($u^\mu u_\mu = -1$), Θ is the Heaviside step function, δ is the delta function, $P^{\mu\nu} := g^{\mu\nu} + u^\mu u^\nu$ and $h^{\mu\nu} := g^{\mu\nu} - n^\mu n^\nu$ are the projection tensors, n^μ is the unit normal of the surface defined by $n_\mu = \nabla_\mu f / (\nabla^\nu f \nabla_\nu f)^{1/2}$, and u^μ and n^μ are orthogonal each other at the surface $u^\mu n_\mu = 0|_{f=0}$.

Projecting the energy–momentum conservation $\nabla_\mu T^{\mu\nu} = 0$ onto u^μ and $P^{\mu\nu}$, we obtain the relativistic continuity and Euler equations as

$$u^\alpha \nabla_\alpha \epsilon + (\epsilon + p)\nabla_\alpha u^\alpha = 0, \tag{4.2}$$

$$(\epsilon + p) u^\alpha \nabla_\alpha u^\mu = -P^{\mu\alpha}\nabla_\alpha p. \tag{4.3}$$

The Young–Laplace relation, obtained by projecting the surface contribution of the energy–momentum conservation onto n^μ, takes the same form as in the non-relativistic case (2.3), but the mean curvature in this case is given by the d-dimensional divergence,

$$\kappa = \nabla_\mu n^\mu. \tag{4.4}$$

The kinematic boundary condition is given by

$$u^\alpha \nabla_\alpha f = 0|_{f=0}. \tag{4.5}$$

As in the non-relativistic case, we introduce cylindrical coordinates, in which the line element of flat spacetime is given by

$$g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}r^2 + r^2\, \mathrm{d}s_n^2 = \eta_{ab}\, \mathrm{d}x^a\, \mathrm{d}x^b + r^2 \gamma_{ij}(\theta)\, \mathrm{d}\theta^i\, \mathrm{d}\theta^j, \tag{4.6}$$

where $x^a = (t, z, r)$ and $\eta_{ab} = \mathrm{diag}(-1, 1, 1)$ is the three-dimensional flat Lorentzian metric. For $d = 3$ we just discard the z-coordinate as in the non-relativistic argument. Assuming that the fluid and surface are axially symmetric around the z-axis, the coordinate dependences of the pressure, velocity field and surface are

$$p = p(t, z, r), \quad u^a = u^a(t, z, r), \quad u^i = 0, \quad f = r - R(t, z). \tag{4.7}$$

The continuity equation (4.2) and Euler equation (4.3) can be written as

$$u^a \partial_a \epsilon + (\epsilon + p)\left(\partial_a u^a + \frac{n}{r} u^r\right) = 0, \tag{4.8}$$

$$(\epsilon + p) u^b \partial_b u_a = -P_a^b \partial_b p, \tag{4.9}$$

where the indices $(a, b, \ldots)$ are raised and lowered by η^{ab} and η_{ab}, respectively. In order to derive the above equations, it is useful to use the Riemann geometry techniques. For example, in the present coordinates, the non-vanishing components of the Christoffel symbol are $\Gamma^a_{ij} = -r\delta^a_r \gamma_{ij}$, $\Gamma^i_{ja} = r^{-1}\delta^i_j \delta^r_a$ and $\Gamma^i_{jk} = {}^{(\gamma)}\Gamma^i_{jk}$, where ${}^{(\gamma)}\Gamma^i_{jk}$ is the Christoffel symbol with respect to γ_{ij}. Using these, the vector derivatives are calculated, that is, $\nabla_a u_b = \partial_a u_b$, $\nabla_i u_j = r u^r \gamma_{ij}$ and $\nabla_\alpha u^\alpha = \partial_a u^a + n r^{-1} u^r$.

The mean curvature (4.4), appearing in the Young–Laplace relation (2.3), is given by

$$\kappa = \frac{n}{R\,[1-(\partial_t R)^2 + (\partial_z R)^2]^{1/2}} - \frac{[1-(\partial_t R)^2]\partial_z^2 R - [1+(\partial_z R)^2]\partial_t^2 R + 2(\partial_t R)(\partial_t \partial_z R)\partial_z R}{[1-(\partial_t R)^2 + (\partial_z R)^2]^{3/2}}. \tag{4.10}$$

Finally, the kinetic boundary condition (4.5) reads

$$u^t \partial_t R + u^z \partial_z R = u^r|_{r=R}. \tag{4.11}$$

4.2. *Linear perturbation of cylinders*

In general, perturbing the relativistic continuity equation (4.2) and Euler equation (4.3) around a static equilibrium where $(p, \epsilon, u^\mu) = (p_0, \epsilon_0, \delta^\mu_t)$, we have

$$\partial_t \delta p + c_s^2(\epsilon_0 + p_0)\nabla_\alpha \delta u^\alpha = 0, \tag{4.12}$$

$$(\epsilon_0 + p_0)\partial_t \delta u^\mu + P^{\mu\alpha}_{(0)} \nabla_\alpha \delta p = 0, \tag{4.13}$$

where $P^{\mu\nu}_{(0)} := g^{\mu\nu} + \delta^\mu_t \delta^\nu_t$ and $c_s^2 := \mathrm{d}p_0/\mathrm{d}\epsilon_0$ is the sound velocity squared (note that $c_s < 1$ from the causality). Eliminating δu^μ from these two equations, one obtains a wave equation for the pressure perturbation,

$$(\partial_t^2 - c_s^2 P^{\alpha\beta}_{(0)} \nabla_\alpha \nabla_\beta)\delta p = 0. \tag{4.14}$$

As in the non-relativistic case in § 2.2, the axially symmetric relativistic system in § 4.1 allows the cylinder as a static equilibrium, where constant pressure p_0 and radius $R = r_0$ satisfy (2.12). Plugging ansatz (2.14) into wave equation (4.14) and using the perturbation of the Young–Laplace relation $\delta p = \sigma \delta\kappa|_{r=R}$, we obtain the radial function in the relativistic case as

$$P(r) = -\frac{[n - (k^2 + \omega^2) r_0^2] r_0^{(n-1)/2}}{n \mathrm{I}_{(n-1)/2}(K r_0)} \frac{\mathrm{I}_{(n-1)/2}(K r)}{r^{(n-1)/2}}, \quad K := \left(k^2 + \frac{\omega^2}{c_s^2}\right)^{1/2}. \tag{4.15}$$

The perturbation of the Euler equation (4.13) in the r-direction reads $(\epsilon_0 + p_0)\partial_t \delta u^r + \partial_r \delta p = 0$. On the other hand, the perturbation of the kinetic boundary condition (4.11) reads $\partial_t \delta R = \delta u^r|_{r=R}$. Eliminating δu^r from these two equations, we obtain

$$(\epsilon_0 + p_0)\partial_t^2 \delta R = -\partial_r \delta p|_{r=R}. \tag{4.16}$$

Plugging (2.13), (2.14) and (4.15) into (4.16), we obtain the dispersion relation for the perturbation of a relativistic compressible cylinder as

$$\omega^2 = \frac{\sigma}{(\epsilon_0 + p_0) r_0^3}[n - (k^2 + \omega^2) r_0^2] K r_0 \frac{\mathrm{I}_{(n+1)/2}(K r_0)}{\mathrm{I}_{(n-1)/2}(K r_0)}. \tag{4.17}$$

It is noted that a similar result for a relativistic compressible fluid with a particular equation of state was obtained in Caldarelli *et al.* (2009).

We introduce the following dimensionless quantities (remember that we have already set the speed of light to unity):

$$\hat{k} := r_0 k, \quad \hat{\omega} := r_0 \omega, \quad \hat{\sigma} := (\epsilon_0 r_0)^{-1} \sigma, \quad \beta := c_s^{-1} > 1. \tag{4.18}$$

Then, the dispersion relation (4.17) is the relation between $\hat{k}$ and $\hat{\omega}$ for which the following function vanishes:

$$F(\hat{\omega}, \hat{k}) := \hat{\omega}^2 - \frac{\hat{\sigma}}{1 + n\hat{\sigma}}[n - (\hat{k}^2 + \hat{\omega}^2)]\hat{K}\frac{\mathrm{I}_{(n+1)/2}(\hat{K})}{\mathrm{I}_{(n-1)/2}(\hat{K})}, \quad \hat{K} := (\hat{k}^2 + \beta^2\hat{\omega}^2)^{1/2}. \tag{4.19}$$

Function $F(\hat{\omega}, \hat{k})$ depends not only on the dimensionless compressibility β but also on the dimensionless surface tension $\hat{\sigma}$. This is contrast to the non-relativistic counterpart (2.22), which depends only on the compressibility β.

5. Analysis of dispersion relation for the relativistic fluid

As explained at the beginning of § 3.1, the homogeneous ($k = 0$) mode has special meaning in that it corresponds to both the homogeneous perturbation of cylinders in $\mathbb{R}^{1,n+2}$ and the spherical perturbation of droplets in $\mathbb{R}^{1,n+1}$. In the first subsection below we look into the $k = 0$ mode, then we proceed to the analysis of general $k \geqslant 0$ perturbations in the second subsection.

5.1. *Instability of drops ($k = 0$ mode)*

Setting $k = 0$ in (4.19), we have

$$F(\hat{\omega}, 0) = \hat{\omega}^2 + \frac{\hat{\sigma}}{1 + n\hat{\sigma}}(\hat{\omega}^2 - n)\beta\hat{\omega}\frac{\mathrm{I}_{(n+1)/2}(\beta\hat{\omega})}{\mathrm{I}_{(n-1)/2}(\beta\hat{\omega})}. \tag{5.1}$$

Let us see the behaviour of $F(\hat{\omega}, 0)$ for small $\hat{\omega}$ by expanding it around $\hat{\omega} = 0$:

$$F(\hat{\omega}, 0) = \left(1 - \frac{n\hat{\sigma}}{(n+1)(1 + n\hat{\sigma})}\beta^2\right)\hat{\omega}^2 + \frac{n\beta^2 + (n+1)(n+3)}{(n+1)^2(n+3)(1 + n\hat{\sigma})}\hat{\sigma}\beta^2\hat{\omega}^4 + O(\hat{\omega}^6). \tag{5.2}$$

From this, one can see that $F(\hat{\omega}, 0)|_{\hat{\omega}=0} = \partial_{\hat{\omega}} F(\hat{\omega}, 0)|_{\hat{\omega}=0} = 0$. On the other hand, one can see $\lim_{\hat{\omega}\to\infty} F(\hat{\omega}, 0) = +\infty$ from (5.1). Thus, if $\partial^2_{\hat{\omega}} F(\hat{\omega}, 0)|_{\hat{\omega}=0} < 0$ holds, $F(\hat{\omega}, 0)$ must have at least one positive zero from continuity. From (5.2), one can see that $\partial^2_{\hat{\omega}} F(\hat{\omega}, 0)|_{\hat{\omega}=0} < 0$ holds if the compressibility parameter is greater than a critical value,

$$\beta > \beta_{c,1} := \left(\frac{(n+1)(1 + n\hat{\sigma})}{n\hat{\sigma}}\right)^{1/2}. \tag{5.3}$$

In this case, the β-dependence of the growth rate near $\beta = \beta_{c,1}$ can be read from (5.2),

$$\hat{\omega} \simeq \left(\frac{4n^3\,(n+3)^2\,\hat{\sigma}^3}{(n+1)(1 + n\hat{\sigma})\,[1 + (2n+3)\hat{\sigma}]^2}\right)^{1/2} (\beta - \beta_{c,1})^{1/2} + O((\beta - \beta_{c,1})^{3/2}). \tag{5.4}$$

The global behaviour of $\hat{\omega}(\beta)$ for several values of $\hat{\sigma}$ is shown in figure 4(*a*). In the present relativistic case, in contrast to the non-relativistic case, the growth rate

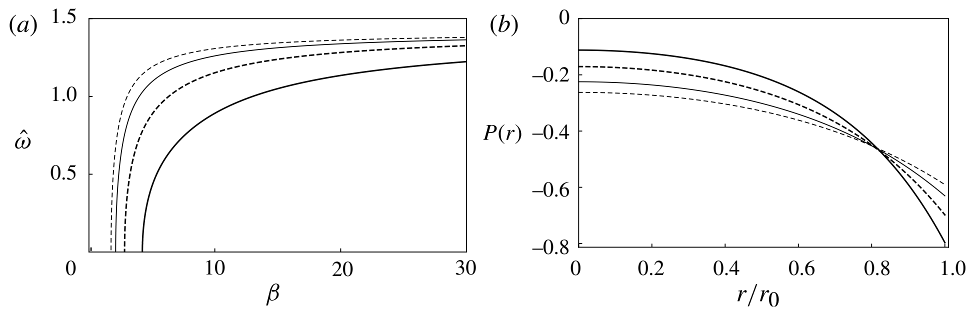

FIGURE 4. (*a*) Compressibility parameter β versus dimensionless growth rate of the radial perturbation of droplets in $\mathbb{R}^{1,3}$ ($n = 2$) for several values of dimensionless surface tension: $\hat{\sigma} = 0.100$ (thick solid), 0.300 (thick dashed), 1.00 (thin solid) and 20.0 (thin dashed). (*b*) Radial function $P(r)$ for the same set of $\hat{\sigma}$ as in panel (*a*). The dimension and compressibility are set, respectively, as $n = 2$ and $\beta/\beta_{c,1} = 1.50$.

asymptotes to a constant in the large-β limit, that is, $\hat{\omega} \simeq \sqrt{n}$ in fact. This can be derived from (4.19). The radial function $P(r)$ is shown in figure 4(*b*).

Getting back to dimensionful quantities with (4.18), the instability criterion (5.3) can be rewritten as

$$r_0 < r_{min} := \frac{n}{n+1} \frac{\sigma}{\epsilon_0 c_s^2} (1 - (n+1)c_s^2). \tag{5.5}$$

Namely, there exists a minimum radius r_{min} below which the drops and cylinders become unstable. See § B.2 for the proof of stability for non-spherical perturbations.

5.2. *Rayleigh–Plateau instability ($k > 0$ modes)*

The behaviour of dispersion relation $\hat{\omega}(\hat{k})$ is quite similar to the non-relativistic case, except that the critical values of compressibility, $\beta_{c,1}$ and $\beta_{c,2}$, depend on $\hat{\sigma}$. A numerical plot of $\hat{\omega} = \hat{\omega}(\hat{k})$ for several values of β is shown in figure 5(*a*). The value of $\hat{\sigma}$ does not affect the qualitative behaviour of the dispersion relation. In order to see that the second critical compressibility $\beta_{c,2}$, above which the $k = 0$ mode is the most unstable one, is only slightly larger than the first critical value $\beta_{c,1}$ in all the range of $\hat{\sigma}$, we numerically plot $\beta_{c,1}$ and $\beta_{c,2}$ for $n = 1$ in figure 5(*b*).

6. Conclusion

We have investigated the stability of spherical drops and cylindrical jets held by the surface tension, in particular, the dependence on the compressibility β or sound velocity c_s of the bulk fluids. For simplicity, we have focused on perfect fluids (i.e. inviscid fluids with no heat transfer) immersed in vacuum, while we consider both the non-relativistic and relativistic fluids in general dimensions, which allows us to treat discs, droplets and cylindrical jets simultaneously in a systematic way.

As the main result, we have shown that there exists a critical compressibility $\beta_{c,1}$ for both non-relativistic and relativistic fluids, (3.3) and (5.3), above which spherical drops are unstable for a spherical perturbation. For given parameters of the fluid and surface, i.e. surface tension σ, sound velocity c_s and density at equilibrium ρ_0 (or ϵ_0 in the relativistic case), the instability criterion poses a lower limit on the droplet size,

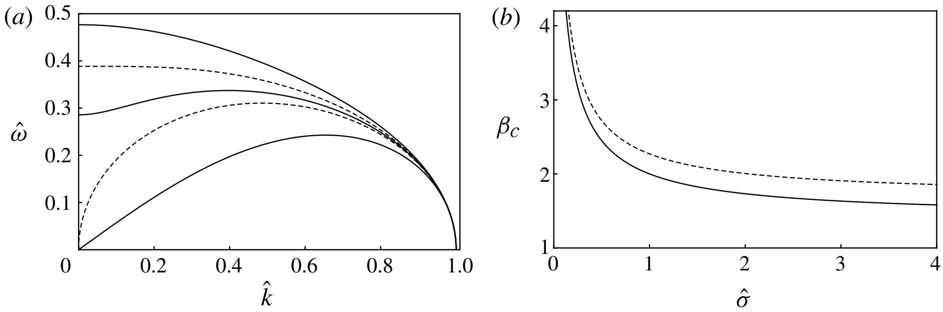

FIGURE 5. (*a*) Dimensionless wavenumber $\hat{k}$ versus dimensionless growth rate $\hat{\omega}$ of the perturbation of a relativistic cylinder in $\mathbb{R}^{1,3}$ ($n=1$) for several values of compressibility: $\beta = 1.00,\ 2.00\ (=\beta_{c,1}),\ 2.13,\ 2.27\ (=\beta_{c,2}),\ 2.45$ from bottom to top. The dimensionless surface tension is fixed to $\hat{\sigma}=1$. (*b*) The $\hat{\sigma}$-dependence of the critical compressibility, $\beta_{c,1}$ (solid) and $\beta_{c,2}$ (dashed), for $n=1$.

$r_{min} \sim \sigma/(\rho_0 c_s^2)$ (see (3.5) and (5.5)), below which any droplets cannot be in stable equilibrium.

We have shown also that, according to the instability of discs and droplets, which corresponds to the instability of cylinders for homogeneous perturbations, the dispersion relation of Rayleigh–Plateau instability exhibits a significant change. Namely, for $\beta > \beta_{c,1}$ cylinders are unstable for perturbations that are homogeneous in the axial direction. Furthermore, such a mode becomes the most unstable one above the second critical compressibility $\beta_{c,2}$, which is slightly larger than $\beta_{c,1}$, in general.

Here, let us stress the significance of the minimum radius r_{min}. In the framework of fluid mechanics, it has been assumed or simply believed that *any* positive finite values can be given to the three quantities σ, c_s and ρ_0 (although they should be correlated with each other if one pursues their origins from a microscopic point of view). We have shown, however, that spherical droplets, which are the most fundamental equilibrium state of localized fluids, exhibit instability for $r_0 < r_{min} \sim \sigma/(\rho_0 c_s^2)$. Therefore, one cannot give values to the three parameters freely in order to describe arbitrarily small-scale dynamics successfully. In other words, the systems defined by the Euler equation and the Young–Laplace relation intrinsically contain the instability, and are not well defined in certain regimes of parameter space.

We have adopted several assumptions for simplicity, such as the absence of viscosity, heat transfer and outer fluids, the constancy of surface tension, and so on. In addition, the instability discovered is just the result of mode analysis, which can never predict the following dynamics. Therefore, there are many directions to proceed by generalizing the analysis in this paper. It would be interesting to see how the viscosity affects the instability. The nonlinear dynamics would be interesting, too, even within the perfect-fluid approximation.

Acknowledgements

The author would like to thank J. Camps, R. Emparan, N. Haddad, T. Harada and N. Shibazaki for stimulating discussions, and the anonymous referees for quite useful suggestions. This work was supported by the Research Center for Measurement in Advanced Science at Rikkyo University, and by the Grant-in-Aid for Scientific

Research Fund of the Ministry of Education, Culture, Sports, Science and Technology, Japan (Young Scientists (B) 22740176).

Appendix A. Comments on r_{min}

After submitting the draft of this paper, the anonymous referees gave the author several useful comments and suggestions about the values of r_{min} in actual physical systems. Here, some of these suggestions are noted for further studies.

A.1. *General order estimate*

On a dimensional basis, one can argue that the critical radius r_{min} is of the order of a microscopic length scale as follows. Let us consider the relativistic case for simplicity. In many physical systems, it would be possible to assume that $c_s^2 = \mathrm{d}p_0/\mathrm{d}\epsilon_0 \sim 1$, which is equivalent to assuming that the compressibility is proportional to the inverse of the energy density, $\epsilon_0^{-1}\,\mathrm{d}\epsilon_0/\mathrm{d}p_0 \sim \epsilon_0^{-1}$. In this case, r_{min} in (5.5) is reduced to $r_{min} \sim \sigma/\epsilon_0$. Here, let us assume further that the surface tension is microscopically proportional to the energy density in the bulk as $\sigma \sim l\epsilon_0$, where l is a microscopic length scale. Such a length scale l could be a mean intermolecular distance. This expectation should be justified in a molecular theory of capillary force (see e.g. Rowlinson & Widom 2002). With these assumptions, one obtains $r_{min} \sim l$, which suggests that the instability is irrelevant since the hydrodynamic description itself breaks down at the scale of l.

A.2. *Liquid-drop model of the nucleus*

It is widely known that many features of the nucleus, such as the global behaviour of binding energy, surface oscillations and nuclear fission, can be understood with the *liquid-drop model*, in which the nucleus is modelled by a liquid drop of an incompressible (at leading order) fluid (Bohr & Mottelson 1969, 1975). However, compressibility, which allows radial oscillations of a drop (i.e. the so-called breathing mode), is important, since it is directly related to the equation of state of nuclear matter, and necessary to the accurate estimate of nuclear properties (radii, masses, giant resonances, etc.). Though the order estimate in § A.1 seems to suggest that the instability found in this paper is irrelevant to nuclei, it would be interesting to compare systematically the parameters in the liquid-drop model and those in this paper. Incidentally, the confine–deconfine phase transition in quantum chromodynamics (QCD) is expected to be of the first order, and so the deconfined phase, i.e. the *quark–gluon plasma* (QGP), could exist as a drop of fluid around the critical temperature. Thus, it would also be interesting to consider the effect of compressibility on the QGP balls.

A.3. *Granular matter*

Recently, it was reported that a kind of *granular matter* such as glass beads of tiny radius exhibit effective compressibility (Boudet, Amarouchene & Kellay 2008) and surface tension (Prado, Amarouchene & Kellay 2011). The existence of surface tension (capillarity) in granular matter might be surprising, since the attractive force between grains is much smaller than other forces at play (gravity, friction, inelasticity). In fact, the surface tension of the granular matter (glass beads) in the experiment (Prado *et al.* 2011) stems not from the attractive forces between the beads but from a strong interaction between the beads and the surrounding air. The dynamics of granular matter cannot be described by hydrodynamics in general, and furthermore the origin of the surface tension is different from that in fluids. Therefore, one cannot apply the result in this paper as is to granular matter. It would be interesting, however, to examine whether or not the possibility that the new capillary instability found in this paper or its analogue is effective in granular matter. To the author's knowledge, the sound velocity of granular matter could be relatively small (e.g. $c_s \sim 1\ \mathrm{m\ s^{-1}}$ in an experiment; Amarouchene & Kellay 2006). Thus, there remains the possibility that $r_{min} \propto c_s^{-2}$ could be large.

Appendix B. Stability of drops for non-spherical perturbations

B.1. *Non-relativistic case*

We show the stability of non-relativistic droplets in $\mathbb{R}^{1,n+1}$ ($n \geqslant 1$) for non-spherical perturbations (and the stability of cylinder in $\mathbb{R}^{1,n+2}$ for homogeneous (in the r-direction) but non-spherical perturbations).

We work in polar coordinates in which the line element of the flat space is given by

$$g_{IJ}\,\mathrm{d}x^I\,\mathrm{d}x^J = \mathrm{d}r^2 + r^2\gamma_{ij}(\theta)\,\mathrm{d}\theta^i\,\mathrm{d}\theta^j. \tag{B 1}$$

In these coordinates, the continuity equation (2.1) and Euler equation (2.2) in the r- and θ^i-directions are

$$(\partial_t + v^r\partial_r + v^i\partial_i)\rho + \rho\left(\partial_r v^r + \frac{n}{r}v^r + \mathrm{D}_i v^i\right) = 0, \tag{B 2}$$

$$\rho((\partial_t + v^r\partial_r + v^i\partial_i)v^r - r\gamma_{ij}v^i v^j) = -\partial_r p, \tag{B 3}$$

$$\rho\left(\partial_t + v^r\partial_r + \frac{v^r}{r} + v^j\mathrm{D}_j\right)v^i = -\frac{\gamma^{ij}}{r^2}\partial_j p, \tag{B 4}$$

where D_i is the covariant derivative compatible with γ_{ij}. If we parametrize the scalar function f as

$$f(t, r, \theta) = r - R(t, \theta_1, \theta_2, \ldots, \theta_n), \tag{B 5}$$

the mean curvature, appearing in the Young–Laplace relation equation (2.3), reads

$$\kappa = \frac{n}{R\,[1 + R^{-2}\,(\mathrm{D}R)^2]^{1/2}} - \frac{R^2\mathrm{D}^2R + (\mathrm{D}^2R - R)\,(\mathrm{D}R)^2 - (\mathrm{D}^iR)(\mathrm{D}^jR)\mathrm{D}_i\mathrm{D}_jR}{R^4\,[1 + R^{-2}\,(\mathrm{D}R)^2]^{3/2}}. \tag{B 6}$$

Here, $(\mathrm{D}R)^2 := \gamma^{ij}(\mathrm{D}_iR)\mathrm{D}_jR$ and $\mathrm{D}^2R := \gamma^{ij}\mathrm{D}_i\mathrm{D}_jR$. Kinetic boundary condition (2.5) is

$$\partial_t R + v^i\partial_i R = v^r|_{r=R}. \tag{B 7}$$

Obviously, equations of motion (B 2)–(B 4) and boundary conditions (2.3) (with (B 6)) and (B 7) allow the spherical drop as a static equilibrium, where the constant pressure p_0 and radius of the sphere $R = r_0$ satisfy (2.12). Now, we consider linear perturbations of this equilibrium resulting from the disturbance of the surface,

$$R(t, \theta) = r_0[1 + \varepsilon\mathrm{e}^{\omega t}Y(\theta)]. \tag{B 8}$$

Here, $Y(\theta_1, \theta_2, \ldots, \theta_n)$ is the harmonic function on the unit n-sphere,

$$[\mathrm{D}^2 + \ell(\ell + n - 1)]Y(\theta) = 0, \quad \ell = 0, 1, 2, \ldots. \tag{B 9}$$

The perturbed pressure to $O(\varepsilon)$ may be written as

$$p(t, r, \theta) = p_0[1 + \varepsilon\mathrm{e}^{\omega t}P(r)Y(\theta)]. \tag{B 10}$$

https://doi.org/10.1017/jfm.2012.142 Published online by Cambridge University Press

Substituting expression (B 10) into wave equation (2.15), we obtain

$$\frac{\mathrm{d}^2 P}{\mathrm{d}r^2} + \frac{n}{r}\frac{\mathrm{d}P}{\mathrm{d}r} - \left(\frac{\omega^2}{c_s^2} + \frac{\ell(\ell+n-1)}{r^2}\right) P = 0. \tag{B 11}$$

With the regularity at the origin and the perturbed Young–Laplace formula $\delta p = \sigma \delta\kappa|_{r=R}$, one finds that the following solves (B 11):

$$P(r) = \frac{(\ell-1)(\ell+n) r_0^{(n-1)/2}}{n \mathrm{I}_{\ell+(n-1)/2}(\omega r_0/c_s)} \frac{\mathrm{I}_{\ell+(n-1)/2}(\omega r/c_s)}{r^{(n-1)/2}}. \tag{B 12}$$

Substituting (2.12), (B 8), (B 10) and (B 12) into (2.19), we obtain

$$\omega^2 = -\frac{\sigma}{\rho_0 r_0^3}(\ell-1)(\ell+n)\left(\ell + \frac{\omega r_0}{c_s}\frac{\mathrm{I}_{\ell+(n+1)/2}(\omega r_0/c_s)}{\mathrm{I}_{\ell+(n-1)/2}(\omega r_0/c_s)}\right). \tag{B 13}$$

If one takes $n=2$, $c_s \to \infty$, and writes $\omega \to \mathrm{i}\Omega$ ($\mathrm{i} = \sqrt{-1}$) in this relation, one reproduces the classic formula (1.1), i.e. the angular frequency of oscillations for the incompressible inviscid droplet immersed in the three-dimensional vacuum (Rayleigh 1894).

If (B 13) has a positive root for a given ℓ ($\geqslant 0$), the spherical droplet is unstable for the perturbation labelled by ℓ. Introducing the dimensionless quantities as in (2.21), the problem is equivalent to finding a positive zero of the following function:

$$F_\ell(\hat{\omega}) := \hat{\omega}^2 + (\ell-1)\ell(\ell+n) + (\ell-1)(\ell+n)\beta\hat{\omega}\frac{\mathrm{I}_{\ell+(n+1)/2}(\beta\hat{\omega})}{\mathrm{I}_{\ell+(n-1)/2}(\beta\hat{\omega})}. \tag{B 14}$$

Note that $F_0(\hat{\omega})$ is nothing but $F(\hat{\omega}, 0)$ in § 3.1, which was shown to have the positive zero. Taking into account the positivity of the modified Bessel function, one can easily see

$$F_\ell(\hat{\omega}) > 0 \quad \text{for } \ell \geqslant 1 \text{ and } \hat{\omega} > 0. \tag{B 15}$$

Thus, there is no positive zero of $F_\ell(\hat{\omega})$ for $\ell \geqslant 1$, proving the stability of the droplet for non-spherical perturbations.

B.2. *Relativistic case*

We show the stability of relativistic droplets in $\mathbb{R}^{1,n+1}$ ($n \geqslant 1$) for non-spherical perturbations (and that of relativistic cylinders in $\mathbb{R}^{1,n+2}$ for non-spherical but homogeneous ($k=0$) perturbations).

We write the flat metric as

$$g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\gamma_{ij}\,\mathrm{d}\theta^i\,\mathrm{d}\theta^j. \tag{B 16}$$

Then, we perturb the spherically symmetric static equilibrium, where the constant pressure p_0 and radius $R = r_0$ satisfy (2.12), with the same ansatz (B 8) and (B 10) as in the non-relativistic case. The mean curvature (4.4) reads

$$\kappa = \frac{n}{R\,[1-(\partial_t R)^2 + R^{-2}\,(\mathrm{D}R)^2]^{1/2}} + \frac{1}{R^4\,[1-(\partial_t R)^2 + R^{-2}\,(\mathrm{D}R)^2]^{3/2}}\{[R^2 + (\mathrm{D}R)^2]R^2\partial_t^2 R \\ - [1-(\partial_t R)^2]R^2\mathrm{D}^2 R + (R - \mathrm{D}^2 R)\,(\mathrm{D}R)^2 + (\mathrm{D}^i R)(\mathrm{D}^j R)\mathrm{D}_i\mathrm{D}_j R\}. \tag{B 17}$$

Substituting expression (B 10) into wave equation (4.14), one finds that the radial function $P(r)$ satisfies the same form of equation as (B 11). With the perturbed kinetic boundary condition, one can fix the integration constant to obtain

$$P(r)=\frac{[(\omega r_0)^2+(\ell-1)(\ell+n)]r_0^{(n-1)/2}}{n\mathrm{I}_{\ell+(n-1)/2}(\omega r_0/c_s)}\frac{\mathrm{I}_{\ell+(n-1)/2}(\omega r/c_s)}{r^{(n-1)/2}}. \tag{B 18}$$

Substituting (B 8), (B 10) and (B 18) into (4.16), one obtains

$$\omega^2=-\frac{\sigma}{(\epsilon_0+p_0)r_0^3}[(\omega r_0)^2+(\ell-1)(\ell+n)]\left(\ell+\frac{\omega r_0}{c_s}\frac{\mathrm{I}_{\ell+(n+1)/2}(\omega r_0/c_s)}{\mathrm{I}_{\ell+(n-1)/2}(\omega r_0/c_s)}\right). \tag{B 19}$$

In terms of the dimensionless quantities in (4.18), to find a positive root of the above equation is equivalent to finding a positive zero of the following function:

$$F_\ell(\hat{\omega}):=\hat{\omega}^2+\frac{\hat{\sigma}}{1+n\hat{\sigma}}[\hat{\omega}^2+(\ell-1)(\ell+n)]\left(\ell+\beta\hat{\omega}\frac{\mathrm{I}_{\ell+(n+1)/2}(\beta\hat{\omega})}{\mathrm{I}_{\ell+(n-1)/2}(\beta\hat{\omega})}\right). \tag{B 20}$$

Observe that $F_0(\hat{\omega})$ is nothing but $F(\hat{\omega},0)$ in § 5.1, which was shown to have a positive zero. One can easily show using the positivity of the modified Bessel functions that

$$F_\ell(\hat{\omega})>0 \quad \text{for } \ell\geqslant 1,\ \hat{\sigma}>0,\ \text{and } \hat{\omega}>0. \tag{B 21}$$

Thus, $F_\ell(\hat{\omega})$ for $\ell\geqslant 1$ has no positive root, proving the stability of relativistic droplets for non-spherical perturbations.

REFERENCES

AMAROUCHENE, Y. & KELLAY, H. 2006 Speed of sound from shock fronts in granular flows. *Phys. Fluids* **18**, 031707.

BECKER, E., HILLER, W. J. & KOWALEWSKI, T. A. 1991 Experimental and theoretical investigation of large-amplitude oscillations of liquid droplets. *J. Fluid Mech.* **231**, 189.

BHATTACHARYYA, S., HUBENY, V. E., MINWALLA, S. & RANGAMANI, M. 2008 Nonlinear fluid dynamics from gravity. *J. High Energy Phys.* **2008** (02), 045.

BOHR, A. & MOTTELSON, B. R. 1969 *Nuclear Structure,* Volume I: *Single-Particle Motion*. W. A. Benjamin.

BOHR, A. & MOTTELSON, B. R. 1975 *Nuclear Structure,* Volume II: *Nuclear Deformations*. W. A. Benjamin.

BOUDET, J. F., AMAROUCHENE, Y. & KELLAY, H. 2008 Shock front width and structure in supersonic granular flows. *Phys. Rev. Lett.* **101**, 254503.

CALDARELLI, M. M., DIAS, O. J. C., EMPARAN, R. & KLEMM, D. 2009 Black holes as lumps of fluid. *J. High Energy Phys.* **2009** (04), 024.

CARDOSO, V. & GUALTIERI, L. 2006 Equilibrium configurations of fluids and their stability in higher dimensions. *Class. Quant. Grav.* **23**, 7151–7198.

CHANDRASEKHAR, S. 1959 The oscillations of a viscous liquid globe. *Proc. Lond. Math. Soc.* **9**, 141.

CHANDRASEKHAR, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Clarendon.

CHANDRASEKHAR, S. 1964 Dynamical instability of gaseous masses approaching the Schwarzschild limit in general relativity. *Phys. Rev. Lett.* **12**, 114.

EGGERS, J. 1997 Nonlinear dynamics and breakup of free-surface flows. *Rev. Mod. Phys.* **69**, 865.

KELVIN, LORD 1890 Oscillations of a liquid sphere. *Math. Phys. Papers* **3**, 384.

LAHIRI, S. & MINWALLA, S. 2008 Plasmarings as dual black rings. *J. High Energy Phys.* **2008** (05), 001.

LAMB, H. 1932 *Hydrodynamics*. Cambridge University Press.

LANDAU, L. D. & LIFSHITZ, E. M. 1987 *Fluid Mechanics*. Butterworth-Heinemann.

MAEDA, K.-I. & MIYAMOTO, U. 2009 Black hole–black string phase transitions from hydrodynamics. *J. High Energy Phys.* **2009** (03), 066.

MISNER, C. W., THORNE, K. S. & WHEELER, J. A. 1974 *Gravitation*. W. H. Freeman.

PLATEAU, J. A. F. 1873 *Statique Expérimentale et Théorique des Liquides Soumis aux Seules Forces Moléculaires*. Gauthiers-Villars.

PRADO, G., AMAROUCHENE, Y. & KELLAY, H. 2011 Experimental evidence of a Rayleigh–Plateau instability in free falling granular jets. *Phys. Rev. Lett.* **106**, 198001.

RANGAMANI, M. 2009 Gravity and hydrodynamics: lectures on the fluid–gravity correspondence. *Class. Quant. Grav.* **26**, 224003.

RAYLEIGH, LORD 1879 On the instability of jets. *Proc. Lond. Math. Soc.* **10**, 4.

RAYLEIGH, LORD 1894 *The Theory of Sound*. Macmillan.

REID, W. H. 1960 The oscillations of a viscous liquid drop. *Q. Appl. Math.* **18**, 86.

ROWLINSON, J. S. & WIDOM, B. 2002 *Molecular Theory of Capillarity*. Dover.

SHAPIRO, S. L. & TEUKOLSKY, S. A. 1983 *Black Holes, White Dwarfs, and Neutron Stars: The Physics of Compact Objects*. Wiley.

WALD, R. M. 1984 *General Relativity*. University of Chicago Press.

WEAST, R. C. (ED.) 1978 *Handbook of Chemistry and Physics*. CRC.