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Subexponential solutions of linear integro-differential equations and transient renewal equations

John A. D. Appleby and **David W. Reynolds**
School of Mathematical Sciences, Dublin City University, Dublin 9, Ireland (john.appleby@dcu.ie; david.reynolds@dcu.ie)

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This paper studies the asymptotic behaviour of the solutions of the scalar integro-differential equation

$$x'(t) = -ax(t) + \int_0^t k(t-s)x(s)\,\mathrm{d}s.$$

The kernel k is assumed to be positive, continuous and integrable. If

$$a > \int_0^\infty k(s)\,\mathrm{d}s,$$

it is known that all solutions x are integrable and $x(t) \to 0$ as $t \to \infty$, but also that $x = 0$ cannot be exponentially asymptotically stable unless there is some $\gamma > 0$ such that

$$\int_0^\infty k(s)\mathrm{e}^{\gamma s}\,\mathrm{d}s < \infty.$$

Here, we restrict the kernel to be in a class of subexponential functions in which $k(t) \to 0$ as $t \to \infty$ so slowly that the above condition is violated. It is proved here that the rate of convergence of $x(t) \to 0$ as $t \to \infty$ is given by

$$\lim_{t\to\infty} \frac{x(t)}{k(t)} = \frac{x(0)}{(a - \int_0^\infty k(s)\,\mathrm{d}s)^2}.$$

The result is proved by determining the asymptotic behaviour of the solution of the transient renewal equation

$$r(t) = h(t) + \int_0^t h(t-s)r(s)\,\mathrm{d}s, \qquad \int_0^\infty h(s)\,\mathrm{d}s < 1.$$

If the kernel h is subexponential, then

$$\lim_{t\to\infty} \frac{r(t)}{h(t)} = \frac{1}{(1 - \int_0^\infty h(s)\,\mathrm{d}s)^2}.$$

1. Introduction

There is a considerable literature (cf. [4–7,11,14]) devoted to the study of the asymptotic behaviour of solutions to the scalar convolution integro-differential equation

$$x'(t) = -ax(t) + \int_0^t k(t-s)x(s)\,\mathrm{d}s. \tag{1.1}$$

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We suppose that $a > 0$ and that k is continuous, integrable and positive on $[0, \infty)$, in which case x is continuously differentiable and does not change sign. If

$$a > \int_0^\infty k(s)\,\mathrm{d}s, \tag{1.2}$$

it is proved in [7] that the following all hold.

(i) Every solution $x(t)$ of (1.1) converges to 0 as $t \to \infty$.

(ii) Every solution x of (1.1) is in $L^1[0, \infty)$.

(iii) The zero solution of (1.1) is asymptotically stable.

(iv) The zero solution of (1.1) is uniformly asymptotically stable.

Brauer [4] established that 0 is uniformly asymptotically stable if

$$a > \int_0^\infty k(s)\,\mathrm{d}s.$$

It is shown in [13] that if

$$a < \int_0^\infty k(s)\,\mathrm{d}s,$$

then $x(t) \to \infty$ exponentially fast as $t \to \infty$. The case

$$a = \int_0^\infty k(s)\,\mathrm{d}s$$

is more subtle: if $s \mapsto sk(s)$ is integrable, then the zero solution is uniformly stable, but not asymptotically stable; if $s \mapsto sk(s)$ is not integrable, the zero solution is asymptotically stable, but not uniformly asymptotically stable. Corduneanu and Lakshmikantham [11] posed the interesting question of whether uniform asymptotic stability of the zero solution of (1.1) always implies its exponential asymptotic stability. This was answered by Murakami [14], who showed that a necessary and sufficient condition for the exponential stability of the zero solution of (1.1) is that there be a number $\gamma > 0$ such that

$$\int_0^\infty k(s)\mathrm{e}^{\gamma s}\,\mathrm{d}s < \infty. \tag{1.3}$$

The question arises of how quickly $x(t) \to 0$ as $t \to \infty$ if the kernel violates (1.3).

In this paper, a class of *subexponential functions* is introduced. The definition of this class is closely related to the hypothesis of a theorem in [9]. The idea of subexponential functions is a development of the definition of *subexponential distributions* introduced by Chistyakov [8]. It turns out that subexponential kernels satisfy $k(t)\mathrm{e}^{\gamma t} \to \infty$ as $t \to \infty$ for every $\gamma > 0$. Hence (1.3) cannot hold, and solutions of (1.1) cannot decay to zero exponentially.

Our method is to represent the solutions of (1.1) in terms of the solution r of

$$r(t) = h(t) + \int_0^t h(t-s)r(s)\,\mathrm{d}s, \tag{1.4}$$

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where

$$\int_0^\infty h(s)\,\mathrm{d}s < 1.$$

If (1.4) is integrated, we obtain a transient renewal equation. In [2] it is proved that if

$$H(t) = \frac{\int_0^t h(s)\,\mathrm{d}s}{\int_0^\infty h(s)\,\mathrm{d}s}$$

is a subexponential distribution, then

$$\lim_{t\to\infty} \frac{\int_t^\infty r(s)\,\mathrm{d}s}{\int_t^\infty h(s)\,\mathrm{d}s} = \frac{1}{(1-\int_0^\infty h(s)\,\mathrm{d}s)^2}. \tag{1.5}$$

It is proved here in theorem 5.2 that if h is a subexponential function, then the rate of decay of $r(t)$ to 0 as $t \to \infty$ is given by

$$\lim_{t\to\infty} \frac{r(t)}{h(t)} = \frac{1}{(1-\int_0^\infty h(s)\,\mathrm{d}s)^2}.$$

The idea of proving the existence of solutions of an integral equation in a space of functions weighted by subexponential functions was used by Chover *et al*. [10].

Theorem 6.2 states that the rate of convergence $x(t) \to 0$ as $t \to \infty$ for solutions of (1.1) is given by

$$\lim_{t\to\infty} \frac{x(t)}{k(t)} = \frac{x(0)}{(a-\int_0^\infty k(s)\,\mathrm{d}s)^2}, \qquad \lim_{t\to\infty} \frac{x'(t)}{x(t)} = 0 \tag{1.6}$$

if k is subexponential. Under the weaker hypothesis that

$$K(t) = \frac{\int_0^t k(s)\,\mathrm{d}s}{\int_0^\infty k(s)\,\mathrm{d}s}$$

be a subexponential distribution function, we find in theorem 6.5 that

$$\lim_{t\to\infty} \frac{\int_t^\infty x(s)\,\mathrm{d}s}{\int_t^\infty k(s)\,\mathrm{d}s} = \frac{x(0)}{(a-\int_0^\infty k(s)\,\mathrm{d}s)^2}, \qquad \lim_{t\to\infty} \frac{x(t)}{\int_t^\infty x(s)\,\mathrm{d}s} = 0. \tag{1.7}$$

This complements the theorem of Burton [6] that if $k(t) \geqslant 0$, k is integrable and x is a non-vanishing integrable solution of (6.1), there is a constant $\beta > 0$ such that

$$\frac{\int_t^\infty x(s)\,\mathrm{d}s}{\int_t^\infty k(s)\,\mathrm{d}s} \geqslant \beta$$

for all $t \geqslant 1$. Wong and Wong [18] is an interesting paper that considers a different class of linear convolution equations and determines precisely the regularly varying asymptotic behaviour of the solution at infinity for a regularly varying kernel. This is another example of the asymptotic behaviour solution of the kernel determining that of the solution.

Teugels [16] proves a converse to the result (1.5); namely, if (1.5) holds, then H must be a subexponential distribution. We establish converse results for both

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theorems 6.2 and 6.5. If (1.6) holds, k must be a subexponential function, and if (1.7) holds, K must be a subexponential distribution.

Our results on the transient renewal equation (1.4) are given in § 5 and those on the integrodifferential equation (1.1) in § 6. The proofs are given therein. For ease of reading, the demonstrations of preparatory and technical lemmata are collected together in § 7.

2. Some notation and subexponential distributions

Firstly, we introduce some notation. $\mathbb{R}^+$ is the set $[0, \infty)$. If F is integrable and G has bounded variation, we put

$$(F \star G)(t) = \int_0^t F(t-s)\,\mathrm{d}G(s), \quad t \geqslant 0.$$

We set $G^{\star 1} = G$ and $G^{\star(n+1)} = G^{\star n} \star G$. Similarly for integrable functions f and g on $\mathbb{R}^+$, the convolution of f with g is defined to be

$$(f * g)(t) = \int_0^t f(t-s)g(s)\,\mathrm{d}s, \quad t \geqslant 0.$$

The n-fold convolution f^{*n} is given by $f^{*1} = f$ and $f^{*(n+1)} = f * f^{*n}$ for $n \geqslant 1$. The two convolutions $\star$ and $*$ are related by

$$(F \star G)(t) = \int_0^t (f * g)(s)\,\mathrm{d}s,$$

if

$$F(t) = \int_0^t f(s)\,\mathrm{d}s \quad \text{and} \quad G(t) = \int_0^t g(s)\,\mathrm{d}s.$$

Chistyakov [8] introduced the class of subexponential distribution functions. An excellent account with applications to age-dependent processes can be found in [2, ch. IV].

Definition 2.1. Let G be a distribution function on $\mathbb{R}$. Then G is *subexponential* if $G(0+) = 0$ and

$$\lim_{t\to\infty} \frac{1 - (G \star G)(t)}{1 - G(t)} = 2.$$

The class of subexponential distribution functions is denoted by $\mathcal{S}$.

Chistyakov [8] showed that if G is a subexponential distribution function, then

$$\lim_{t\to\infty} \frac{1 - G(t-s)}{1 - G(t)} = 1 \tag{2.1}$$

uniformly for s in compact intervals of $\mathbb{R}^+$. Chistyakov [8] demonstrated that a consequence of (2.1) is that subexponential distribution functions have ‘heavy tails’ that decay more slowly than any exponential functions.

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Lemma 2.2. *Let G be in $\mathcal{S}$. Then, for all $\gamma > 0$,*

$$\lim_{t\to\infty}(1 - G(t))\mathrm{e}^{\gamma t} = \infty.$$

We record some properties of subexponential distributions, which are used in later proofs. It is important that $\mathcal{S}$ is closed under asymptotic equivalence. The following result, which is alluded to in [9], is theorem 3 of [16].

Lemma 2.3. *Let G be in $\mathcal{S}$, and F a distribution function satisfying $F(0+) = 0$ and*

$$\lim_{t\to\infty}\frac{1 - F(t)}{1 - G(t)} > 0.$$

Then F is also in $\mathcal{S}$.

Next we state lemma 2.5.2 of [15].

Lemma 2.4. *Let G be in $\mathcal{S}$ and F a distribution function satisfying*

$$\lim_{t\to\infty}\frac{1 - F(t)}{1 - G(t)} = \lambda \geqslant 0.$$

Then

$$\lim_{t\to\infty}\frac{1 - (F \star G)(t)}{1 - G(t)} = 1 + \lambda$$

and $F \star G$ is in $\mathcal{S}$.

The following lemma is useful in demonstrating a converse to our result on the convergence of the tails of solutions of the integrodifferential equation (1.1). Its proof is given in § 7.

Lemma 2.5. *Let F and G be distribution functions such that $F(0+) = 0$ and G satisfies (2.1). If*

$$\lim_{t\to\infty}\frac{1 - F(t)}{1 - G(t)} = \lambda > 0, \qquad \lim_{t\to\infty}\frac{1 - (F \star G)(t)}{1 - G(t)} = \nu,$$

then

$$\lim_{t\to\infty}\frac{1 - (F \star F)(t)}{1 - F(t)} = 1 + \nu - \lambda.$$

3. Subexponential functions

3.1. Definition of subexponential functions

Our definition of subexponential functions is a based on the hypotheses of theorem 3 of [9].

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Definition 3.1. A *subexponential function* is a continuous mapping $h : \mathbb{R}^+ \to \mathbb{R}$, with $h(t) > 0$ for all $t > 0$, h in $L^1(\mathbb{R}^+)$ and

$$\lim_{t\to 0+} \frac{h^{*2}(t)}{h(t)} = 0, \tag{US0}$$

$$\lim_{t\to\infty} \frac{h^{*2}(t)}{h(t)} = 2\int_0^\infty h(s)\,\mathrm{d}s, \tag{US1}$$

$$\lim_{t\to\infty} \frac{h(t-s)}{h(t)} = 1 \quad \text{for each fixed } s > 0. \tag{US2}$$

A *positive* subexponential function is a subexponential function that satisfies $h(t) > 0$ for all $t \geqslant 0$. The class of subexponential functions is denoted by $\mathcal{U}$.

Definition 3.2. A continuously differentiable subexponential function $h : \mathbb{R}^+ \to \mathbb{R}$, which, in addition, satisfies the condition

$$\lim_{t\to\infty} \frac{h'(t)}{h(t)} = 0, \tag{US2b}$$

is called a *smooth subexponential* function.

Some facets of definition 3.1 should be further discussed. Condition (US2) is equivalent to $t \mapsto h(\log t)$ being *slowly varying*. Due to Karamata's uniform convergence theorem, equation (US2) implies the condition

$$\lim_{t\to\infty} \sup_{0\leqslant s\leqslant A} \left| \frac{h(t-s)}{h(t)} - 1 \right| = 0 \tag{US2a}$$

for each $A > 0$. Thus (US2) and (US2a) are equivalent.

Chover *et al*. [9] use complicated Banach algebra techniques to show that if $h : \mathbb{R}^+ \to \mathbb{R}$ is a positive continuous function in $L^1(\mathbb{R}^+)$, satisfying (US2) and

$$\lim_{t\to\infty} \frac{h^{*2}(t)}{h(t)} = c, \tag{US1a}$$

then

$$c = 2\int_0^\infty h(s)\,\mathrm{d}s.$$

However, we prefer to require condition (US1) in definition 3.1 rather than (US1a).

Equation (US0) is included because we do not only consider positive subexponential functions but allow $h(0) = 0$: for continuous h with $h(0) > 0$, equation (US0) is always satisfied. If h is C^1 at 0, $h(0) = 0$ and $h'(0) > 0$, then (US0) is satisfied. For convenience, we set $h^{*n}(0)/h(0) = 0$. By (US0) and (US1), $t \mapsto h^{*2}(t)/h(t)$ is a bounded continuous function on $\mathbb{R}^+$.

If h satisfies (US2b), it follows from the identity

$$\frac{h(t)}{h(t-s)} - 1 = \exp\left\{\int_{t-s}^{t} \frac{h'(\tau)}{h(\tau)}\,\mathrm{d}\tau\right\} - 1$$

that h satisfies (US2a). Thus (US2b) is a stronger condition than (US2).

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Our nomenclature of a subexponential function is justified by the fact that if h satisfies (US2),

$$\lim_{t\to\infty} h(t)e^{\gamma t} = \infty, \quad \text{for all } \gamma > 0. \tag{3.1}$$

3.2. Criteria for functions to be subexponential

Next we consider conditions that can help determine whether a function belongs to $\mathcal{U}$.

Recall that a measurable function h is said to be *regularly varying* at infinity if

$$\lim_{t\to\infty} \frac{h(\gamma t)}{h(t)} = \gamma^{\alpha}$$

for some finite α and all $\gamma > 0$ (cf. [3]). Examples of members of this class include functions h with $h(t) \sim t^{\beta}$, or $h(t) \sim t^{\beta}\log t$, both as $t \to \infty$. We say that a function h is of *dominated variation* at infinity if

$$\limsup_{t\to\infty} \sup_{\gamma\in[1/2,1]} \frac{h(\gamma t)}{h(t)} < \infty.$$

This definition is motivated by the notion of distribution functions with *tails of dominated variation* (cf. [3, Appendix 4]). A distribution function F is said to have tails of dominated variation if

$$\limsup_{t\to\infty} \frac{1-F(t/2)}{1-F(t)} < \infty.$$

We elucidate the relationships between $\mathcal{U}$ and functions with these properties. The proof is relegated to § 7.

Proposition 3.3. *If h is a positive continuous integrable function that is regularly varying at infinity, then it is also subexponential. Furthermore, a positive continuous integrable function that is of dominated variation at infinity, and, in addition, satisfies (US2), is subexponential.*

This result has parallels for distribution functions. In [17, § 8, ch. VIII], Feller shows that if the tail of a distribution function F is regularly varying, then F is subexponential, while Goldie [12] proves that if a distribution function F supported on $\mathbb{R}^+$ satisfies (2.1) and has tails of dominated variation, then F is a subexponential distribution.

Note that the class of continuous, positive and integrable regularly varying functions is a strictly smaller class than the class of positive subexponential functions. For example, the positive continuous integrable functions that behave according as $h(t) \sim \mathrm{e}^{-t^{\alpha}}$, for $\alpha \in (0,1)$, or $h(t) \sim \mathrm{e}^{-t/\log^2 t}$, both as $t \to \infty$, are subexponential, but neither are of regular, nor dominated, variation.

Note also that if h has all the properties of subexponential functions except (US2) and is decreasing, then it must *a fortiori* satisfy (US2). The proof is similar to [8, theorem 1] or [2, lemma 3].

Chover *et al.* provide in [9, remark 1] a condition for a non-negative continuous integrable function h on $\mathbb{R}^+$ to satisfy (US1). Here we present another that corresponds to [16, theorem 2].

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PROPOSITION 3.4. *Let h be a non-negative continuous integrable function on $\mathbb{R}^+$. Suppose that*

(i) $\lim_{t\to\infty} h(t) = 0$ *and* $t \mapsto -\log h(t)$ *is asymptotically concave;*

(ii) *there is a function g such that* $0 < g(t) \to \infty$ *and* $t - g(t) \to \infty$ *as* $t \to \infty$ *and*

$$\lim_{t\to\infty} \frac{h(t-g(t))}{h(t)} = 1;$$

(iii) $th(g(t)) \to 0$ *as* $t \to \infty$.

Then h satisfies (US1).

This can be used to show that positive continuous functions h with $h(t) \sim \mathrm{e}^{-t^\alpha}$ as $t \to \infty$ are subexponential for $\alpha \in (0, 1)$ (put $g(t) = t^{1-\alpha-\beta}$ for $0 < \beta < 1 - \alpha$).

3.3. Some important lemmata

We end this section with some remarks and technical lemmata.

REMARK 3.5. If h is subexponential, there is a constant $0 < M_h < \infty$ such that

$$\sup_{t\geqslant 0} \frac{h^{*2}(t)}{h(t)} = M_h. \tag{3.2}$$

Athreya and Ney [2, lemma 7, § 4, ch. IV] include a result of Kesten's, providing a uniform bound for

$$\sup_{t\geqslant 0} \frac{1 - G^{*n}(t)}{1 - G(t)},$$

where G is a subexponential distribution function. It allows the use of the dominated convergence theorem to obtain limits. The following lemma is the generalization of Kesten's result for subexponential functions.

LEMMA 3.6. *Let h be subexponential. For each $\epsilon > 0$, there is a constant $\kappa > 0$, independent of n, such that, for all $n \geqslant 2$,*

$$\sup_{t\geqslant 0} \frac{h^{*n}(t)}{h(t)} \leqslant \kappa(1+\epsilon)^n \mu^n,$$

where

$$\mu = \int_0^\infty h(s)\,\mathrm{d}s. \tag{3.3}$$

The next lemma is also needed, and is used in the proof of the above vital result.

LEMMA 3.7. *Let h be subexponential. For each $\eta > 0$, there is a $B > 0$, independent of n, such that, for all $n \geqslant 2$,*

$$\sup_{0\leqslant t\leqslant B} \frac{h^{*n}(t)}{h(t)} \leqslant \eta^{n-1}.$$

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We later establish converses of our main results, proving that the kernel k in the integro-differential equation (1.1) is subexponential. The following lemma is used.

LEMMA 3.8. *Let f and g be positive continuous functions on $\mathbb{R}^+$. Suppose further that g is integrable, satisfies (US2) and*

$$\lim_{t\to\infty}\frac{f(t)}{g(t)}=\lambda>0,\qquad \lim_{t\to\infty}\frac{(f*g)(t)}{g(t)}=\nu.$$

Then f is integrable, satisfies (US2) and

$$\lim_{t\to\infty}\frac{(f*f)(t)}{f(t)}=\nu+\int_0^\infty (f(s)-\lambda g(s))\,\mathrm{d}s.$$

4. Spaces of functions weighted by subexponential functions

We now establish some properties of subexponential functions and spaces of bounded continuous functions weighted by subexponential functions. Proofs are collected together in § 7.

Let h be a subexponential function on $\mathbb{R}^+$. Then $BC_h(\mathbb{R}^+)$ is defined to be the space of functions f on $\mathbb{R}^+$ such that $f=\phi h$ for some bounded continuous function ϕ on $\mathbb{R}^+$. By convention, we write $\phi=f/h$ and understand $f(0)/h(0)=\phi(0)$. $BC_h(\mathbb{R}^+)$ is usually abbreviated to BC_h here. It is a Banach space if equipped with the norm

$$\|f\|_h=M_h\sup_{t\geqslant 0}\left|\frac{f(t)}{h(t)}\right|,$$

where M_h is defined in (3.2). We denote by BC_h^l the closed subspace of functions in BC_h for which

$$L_hf:=\lim_{t\to\infty}\frac{f(t)}{h(t)}$$

exists. $L:BC_h^l\to\mathbb{R}$ is a bounded linear operator on BC_h^l. BC_h^0 is defined to be the closed subspace of functions in BC_h^l for which $L_hf=0$. By definition 3.1, $h*h=h^{*2}$ is in BC_h^l and

$$L_h(h^{*2})=2\int_0^\infty h(s)\,\mathrm{d}s.$$

The following result is partly based on [9, lemma 1].

THEOREM 4.1. *Suppose that h is a subexponential function. Then BC_h is a commutative Banach algebra with the convolution as product, and BC_h^l and BC_h^0 are subalgebras. If f and g are both in BC_h^l,*

$$L_h(f*g)=L_hf\int_0^\infty g(s)\,\mathrm{d}s+L_hg\int_0^\infty f(s)\,\mathrm{d}s. \tag{4.1}$$

A simple induction argument using theorem 4.1 establishes the following result.

COROLLARY 4.2. *Let h be subexponential. Then h^{*n} is in BC_h^l for every $n\geqslant 2$ and $L_h(h^{*n})=n\mu^{n-1}$, where μ is defined as in (3.3).*

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It is important later to prove that the resolvents of subexponential functions are subexponential. The following lemma is employed. It is an analogue of lemma 2.3 for subexponential functions.

LEMMA 4.3. *If f is in BC^l_h and $L_h f \neq 0$, then f satisfies (US1) and (US2). Moreover, if, in addition, f is positive and $\lim_{t\to 0+} f(t)/h(t) > 0$, then f is subexponential.*

A corollary of this lemma is the following observation, which corresponds to lemma 2.4.

PROPOSITION 4.4. *Let h be in $\mathcal{U}$. Suppose that f is in $C(\mathbb{R}^+)\cap L^1(\mathbb{R}^+)$, positive on $(0,\infty)$ and*

$$\lim_{t\to 0}\frac{f(t)}{h(t)} > 0, \qquad \lim_{t\to \infty}\frac{f(t)}{h(t)} > 0.$$

Then f is also in $\mathcal{U}$.

5. Subexponential solutions of transient renewal equations

This section considers solutions of the linear scalar convolution equation

$$r(t) = h(t) + \int_0^t h(t-s)r(s)\,\mathrm{d}s, \quad t \geqslant 0, \tag{5.1}$$

with h subexponential and

$$\mu = \int_0^\infty h(s)\,\mathrm{d}s < 1. \tag{5.2}$$

Then r is positive on $(0,\infty)$, continuous, integrable and

$$\int_0^\infty r(s)\,\mathrm{d}s = \frac{\mu}{1-\mu}.$$

It can also be represented by the Neumann series

$$r(t) = \sum_{n=1}^\infty h^{(*n)}(t). \tag{5.3}$$

r is called the *resolvent* of h, since every solution of

$$y(t) = f(t) + \int_0^t y(t-s)h(s)\,\mathrm{d}s, \quad t \geqslant 0, \tag{5.4}$$

can be represented as

$$y(t) = f(t) + \int_0^t r(t-s)f(s)\,\mathrm{d}s. \tag{5.5}$$

Integration of (5.1) yields the *renewal equation*

$$U(t) = \mu H(t) + \mu\int_0^t U(t-s)\,\mathrm{d}H(s), \tag{5.6}$$

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where

$$H(t) = \frac{1}{\mu}\int_0^t h(s)\,\mathrm{d}s, \qquad U(t) = \int_0^t r(s)\,\mathrm{d}s. \tag{5.7}$$

If (5.2) holds, equation (5.6) is a *transient renewal equation*. Clearly, $U(\infty) = \mu(1-\mu)^{-1}$.

A sharp result is known on the convergence of the tails of the distributions in (5.6). The fact that (i) implies (iii) follows from [2, theorem 3, § 4, ch. 4]; the rest of the result follows from [16, theorem 4].

THEOREM 5.1. *Let H be a distribution function with $H(0+) = 0$. Suppose that $0 < \mu < 1$. Let U be the solution of (5.6). Then the following statements are equivalent.*

(i) *H is in $\mathcal{S}$.*

(ii) *$U/U(\infty)$ is in $\mathcal{S}$.*

(iii) *The rate of convergence of $U(t) \to U(\infty)$ as $t \to \infty$ is given by*

$$\lim_{t\to\infty} \frac{U(\infty) - U(t)}{1 - H(t)} = \frac{\mu}{(1-\mu)^2}. \tag{5.8}$$

For solutions r of (5.1), equation (5.8) is equivalent to

$$\lim_{t\to\infty} \frac{\int_t^\infty r(s)\,\mathrm{d}s}{\int_t^\infty h(s)\,\mathrm{d}s} = \frac{1}{(1-\mu)^2}.$$

Here we establish a result on the convergence of the densities of the distributions, rather than their tails. The only other theorem of this kind that we know of is theorem $2'$ of [10].

THEOREM 5.2. *Let h be a subexponential function satisfying (5.2). Then the resolvent r defined by (5.1) is in BC_h^l and*

$$L_h r = \frac{1}{(1-\mu)^2}. \tag{5.9}$$

Also, r is subexponential.

Moreover, if f is in BC_h, the solution y of (5.4) is also in BC_h. If f is in BC_h^l, then y is in BC_h^l and

$$L_h y = \frac{1}{1-\mu} L_h f + \frac{1}{(1-\mu)^2}\int_0^\infty f(s)\,\mathrm{d}s. \tag{5.10}$$

Proof. By the representation (5.3) for r, corollary 4.2 and the uniform convergence implied by lemma 3.6,

$$\lim_{t\to\infty} \frac{r(t)}{h(t)} = \lim_{t\to\infty} \sum_{n=1}^\infty \frac{h^{(*n)}(t)}{h(t)} = \sum_{n=1}^\infty \lim_{t\to\infty} \frac{h^{(*n)}(t)}{h(t)} = \sum_{n=1}^\infty n\mu^{n-1} = \frac{1}{(1-\mu)^2}.$$

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To prove that r is subexponential, we show that $r(t)/h(t) \to 1$ as $t \to 0+$. It follows that r is in BC_h^l. Since $L_h r > 0$, lemma 4.3 then asserts that r is subexponential. Choose $0 < \eta < 1$. By lemma 3.7 and (5.3),

$$\sup_{0 \leqslant s \leqslant B} \frac{r(s)}{h(s)} \leqslant \frac{1}{1-\eta},$$

where B is the number whose existence is asserted in lemma 3.7. For $0 < t \leqslant B$,

$$\frac{(r*h)(t)}{h(t)} \leqslant \frac{1}{1-\eta} \frac{h^{*2}(t)}{h(t)}.$$

Because $r = h + r*h$, we immediately see that $r(t)/h(t) \to 1$ as $t \to 0+$.

The properties of the solution y of (5.4) follow from the representation (5.5) and theorem 4.1. A simple calculation using (4.1) establishes (5.10). □

6. Linear integro-differential equations

In this section we consider the asymptotic stability of the scalar linear Volterra integro-differential equation

$$x'(t) = -ax(t) + \int_0^t k(t-s)x(s)\,\mathrm{d}s + f(t), \quad x(0) = x_0, \tag{6.1}$$

under the assumption that $k > 0$ on $\mathbb{R}^+$, k is in $C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ and f is in $C(\mathbb{R}^+)$. It is convenient to introduce the differential resolvent z, which is the solution of

$$z'(t) = -az(t) + \int_0^t k(t-s)z(s)\,\mathrm{d}s, \quad z(0) = 1. \tag{6.2}$$

It is easily shown that $z(t) > 0$ for all $t \geqslant 0$. If k is in $C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, $k > 0$ on $\mathbb{R}^+$ and

$$a > \int_0^\infty k(s)\,\mathrm{d}s,$$

theorem 1 of [7] says that z is in $L^1(\mathbb{R}^+)$ and $z(t) \to 0$ as $t \to \infty$. The significance of the differential resolvent is that the solution of (6.1) can be expressed as

$$x(t) = z(t)x_0 + \int_0^t z(t-s)f(s)\,\mathrm{d}s. \tag{6.3}$$

We represent the differential resolvent z solving (6.2) in terms of the resolvent of a transient renewal equation.

Proposition 6.1. *Let k be in $C(\mathbb{R}^+)$. The unique continuous solution of (6.2) satisfies*

$$z = e + e * r, \tag{6.4}$$

where $e(t) = \mathrm{e}^{-at}$ for $t \geqslant 0$, $h = e * k$ and r is the resolvent given by (5.1).

Proof. See the beginning of the proof of theorem 2 of [1]. □

The next result is the main result of this section.

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THEOREM 6.2. *Let k be a positive subexponential function. Suppose that*

$$a > \int_0^\infty k(s)\,\mathrm{d}s.$$

Then the differential resolvent z, given by (6.2), is in BC_k^l and satisfies

$$\lim_{t\to\infty} \frac{z(t)}{k(t)} = \frac{1}{(a - \int_0^\infty k(s)\,\mathrm{d}s)^2}, \qquad \lim_{t\to\infty} \frac{z'(t)}{z(t)} = 0. \tag{6.5}$$

Moreover, z is subexponential.

Proof. Firstly, it is proved that $h = e * k$ is subexponential. Clearly, $h(t) > 0$ on $(0, \infty)$, h is continuous and (5.2) holds, where

$$\mu = \frac{1}{a} \int_0^\infty k(s)\,\mathrm{d}s.$$

By hypothesis, $\mu < 1$. By (3.1),

$$\frac{e(t)}{k(t)} = \frac{1}{k(t)\mathrm{e}^{at}} \to 0 \quad \text{as } t \to \infty.$$

Since also $e(t)/k(t) \to 1/k(0) > 0$ as $t \to 0+$, e is in BC_k^0. By theorem 4.1, $h = e * k$ is in BC_h^l and

$$L_k h = L_k e \int_0^\infty k(s)\,\mathrm{d}s + \int_0^\infty e(s)\,\mathrm{d}s = \frac{1}{a}. \tag{6.6}$$

By lemma 4.3, h has the properties (US1) and (US2). Clearly, h is continuous. For each $\epsilon > 0$, there is a number $0 < \delta \leqslant 1$ such that $0 < t < \delta$ implies that $k^{*2}(t)/k(t) < \epsilon$. Let $0 < t < \delta$. Since $e^{*2}(t) = t\mathrm{e}^{-at}$,

$$\frac{h^{*2}(t)}{h(t)} = \frac{\int_0^t k^{*2}(t-s)s\mathrm{e}^{-as}\,\mathrm{d}s}{\int_0^t k(t-s)\mathrm{e}^{-as}\,\mathrm{d}s} < \epsilon \frac{\int_0^t k(t-s)s\mathrm{e}^{-as}\,\mathrm{d}s}{\int_0^t k(t-s)\mathrm{e}^{-as}\,\mathrm{d}s} \leqslant \epsilon t < \epsilon\delta \leqslant \epsilon.$$

This shows that h satisfies (US0). Hence h is subexponential. Theorem 4.1 can now be applied.

To complete the proof, observe that by (4.1), (5.9), (6.4) and (6.6),

$$\begin{aligned} L_k z &= L_k e + L_k(r * e) \\ &= L_k h L_h(r * e) \\ &= \frac{1}{a}\biggl(L_h r \int_0^\infty e(s)\,\mathrm{d}s + L_h e \int_0^\infty r(s)\,\mathrm{d}s\biggr) \\ &= \frac{1}{a^2(1-\mu)^2}, \end{aligned}$$

because $L_k e = L_h e = 0$. This result, equation (4.1) and the observation that

$$\int_0^\infty z(s)\,\mathrm{d}s = \frac{1}{a(1-\mu)} \tag{6.7}$$

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together imply

$$L_k z' = L_k(-az + k * z) = -aL_k z + L_k z \int_0^\infty k(s)\,\mathrm{d}s + \int_0^\infty z(s)\,\mathrm{d}s = 0.$$

Since $\lim_{t\to 0+} z(t)/k(t) = k(0)^{-1} > 0$ and $L_k z > 0$, lemma 4.3 implies that z is subexponential. Clearly, z is in BC_k^l. □

COROLLARY 6.3. *Let k be a positive subexponential function. Suppose that*

$$a > \int_0^\infty k(s)\,\mathrm{d}s.$$

For every f in BC_k^l, the solution x of (6.1) is in BC_k^l and

$$L_k x = \frac{1}{(a - \int_0^\infty k(s)\,\mathrm{d}s)^2}\left(x_0 + \int_0^\infty f(s)\,\mathrm{d}s\right) + \frac{L_k f}{a - \int_0^\infty k(s)\,\mathrm{d}s}.$$

Proof. It has been proved that z is in BC_k^l and that (6.7) holds. Hence theorem 4.1 can be applied to the representation (6.3) to infer that x is in BC_h^l and obtain the above formula for $L_k x$. □

There is a converse to theorem 6.2, which shows that the hypothesis that k be subexponential is required for the conclusion to hold.

THEOREM 6.4. *Let k be a continuous positive function in $L^1(\mathbb{R}^+)$ and z the differential resolvent defined in (6.2). Suppose that*

$$a > \int_0^\infty k(s)\,\mathrm{d}s.$$

Then

(i) *k is subexponential;*

(ii) *z satisfies the conditions (6.5),*

are equivalent, and either implies that z is a smooth subexponential function.

Proof. Theorem 6.2 asserts that if k is subexponential, then z has the properties in (6.5). We suppose now that z obeys (6.5).

It has been observed that $\lim_{t\to\infty} z'(t)/z(t) = 0$ implies that z has the property (US2). Since $\lim_{t\to\infty} z(t)/k(t) > 0$, k must also satisfy (US2). It follows from (6.2) and (6.5) that

$$\lim_{t\to\infty} \frac{(k * z)(t)}{z(t)} = a.$$

Since $z(t) \to 0$ as $t \to \infty$ and z is positive, the integration of (6.2) leads to (6.7). By applying lemma 3.8, we see that

$$\lim_{t\to\infty} \frac{(k * k)(t)}{k(t)} = a + a\mu - \frac{a^2(1-\mu)^2}{a(1-\mu)} = 2a\mu = 2\int_0^\infty k(s)\,\mathrm{d}s.$$

□

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We can characterize for solutions of (6.2) the decay rate of the tail of the integral

$$\int_t^\infty z(s)\,\mathrm{d}s \quad \text{as } t \to \infty.$$

Let

$$K(t) = \frac{\int_0^t k(s)\,\mathrm{d}s}{\int_0^\infty k(s)\,\mathrm{d}s}, \quad Z(t) = \frac{\int_0^t z(s)\,\mathrm{d}s}{\int_0^\infty z(s)\,\mathrm{d}s}, \quad t \geqslant 0. \tag{6.8}$$

We also use the notation in (5.7).

THEOREM 6.5. *Let k be a positive integrable function on $(0, \infty)$. Suppose that*

$$a > \int_0^\infty k(s)\,\mathrm{d}s.$$

If K defined in (6.8) is in $\mathcal{S}$, the differential resolvent z given by (6.2) satisfies

$$\lim_{t\to\infty} \frac{\int_t^\infty z(s)\,\mathrm{d}s}{\int_t^\infty k(s)\,\mathrm{d}s} = \frac{1}{(a - \int_0^\infty k(s)\,\mathrm{d}s)^2}, \qquad \lim_{t\to\infty} \frac{z(t)}{\int_t^\infty z(s)\,\mathrm{d}s} = 0. \tag{6.9}$$

Moreover, Z given in (6.8) is in $\mathcal{S}$.

Proof. Firstly, we note that (6.8) implies that

$$\frac{\int_t^\infty z(s)\,\mathrm{d}s}{\int_t^\infty k(s)\,\mathrm{d}s} = \frac{1}{a^2\mu(1-\mu)} \frac{1 - Z(t)}{1 - K(t)}. \tag{6.10}$$

Also, it is a consequence of (6.4) that

$$\frac{1 - Z(t)}{1 - K(t)} = (1-\mu)\frac{1 - E(t)}{1 - K(t)} + (1-\mu)\frac{U(\infty) - (E \star U)(t)}{1 - K(t)}, \tag{6.11}$$

where $E(t) = 1 - \mathrm{e}^{-at}$. The limit on the right-hand side of (6.10) is evaluated by considering each of the terms

$$\frac{1 - E(t)}{1 - K(t)}, \quad \frac{U(\infty) - (E \star U)(t)}{U(\infty) - U(t)}, \quad \frac{U(\infty) - U(t)}{1 - H(t)} \quad \text{and} \quad \frac{1 - H(t)}{1 - K(t)}$$

as $t \to \infty$.

Integration of $h = e * k$ leads to the formula $H = E \star K$ for the distribution defined in (5.7). We infer from lemma 2.2 that

$$\frac{1 - E(t)}{1 - K(t)} = \frac{1}{\mathrm{e}^{at}(1 - K(t))} \to 0 \tag{6.12}$$

as $t \to \infty$. Therefore, from lemma 2.4,

$$1 = \lim_{t\to\infty} \frac{1 - (E * K)(t)}{1 - K(t)} = \lim_{t\to\infty} \frac{1 - H(t)}{1 - K(t)}, \tag{6.13}$$

and H is in $\mathcal{S}$. By theorem 5.1, $U(\cdot)/U(\infty)$ is in $\mathcal{S}$ and (5.8) holds. Therefore, it follows from lemma 2.2 that

$$\lim_{t\to\infty} \frac{1 - E(t)}{U(\infty) - U(t)} = 0.$$

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Lemma 2.4 implies that

$$1 = \lim_{t\to\infty} \frac{U(\infty) - (E \star U)(t)}{U(\infty) - U(t)}. \tag{6.14}$$

It follows from (6.11), (6.12), (6.13), (5.8) and (6.14) that

$$\lim_{t\to\infty} \frac{1 - Z(t)}{1 - K(t)} = \frac{\mu}{1-\mu},$$

and hence (6.9), with the aid of (6.10). Also, we can infer from this limit and lemma 2.4 that

$$\lim_{t\to\infty} \frac{1 - (K \star Z)(t)}{1 - Z(t)} = 1 + \frac{1-\mu}{\mu} = \frac{1}{\mu}. \tag{6.15}$$

Manipulation of (6.2) gives

$$(1-\mu)z(t) = 1 - Z(t) - \mu(1 - (K \star Z)(t)). \tag{6.16}$$

Due to (6.15), this implies the second equation in (6.9). □

THEOREM 6.6. *Let k be a continuous positive function in $L^1(\mathbb{R}^+)$ and z the differential resolvent defined in (6.2). Suppose that*

$$a > \int_0^\infty k(s)\,\mathrm{d}s.$$

Then the following are equivalent.

(i) *K is a subexponential distribution.*

(ii) *z satisfies the conditions (6.9).*

Moreover, either implies that Z is in $\mathcal{S}$.

Proof. The fact that (i) implies (ii) has been established in theorem 6.5. Assume now that (ii) is true. By hypothesis,

$$\lim_{t\to\infty} \frac{1 - Z(t)}{1 - K(t)} = \frac{\mu}{1-\mu}, \qquad \lim_{t\to\infty} \frac{z(t)}{1 - Z(t)} = 0.$$

Therefore, since (6.16) holds,

$$\lim_{t\to\infty} \frac{1 - (K \star Z)(t)}{1 - Z(t)} = \frac{1}{\mu}.$$

It follows from lemma 2.5 that K is a subexponential distribution. □

7. Some proofs

In this section we gather the proofs of some lemmata from previous sections.

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7.1. Proof of lemma 2.5

We consider

$$\frac{1-(F\star F)(t)}{1-F(t)} = 1 + \int_0^t \frac{1-F(t-s)}{1-F(t)}\,\mathrm{d}F(s).$$

It can be shown from the identity

$$\frac{1-F(t-s)}{1-F(t)} = \frac{1-F(t-s)}{1-G(t-s)}\frac{1-G(t-s)}{1-G(t)}\frac{1-G(t)}{1-F(t)}$$

that $F(t-s)/F(t) \to 1$ as $t \to \infty$ uniformly for s in compact subsets of $\mathbb{R}^+$. Hence, for any $A > 0$,

$$\int_{t-A}^t \frac{1-F(t-s)}{1-F(t)}\,\mathrm{d}F(s) \leqslant \frac{F(t)-F(t-A)}{1-F(t)} = \frac{1-F(t-A)}{1-F(t)} - 1 \to 0 \tag{7.1}$$

as $t \to \infty$. Also,

$$\begin{aligned}\int_0^{t-A} \frac{1-F(t-s)}{1-F(t)}\,\mathrm{d}F(s) &= \int_0^{t-A} \frac{1-F(t-s)}{1-G(t-s)}\frac{1-G(t-s)}{1-F(t)}\,\mathrm{d}F(s)\\ &= \int_0^{t-A}\biggl\{\frac{1-F(t-s)}{1-G(t-s)} - \lambda\biggr\}\frac{1-G(t-s)}{1-F(t)}\,\mathrm{d}F(s)\\ &\qquad + \lambda\int_0^{t-A}\frac{1-G(t-s)}{1-F(t)}\,\mathrm{d}F(s).\end{aligned} \tag{7.2}$$

But

$$\begin{aligned}&\int_0^{t-A} \frac{1-G(t-s)}{1-F(t)}\,\mathrm{d}F(s)\\ &\qquad = \frac{1-(F\star G)(t)}{1-F(t)} - 1 - \int_{t-A}^t \frac{1-G(t-s)}{1-F(t)}\,\mathrm{d}F(s) \to \frac{\nu}{\lambda} - 1\end{aligned}$$

as $t \to \infty$, because, by (7.1),

$$\biggl|\int_{t-A}^t \frac{1-G(t-s)}{1-F(t)}\,\mathrm{d}F(s)\biggr| \leqslant \sup_{0\leqslant\tau\leqslant A}\frac{1-G(\tau)}{1-F(\tau)}\int_{t-A}^t \frac{1-F(t-s)}{1-F(t)}\,\mathrm{d}F(s) \to 0$$

as $t \to \infty$. Also, given $\epsilon > 0$, there is $A > 0$ such that

$$\biggl|\frac{1-F(\tau)}{1-G(\tau)} - \lambda\biggr| < \epsilon$$

if $\tau > A$. Hence, for $t \geqslant A$,

$$\begin{aligned}&\biggl|\int_0^{t-A}\biggl\{\frac{1-F(t-s)}{1-G(t-s)} - \lambda\biggr\}\frac{1-G(t-s)}{1-F(t)}\,\mathrm{d}F(s)\biggr|\\ &\qquad \leqslant \epsilon\int_0^{t-A}\frac{1-G(t-s)}{1-F(t)}\,\mathrm{d}F(s) \to \epsilon\biggl(\frac{\nu}{\lambda} - 1\biggr) \quad \text{as } t \to \infty.\end{aligned}$$

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Since ϵ is arbitrarily small, it follows from (7.1) and (7.2) that

$$\int_0^{t-A} \frac{1-F(t-s)}{1-F(t)}\,\mathrm{d}F(s) \to \nu - \lambda$$

as $t \to \infty$, completing the proof.

7.2. Proof of proposition 3.3

Clearly, if h is regularly varying at infinity, it is also of dominated variation. By positivity, equation (US0) is automatic. Equation (US2) is true by hypothesis. Next we note that

$$\lim_{t\to\infty} \frac{\int_0^{t/2} h(t-s)h(s)\,\mathrm{d}s}{h(t)} = \int_0^\infty h(s)\,\mathrm{d}s$$

is equivalent to (US1). Define

$$C = \limsup_{t\to\infty} \sup_{\gamma\in[1/2,1]} \frac{h(\gamma t)}{h(t)}. \tag{7.3}$$

For every $\epsilon > 0$, there exists $A > 0$ such that

$$\int_A^\infty h(s)\,\mathrm{d}s < \frac{\epsilon}{2C} \leqslant \tfrac{1}{2}\epsilon. \tag{7.4}$$

Consider, for $t > 2A$, the identity

$$\begin{aligned}\frac{1}{h(t)}\int_0^{t/2} h(t-s)h(s)\,\mathrm{d}s - \int_0^\infty h(s)\,\mathrm{d}s \\ = \int_0^A \left(\frac{h(t-s)}{h(t)} - 1\right)h(s)\,\mathrm{d}s + \int_A^{t/2} \frac{h(t-s)}{h(t)} h(s)\,\mathrm{d}s - \int_A^\infty h(s)\,\mathrm{d}s.\end{aligned}$$

From (7.3), (7.4), (US2) and the fact that h is of dominated variation

$$\limsup_{t\to\infty}\left|\frac{1}{h(t)}\int_0^{t/2} h(t-s)h(s)\,\mathrm{d}s - \int_0^\infty h(s)\,\mathrm{d}s\right| < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we conclude the proof.

Note that if h is regularly varying, the *representation theorem* yields (US2) directly. See [3, theorem 1.3.1] for slowly varying functions.

7.3. Proof of lemma 3.6

Let $\epsilon > 0$. We define

$$\alpha_n = \sup_{t\geqslant 0} \frac{h^{*n}(t)}{h(t)}.$$

The idea of the proof is to infer from

$$\begin{aligned}\alpha_{n+1} \leqslant \sup_{0\leqslant t\leqslant T} \int_0^t \frac{h^{*n}(s)h(t-s)}{h(t)}\,\mathrm{d}s \\ + \sup_{t\geqslant T} \int_0^A \frac{h^{*n}(s)h(t-s)}{h(t)}\,\mathrm{d}s + \sup_{t\geqslant T} \int_A^t \frac{h^{*n}(s)h(t-s)}{h(t)}\,\mathrm{d}s\end{aligned}$$

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an inequality of the form

$$\alpha_{n+1} \leqslant c + (1+\epsilon)\mu\alpha_n, \quad n \geqslant 2. \tag{7.5}$$

The result then easily follows.

Due to (US1) and (US2), there are numbers $0 < A < T$ such that

$$\left|\int_0^t \frac{h(s)h(t-s)}{h(t)}\,\mathrm{d}s - \int_0^A h(s)\,\mathrm{d}s\right| < \mu(1+\tfrac{1}{2}\epsilon), \quad t \geqslant A,$$

$$\sup_{0\leqslant s\leqslant A}\left|\frac{h(t-s)}{h(t)} - 1\right| \leqslant \tfrac{1}{2}\epsilon, \qquad t \geqslant T.$$

Since

$$\int_A^t \frac{h(t-s)h(s)}{h(t)}\,\mathrm{d}s = \int_0^t \frac{h(t-s)h(s)}{h(t)}\,\mathrm{d}s - \int_0^A h(s)\,\mathrm{d}s - \int_0^A h(s)\left\{\frac{h(t-s)}{h(t)} - 1\right\}\mathrm{d}s,$$

it easily follows that

$$\sup_{t\geqslant T}\int_A^t \frac{h(t-s)h(s)}{h(t)}\,\mathrm{d}s \leqslant (1+\epsilon)\mu,$$

and hence that

$$\sup_{t\geqslant T}\int_A^t \frac{h^{*n}(s)h(t-s)}{h(t)}\,\mathrm{d}s \leqslant (1+\epsilon)\mu\alpha_n. \tag{7.6}$$

Let $0 < \eta < 1$ and $0 < B < A$ be the corresponding number mentioned in lemma 3.7. Then, for $0 \leqslant t \leqslant B$,

$$\int_0^t \frac{h^{*n}(s)h(t-s)}{h(t)}\,\mathrm{d}s = \int_0^t \frac{h^{*n}(s)}{h(s)}\frac{h(s)h(t-s)}{h(t)}\,\mathrm{d}s \leqslant \eta^{n-1}\frac{h^{*2}(t)}{h(t)} \leqslant M_h.$$

Similarly for $B \leqslant t \leqslant T$,

$$\begin{aligned}\int_0^t \frac{h^{*n}(s)h(t-s)}{h(t)}\,\mathrm{d}s &= \int_0^B \frac{h^{*n}(s)}{h(s)}\frac{h(t-s)h(s)}{h(t)}\,\mathrm{d}s + \int_B^t \frac{h^{*n}(s)h(t-s)}{h(t)}\,\mathrm{d}s \\ &\leqslant \eta^{n-1}\frac{h^{*2}(t)}{h(t)} + \frac{\max_{0\leqslant t<\infty} h(t)}{\min_{B\leqslant t\leqslant T} h(t)}\int_B^t h^{*n}(s)\,\mathrm{d}s \\ &\leqslant M_h + \frac{\max_{0\leqslant t<\infty} h(t)}{\min_{B\leqslant t\leqslant T} h(t)} := c_1.\end{aligned}$$

Hence

$$\sup_{0\leqslant t\leqslant T}\int_0^t \frac{h^{*n}(s)h(t-s)}{h(t)}\,\mathrm{d}s \leqslant c_1.$$

Similarly,

$$\sup_{t\geqslant T}\int_0^A \frac{h^{*n}(s)h(t-s)}{h(t)}\,\mathrm{d}s \leqslant c_2,$$

where c_2 is independent of n. Equation (7.5) has been established with $c = c_1 + c_2$, completing the proof.

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7.4. Proof of lemma 3.7

The result follows from the inequality

$$\frac{h^{*(n+1)}(t)}{h(t)} = \int_0^t \frac{h^{*n}(s)}{h(s)} \frac{h(t-s)h(s)}{h(t)} \, \mathrm{d}s \leqslant \sup_{0<s\leqslant t} \frac{h^{*n}(s)}{h(s)} \frac{h^{*2}(t)}{h(t)},$$

and by choosing B such that

$$\sup_{0\leqslant t\leqslant B} h^{*2}(t)/h(t) \leqslant \eta.$$

7.5. Proof of lemma 3.8

The fact that $\lim_{t\to\infty} f(t)/g(t) > 0$ implies that f is in L^1. Also, it follows from

$$\frac{f(t-s)}{f(t)} - 1 = \frac{f(t-s)/g(t-s)}{f(t)/g(t)} \left(\frac{g(t-s)}{g(t)} - 1 \right) + \left(\frac{f(t-s)/g(t-s)}{f(t)/g(t)} - 1 \right)$$

that f has the property (US2). The remainder of the proof involves the identity

$$\frac{(f*f)(t)}{f(t)} - \lambda \frac{(f*g)(t)}{f(t)} - \int_0^t (f(s) - \lambda g(s)) \, \mathrm{d}s = \int_0^t \left(\frac{f(t-s)}{f(t)} - 1 \right) (f(s) - \lambda g(s)) \, \mathrm{d}s.$$

We demonstrate that the right-hand side tends to 0 as $t \to \infty$, thus establishing the lemma. Let $\epsilon > 0$. There is a number $T > 0$ such that $|f(t)/g(t) - \lambda| \leqslant \epsilon$ for all $t \geqslant T$. Hence for $t \geqslant T$

$$\begin{aligned} \left| \int_T^t \left(\frac{f(t-s)}{f(t)} - 1 \right) (f(s) - \lambda g(s)) \, \mathrm{d}s \right| &= \left| \int_T^t \left(\frac{f(t-s)}{f(t)} - 1 \right) \left(\frac{f(s)}{g(s)} - \lambda \right) g(s) \, \mathrm{d}s \right| \\ &\leqslant \epsilon \int_T^t \left| \frac{f(t-s)g(s)}{f(t)} - g(s) \right| \mathrm{d}s \\ &\leqslant \epsilon \left\{ \frac{(f*g)(t)}{f(t)} + \int_0^t g(s) \, \mathrm{d}s \right\} \\ &\to \epsilon \left\{ \frac{\nu}{\lambda} + \int_0^\infty g(s) \, \mathrm{d}s \right\} \quad \text{as } t \to \infty. \end{aligned}$$

Also, because f satisfies (US2a),

$$\left| \int_0^T \left(\frac{f(t-s)}{f(t)} - 1 \right) (f(s) - \lambda g(s)) \, \mathrm{d}s \right| \leqslant \sup_{0\leqslant s\leqslant T} \left| \frac{f(t-s)}{f(t)} - 1 \right| \int_0^T (f(s) - \lambda g(s)) \, \mathrm{d}s,$$

which tends to zero as $t \to \infty$. It has been proved that

$$\limsup_{t\to\infty} \left| \int_0^t \left(\frac{f(t-s)}{g(t)} - 1 \right) (f(s) - \lambda g(s)) \, \mathrm{d}s \right| \leqslant \epsilon \left(\frac{\nu}{\lambda} + \int_0^\infty g(s) \, \mathrm{d}s \right).$$

Since ϵ is arbitrarily small, the proof of the lemma is complete.

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7.6. Proof of theorem 4.1

Let f and g be in BC_h. Then

$$M_h\frac{|(f*g)(t)|}{h(t)} \leqslant M_h\int_0^t\left|\frac{f(t-s)}{h(t-s)}\right|\left|\frac{g(s)}{h(s)}\right|\frac{h(t-s)h(s)}{h(t)}\,\mathrm{d}s \leqslant \|f\|_h\|g\|_h\frac{1}{M_h}\frac{h^{*2}(t)}{h(t)}. \tag{7.7}$$

Hence (3.2) implies that $\|f*g\|_h \leqslant \|f\|_h\|g\|_h$.

Suppose that $g \in BC_h^0$. Clearly, g is in $L^1(\mathbb{R}^+)$. Let $\epsilon > 0$. There is a number $A > 0$ such that

$$\int_A^\infty |g(s)|\,\mathrm{d}s < \epsilon, \quad \left|\frac{g(t)}{h(t)}\right| < \epsilon \quad \text{for all } t > A.$$

Suppose that $t > A$. Since

$$\begin{aligned}\frac{(g*h)(t)}{h(t)} - \int_0^\infty g(s)\,\mathrm{d}s &= \frac{1}{h(t)}\int_A^t \frac{g(s)}{h(s)}h(s)h(t-s)\,\mathrm{d}s \\ &\quad + \int_0^A\left(\frac{h(t-s)}{h(t)}-1\right)g(s)\,\mathrm{d}s - \int_A^\infty g(s)\,\mathrm{d}s,\end{aligned}$$

we see that

$$\begin{aligned}&\left|\frac{(g*h)(t)}{h(t)} - \int_0^\infty g(s)\,\mathrm{d}s\right| \\ &\quad\leqslant \sup_{A\leqslant s\leqslant t}\left|\frac{g(s)}{h(s)}\right|\frac{1}{h(t)}\int_0^t h(s)h(t-s)\,\mathrm{d}s + \sup_{0\leqslant s\leqslant A}\left|\frac{h(t-s)}{h(t)}-1\right|\int_0^A|g(s)|\,\mathrm{d}s + \epsilon \\ &\quad< \epsilon + \frac{\epsilon}{h(t)}\int_0^t h(s)h(t-s)\,\mathrm{d}s + \sup_{0\leqslant s\leqslant A}\left|\frac{h(t-s)}{h(t)}-1\right|\int_0^A|g(s)|\,\mathrm{d}s.\end{aligned}$$

By taking the limit superior of each side as $t \to \infty$, it is seen that (US1) and (US2) imply

$$\limsup_{t\to\infty}\left|\frac{(g*h)(t)}{h(t)} - \int_0^\infty g(s)\,\mathrm{d}s\right| \leqslant 2\epsilon\int_0^\infty h(s)\,\mathrm{d}s + \epsilon,$$

establishing that

$$L_h(g*h) = \int_0^\infty g(s)\,\mathrm{d}s \tag{7.8}$$

for all $g \in BC_h^0$. If $f \in BC_h^l$, then $f(t) = (L_hf)h(t) + \tilde{f}(t)$ for some $\tilde{f} \in BC_h^0$. The linearity of L implies that

$$\begin{aligned}L_h(f*h) &= (L_hf)L_h(h*h) + L_h(\tilde{f}*h) \\ &= 2L_hf\int_0^\infty h(s)\,\mathrm{d}s + \int_0^\infty \tilde{f}(s)\,\mathrm{d}s \\ &= 2L_hf\int_0^\infty h(s)\,\mathrm{d}s + \int_0^\infty f(s)\,\mathrm{d}s - L_hf\int_0^\infty h(s)\,\mathrm{d}s \\ &= L_hf\int_0^\infty h(s)\,\mathrm{d}s + \int_0^\infty f(s)\,\mathrm{d}s.\end{aligned} \tag{7.9}$$

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We now prove $L_h(g*g)=0$ for all $g\in BC_h^0$. Let $\epsilon>0$. There is a number $A>0$ such that $t>A$ implies

$$\left|\frac{g(t)}{h(t)}\right|<\epsilon.$$

Observe that if $t>A$,

$$\begin{aligned}\left|\frac{1}{h(t)}\int_A^t g(t-s)g(s)\,\mathrm{d}s\right| &\leqslant \frac{1}{h(t)}\int_A^t\left|\frac{g(s)}{h(s)}\right||g(t-s)|h(s)\,\mathrm{d}s\\ &\leqslant \frac{\epsilon}{h(t)}\int_A^t|g(t-s)|h(s)\,\mathrm{d}s\\ &\leqslant \epsilon\frac{(|g|*h)(t)}{h(t)}.\end{aligned}$$

Hence, by (7.8),

$$\limsup_{t\to\infty}\left|\frac{1}{h(t)}\int_A^t g(t-s)g(s)\,\mathrm{d}s\right|\leqslant\epsilon\int_0^\infty|g(s)|\,\mathrm{d}s. \tag{7.10}$$

Next we note that, for $t>2A$,

$$\begin{aligned}\left|\frac{1}{h(t)}\int_0^A g(t-s)g(s)\,\mathrm{d}s\right| &= \left|\int_0^A\left\{\frac{g(t-s)}{h(t-s)}\left(\frac{h(t-s)}{h(t)}-1\right)+\frac{g(t-s)}{h(t-s)}\right\}g(s)\,\mathrm{d}s\right|\\ &\leqslant\left(1+\sup_{0\leqslant s\leqslant A}\left|\frac{h(t-s)}{h(t)}-1\right|\right)\int_0^A\frac{|g(t-s)|}{h(t-s)}|g(s)|\,\mathrm{d}s\\ &<\epsilon\left(1+\sup_{0\leqslant s\leqslant A}\left|\frac{h(t-s)}{h(t)}-1\right|\right)\int_0^\infty|g(s)|\,\mathrm{d}s.\end{aligned}$$

By (US2), we have

$$\limsup_{t\to\infty}\left|\frac{1}{h(t)}\int_0^A g(t-s)g(s)\,\mathrm{d}s\right|\leqslant\epsilon\int_0^\infty|g(s)|\,\mathrm{d}s. \tag{7.11}$$

By putting (7.10) and (7.11) together, we see that

$$L_h(g*g)=0. \tag{7.12}$$

If f and g are both in BC_h^0, this result can be applied to $L_h((f+g)*(f+g))$ to infer that $L_h(f*g)=0$ and hence that BC_h^0 is a subalgebra. This fact and (7.9) imply that BC_h^l is also a subalgebra.

Finally, to establish (4.1), consider two functions f and g in BC_h^l and express them in the form

$$f=(L_hf)h+\tilde{f},\qquad g=(L_hg)h+\tilde{g}.$$

The formula (4.1) is derived by applying (7.8) and (7.12) to $L_h(f*g)$.

7.7. Proof of lemma 4.3

Since

$$\frac{f(t-s)}{f(t)}=\frac{f(t-s)}{h(t-s)}\cdot\frac{h(t-s)}{h(t)}\cdot\frac{h(t)}{f(t)}$$

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and h satisfies (US2), f must also satisfy (US2). Put $\lambda = L_h f$. By writing

$$f * f = (f - \lambda h) * (f - \lambda h) + 2\lambda f * h - \lambda^2 h * h,$$

and noting that $L_h h = 1$ and $L_h(f - \lambda h) = 0$, we see that theorem 4.1 implies

$$L_h(f * f) = 2\lambda \int_0^\infty f(s)\,\mathrm{d}s.$$

Because $\lambda \neq 0$, f then satisfies (US1). It follows from (7.7) that $(f * f)(t)/h(t) \to 0$ as $t \to 0+$. Since $\lim_{t\to 0+} f(t)/h(t) > 0$, f satisfies (US0), as claimed.

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