Ergod. Th. & Dynam. Sys. (2013), **33**, 455–474
doi:10.1017/S0143385711001027

On Finsler surfaces without conjugate points

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(*Received* 5 *January* 2011 *and accepted in revised form* 27 *October* 2011)

Abstract. If (M, F) is a C^4 compact Finsler surface of genus at least two without conjugate points, we show that the first integrals of the geodesic flow are constant. Using this fact, we show that if (M, F) is also of Landsberg type then (M, F) is Riemannian. The connection between the absence of conjugate points and the Riemannian character of the Finsler metric has some remarkable consequences concerning rigidity.

1. *Introduction*

Finsler spaces are natural generalizations of Riemannian spaces: a Finsler metric in a smooth manifold is essentially a convex, non-degenerate function defined in the tangent space of the manifold with values in $\mathbb{R}$ (see below for a rigorous definition). However, Finsler spaces include a class of spaces that is much wider than Riemannian ones. In fact, the Hamiltonian flow at a suitably high energy level of a smooth, convex, superlinear Hamiltonian can be reparametrized in a way that it becomes the geodesic flow of a Finsler metric [**6**]. So the study of the geodesic flow of Finsler spaces is relevant in classical mechanics; it is just a Hamiltonian flow at a supercritical energy level of the Hamiltonian.

The main motivation of this paper is a rigidity result [**15, 17**] concerning Riemannian geodesic flows and magnetic flows in surfaces which preserve $C^{2,1}$ codimension-one foliations ($C^{2,1}$ means C^2 with Lipschitz C^2 derivatives). Under such a hypothesis in the dynamics of the flows it is proved in [**17**] that the magnetic flow has constant, non-positive curvature and Lorentz force of constant norm. In particular, the magnetic flow is an algebraic model: either the surface is a torus and the flow is a flat, Riemannian geodesic flow, or the surface is hyperbolic and the magnetic flow is either Anosov or a horocycle flow. The relationship between rigidity and a dynamical system preserving smooth foliations goes back to the work of Ghys [**13**] and Hurder and Katok [**18**], who studied the rigidity of Anosov flows under such assumptions. The main result of the paper is closely related to the subject.

THEOREM 1. *Let (M, F) be a C^4 Finsler metric in a compact surface of genus greater than one, without conjugate points. Then every first integral of the Finsler geodesic flow is constant.*

Applying Theorem 1 and Finsler geometry we get the following theorem.

THEOREM 2. *Let (M, F) be a C^4 Finsler metric in a compact surface of genus greater than one, without conjugate points. If (M, F) is a Landsberg metric then (M, F) is Riemannian.*

The link between Theorems 1, 2, and the rigidity results mentioned above is made by a beautiful theorem proved by Mañé [**21**]: geodesic flows which preserve continuous, Lagrangian bundles have no conjugate points. So if we assume that the geodesic flow of the compact surface preserves a smooth, codimension-one foliation, the tangent space of the leaves is automatically an invariant, Lagrangian subbundle and, therefore, the metric has no conjugate points (see [**17**] for details).

The proof of Theorem 1 is essentially the content of the paper. Let us give a brief summary of results. A good starting point to motivate the relevance of Theorem 1 is that it is already known for Riemannian surfaces of higher genus. Indeed, the geodesic flow of compact Riemannian surfaces without conjugate points and genus at least two is transitive. This important feature of two-dimensional geometry without conjugate points was proved by Eberlein [**9**] as a part of his beautiful work about visibility manifolds. So the first integrals of the geodesic flow are obviously constant in this case. A close look in Eberlein's proof shows the crucial role of the reversibility of the Riemannian metric: the fact that a minimizing geodesic with reversed parametrization is also minimizing is essential in many steps of the proof. This is obviously not the case for a general Finsler metric; the non-reversibility issue is behind many subtle arguments in the paper.

Section 2 contains some preliminaries about Finsler geometry: the Chern–Rund connection, Jacobi fields and Cartan's structural equations. Section 3 deals with a problem that is of interest in its own right: the divergence of Jacobi vector fields and geodesic rays in the universal covering of a surface without conjugate points. The divergence of such objects is well known in the universal covering of Riemannian compact surfaces without conjugate points, but in higher dimensions there are still many interesting open problems concerning divergence. For instance, Jacobi fields diverge in geodesics without conjugate points, but if the dimension of the manifold is at least three the divergence is not uniform; it might depend on the geodesic (we give more details in §3). This was observed by Eberlein [**9**] for Riemannian manifolds and by Contreras and Iturriaga [**5**] for regular energy levels of convex Hamiltonians. So in §3 we show that both Jacobi fields vanishing at one point and geodesic rays diverge uniformly, as in the Riemannian case, in the universal covering of a compact Finsler surface without conjugate points.

Section 4 is the core of the paper: we are not able to show the transitivity of the geodesic flow of compact Finsler surfaces without conjugate points and higher genus because reversibility is crucial in Eberlein's work. To get around this difficulty we generalize the construction by [**16**] of central stable sets for Riemannian surfaces without conjugate points and higher genus, and show that the collection of such sets forms a minimal foliation of the unit tangent bundle. The proof of this fact relies strongly on the divergence properties of geodesic rays (§3) and on Morse's work about globally minimizing geodesics [**23**]. By the way, we think that the transitivity of the geodesic flow of compact reversible Finsler surfaces without conjugate points and genus at least two follows from the results of §4 (see, for instance, Egloff [**10**] for a treatment of reversible Finsler surfaces of non-positive curvature).

In §5 we show that a first integral of the geodesic flow is constant in the central stable leaf of a hyperbolic closed orbit. This feature, together with the existence of closed hyperbolic orbits of geodesic flows of compact surfaces of higher genus (Katok) and the density of any central leaf, concludes the proof of Theorem 1. Section 6 is the proof of Theorem 2, and §7 contains some further applications of Theorem 1 regarding the work of Ikeda and Foulon about Finsler metrics.

We would like to make some remarks to frame Theorem 2 in the context of Finsler geometry. There is a long-standing problem in the theory of Finsler metrics considered by some authors as one of the main questions in the theory: are there Landsberg metrics which are not Berwald metrics (see [**4**] for the definitions)? Berwald metrics are Landsberg metrics, but the converse of this assertion is an open problem. Since it is known (see [**4**], for instance) that Berwald compact surfaces are either locally Minkowskian (and the surface is the torus) or Riemannian ([**26**], Theorem 2.4), this problem in the case of surfaces of higher genus amounts to one of whether there are Landsberg compact surfaces that are not Riemannian. Paternain showed that analytic Landsberg metrics in compact surfaces of higher genus are Riemannian [**24**].

This problem has proved to be very hard (recently Szabó [**27**] claimed that C^4 Landsberg metrics were Berwald, but Matveev [**22**] found a gap in the proof), and R. Bryant found some non-smooth examples of what he calls generalized Finsler surfaces, which are Landsberg and not Riemannian (see [**2**]). Such examples are called unicorns by many specialists in the theory, because they are not true Finsler metrics. Nevertheless, these examples have become relevant in the theory as long as there are no available true Landsberg, non-Riemannian surfaces of higher genus. Our contribution to the subject is the following consequence of Theorem 2: if there exists a Landsberg compact surface of higher genus that is not Riemannian, then it must have conjugate points.

The main idea of the proof of Theorem 2 is to use the characterization of Landsberg metrics in terms of the Cartan structural equations. From the three generalized curvatures appearing in Cartan equations, the so-called Cartan tensor or I-curvature (see §2) has special properties. In the case of Landsberg metrics it is a first integral of the geodesic flow. Using Theorem 1, this yields Theorem 2, since the I-curvature must vanish when it is constant and this characterizes Riemannian metrics.

2. *Preliminaries*

2.1. *Finsler spaces, Landsberg metrics.* In this section we follow [**4**] as our main reference.

Let M be an n-dimensional, C^∞ manifold, let T_pM be the tangent space at $p \in M$, and let TM be its tangent bundle. In local coordinates, an element of T_xM can be expressed as a pair (x, y), where y is a vector tangent to x. Let $TM_0 = \{(x, y) \in TM;\ y \neq 0\}$ be the complement of the zero section. A C^k ($k \geq 2$) *Finsler structure* on M is a function $F : TM \to [0, +\infty)$ with the following properties:

(i) F is C^k on TM_0;

(ii) F is positively homogeneous of degree one in y, where $(x, y) \in TM$, that is,

$$F(x, \lambda y) = \lambda F(x, y) \quad \text{for all } \lambda > 0;$$

(iii) the Hessian matrix of $F^2 = F \cdot F$,

$$g_{ij} = \frac{1}{2} \frac{\partial^2}{\partial y^i \partial y^j} F^2,$$

is positive definite on TM_0.

A C^k *Finsler manifold* (or just a Finsler manifold) is a pair (M, F) consisting of a C^∞ manifold M and a C^k Finsler structure F on M.

Given a Lipschitz continuous curve $c : [a, b] \to M$ on a Finsler manifold (M, F), we define the Finsler length of c as

$$L_F(c) := \int_a^b F\left(c(t), \frac{dc}{dt}(t)\right) dt,$$

L_F gives rise to a function $d = d_F : M \times M \longrightarrow [0, \infty)$ by

$$d_F(p, q) := \inf_c L_F(c) = \inf_c \int_a^b F\left(c(t), \frac{dc}{dt}(t)\right) dt,$$

where the infimum is taken over all Lipschitz continuous curves $c : [0, 1] \to M$ with $c(0) = p$ and $c(1) = q$. It is clear that

$$d_F(p, q) \leq d_F(p, r) + d_F(r, q)$$

and

$$d_F(p, q) = 0 \Leftrightarrow p = q,$$

but since F is not absolutely homogeneous or reversible, $d_F(p, q)$ might not be equal to $d_F(q, p)$, and therefore d_F might not be a distance. If M is compact, there exists $C > 0$ such that $(1/C) d_F(p, q) \leq d_F(q, p) \leq C d_F(p, q)$ for every $p, q \in M$. There are many natural ways to get a true distance from d_F, all of them equivalent when the manifold M is compact due to the above assertion [**4**].

Therefore, throughout this paper we shall use d_F as a true distance without loss of generality. The non-reversibility of the Finsler metric does not pose relevant obstructions to obtaining Finsler versions of many well-known results about the global geometry of Riemannian geodesics.

The local theory of geodesics is just the local theory of existence and uniqueness of solutions of the Euler–Lagrange equation. We shall assume throughout this paper that geodesics have unit speed.

The Finsler manifold (M, F) naturally induces a Finsler structure in the universal covering $\tilde{M}$ of M, just by pulling back the Finsler structure F to the tangent space of $\tilde{M}$ by the covering map. Let us denote this Finsler manifold by $(\tilde{M}, \tilde{F})$.

A geodesic $\sigma : [a, b] \longrightarrow \tilde{M}$ is called *forward minimizing*, or simply minimizing, if $L_{\tilde{F}}(\sigma) \leq L_{\tilde{F}}(c)$ for all rectifiable curves $c : [a, b] \longrightarrow \tilde{M}$ such that $c(a) = \sigma(a)$, $c(b) = \sigma(b)$ (this implies that $\sigma : [s, t] \longrightarrow \tilde{M}$ is also minimizing for every $a \leq s \leq t \leq b$). Notice that, in general, a minimizing geodesic σ might fail to be minimizing if one reverses its orientation, because the Finsler metric might not be reversible. That is the meaning of the word 'forward' in the term 'forward minimizing'; the minimization property is attached to an orientation of the geodesic.

Let (M, F) be a positively complete Finsler manifold and let $x \in M$. For a non-vanishing vector $y \in T_xM$, we shall denote by $\sigma_{(x,y)}(t)$ the geodesic with initial conditions $\sigma_{(x,y)}(0) = x$ and $\sigma'_{(x,y)}(0) = y$. The *exponential map* at x, $\exp_x : T_xM \to M$, is defined as usual: $\exp_x(y) := \sigma_{(x,y)}(1)$.

2.2. *Chern–Rund connection (or Chern connection) and Jacobi fields.* Let T_x^*M be the cotangent space at x, and let T^*M be the cotangent bundle of M. Take local coordinates $(x_1, x_2, \ldots, x_n)$ for M, and let $b_i = \partial/\partial x^i$, dx_i, $i = 1, 2, \ldots, n$, be the corresponding basis for TM and T^*M, respectively. The so-called fundamental tensor of the Finsler metric is given by

$$g_{ij(x,y)}\, dx^i \otimes dx^j$$

where $g_{ij(x,y)} = (\frac{1}{2}F^2)_{ij}(x, y)$, that is, g_{ij} is the ij-entry of the Hessian of $\frac{1}{2}F^2$.

The fundamental tensor is very convenient to study Finsler Jacobi fields: vector fields defined along a geodesic obtained by differentiating C^2 variations of the geodesic. The second variation formula for the Finsler length gives the Jacobi equation of the metric, whose solutions are just the Jacobi fields. This equation is more complicated than the Riemannian Jacobi equation in general (see, for instance, [**5**] for a Hamiltonian expression of the Jacobi equation). Using the fundamental tensor, it is possible to define a sort of covariant differentiation along geodesics that is 'almost compatible' with the fundamental tensor (see [**4**] for details), such that the Jacobi equation of the geodesic acquires a Riemannian form. In the next lemma, suitable for the applications in the present paper, we summarize some basic properties of this connection (see [**4, 25**] for details).

LEMMA 2.1. *Let (M, F) be a C^4 Finsler manifold, let $\sigma(t)$ be a C^∞ curve, and*

$$\sigma(t, u) : \triangle = \{(t, u); 0 \leq t \leq r, -\varepsilon < u < \varepsilon\} \longrightarrow M$$

be a C^2 variation of $\sigma(t, 0) = \sigma(t)$ by C^∞ curves. Then, in the tangent space $T_{\sigma(t,u)}M$, the inner product

$$g_T := g_{ij(\sigma(t,u),T(t,u))}\, dx^i \otimes dx^j,$$

where

$$T = T(t, u) := \sigma_* \frac{\partial}{\partial t} = \frac{\partial \sigma}{\partial t},$$

satisfies the following properties.

(1) $g_T(T, T) = F^2(T)$.

(2) *$\sigma(t)$ is a Finslerian geodesic if and only if*

$$\frac{d}{dt} g_T(V, W) = g_T(D_T V, W) + g_T(V, D_T W)$$

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where V and W are two arbitrary vector fields along σ. The operator $D_T = d/dt$ is called covariant differentiation with reference vector T.

(3) *In particular, Finslerian geodesics satisfy*

$$D_T\left[\frac{T}{F(T)}\right] = 0.$$

The constant speed Finslerian geodesics $F(v) = c$ are the solutions of

$$D_T T = 0,$$

(4) *Assume that $\sigma(t)$ is a unit speed geodesic. Then a Jacobi field $J(t) = (\partial\sigma/\partial u)(t, 0)$ along $\sigma(t)$ satisfies*

$$D_T D_T J + R(J, T)T = 0,$$

where R is the Jacobi tensor of the Finsler metric (we shall denote as usual $J'' = D_T D_T J$, $J' = D_T J$). When $\dim(M) = 2$,

$$R(y, u)u = K(y)[g_y(y, y)u - g_y(y, u)y], \quad y, u \in T_x M \backslash \{0\}$$

where $K(y)$ is the Gaussian curvature, which coincides as well with the flag curvature.

(5) *Let $\sigma(t)$ be a unit speed geodesic. Then, if $J(t)$ is a Jacobi field along $\sigma(t)$, the component $J_\perp(t)$ of $J(t)$ that is perpendicular to $\sigma'(t)$ with respect to g_T satisfies the scalar Jacobi equation*

$$J_\perp'' + K J_\perp = 0.$$

Moreover, if $g_T(T, J(t_0)) = g_T(T, J'(t_0)) = 0$ at some point t_0, then $g_T(T, J) = 0$ at every point.

Throughout this paper, all covariant differentiations will be carried out with reference vector T. Lemma 2.1 reduces many Finsler problems concerning Jacobi fields to Riemannian ones. We shall often call the inner product g_T the *adapted Riemannian metric*.

Definition 2.1. A Finsler space (M, F) is said to be of Landsberg type if for every smooth curve $c : [a, b] \longrightarrow M$ the parallel transport along c with respect to the Chern–Rund connection is a linear isometry between $(T_{c(a)}M, g_{c'(a)})$ and $(T_{c(b)}M, g_{c'(b)})$.

2.3. *Conjugate points.* We say that q is *conjugate* to p along a geodesic σ if there exists a non-zero Jacobi field J along σ which vanishes at p and q. We say that (M, F) *has no conjugate points* if no geodesic has conjugate points. The following result taken from [**4**] (Proposition 7.1.1) has a similar, well-known counterpart in Riemannian geometry.

PROPOSITION 2.2. *Let $\sigma(t) = \exp_p(tv)$, $0 \le t \le r$, be a unit speed geodesic. Then the following statements are all equivalent.*

(1) *The point $q = \sigma(r)$ is not conjugate to $p = \sigma(0)$ along σ.*

(2) *Any Jacobi field defined along σ that vanishes at p and q must be identically zero.*

(3) *Given any $V \in T_p M$ and $W \in T_q M$, there exists a unique Jacobi field $J : [0, r] \longrightarrow TM$ defined along σ such that $J(0) = V$, $J(r) = W$.*

(4) *The derivative* $(\exp_p)_*$ *of the exponential map* $\exp_p$ *is non-sigular at* rv.
(5) *Each geodesic* $\gamma : \mathbb{R} \longrightarrow \tilde{M}$ *is minimizing.*

In $T_x M$, we define the *tangent spheres*

$$S_x(r) := \{y \in T_x M;\ F(x, y) = r\}$$

of radius r. For r small enough, $\exp_x[S_x(r)]$ is diffeomorphic to $S_x(r)$. The image set $\exp_x[S_x(r)]$ is called a *geodesic sphere* in M centred at x. A natural generalization of Gauss's lemma is available for Finsler spheres (see, for instance, [**4**]).

2.4. *Cartan's structural equations.* Here we recall briefly Cartan's structural equations for Finsler metrics because they give us a shortcut to define all Finsler curvature tensors (for details we refer to [**4**]). The tangent bundle of $T_1 M$ has a natural oriented frame of vectors e_1, e_2, e_3, where e_2 is the unit vector tangent to the geodesic flow and e_3 is tangent to the vertical bundle. The vectors e_1, e_2 are chosen in a way that they are orthonormal in each $T_p M$ with respect to the adapted Riemannian metric $g_T := g_{ij} dx^i \otimes dx^j$ already defined in the previous subsection. The partial derivatives of a function $f : T_1 M \longrightarrow \mathbb{R}$ with respect to the vectors fields e_i will be denoted by f_i. The structural equations of the Finsler metric are written in terms of a dual basis of the vectors e_i, a frame of 1-forms ω^i, $i = 1, 2, 3$, in the following way:

$$d\omega^1 = -I\omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^3, \tag{S1}$$

$$d\omega^2 = -\omega^1 \wedge \omega^3, \tag{S2}$$

$$d\omega^3 = K\omega^1 \wedge \omega^2 - J\omega^1 \wedge \omega^3. \tag{S3}$$

The scalar K is the *Gaussian curvature* of the Finsler surface, J is called the *Landsberg scalar* and I the Cartan scalar. The three scalars I, J, K are all functions on $T_1 M$.

It is possible to characterize Finsler metrics which are Riemannian in terms of the above functions. Indeed, I vanishes everywhere if and only if the Finsler structure is Riemannian. Moreover, a Finsler surface (M, F) is *Landsberg* if $J = 0$.

Is possible to show, using the Bianchi identities, that $J = 0$ if and only if I is constant along the orbits of the geodesic flow. The vectors e_i give rise to smooth local vector fields, also denoted by e_i.

3. *Divergence of Jacobi fields and of geodesic rays*

The purpose of this section is to show that geodesic rays in the universal covering of a Finsler compact surface without conjugate points diverge as in the Riemannian case. The divergence of geodesic rays for Riemannian surfaces is a well-known result due to Green [**14**]. We shall show that Green's ideas can be adapted to the Finsler case.

We recall Green's proof step by step. First of all, the absence of conjugate points in a geodesic allows the construction of asymptotic Jacobi fields. More precisely, let $\gamma(t)$ be a geodesic without conjugate points where the norm of the curvature operator is bounded from below by a constant K_0. Let $V \in T_{\gamma(0)} M$ be linearly independent of $\gamma'(0)$, and let $J_r(t)$ be the Jacobi field whose boundary conditions are

$$J_r(0) = V, \quad J_r(r) = 0.$$

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Since there are no conjugate points in γ, J_r exists and is unique. Moreover, the following lemma holds.

LEMMA 3.1. *There exists the limit*

$$\lim_{r \to +\infty} J_r(t) = J_V^+(t)$$

which is a Jacobi field that never vanishes if $V \neq 0$.

This lemma is a Finsler version [**11**] of a well-known result of E. Hopf.

Such asymptotic Jacobi fields will be called *Green Jacobi fields*. Their natural lifts to the tangent space of the unit tangent bundle will be referred to as *Green subspaces* (see §5). Those obtained as limits when $r \to +\infty$ are often called *stable Jacobi fields* (respectively, stable subspaces). Those obtained as limits when $r \to -\infty$ are usually called *unstable Jacobi fields*.

Asymptotic Jacobi fields give rise to solutions of the Riccati equation

$$u'(t) + u^2(t) + K_{\gamma'}(t) = 0$$

defined for every $t \in \mathbb{R}$ (see Foulon [**11**], for instance). So the theory of the Riccati equation holds for Finsler asymptotic Jacobi fields, and this is enough to show the divergence of Jacobi fields vanishing at just one point. Our next statement is not exactly as in Green's paper, but it is a straightforward consequence of Green's lemma about the divergence of Jacobi fields for Riemannian surfaces.

PROPOSITION 3.2. *Let* $L > 0$*, and let* $K : \mathbb{R} \longrightarrow \mathbb{R}$ *be a continuous function such that* $\| K \|_\infty \leq L$*. Suppose that the Jacobi equation*

$$J'' + KJ = 0 \tag{1}$$

has no conjugate points. Then the solutions $J(t)$ *with* $J(0) = 0$*,* $J'(0) \neq 0$ *diverge: given* $\varepsilon > 0$*,* $R > 0$*, there exists* $r = r(L, \varepsilon, R) > 0$ *such that for every solution of (1) with* $\|J'(0)\| \geq \varepsilon$*,*

$$\|J(t)\| \geq R \quad \textit{for all } t \geq r.$$

COROLLARY 3.3. *Let* (M, F) *be a compact Finsler space without conjugate points. Then Jacobi fields which vanish at just one point diverge in the sense of Proposition 3.2.*

Proof. Let γ be a geodesic of (M, F). As already observed in Lemma 2.1(5), the Jacobi equation along γ in the adapted Riemannian metric $g_{\gamma'(t)}$ takes the form $J''(t) + K_{\gamma'}(t)J(t) = 0$, where $K_{\gamma'}(t)$ is a function depending on the Gaussian curvature of (M, F) along γ. Then, the relevant Jacobi fields are those which are perpendicular (in the adapted metric) to $\gamma'(t)$. Since the surface M is compact, the curvature is bounded, and hence the functions $K_\gamma(t)$ are uniformly bounded in $\mathbb{R}$. So we can apply Proposition 3.2 and the corollary holds. □

Remark. In higher dimensions, Jacobi fields which vanish somewhere in compact Finsler metrics with no conjugate points diverge too [**5**]. However, the divergence might not be uniform as in Proposition 3.2. That is, the number r might depend on the geodesic.

LEMMA 3.4. *$S_t = \{\exp_p(tc(s));\ t \geq 0,\ s \in [0, a]\}$ has bounded geometry for $t \geq 2$, that is, the normal curvatures (with respect to the adapted metric) of large spheres are bounded by a uniform constant.*

Proof. The proof of the lemma reduces to the Riemannian proof. In fact, since M is compact the Gaussian curvature K is bounded. Therefore, the following claim implies the lemma.

CLAIM. *The second fundamental form of $S_r(x)$ at the point $\exp_x(rv)$ is the Riccati equation solution at $t = r$.*

For the proof of the claim, see [**25**, Lemma 14.4.2]. □

PROPOSITION 3.5. (Divergence of geodesic rays) *Let (M, F) be a C^4 closed Finsler surface without conjugate points, $(\tilde{M}, \tilde{F})$ the lift of F to the universal covering. Then geodesic rays diverge uniformly in $(\tilde{M}, \tilde{F})$. That is, given $\epsilon > 0$, $R > 0$, there exists $r > 0$ such that, for every pair of geodesic rays $\gamma_\theta : [0, +\infty) \longrightarrow \tilde{M}$, $\gamma_\eta : [0, +\infty) \longrightarrow \tilde{M}$, with $\gamma(0) = \beta(0)$, parametrized by arc length, and $\tilde{F}(\theta, \eta) \geq \epsilon$,*

$$\inf\{d_{\tilde{F}}(\gamma_\theta(t), \gamma_\eta(t)), d_{\tilde{F}}(\gamma_\eta(t), \gamma_\theta(t))\} \geq R$$

for every $t \geq r$.

Proof. The proof reduces to Green's argument [**14**]. Suppose by contradiction that we have $p \in \tilde{M}$, and two geodesics $\gamma_{(p,v)}$, $\gamma_{(p,w)}$, such that

$$d_{\tilde{F}}(\gamma_{(p,v)}(t), \gamma_{(p,w)}(t)) \leq C \quad \text{for all } t \geq 0$$

and

$$d_{\tilde{F}}(\gamma_{(p,w)}(t), \gamma_{(p,v)}(t)) \leq C \quad \text{for all } t \geq 0. \tag{2}$$

Let us assume that the pair (v, w) has the canonical orientation of $\tilde{M}$.

The set

$$\{\gamma_{(p,v)}(t);\ t \geq 0\} \cup \{\gamma_{(p,w)}(t);\ t \geq 0\}$$

bounds a geodesic cone

$$S = \{\exp_p(tc(s));\ t \geq 0,\ s \in [0, a]\} \in \tilde{M},$$

where $c : [0, a] \to T_p\tilde{M}$, of normal vectors $c(s)$ such that $c(0) = v$, $c(a) = w$, and $\|c'(s)\| = 1$ for every $s \in [0, a]$.

The set $S_t = \{\exp_p(tc(s));\ s \in [0, a]\}$ is an arc of the sphere $S_t(p)$. Assumption (2) and Lemma 3.4 imply that there exists a constant L_0 such that

$$\int_0^a \|S_t'(s)\|\, ds \leq L_0 \quad \text{for all } t \geq 0.$$

But

$$\int_0^a \|S_t'(s)\|\, ds = \int_0^a \left\| \frac{d}{ds} \exp_p(tc(s)) \right\| ds = \int_0^a \|J_s(t)\|\, ds,$$

https://doi.org/10.1017/S0143385711001027 Published online by Cambridge University Press

where $J_s(t) = (d/ds) \exp_p(tc(s))$ is a Jacobi field because it is a variation $f_t(s)$ by geodesics. This Jacobi field vanishes at $t = 0$, and $\|J_s'(0)\| = \|c'(s)\| = 1$. Therefore,

$$\int_0^a \|J_s(t)\| \, ds \le L_0 \quad \text{for all } t \ge 0.$$

Applying the mean value theorem for integrals, given $t > 0$, there exists $s_t \in [0, a]$ such that

$$\|J_{s_t}(t)\| \le a \cdot L_0.$$

This contradicts Proposition 3.2. □

4. *Global geometry of geodesics and central foliations*

The goal of this section is to show that the geodesic flow of every compact Finsler surface without conjugate points has central stable and central unstable foliations. We shall follow the ideas in [**16**] applied to study Riemannian surfaces; in particular, we shall show that each central foliation is C^0 conjugated to a hyperbolic central foliation.

4.1. *Morse's work and shadowing of geodesics by hyperbolic geodesics.*

In this subsection we recall briefly some applications of the celebrated work of Morse [**23**] about quasi-geodesics in surfaces to Finsler geodesics.

We say that a Lipschitz continuous curve $c : [a, b] \to \tilde{M}$ is a *forward* (A, B)*-quasi-geodesic* or just quasi-geodesic if

$$\frac{1}{A} \cdot d_{\tilde{F}}(c(s), c(t)) - B \le L_{\tilde{F}}(c|_{[s,t]}) \le A \cdot d_{\tilde{F}}(c(s), c(t)) + B \quad \text{for all } s \le t \in [a, b].$$

The word 'forward' appears here for the same reason it appeared in the notion of minimizing geodesics: Finsler metrics might not be reversible. Observe that a geodesic $\gamma : I \longrightarrow M$ is forward minimizing if it is a $(1, 0)$-forward quasi-geodesic. From the work of Morse [**23**] about minimizing geodesics in surfaces we have the following theorem.

THEOREM 4.1. (Morse) *Let (M, g) be a Riemannian metric in a compact surface M of genus greater than one. Let $(\tilde{M}, \tilde{g})$ be the pullback of a Riemannian metric g in M to the universal covering $\tilde{M}$. Given $A \ge 1$, $B \ge 0$, there exists $D > 0$ such that every forward (A, B)-quasi-geodesic $c : [t_1, t_2] \longrightarrow \tilde{M}$ of $(\tilde{M}, \tilde{g})$ is within a hyperbolic distance D from the hyperbolic geodesic joining $c(t_1)$ to $c(t_2)$ in the hyperbolic plane $(\tilde{M}, g_0)$.*

Let us define

$$\bar{d} : \tilde{M} \times \tilde{M} \to \mathbb{R}, \quad \bar{d}(p, q) = \tfrac{1}{2}[d_{\tilde{F}}(p, q) + d_{\tilde{F}}(q, p)],$$

which is a distance in $\tilde{M}$.

The compactness of M implies the following corollary.

COROLLARY 4.2. *Let (M, F) be a compact C^2 Finsler surface. There exists $\lambda \ge 1$ such that every forward minimizing geodesic $c : I \longrightarrow \tilde{M}$ is a $(\lambda, 0)$-quasi-geodesic of $(\tilde{M}, \bar{d})$.*

So in combination with Morse's theorem we get the following proposition.

PROPOSITION 4.3. *Given a C^2 Finsler metric (M, F) in a compact surface of genus greater than one, there exists $Q > 0$ such that every minimizing geodesic $c : [a, b] \longrightarrow \tilde{M}$ is in a Q-tubular neighbourhood of a hyperbolic geodesic.*

Notice that Proposition 4.3 applies to the geodesics $c_{p,q}$ (from p to q) and $c_{q,p}$ (from q to p) for every $p, q \in \tilde{M}$. In particular, both geodesics (if different from each other) are in the Q-tubular neighbourhood of the hyperbolic geodesic joining p to q.

4.2. *Central foliations for Finsler surfaces without conjugate points.* Let (M, F) be a complete C^2 Finsler surface with no conjugate points. Two Finsler geodesics $\gamma(t)$, $\beta(t)$ in $\tilde{M}$ are called *(forward) asymptotic* if there exists $C > 0$ such that

$$d_{\tilde{F}}(\gamma(t), \beta(t)) \leq C \quad \text{for all } t \geq 0$$

and

$$d_{\tilde{F}}(\beta(t), \gamma(t)) \leq C \quad \text{for all } t \geq 0.$$

Analogously, $\gamma(t)$, $\beta(t)$ are called *(backward) asymptotic* if there exists $C > 0$ such that

$$d_{\tilde{F}}(\gamma(t), \beta(t)) \leq C \quad \text{for all } t \leq 0$$

and

$$d_{\tilde{F}}(\beta(t), \gamma(t)) \leq C \quad \text{for all } t \leq 0.$$

Combining Proposition 4.3 and the divergence of geodesic rays (Proposition 3.5), we get the following proposition.

PROPOSITION 4.4. *Let (M, F) be a C^4 Finsler compact surface of genus greater than one, without conjugate points. There exists $Q > 0$ such that given $(p, v) \in T_1\tilde{M}$, $x \in \tilde{M}$, there exists a unique minimizing geodesic $\gamma_{(x,w(x,v))} : [0, \infty) \to \tilde{M}$ such that:*

(1) $\gamma_{(x,w(x,v))}(0) = x$;

(2) *if $\gamma^0_{(p,v)}$ is the hyperbolic geodesic in $\tilde{M}$ that satisfies $\gamma^0_{(p,v)}(0) = p$ and ${\gamma^0_{(p,v)}}'(0) = v$, then*

$$d_0(\gamma_{(x,w(x,v))}(t), \gamma^0_{(p,v)}) \leq Q$$

for every $t \geq 0$, where d_0 is the hyperbolic distance and $d_0(\gamma(t), \beta)$ is the hyperbolic distance from $\gamma(t)$ to the geodesic β;

(3) *the map $x \mapsto w(x, v)$ is continuous for every $(p, v) \in T_1\tilde{M}$;*

(4) *the geodesic $\gamma_{(x,w(x,v))}$ extends to a unique geodesic $\tilde{\gamma}_{(x,w(x,v))} : (-\infty, \infty) \to \tilde{M}$ (i.e., $\gamma_{(x,w(x,v))}(t) = \tilde{\gamma}_{(x,w(x,v))}(t)$ for every $t \geq 0$) that is forward minimizing.*

Proof. Take a sequence of minimizing geodesics $c_{x,t} : [0, r_p] \to \tilde{M}$, $t > 0$, for the Finsler metric, where $c_{x,t}(0) = x$ and $c_{x,t}(r_p) = \gamma^0_{p,v}(t)$. Since they are all in a Q-tubular neighbourhood of $\gamma^0_{(p,v)}$ they have a convergent subsequence. A limit $\gamma_{(x,w(x,v))}$ is a minimizing geodesic for F too, and it has to be the only geodesic starting at x and asymptotic to $\gamma^0_{(p,v)}$ by the divergence of geodesic rays of F. This immediately implies the continuity of $x \to w(x, v)$ for each $(p, v) \in T_1\tilde{M}$. Now, if we consider an exhausting sequence of compact balls $B_n(x)$ of hyperbolic radius $n \in \mathbb{N}$, the geodesics $\gamma_{(y,w(y,v))}$

https://doi.org/10.1017/S0143385711001027 Published online by Cambridge University Press

for $y \in B_m(x)$ must extend the geodesics $\gamma_{(y,w(y,v))}$ for $y \in B_n(x)$ and every $n < m$. This follows from the divergence of geodesic rays once more. Thus, item (4) in the statement holds and we conclude the proof of the proposition. □

Using Proposition 4.4 we can define a compactification $\tilde{M}(\infty)$ of $\tilde{M}$ as in the Riemannian case. Indeed, Morse's work allows us to define asymptotic classes $[\gamma]$ of geodesics γ in $\tilde{M}$, whose collection will be denoted by $\partial\tilde{M}(\infty)$. Then, the compactification

$$\tilde{M}(\infty) = \tilde{M} \cup \partial\tilde{M}(\infty)$$

is a topological space endowed with the Finsler version of the cone topology: given $p \in \tilde{M}$ and $T_p^1\tilde{M} = \{\tilde{F}(p, v) = 1\}$, the open sets of $\partial\tilde{M}(\infty)$ are generated by the asymptotic classes of geodesics whose initial conditions are contained in open subsets of $T_p^1\tilde{M}$. That is, $V \subset T_p^1\tilde{M}$ is open if and only if $[V] = \{[\gamma(p, v)], (p, v) \in V\}$ is open in $\partial\tilde{M}(\infty)$. By Proposition 4.4, given an asymptotic class $[\gamma]$, we have continuous dependence on $p \in \tilde{M}$ of the geodesic rays starting at p with the same asymptotic class. This yields that the topology we have just defined in $\partial\tilde{M}(\infty)$ does not depend on the point $p \in \tilde{M}$.

For each $\theta = (q, v) \in T_1\tilde{M}$, $\|v\| = 1$, if β is the geodesic through $p \in \tilde{M}$, say $\beta(0) = p$ that is asymptotic to γ_θ, let us denote the unit vector $\beta'(0)$ by $X_\theta^{cs}(p)$. The *centre stable set* of $\theta = (q, v) \in T_1\tilde{M}$ is defined by

$$\tilde{\mathscr{F}}^{cs}(\theta) = \{(p, X_\theta^{cs}(p));\ p \in \tilde{M}\}.$$

We shall denote by $\tilde{\mathscr{F}}^{cs}$ the collection of centre stable sets in the space $T_1\tilde{M}$.

The *centre stable set* of $\theta = (q, v) \in T_1M$ is defined by $\mathscr{F}^{cs}(\theta) = \hat{\pi}(\tilde{\mathscr{F}}^{cs}(\tilde{\theta}))$, where $\hat{\pi}(\tilde{\theta}) = \theta$.

We list in the next lemma some of the most important basic properties of centre stable sets. Since we are considering just forward minimizing geodesics, the proof of the lemma is completely analogous to [**16**, Proof of Lemma 2.1] and we leave the details to the reader.

LEMMA 4.5. *Let (M, F) be a C^4 closed, oriented C^4 Finsler surface without conjugate points. Then the following assertions hold.*

(1) *The family of sets $\tilde{\mathscr{F}}^{cs} = \bigcup_{\theta \in T_1\tilde{M}} \tilde{\mathscr{F}}^{cs}(\theta)$ is a collection of C^0 submanifolds which are either disjoint or coincident.*

(2) *The sets $\tilde{\mathscr{F}}^{cs}(\theta)$, $\theta \in T_1\tilde{M}$, depend continuously on θ, uniformly on compact subsets of $T_1\tilde{M}$, and hence the collection $\mathscr{F}^{cs} = \bigcup_{\theta \in T_1 M} \mathscr{F}^{cs}(\theta)$ is a continuous foliation by C^0 leaves of $T_1\tilde{M}$.*

(3) *Given $p \in \tilde{M}$, there exists a homeomorphism*

$$\Psi_p : \tilde{M} \times \tilde{V}_p \longrightarrow T_1\tilde{M}$$

such that

$$\Psi_p(\tilde{M} \times \{(p, v)\}) = \tilde{\mathscr{F}}^{cs}(p, v).$$

In particular, the collections $\mathscr{F}^{cs}$, $\tilde{\mathscr{F}}^{cs}$, are continuous foliations, and the space of leaves of $\tilde{\mathscr{F}}^{cs}(\theta)$ is homeomorphic to the vertical fibre $\tilde{V}_p$ for any $p \in \tilde{M}$.

(4) *There exists a homeomorphism*

$$\Psi_\infty : T_1\tilde{M} \longrightarrow \tilde{M} \times \partial\tilde{M}(\infty)$$

such that

$$\Psi_\infty(\tilde{\mathscr{F}}^{cs}_\omega) = \tilde{M} \times \{\omega\},$$

where $\tilde{\mathscr{F}}^{cs}_\omega$ *is the centre stable leaf such that all the orbits in the leaf project into geodesics of* $\tilde{M}$ *whose* ω*-limit is* $\omega \in \partial\tilde{M}(\infty)$.

LEMMA 4.6. [**23**] *Let* $\mathscr{F}$ *be the central leaf of a lift* $\tilde{\gamma}$ *of a closed geodesic* γ. *Then we have two possibilities:*

(1) *either* γ *is unique in its homotopy class and hence every orbit in* $\mathscr{F}^{cs}$ *is strongly asymptotic to the orbit* $(\gamma(t), \gamma'(t))$ *of* γ: $\lim_{t\to+\infty} d(\tilde{\gamma}(t), \tilde{\beta}(t)) = 0$ *(up to a reparametrization of* β*) for every geodesic* β *asymptotic to* γ*;*

(2) *or there exists an invariant annulus in* $\mathscr{F}$ *bounded by two orbits homotopic to* γ *with the same period.*

4.3. *A fibre preserving* C^0 *conjugacy between central foliations on* $K = -1$ *and central foliations.* We recall that if $K = -1$ then the Finsler metric is a Riemannian metric (a result due to Akbar-Zadeh [**1**]). The central foliations of a metric of constant negative curvature will be called *hyperbolic central foliations*.

The main result of this subsection is a generalization for Finsler surfaces of [**16**, Proposition 3.1].

PROPOSITION 4.7. *Let* $M = (M, F)$ *be a closed* C^4 *Finsler surface without conjugate points and genus greater than one. Then there exist a hyperbolic metric* g_0 *in* M *and a fibre preserving homeomorphism* $H : T_1M \longrightarrow T_1M_0$*, such that the image of each centre stable leaf of* T_1M *is a centre stable leaf of* T_1M_0.

We shall briefly recall the construction of the conjugacy since the argument in [**16**] carried over to the Finsler case with no major changes. Let us use the notation $\tilde{M}$ and $\tilde{M}_0$ to designate the universal covering of M and M_0 endowed with the metrics $\tilde{F}$ and $\tilde{g}_0$, respectively. We can assume that the universal covering of M is the unit disc D. We determine the constant curvature structure M_0 by lifting the fundamental group of M to a discrete group of isometries of the hyperbolic disc and then considering the hyperbolic metric in M obtained by the quotient of D by the action of this group of isometries. In this way, the fundamental groups of M and M_0 will coincide with just one discrete group acting in D, and the boundaries at infinity of $\tilde{M}$ and $\tilde{M}_0$ will be the same via the natural identification induced by the work of Morse.

According to the previous subsection, we can parametrize central leaves by their endpoints at infinity. So for ease of notation in this section we shall denote by $\mathscr{F}$, $\tilde{\mathscr{F}}$ the central foliations of the geodesic flows of (M, F) and its covering $(D, \tilde{M})$ respectively, and by $\mathscr{F}^0$, $\tilde{\mathscr{F}}^0$ the central foliations of the geodesic flows of (M, F_0) and its covering $(D, \tilde{F}_0)$ respectively.

The central vector fields in $\tilde{M}$, $\tilde{M}_0$ defined in the previous subsection will be denoted by $X_{\omega(\theta)}$, $X^0_{\omega(\sigma)}$ respectively, where $\theta \in T_1\tilde{M}$, $\sigma \in T_1\tilde{M}_0$, and $\omega(\theta)$, $\omega(\sigma) \in \partial D(\infty)$ are respectively the $\tilde{F}$-ω-limit of θ and the $\tilde{F}_0$-ω-limit of σ. We shall assign to each leaf $\tilde{\mathscr{F}}(\theta)$, for $\theta \in T_1\tilde{M}$, the notation

$$\tilde{\mathscr{F}}(\theta) = \tilde{\mathscr{F}}_{\omega(\theta)}.$$

As in [**16**], for the construction of the conjugacy between the central foliations we construct first a fibre preserving conjugacy between the lifts of these foliations in $T_1\tilde{M}$, $T_1\tilde{M}_0$. Let

$$\tilde{H} : T_1\tilde{M} \longrightarrow T_1\tilde{M}_0$$

be the map given by

$$\tilde{H} = (\Psi^0_\infty)^{-1} \circ \Psi_\infty.$$

This map is a fibred map, that is, $\tilde{H}(x, v) \in V^0_x$ for every $x \in D$, and observe that the $\tilde{F}$-ω-limit of (x, v) and the $\tilde{F}_0$-ω-limit of $\tilde{H}(x, v)$ are the same. Notice that the restriction of $\tilde{H}$ to the fibre $\tilde{V}_x$ is given by

$$\tilde{H}|_{\tilde{V}_x}(v) = (P^0_x)^{-1} \circ P_x(v),$$

where $P^0_x : \tilde{V}^0_x \longrightarrow \partial D(\infty)$ associates to $v \in \tilde{V}^0_x$ the $\tilde{F}_0$-ω-limit of (x, v). The following result is a summary of properties of the above construction, whose proofs in the Riemannian case can be found in [**16**]. The proofs in the Finsler case can be transposed from the Riemannian case without any changes.

LEMMA 4.8. *The map $\tilde{H}$ is a homeomorphism which preserves the lifted central foliations. Moreover, $\tilde{H}$ is equivariant by the action of the fundamental group of M in the spaces $T_1\tilde{M}$, $T_1\tilde{M}_0$.*

The existence of a continuous conjugacy between the central foliations of T_1M and T_1M_0 follows from the previous lemma, because the equivariance of $\tilde{H}$ by the action of the fundamental group allows us to push forward $\tilde{H}$ by the covering map $\tilde{\pi} : T_1\tilde{M} \longrightarrow T_1M$. In this way we get a homeomorphism from T_1M to T_1M_0 preserving the central foliation of the geodesic flow. For details we refer the reader to [**16**].

5. *First integrals of the Finsler geodesic flow*

From now on we assume that the Finsler surface (M, F) is closed and C^4. In this section we show Theorem 1.

THEOREM 1. *Let (M, F) be a C^4 Finsler metric in a compact surface of genus greater than one, without conjugate points. Then every first integral of the Finsler geodesic flow is constant.*

Our approach is dynamical and inspired by a paper by Paternain [**24**]: we shall show that every first integral of the geodesic flow is constant. This is done by Paternain for analytic Landsberg metrics in compact surfaces of genus greater than one, but we show that the same holds in our setting. For the proof of Theorem 1 we shall need to introduce the so-called Green bundles. We devote the next subsection to the subject for the sake of completeness.

5.1. *Green bundles.* A remarkable property of geodesics without conjugate points is the existence of the so-called Green bundles, which were defined by Hopf and Green for Riemannian metrics in terms of the solutions of the Riccati equation (§3), and by Foulon [**11**] for Finsler metrics in surfaces (see, for instance, [**5**] for a Hamiltonian definition).

PROPOSITION 5.1. (Green bundles) *Let (M, F) be a Finsler surface without conjugate points. Then there exist, for each $\theta \in T_1 M$, two invariant subspaces $E^s(\theta)$ and $E^u(\theta)$ of $T T_1 M$, defined by*

$$E^s(\theta) = \lim_{\tau \to +\infty} D\varphi_{-t}(V_{\varphi_t(\theta)}),$$
$$E^u(\theta) = \lim_{\tau \to -\infty} D\varphi_{-t}(V_{\varphi_t(\theta)}),$$

where $D\varphi_t$ is the differential of the geodesic flow, and V_θ is the vertical subspace at θ. The distributions $E^s(\theta)$ and $E^u(\theta)$ are measurable, transverse to the vertical subbundle V, and transverse to the geodesic vector field.

Green bundles are always transversal to the vertical subspace in $T T_1 M$ and have dimension $n - 1$, where $n = \dim(M)$. They are invariant Lagrangian subbundles defined in the set of globally (forward) minimizing orbits (see [**5**], for instance). Moreover, the Green subbundles $E^s(\theta)$, $E^u(\theta)$ are given as graphs of linear operators given by two distinguished solutions u^s_θ, u^u_θ of the Riccati equation (§3), as in the Riemannian case (see [**11**]). Such solutions are defined for every $t \in \mathbb{R}$ along the geodesic γ_θ.

The next lemma establishes a link between the centre stable sets constructed in the previous section and the dynamical centre stable set of a hyperbolic closed orbit.

LEMMA 5.2. *Let (M, F) be a C^2 compact Finsler manifold. Let $\phi_t(\theta)$ be a closed hyperbolic orbit of the geodesic flow of (M, F) without conjugate points (that is, the subjacent geodesic γ_θ has no conjugate points). Then:*

(1) *the Green subspaces $E^s(\theta)$, $E^u(\theta)$ are respectively the dynamical stable and unstable invariant subspaces of θ;*
(2) *the dynamical centre stable and centre unstable submanifolds of θ are local Lagrangian graphs of the canonical projection;*
(3) *if (M, F) is a compact surface without conjugate points, the centre stable set of θ coincides with the dynamical centre stable set of θ.*

The proof of item (1) is not difficult and can be found in [**11**] or [**5**], for instance. Item (2) follows from the fact that invariant submanifolds of θ are tangent along the orbit of θ to a subbundle that is transversal to the vertical subspace, so by continuity this holds in a neighbourhood of the orbit of θ. Therefore, the invariant submanifolds of the orbit of θ are transversal to the vertical fibres of $T_1 M$ as well, and hence they are local graphs of the canonical projection. The proof of item (3) follows from Proposition 3.5 and the existence of centre stable sets. Indeed, the hyperbolicity of the orbit of θ implies that the orbits in the dynamical centre stable set of θ correspond to geodesics which are strongly asymptotic to γ_θ. Then such geodesics are in the centre stable set of θ, $\mathscr{F}(\theta)$, as we claimed.

5.2. *A first integral is constant on the centre stable leaf of a hyperbolic orbit.* The purpose of this subsection is to show a preliminary version of Theorem 1, namely, the following proposition.

PROPOSITION 5.3. *Let $f : TM \longrightarrow \mathbb{R}$ be a first integral of the geodesic flow of a C^4 Finsler compact surface without conjugate points and genus greater than 1. Let θ be a hyperbolic periodic point of the geodesic flow. Then f is constant in $\mathscr{F}^{cs}_{\theta}$.*

We shall subdivide the proof into several lemmas. The main issue of the proof is the existence or not of an annulus in $\mathscr{F}^{cs}_{\theta}$ bounded by two different bi-asymptotic geodesics.

LEMMA 5.4. *Suppose that (M, F) is a Finsler surface without conjugate points. Let $\theta \in T_1M$ be a periodic point in a hyperbolic closed orbit. Then there is no annulus in M bounded by bi-asymptotic orbits in the central stable leaf $\mathscr{F}^{cs}_{\theta}$ of θ. That is, every pair of orbits in $\mathscr{F}^{cs}(\theta)$ is strongly forward asymptotic.*

Proof. In the case of Riemannian surfaces, the lemma follows from the fact that an annulus of bi-asymptotic orbits containing a closed orbit is foliated by closed orbits of the same period in the same homotopy class. This is proved by Morse [**23**], and the argument extends to reversible Finsler metrics. If the metric is not reversible, there are some technical problems in showing this result, posed by the 'orientation' of the minimizing properties of geodesics. Although we think this should be true in the non-reversible Finsler case as well, we prefer to give a more direct proof of the lemma, using the hyperbolicity of the closed orbit, instead of involving ourselves with a generalization of Morse's work for Finsler surfaces.

So let γ_θ be a closed hyperbolic geodesic that is forward minimizing, let $\tilde{\theta}$ be a lift of θ in $T_1\tilde{M}$, and let us suppose that there exists a closed, forward minimizing geodesic β in M such that the orbit of β is in the centre stable manifold of θ. So β is homotopic to γ_θ and the no conjugate points condition implies that they have the same period (see Morse [**23**]). Let $\tilde{\beta}$ be a lift of β that is bi-asymptotic to $\gamma_{\tilde{\theta}}$.

Let $\tilde{A}$ be the strip in $\tilde{M}$ bounded by $\gamma_{\tilde{\theta}}$ and $\tilde{\beta}$. The dynamical centre stable and centre unstable manifolds of the hyperbolic orbit $O(\theta)$ of θ induce dynamical centre stable and centre unstable manifolds for the orbit of $\tilde{\theta}$. Since the surface has no conjugate points, the strongly asymptotic orbits in the centre stable manifold of $\tilde{\theta}$ project, by the canonical projection, onto strongly asymptotic geodesics of $\gamma_{\tilde{\theta}}$ (Lemma 5.2). By the results in the previous section, the strip $\tilde{A}$ contains a strip of strongly asymptotic geodesics of $\gamma_{\tilde{\theta}}$ coming from the canonical projections of the orbits in the centre stable manifold of $\tilde{\theta}$. In fact, by Proposition 3.5, the minimizing geodesics in $\tilde{M}$ which are strongly (forward) asymptotic to $\gamma_{\tilde{\theta}}$ cannot meet the geodesic $\tilde{\beta}$. If the metric were reversible, we could claim the same above statement for the projections in $\tilde{A}$ of the geodesics in the centre unstable manifold of $\tilde{\theta}$ (because such geodesics would be backward minimizing too, and hence they cannot meet $\tilde{\eta}(t) = \tilde{\beta}(-t)$ which would be backward minimizing as well). So we have to work a little more to deal with the projections of centre unstable orbits of $\tilde{\theta}$ in $\tilde{A}$.

By Lemma 5.2, the dynamical centre unstable set of the orbit of θ is a smooth submanifold that is a local Lagrangian graph of the canonical projection. That is, the canonical projection of the centre unstable manifold of θ gives rise to a local flow whose

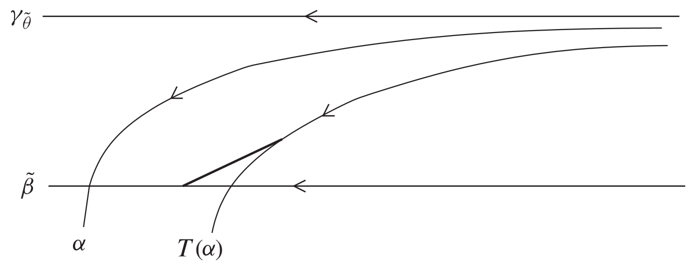

FIGURE 1. Strip of geodesics, shortcut argument, and Morse's lemma.

orbits are backward asymptotic geodesics of γ_θ. This implies that in $\tilde M$, the canonical projection of the centre unstable manifold of the orbit of $\gamma_{\tilde\theta}$ is a flow by backward asymptotic, forward minimizing geodesics. Since this flow projects into an open subset $W \subset \tilde M$ containing $\gamma_{\tilde\theta}$ in its interior, $\tilde A$ contains an open subset $W' \subset W \cap \tilde A$ foliated by unstable geodesics: there exists a collection Γ of (forward minimizing) geodesics which are backward asymptotic to $\gamma_{\tilde\theta}$ such that

$$W' = \bigcup_{\alpha\in\Gamma} \alpha(-\infty, t(\alpha)),$$

where $t(\alpha)$ is the supremum of $t \in \mathbb{R}$ such that $\alpha(t) \in \tilde A$ for every $t \le t(\alpha)$.

CLAIM. *If $\alpha \in \Gamma$, then $\alpha(-\infty, \infty)$ is contained in the strip $\tilde A$. In other words, $t(\alpha) = \infty$ if $\alpha \in \Gamma$.*

The claim is a consequence of the next result, whose proof essentially follows from the work of Morse [**23**].

SUBLEMMA. *If $\alpha : \mathbb{R} \longrightarrow \tilde M$ is a minimizing geodesic such that:*
(1) $\alpha(-\infty, 0]$ *is contained in the substrip of $\tilde A$ bounded by $\gamma_{\tilde\theta}$ and $\tilde\beta$; and*
(2) $\lim_{t\to-\infty} d_{\tilde F}(\gamma_{\tilde\theta}(t), \alpha(t)) = 0$,
then $\alpha(t) \in \tilde A$ for every $t \in \mathbb{R}$.

By contradiction, assume that α is not totally contained in the strip bounded by $\gamma_{\tilde\theta}$ and $\tilde\beta$. Assume that $\alpha(0) = \tilde\beta(0) \in \tilde\beta$ and that $\alpha(t)$ is contained in the strip bounded by $\gamma_{\tilde\theta}$ and $\tilde\beta$ for every $t < 0$. Take any Riemannian metric in the surface M and lift it to $\tilde M$. If the angle between $\alpha'(0)$ and $\tilde\beta'(0)$ is not zero in this Riemannian metric, then a shortcut argument (see Figure 1) shows that $\alpha(-\infty, 0]$ is not minimizing for the Finsler metric. This contradicts the absence of conjugate points in the Finsler surface, so α must be contained in the strip $\tilde A$ as we claimed.

In Figure 1, T is a deck transformation preserving $\gamma_{\tilde\theta}$ and $\tilde\beta$. Notice that in the figure we represent a transversal intersection of the geodesics α and $\tilde\beta$, but since the Finsler metric might not be reversible, they might have a non-transversal intersection: $\alpha'(0)$ might be tangent to $-\tilde\beta'(0)$. Nevertheless, a shortcut argument proceeds as well as in the case of transversal intersection, as shown in [**7**, Proof of Lemma 2.1]. The sublemma implies that the projections of the orbits in the dynamical centre unstable manifold of $\tilde\theta$ which meet the strip $\tilde{A}$ are bi-asymptotic to $\gamma_{\tilde{\theta}}$. And by the divergence of geodesic rays such orbits must be subsets of the centre stable leaf of $\tilde{\theta}$. We thus obtain a strip $\hat{A} \subset \tilde{A}$ bounded by $\gamma_{\tilde{\theta}}$ containing an open subset where the strongly forward asymptotic geodesics of $\gamma_{\tilde{\theta}}$ are backward strongly asymptotic to $\gamma_{\tilde{\theta}}$, proving the claim.

To finish the proof of the lemma, observe that the claim contradicts the hyperbolicity of θ unless the strip $\tilde{A}$ reduces to $\gamma_{\tilde{\theta}}$. Indeed, the transversality of dynamical stable and unstable manifolds of hyperbolic orbits implies that the canonical projections of forward asymptotic orbits of θ are always transversal to the projections of backward asymptotic orbits of θ. □

Proof of Proposition 5.3. By Lemma 5.4 every pair of orbits in the centre stable leaf of a hyperbolic closed orbit is strongly asymptotic, so the distance between any two such orbits tends to zero as time goes to $+\infty$. Let θ be a periodic point, and $\eta \in \mathscr{F}^{cs}_{\theta}$. If f is a continuous first integral of the geodesic flow, its value in the orbit of η is a constant $c(\eta)$, which, by continuity, approaches the constant $c(\theta) = f(\theta)$ attained at the orbit of θ. Hence, $c(\eta) = c(\theta)$ for every $\eta \in \mathscr{F}^{cs}_{\theta}$, thus proving the proposition. □

5.3. *Proof of Theorem 1.* The proof of Theorem 1 is a consequence of the results of §4, Proposition 5.3, and the following well-known result due to Katok [**20**].

KATOK'S THEOREM. *A $C^{1+\alpha}$, $\alpha > 0$, flow on a three-dimensional manifold, with positive topological entropy, has a hyperbolic closed orbit with a transverse homoclinic point.*

Applied to geodesic flows of compact surfaces of genus greater than one, Katok's theorem gives us the existence of a hyperbolic closed orbit. Indeed, the exponential growth of the fundamental group of M implies that the topological entropy of the geodesic flow is positive (this was shown by Dinaburg [**8**] for Riemannian metrics but the proof extends to Finsler surfaces as observed, for instance, in [**24**]). By Katok's theorem, there is always a periodic hyperbolic orbit, and the conjugacy of the central foliation with a hyperbolic central one (Proposition 4.7) yields that each central leaf is dense in T_1M. By Proposition 5.3, every first integral is constant in T_1M.

6. *Landsberg metrics without conjugate points are Riemannian*

In this section we show Theorem 2.

THEOREM 2. *Let (M, F) be a C^4 Finsler metric in a compact surface of genus greater than one, without conjugate points. If (M, F) is a Landsberg metric then (M, F) is Riemannian.*

Let I be the function defined in §2.4. We recall that, (M, F) is of Landsberg type if and only if $I_2 = 0$. In the notation of §2.4, I_2 is the derivative of I with respect to the geodesic vector field, so $I_2 = 0$ if and only if I is a first integral of the geodesic flow. Therefore, using Theorem 1 we get the following corollary.

COROLLARY 6.1. *Let (M, F) be a C^4 Finsler metric in a compact surface of genus greater than one, without conjugate points. Then $I_2 = 0$ if and only if the invariant I is constant in T_1M.*

Therefore, assuming that (M, F) is of Landsberg type, we conclude that I is constant. The following lemma implies that this constant is zero (see [**4**]).

LEMMA 6.2. *Let (M, F) be a Landsberg surface. Let us suppose that I is constant in T_1M. Then, this constant is equal to zero.*

Since $I = 0$ if and only if (M, F) is a Riemannian surface, we get Theorem 2.

7. *Further applications: around the works of Foulon and Ikeda*

In this section we give some further applications of our results on C^4 Finsler metrics without conjugate points, motivated by some papers by Foulon [**12**] and Ikeda [**19**]. In these papers, the condition $K_2 = 0$, that is, K is constant in the direction of the geodesic flow, is considered.

First of all, let us recall from Foulon [**12**] (see also [**3**]) that $K_2 = 0$ if the Finsler structure satisfies the following three conditions:

(1) reversibility;
(2) C^3 differentiability away from the zero section of TM;
(3) it is *locally symmetric*, that is, geodesic reflection at any point is a local isometry.

PROPOSITION 7.1. *Let (M, F) be a C^4 compact Finsler surface of genus greater than one, without conjugate points. If $K_2 = 0$ then K is constant.*

Proof. By our assumptions, K is a continuous first integral of the geodesic flow, so, from Theorem 1, K is a constant. □

Next, let us recall the following theorem due to Ikeda [**19**].

THEOREM 7.2. *Let (M, F) be a connected Landsberg surface. Suppose that*

$$K_2 = 0.$$

Then:

(i) *K is constant;*
(ii) *(M, F) must be Riemannian whenever that constant is non-zero.*

Our results lead to the following lemma.

LEMMA 7.3. *Let (M, F) be a compact Finsler surface of genus greater than one, without conjugate points. If $K_2 = 0$, $J_2 = 0$, then K is constant and F must be Riemannian whenever that constant is non-zero.*

Proof. As $J_2 = 0$ the Bianchi identity (obtained from $(S3)$)

$$K_3 + KI + J_2 = 0,$$

where K_3 is the derivative with respect to the vertical direction, becomes

$$K_3 + KI = 0.$$

Since $K_2 = 0$, the previous proposition implies that K is constant, so

$$KI = 0.$$

Therefore, if the constant curvature K is non-zero, we conclude that $I = 0$ and (M, F) must be Riemannian. □

Acknowledgements. The second author was partially supported by CNPq, Pronex de Geometria (Brazil), and the École Normale Supérieure de Paris.

REFERENCES

[1] H. Akbar-Zadeh. Sur les espaces de Finsler à courbures sectionnelles constantes. *Acad. Roy. Belg. Bull. C1. Sci.* **74**(5) (1988), 281–322.

[2] D. Bao. On two curvature-driven problems in Riemann–Finsler geometry. *Finsler Geometry, Sapporo 2005. Part II: Hunting Unicorns and Other Topics of Global Finsler Geometry (Advanced Studies in Pure Mathematics, 48).* Eds. S. V. Sabau and H. Shimada. Mathematical Society of Japan, Tokyo, 2007, pp. 19–72.

[3] D. Bao, S. S. Chern and Z. Shen. Rigidity issues on Finsler surfaces. *Rev. Roumaine Math. Pures Appl.* **42** (1997), 707–735.

[4] D. Bao, S.-S Chern and Z. Shen. *An Introduction to Riemann–Finsler Geometry*. Springer, New York, 2000.

[5] G. Contreras and R. Iturriaga. Convex Hamiltonians without conjugate points. *Ergod. Th. & Dynam. Sys.* **19** (1999), 901–952.

[6] G. Contreras, R. Iturriaga, G. P. Paternain and M. Paternain. Lagrangian graphs, minimizing measures and Mañé's critical values. *Geom. Funct. Anal.* **8** (1998), 788–809.

[7] M. J. Dias Carneiro and R. Ruggiero. On Birkhoff theorems for Lagrangian invariant tori with closed orbits. *Manuscripta Math.* **119** (2006), 411–432.

[8] E. I. Dinaburg. On the relations among various entropy characteristics of dynamical systems. *Math. USSR Izv.* **5** (1971), 337–378.

[9] P. Eberlein. Geodesic flow in certain manifolds without conjugate points. *Trans. Amer. Math, Soc.* **167** (1972), 151–170.

[10] D. Egloff. On the dynamics of uniform Finsler manifolds of negative flag curvature. *Ann. Global. Anal. Geom.* **15** (1997), 101–116.

[11] P. Foulon. Estimation de l'entropie des systèmes lagrangiens sans points conjugués. *Ann. Inst. Henri Poincaré Phys. Théor.* **57** (1992), 117–146.

[12] P. Foulon. Locally symmetric Finsler spaces in negative curvature. *C. R. Acad. Sci. Paris, Ser. I* **324** (1997), 1127–1132.

[13] E. Ghys. Flots d'Anosov dont les feuilletages stables sont différentiables. *Ann. Sci. Éc. Norm. Supér.* **20** (1987), 251–270.

[14] L. W. Green. Surfaces without conjugate points. *Trans. Amer. Math. Soc.* **76** (1954), 529–546.

[15] J. B. Gomes and R. O. Ruggiero. Rigidity of surfaces whose geodesic flows preserve smooth foliations of codimension 1. *Proc. Amer. Math. Soc.* **135** (2007), 507–515.

[16] J. B. Gomes and R. O. Ruggiero. Uniqueness of central foliations of geodesic flows for compact surfaces without conjugate points. *Nonlinearity* **20** (2007), 497–515.

[17] J. B. Gomes and R. O. Ruggiero. Rigidity of magnetic flows for compact surfaces. *Comptes Rendus Acad. Sci. Paris, Ser. I* **346** (2008), 313–316.

[18] S. Hurder and A. Katok. Differentiability, rigidity and Godbillon-Vey classes for Anosov flows. *Publ. Math. Inst. Hautes Etudes Sci.* **72** (1990), 5–61.

[19] F. Ikeda. On two-dimensional Landsberg spaces. *Tensor (N.S.)* **33** (1979), 43–48.

[20] A. Katok. Lyapunov exponents, entropy and periodic orbits for diffeomorphisms. *Publ. Math. Inst. Hautes Études Sci.* **51** (1980), 137–173.

[21] R. Mañé. On a theorem of Klingenberg. *Dynamical Systems and Bifurcation Theory*. Eds. M. I. Camacho, M. J. Pacífico and F. Takens. Longman Scientific & Technical, New York, 1987, pp. 319–345.

[22] V. S. Matveev. On 'All regular Landsberg metrics are always Berwald' by Z. I. Szabó. *Balkan J. Geom. Appl.* **14**(2) (2009), 50–52.

[23] H. M. Morse. A fundamental class of geodesics on any closed surface of genus greater than one. *Trans. Amer. Math. Soc.* **26** (1924), 25–60.

[24] G. P. Paternain. Finsler structures on surfaces with negative Euler characteristic. *Houston J. Math.* **23** (1997), 421–426.

[25] Z. Shen. *Lectures on Finsler Geometry*. World Scientific Publishing, Singapore, 2001.

[26] Z. I. Szabó. Positive definite Berwald spaces (Structure theorems on Berwald spaces). *Tensor (N.S.)* **35** (1981), 25–39.

[27] Z. I. Szabó. All regular Landsberg metrics are Berwald. *Ann. Glob. Anal. Geom.* **34** (2008), 381–386.