Euro. Jnl of Applied Mathematics (2019), *vol.* 30, *pp.* 338–357. © Cambridge University Press 2018 338
doi:10.1017/S0956792518000104

A hunter-gatherer–farmer population model: Lie symmetries, exact solutions and their interpretation

R. M. CHERNIHA and V. V. DAVYDOVYCH

Institute of Mathematics, NAS of Ukraine, 3 Tereshchenkivs'ka Street, 01004 Kyiv, Ukraine
emails: r.m.cherniha@gmail.com, davydovych@imath.kiev.ua

(*Received 7 August 2017; revised 3 February 2018; accepted 5 February 2018; first published online 27 February 2018*)

The Lie symmetry classification of the known three-component reaction–diffusion system modelling the spread of an initially localized population of farmers into a region occupied by hunter-gatherers is derived. The Lie symmetries obtained for reducing the system in question to systems of ordinary differential equations (ODEs) and constructing exact solutions are applied. Several exact solutions of travelling front type are also found, their properties are identified and biological interpretation is discussed.

Key words: Reaction–diffusion system, diffusive Lotka–Volterra system, Lie symmetry, exact solution, travelling front

1 Introduction

It is widely accepted nowadays that non-linear reaction–diffusion (RD) systems are governing equations for many well-known non-linear second-order models used to describe various processes in biology [7, 25, 28], physics [2, 34], chemistry [4] and ecology [30]. The remarkable Turing paper [38] should be mentioned as a pioneering work in this direction.

At the present time, one may claim that non-linear RD systems have been extensively studied by means of different mathematical methods, including symmetry-based (group-theoretical) methods during the last decades. However, the progress is still insufficient, in particular, Lie symmetries are not completely described for many RD systems arising in applications because of principal and technical difficulties. For example, although finding Lie symmetries of the two-component RD systems was initiated about 35 years ago [40], this Lie symmetry classification problem (the terminology 'group classification problem' is also used in this context) was finished only in the 2000s in papers [12–14, 29] (for constant diffusivities) and [15, 16, 24] (for non-constant diffusivities).

In the case of non-linear RD systems with the cross-diffusion, the problem is still open except the case when the system in question involves a constant cross-diffusion only [29]. Notably, Lie symmetries of some non-linear RD systems with correctly specified forms of cross-diffusion arising in real-world applications were studied in [11, 17, 36, 37].

In contrast to the two-component systems, the multi-component RD systems (i.e., those consisting of three and more equations) were not widely examined by symmetry-based methods. To the best of our knowledge, the most general results for the multi-component RD systems (under essential restrictions on the structure of diffusion coefficients) were derived in [16]. There are also some studies (see, e.g., [9]) devoted to the Lie symmetry search of the multi-component RD systems involving only correctly specified functions (i.e., there are no arbitrary functions as parameters). Becausecomplete Lie symmetry classification of the general class of multi-component RD systems is an extremely difficult problem, it is reasonable to restrict ourselves to some systems arising in real world applications.

In this paper, we examine the three-component model introduced in [3] for describing the spread of an initially localized population of farmers into a region occupied by hunter-gatherers. Individuals migrate at random, and some hunter-gatherers are transformed into converted farmers. It is assumed that three populations (initial farmers, hunter-gatherers and converted farmers) are growing logistically and there is no intermarriage among the three populations [3]. Under the assumptions indicated above, the spread and interaction between farmers and hunter-gatherers can be modelled as an RD process. The corresponding non-linear RD system has the form

$$\begin{aligned} F_t &= d_f F_{xx} + r_f F\left(1-(e_1F+e_2C)/K\right), \\ C_t &= d_c C_{xx} + r_c C\left(1-(e_1F+e_2C)/K\right) + e_1FH + e_2CH, \\ H_t &= d_h H_{xx} + r_h H\left(1-H/L\right) - e_1FH - e_2CH, \end{aligned} \tag{1.1}$$

where $F(t,x)$, $C(t,x)$ and $H(t,x)$ are densities of the three populations of initial farmers, converted farmers and hunter-gatherers, respectively. Parameters d_f, d_c and d_h are the positive diffusion constants; r_f, r_c and r_h are the intrinsic growth rates of initial farmers, converted farmers and hunter-gatherers, respectively; K and L are the carrying capacities of farmers and hunter-gatherers, respectively; and e_1 and e_2 are the conversion rates of hunter-gatherers to initial and converted farmers, respectively. Parameters e_2, r_c and r_h are assumed to be non-negative, while all other parameters are assumed to be positive. We note that the equalities $e_1 = e_2$ and $d_f = d_c = d_h$ are assumed in [3]. In our opinion, it is very unlikely that the three populations of initial farmers, converted farmers and hunter-gatherers have the same diffusivity in space; hence, their diffusivities should be assumed arbitrary, i.e., the equality $d_f = d_c = d_h$ can take place only in a special case.

The non-linear RD system (1.1) can be simplified using the following re-scaling of the variables:

$$F \to \frac{K}{e_1}u,\ C \to \frac{KL}{r_f}v,\ H \to Lw,\ t \to \frac{1}{r_f}t,\ x \to \sqrt{\frac{1}{r_f}}\,x, \tag{1.2}$$

and introducing new notation

$$a_1 = \frac{e_2L}{r_f},\ a_2 = \frac{r_c}{r_f},\ a_3 = \frac{r_h}{r_f},\ a_4 = \frac{K}{r_f} \neq 0,\ a_5 = \frac{e_2KL}{r_f^2},\ d_f = d_1,\ d_c = d_2,\ d_h = d_3.$$

https://doi.org/10.1017/S0956792518000104 Published online by Cambridge University Press

Re-scaling (1.2) in symmetry analysis is called the equivalence transformation of system (1.1). Transformation (1.2) reduce system (1.1) to the equivalent form

$$\begin{aligned} u_t &= d_1u_{xx} + u(1-u-a_1v), \\ v_t &= d_2v_{xx} + a_2v(1-u-a_1v) + uw + a_1vw, \\ w_t &= d_3w_{xx} + a_3w(1-w) - a_4uw - a_5vw. \end{aligned} \tag{1.3}$$

Hereafter, (1.3) is called the hunter-gatherer–farmer (HGF) system and one is the main object of investigation in this paper. We naturally assume that $a_4 \neq 0$ (otherwise, $K = 0$ in (1.1)) and $d_1d_2d_3 \neq 0$.

The paper is organized as follows: In Section 2, the Lie symmetry classification of the HGF system (1.3) is derived. In Section 3, the most important (from applicability point of view) cases of system (1.3) with non-trivial Lie symmetries are examined. In particular, non-trivial Lie ansätze are derived and applied for reducing the systems in question to systems of ODEs. The reduced systems are analyzed in order to construct exact solutions. In Section 4, the travelling fronts (TFs) of the HGF system (1.3) with correctly specified coefficients are constructed in explicit forms. The properties of TFs obtained are analysed and some biological interpretation is presented. Finally, we briefly discuss the result obtained and present some conclusions in the last section.

2 Main theorem

To find Lie invariance operators, one needs to consider system (1.3) as the manifold

$$\mathcal{M} = \{S_1 = 0, S_2 = 0, S_3 = 0\},$$

where

$$\begin{aligned} S_1 &\equiv d_1u_{xx} - u_t + u(1-u-a_1v), \\ S_2 &\equiv d_2v_{xx} - v_t + a_2v(1-u-a_1v) + uw + a_1vw, \\ S_3 &\equiv d_3w_{xx} - w_t + a_3w(1-w) - a_4uw - a_5vw, \end{aligned}$$

in the prolonged space of the variables

$$t, x, \; u, v, w, \; u_t, v_t, w_t, u_x, v_x, w_x, \; u_{xx}, v_{xx}, w_{xx}, u_{xt}, v_{xt}, w_{xt}, u_{tt}, v_{tt}, w_{tt}.$$

According to the Lie invariance criterion, system (1.3) is invariant under the Lie group generated by the infinitesimal operator:

$$X = \xi^0(t,x,u,v,w)\partial_t + \xi^1(t,x,u,v,w)\partial_x + \\ \eta^1(t,x,u,v,w)\partial_u + \eta^2(t,x,u,v,w)\partial_v + \eta^3(t,x,u,v,w)\partial_w,$$

if the following Lie's invariance conditions are satisfied:

$$\underset{2}{X}(S_1)\Big|_{\mathcal{M}} = 0, \quad \underset{2}{X}(S_2)\Big|_{\mathcal{M}} = 0, \quad \underset{2}{X}(S_3)\Big|_{\mathcal{M}} = 0, \tag{2.1}$$

where the operator $\underset{2}{X}$ is the second prolongation of the operator X (see, e.g., [5, 6, 10, 19, 31, 32]).

Obviously, system (1.3) admits the Lie algebra with the basic operators

$$P_t = \partial_t,\ P_x = \partial_x, \tag{2.2}$$

because one is invariant with respect to the time and space translations. It can be easily shown that (2.2) is the principal (trivial) algebra of system (1.3), i.e., this is maximal invariance algebra of this system with arbitrary coefficients a_j and d_k. To find all possible extensions of principal algebra in the case of the system (1.3), one needs to apply the invariance criterion (2.1) and to solve the corresponding system of determining equations (DEs). Omitting rather standard calculations (nowadays they can be done using Maple, Mathematica, etc.), we present the DE system obtained:

$$\xi^0_x = \xi^0_u = \xi^0_v = \xi^0_w = \xi^1_u = \xi^1_v = \xi^1_w = 0, \tag{2.3}$$

$$\eta^k_{uu} = \eta^k_{uv} = \eta^k_{vv} = \eta^k_{ww} = \eta^k_{uw} = \eta^k_{vw} = 0,\ k = 1, 2, 3, \tag{2.4}$$

$$\eta^1_{xv} = \eta^1_{xw} = \eta^2_{xu} = \eta^2_{xw} = \eta^3_{xu} = \eta^3_{xv} = 0, \tag{2.5}$$

$$\begin{gathered}(d_1 - d_2)\eta^1_v = (d_1 - d_3)\eta^1_w = (d_1 - d_2)\eta^2_u = \\ (d_2 - d_3)\eta^2_w = (d_1 - d_3)\eta^3_u = (d_2 - d_3)\eta^3_v = 0,\end{gathered} \tag{2.6}$$

$$2\xi^1_x - \xi^0_t = 0,\ 2d_1\eta^1_{xu} + \xi^1_t = 0,\ 2d_2\eta^2_{xv} + \xi^1_t = 0,\ 2d_3\eta^3_{xw} + \xi^1_t = 0, \tag{2.7}$$

$$\eta^1 C^1_u + \eta^2 C^1_v + \eta^3 C^1_w + (2\xi^1_x - \eta^1_u)C^1 = \eta^1_t - d_1\eta^1_{xx} + \frac{d_1}{d_2}\eta^1_v C^2 + \frac{d_1}{d_3}\eta^1_w C^3, \tag{2.8}$$

$$\eta^1 C^2_u + \eta^2 C^2_v + \eta^3 C^2_w + (2\xi^1_x - \eta^2_v)C^2 = \eta^2_t - d_2\eta^2_{xx} + \frac{d_2}{d_1}\eta^2_u C^1 + \frac{d_2}{d_3}\eta^2_w C^3, \tag{2.9}$$

$$\eta^1 C^3_u + \eta^2 C^3_v + \eta^3 C^3_w + (2\xi^1_x - \eta^3_w)C^3 = \eta^3_t - d_3\eta^3_{xx} + \frac{d_3}{d_1}\eta^3_u C^1 + \frac{d_3}{d_2}\eta^3_v C^2, \tag{2.10}$$

where

$$\begin{aligned} C^1 &= u(1 - u - a_1 v), \\ C^2 &= a_2 v(1 - u - a_1 v) + uw + a_1 vw, \\ C^3 &= a_3 w(1 - w) - a_4 uw - a_5 vw. \end{aligned} \tag{2.11}$$

Now, we want to find all possible values of the coefficients a_j and d_k leading to extensions of the principal algebra (2.2). It means that all inequivalent solutions of the system of DEs (2.3)–(2.10) (under restrictions (2.11) on the functions C^1, C^2 and C^3) should be constructed. As a result, the following statement was proved.

Theorem 2.1 *The HGF system (1.3) with* $a_4 d_1 d_2 d_3 \neq 0$ *admits a non-trivial Lie algebra of symmetries if and only if its reaction terms have the forms listed in the second column of Table 1. The corresponding Lie symmetry operators generating the maximal algebra of invariance are listed in the last column of Table 1.*

Sketch of the proof. In order to prove the theorem, one needs to solve the system of DEs (2.3)–(2.10) with the functions C^k ($k = 1, 2, 3$) from (2.11). Although this is a

Table 1. *Lie symmetry operators of the HGF system (1.3)*

	Reaction terms	Restrictions	Lie symmetries
(1)	$u(1-u)$ $a_2v(1-u)+uw$ $-a_4uw$	$a_2 \neq 0$	$\partial_t,\ \partial_x,\ I = v\partial_v + w\partial_w$
(2)	$u(1-u)$ uw $a_3w(1-w)-a_4uw$	$a_3 \neq 0$	$\partial_t,\ \partial_x,$ $X^\infty = P(t,x)\partial_v,$ $P_t = d_2P_{xx}$
(3)	$u(1-u)$ uw $-a_4uw$		$\partial_t,\ \partial_x,\ I,\ X^\infty$
(4)	$u(1-u-a_1v)$ $v(1-u-a_1v)+uw+a_1vw$ $a_3w(1-w)-a_4uw-a_1a_4vw$	$d_1 = d_2$ $a_1 \neq 0$	$\partial_t,\ \partial_x,$ $Q_1 = -a_1u\partial_u + u\partial_v$
(5)	$u(1-u)$ $v(1-u)+uw$ $a_3w(1-w)-a_4uw$	$d_1 = d_2$ $a_3 \neq 0$	$\partial_t,\ \partial_x,$ $u\partial_v,\ Q_2 = e^t(u-1)\partial_v$
(6)	$u(1-u)$ $v(1-u)+uw$ $-a_4uw$	$d_1 = d_2$	$\partial_t,\ \partial_x,\ u\partial_v,\ I,\ Q_2$
(7)	$u(1-u)$ $a_4v(1-u)+uw$ $-a_4uw$	$d_2 = d_3$	$\partial_t,\ \partial_x,\ e^{a_4t}w\partial_v,\ I$
(8)	$u(1-u)$ uw $-a_4uw$	$d_2 = d_3$	$\partial_t,\ \partial_x,$ $w\partial_v - a_4w\partial_w,\ I,\ X^\infty$
(9)	$u(1-u-a_1v)$ $v(1-u-a_1v)+uw+a_1vw$ $-a_4uw-a_1a_4vw$	$d_1 = d_2 = d_3$ $a_1 \neq 0$	$\partial_t,\ \partial_x,\ Q_1,$ $e^t\left(\frac{a_4-1}{a_1}u+(a_4-1)v+\right.$ $\left.w+\frac{1-a_4}{a_1}\right)\left(\partial_u-\frac{1}{a_1}\partial_v\right)$

https://doi.org/10.1017/S0956792518000104 Published online by Cambridge University Press

Table 1. *Continued*

	Reaction terms	Restrictions	Lie symmetries
(10)	$u(1-u)$ $a_2v(1-u)+uw$ $-uw$	$d_1=d_2=d_3$ $a_2\neq 0,\ a_2\neq 1$	$\partial_t,\ \partial_x,\ I,$ $u\partial_v+(a_2-1)(u-1)\partial_w$
(11)	$u(1-u)$ $v(1-u)+uw$ $-uw$	$d_1=d_2=d_3$	$\partial_t,\ \partial_x,$ $u\partial_v,\ we^t\partial_v,\ I,\ Q_2$
(12)	$u(1-u)$ uw $-uw$	$d_1=d_2=d_3$	$\partial_t,\ \partial_x,\ w\partial_v-w\partial_w,$ $u\partial_v+(1-u)\partial_w,$ $ue^{-t}(\partial_v-\partial_w),\ I,\ X^\infty$

standard routine at the present time, all possible special cases (not some of them) should be identified and examined in order to obtain a full Lie symmetry classification.

It can be noted that the forms of the functions ξ^0, ξ^1 and η^k $(k=1,2,3)$ can be defined independently on the functions C^k. In fact, equations (2.3)–(2.5) can be easily integrated:

$$\begin{aligned}
&\xi^0=\xi^0(t),\ \xi^1=\xi^1(t,x),\\
&\eta^1=r^1(t,x)u+q^1(t)v+h^1(t)w+p^1(t,x),\\
&\eta^2=r^2(t,x)v+q^2(t)u+h^2(t)w+p^2(t,x),\\
&\eta^3=r^3(t,x)w+q^3(t)u+h^3(t)v+p^3(t,x),
\end{aligned}$$

where ξ^0, ξ^1, r^k, q^k, h^k and p^k $(k=1,2,3)$ are to-be-determined functions.

Now, we analyse equation (2.6). It can be easily seen that five different cases should be examined depending on diffusion coefficients, namely: (I) d_k are arbitrary positive constants, (II) $d_1=d_2$, (III) $d_1=d_3$, (IV) $d_2=d_3$ and (V) $d_1=d_2=d_3$.

Let us examine case (I). Because the diffusivities d_k $(k=1,2,3)$ are arbitrary constants, equation (2.6) immediately produces $q^k=h^k=0,\ k=1,2,3$. Equations (2.8)–(2.10) can be split with respect to the variables u,v,w and their products $uv,uw,vw,u^2,\ v^2,w^2$. As a result, the system of DEs (2.3)–(2.10) reduces to the form

$$a_1p^1=0,\ p^1+a_1p^2=0,\ -a_2p^2+p^3=0,\ a_4p^3=0,\ a_5p^3=0,\ r^2=r^3, \tag{2.12}$$

$$2\xi^1_x-\xi^0_t=0, r^1+2\xi^1_x=0, 2d_1r^1_x+\xi^1_t=0, 2d_2r^2_x+\xi^1_t=0,\ 2d_3r^3_x+\xi^1_t=0, \tag{2.13}$$

$$a_1(r^2+2\xi^1_x)=0,\ a_3(r^2+2\xi^1_x)=0,\ a_5(r^2+2\xi^1_x)=0, \tag{2.14}$$

$$d_1r^1_{xx}-r^1_t+2\xi^1_x-2p^1-a_1p^2=0, \tag{2.15}$$

$$d_2r^2_{xx}-r^2_t+2a_2\xi^1_x-a_2p^1-2a_1a_2p^2+a_1p^3=0, \tag{2.16}$$

$$d_3r^3_{xx}-r^3_t+2a_3\xi^1_x-a_4p^1-a_5p^2-2a_3p^3=0, \tag{2.17}$$

$$d_1p^1_{xx}-p^1_t+p^1=0,\ d_2p^2_{xx}-p^2_t+a_2p^2=0,\ d_3p^3_{xx}-p^3_t+a_3p^3=0. \tag{2.18}$$

Because (2.12) is the set of algebraic equations, we find $p^1 = p^3 = 0$ and $a_1a_2p^2 = 0$, while the overdetermined system (2.13) leads to

$$\xi^0_{tt} = \xi^1_{xx} = \xi^1_t = r^1_x = r^1_t = r^2_x = 0.$$

Hence, equation (2.15) produces $\xi^1_x = 0$. Having $\xi^1_x = 0$, equations (2.14) give $r^2 = r^3 = 0$ provided $a_1^2 + a_3^2 + a_5^2 \neq 0$. In this case, one can find non-trivial Lie symmetry only under the restriction $p^2 \neq 0$; hence, $a_1 = a_2 = a_5 = 0$. Thus, the general solution of (2.12)–(2.18) has the form

$$\xi^0 = c_0,\ \xi^1 = c_1,\ p^1 = p^3 = 0,\ r^1 = r^2 = r^3 = 0,\ p^2 = P(t,x),$$

hereafter, c_k $(k = 0, 1, \dots)$ is arbitrary constant, while the function $P(t,x)$ is an arbitrary solution of equation $P_t = d_2P_{xx}$); therefore, Case 2 of Table 1 is obtained.

In the case $a_1 = a_3 = a_5 = 0$, we obtain Cases 1 and 3 of Table 1. Thus, case I) is completely examined.

Now, we turn to case (II). Having done a preliminary analysis, we find

$$\begin{aligned} &q^1 = q^3 = h^k = 0,\ k = 1,2,3, \\ &p^1 = p^3 = 0,\ a_1p^2 = 0, \end{aligned} \tag{2.19}$$

and derive the system of DEs

$$2\xi^1_x - \xi^0_t = 0,\ 2d\eta^1_{xu} + \xi^1_t = 0,\ 2d\eta^2_{xv} + \xi^1_t = 0,\ 2d_3\eta^3_{xw} + \xi^1_t = 0, \tag{2.20}$$

$$dr^1_{xx} - r^1_t + 2\xi^1_x = 0, dr^2_{xx} - r^2_t + 2a_2\xi^1_x = 0, d_3r^3_{xx} - r^3_t + 2a_3\xi^1_x - a_5p^2 = 0, \tag{2.21}$$

$$a_1(r^2 + 2\xi^1_x) = 0,\ a_5(r^2 + 2\xi^1_x) = 0,\ a_1(r^3 + 2\xi^1_x) = 0,\ a_3(r^3 + 2\xi^1_x) = 0, \tag{2.22}$$

$$(-1 + a_2)\, q^2 = 0,\ a_1\,(-1 + 2a_2)\, q^2 + a_2\left(r^1 + 2\xi^1_x\right) = 0, \tag{2.23}$$

$$a_5q^2 + a_4\left(r^1 + 2\xi^1_x\right) = 0,\ a_1q^2 + r^1 + 2\xi^1_x = 0, \tag{2.24}$$

$$a_1q^2 + r^1 - r^2 + r^3 + 2\xi^1_x = 0, \tag{2.25}$$

$$dq^2_{xx} - q^2_t + (a_2 - 1)\, q^2 - a_2p^2 = 0, \tag{2.26}$$

$$dp^2_{xx} - p^2_t + a_2p^2 = 0, \tag{2.27}$$

for finding all other functions.

It can be seen from (2.19) that new non-trivial Lie symmetries can exist only if $q^2 \neq 0$ (otherwise, one obtains the result of case (I)). Thus, the first equation of (2.23) immediately produces $a_2 = 1$, while restriction $a_5 = a_1a_4$ follows from the compatibility condition of equations (2.24).

The further analysis of the system of DEs (2.20)–(2.27) depends on the value of constant a_1.

If $a_1 \neq 0$, then $p^2 = 0$ and $r^2 = r^3 = -2\xi^1_x$. As a result, equations (2.20) and (2.21) produce $r^2 = r^3 = 0,\ r^1 = const,\ \xi^0 = c_0,\ \xi^1 = c_1$. The last unknown function q^2 can be found from (2.24). Hence,

$$\xi^0 = c_0,\ \xi^1 = c_1,\ \eta^1 = -a_1c_3u,\ \eta^2 = c_3u,\ \eta^3 = 0,$$

and Case 4 of Table 1 is obtained. In a quite similar way, one examines the sub-case $a_1 = 0$ and arrives at Cases 5 and 6 of Table 1.

The examination of the system of DEs in case (III) does not lead to new system of the form (1.3) with non-trivial Lie symmetries.

Analysis of case (IV) leads to systems and Lie symmetries listed in Cases 7 and 8 of Table 1, while case (V) produces Cases 9–12 of Table 1. The relevant calculations are omitted here.

The sketch of the proof is now completed. □

3 Reduction of the HGF system to ODE systems

In this section, we present examples of reductions of the HGF system (1.3) to ODE systems using the Lie symmetries obtained. If one compares system (1.3) with the reaction terms arising in Table 1 with its general form (1.3), then one realizes that Cases 4, 5 and 9 are the most interesting from the applicability point of view. Notably, Cases 4 and 9 only involve the HGF system with $a_1 \neq 0$, and this means that there is an interaction between initial farmers and converted farmers. Case 5 corresponds to the most general HGF system among those with $a_1 = 0$. In fact, all the other cases of Table 1 lead to the systems of the form (1.3) with too many zero coefficients; hence, it is unlikely that such systems can describe adequately the spread and interaction between farmers and hunter-gatherers. For this reason, we consider the systems from Cases 4, 5 and 9 of Table 1 and the relevant linear combinations of the Lie symmetries involving non-trivial operators. The case of the Lie symmetry operators leading to plane wave solutions, especially TFs, is examined separately in Section 4.

3.1 Case 4 of Table 1

First of all, we note that one diffusivity, e.g., d_1, can be set 1 in (1.3) without losing a generality; hence, the system from Case 4 of Table 1 has the form

$$\begin{aligned} u_t &= u_{xx} + u(1 - u - a_1 v), \ \ a_1 \neq 0, \\ v_t &= v_{xx} + v(1 - u - a_1 v) + uw + a_1 vw, \\ w_t &= dw_{xx} + a_3 w(1 - w) - a_4 uw - a_1 a_4 vw. \end{aligned} \tag{3.1}$$

Let us consider two essentially different linear combinations of the Lie symmetry operators of system (3.1)

$$X = \partial_t + \alpha \partial_x - \beta a_1 u \partial_u + \beta u \partial_v \tag{3.2}$$

and

$$X = \partial_x - \beta a_1 u \partial_u + \beta u \partial_v. \tag{3.3}$$

Hereafter, α and $\beta \neq 0$ are arbitrary constants.

Solving the characteristic equation

$$\frac{dt}{1} = \frac{dx}{\alpha} = \frac{du}{-\beta a_1 u} = \frac{dv}{\beta u}$$

corresponding to operator (3.2), one obtains the ansatz

$$\begin{array}{l} u = e^{-\beta a_1 t} U(\omega), \ \omega = x - \alpha t, \\ v = V(\omega) - \frac{1}{a_1} e^{-\beta a_1 t} U(\omega), \\ w = W(\omega), \end{array} \tag{3.4}$$

where U, V and W are new unknown functions. Substituting ansatz (3.4) into (3.1), we arrive at the system of ODEs

$$\begin{array}{l} U'' + \alpha U' + U(1 + a_1\beta - a_1 V) = 0, \\ V'' + \alpha V' + V(1 - a_1 V + a_1 W) = 0, \\ dW'' + \alpha W' + a_3 W(1 - W) - a_1 a_4 V W = 0. \end{array} \tag{3.5}$$

One sees that the reduced system (3.5) is non-linear and the problem of constructing its exact solutions is still a difficult task. However, we were able to note the three special cases:

$$\text{(i) } d = a_3 = 1, \quad \text{(ii) } d = 1, \ a_3 = 0, \quad \text{(iii) } d = 1, \ a_4 = 1 + a_1 + a_3, \ a_3 \neq 0,$$

when the functions V and W can be found from the second and the third ODEs of system (3.5), and then U can be extracted from the first ODE of (3.5). In fact, if one assumes that the components V and W have the same structure as the well-known solution of the Fisher equation [1] (see formula (3.18) below), then the case (i) leads to the exact solution

$$\begin{array}{l} u = e^{-\beta a_1 t} U(\omega), \ \omega = x - \dfrac{5}{\sqrt{6}}\, t, \\ v = \dfrac{\kappa}{4a_1}\left(1 - \tanh\left[\dfrac{1}{2\sqrt{6}}\,\omega\right]\right)^2 - \dfrac{1}{a_1}\, e^{-\beta a_1 t} U(\omega), \\ w = \dfrac{1 - a_4}{4(1 + a_1 a_4)}\left(1 - \tanh\left[\dfrac{1}{2\sqrt{6}}\,\omega\right]\right)^2, \end{array}$$

the case (ii) gives the solution

$$\begin{array}{l} u = e^{-\beta a_1 t} U(\omega), \ \omega = x - \dfrac{5\sqrt{a_4}}{\sqrt{6}}\, t, \\ v = \dfrac{1}{4a_1}\left(1 - \tanh\left[\dfrac{\sqrt{a_4}}{2\sqrt{6}}\,\omega\right]\right)^2 - \dfrac{1}{a_1}\, e^{-\beta a_1 t} U(\omega), \\ w = \dfrac{1 - a_4}{4a_1}\left(\left(1 - \tanh\dfrac{\sqrt{a_4}}{2\sqrt{6}}\,\omega\right)^2 - 4\right), \end{array}$$

and the case (iii) leads to the solution

$$u = e^{-\beta a_1 t} U(\omega),\ \omega = x - \frac{5\sqrt{1+a_1}}{\sqrt{6}}\, t,$$
$$v = \frac{1}{4a_1}\left(1 - \tanh\left[\frac{\sqrt{1+a_1}}{2\sqrt{6}}\,\omega\right]\right)^2 - \frac{1}{a_1}\, e^{-\beta a_1 t} U(\omega),$$
$$w = 1 - \frac{1}{4}\left(1 - \tanh\frac{\sqrt{1+a_1}}{2\sqrt{6}}\,\omega\right)^2.$$

Here, $\kappa = \frac{1+a_1}{1+a_1 a_4}$, while the function U is an arbitrary solution of the linear ODE

$$U'' + \alpha U' + U\left(1 + a_1\beta - \kappa_1\left(1 - \tanh\left[\frac{\kappa_2}{2\sqrt{6}}\,\omega\right]\right)^2\right) = 0, \tag{3.6}$$

where

$$\kappa_1 = \begin{cases} \frac{1}{4}\kappa & \text{in case (i)},\\ \frac{1}{4} & \text{in case (ii)},\\ \frac{1}{4} & \text{in case (iii)}, \end{cases} \qquad \kappa_2 = \begin{cases} 1 & \text{in case (i)},\\ \sqrt{a_4} & \text{in case (ii)},\\ \sqrt{1+a_1} & \text{in case (iii)}. \end{cases}$$

Ansatz corresponding to operator (3.3) and the reduced system for system (3.1) have the forms

$$u = U(t)e^{-\beta a_1 x}, \quad v = V(t) - \tfrac{e^{-\beta a_1 x}}{a_1}\, U(t), \quad w = W(t) \tag{3.7}$$

and

$$\begin{aligned} &U' + U(a_1 V - 1 - \beta^2 a_1^2) = 0,\\ &V' + V(a_1 V - a_1 W - 1) = 0,\\ &W' + W(a_3 W + a_1 a_4 V - a_3) = 0. \end{aligned} \tag{3.8}$$

We have solved system (3.8) assuming that the functions V and W are linearly dependent. In a such way, three different cases

$$\text{(i) } a_3 = 1, \quad \text{(ii) } a_3 = 0, \quad \text{(iii) } a_4 = 1 + a_1 + a_3,\ a_3 \neq 0$$

hold. Thus, case (i) gives the exact solution

$$U = \frac{\delta_2 \exp\left(t + \beta^2 a_1^2 t\right)}{(1 - \delta_1 + \delta_1 e^t)^{\kappa}},\ V = \frac{\kappa\delta_1 e^t}{a_1\,(1 - \delta_1 + \delta_1 e^t)},\ W = \frac{\kappa\delta_1(1 - a_4)e^t}{(1 + a_1)\,(1 - \delta_1 + \delta_1 e^t)}, \tag{3.9}$$

the case (ii) leads to the solution

$$U = \frac{\delta_2 \exp\left(t + \beta^2 a_1^2 t\right)}{(1 - \delta_1 + \delta_1 e^{a_4 t})^{\frac{1}{a_4}}},\ V = \frac{\delta_1 e^{a_4 t}}{a_1\,(1 - \delta_1 + \delta_1 e^{a_4 t})},\ W = \frac{(1 - a_4)(\delta_1 - 1)}{a_1\,(1 - \delta_1 + \delta_1 e^{a_4 t})}, \tag{3.10}$$

while case (iii) gives the solution

$$U = \frac{\delta_2 \exp\left(t + \beta^2 a_1^2 t\right)}{\left(1 - \delta_1 + \delta_1 e^{(1+a_1)t}\right)^{\frac{1}{1+a_1}}}, \ V = \frac{\delta_1 e^{(1+a_1)t}}{a_1\left(1 - \delta_1 + \delta_1 e^{(1+a_1)t}\right)}, \ W = \frac{1-\delta_1}{1 - \delta_1 + \delta_1 e^{(1+a_1)t}} \tag{3.11}$$

of system (3.8). Here, δ_1 and δ_2 are arbitrary positive constants, while $\kappa = \frac{1+a_1}{1+a_1 a_4}$.

Let us consider the most interesting solution (3.9) from the applicability point of view in detail. In fact, the first component U in solutions (3.10) and (3.11) tends to infinity as $t \to +\infty$, while U is bounded in (3.9) for correctly specified κ. Substituting (3.9) into ansatz (3.7), the three-parameter family of exact solutions

$$\begin{aligned} u &= \frac{\delta_2 \exp\left((1+\beta^2 a_1^2)t - \beta a_1 x\right)}{(1-\delta_1+\delta_1 e^t)^\kappa}, \\ v &= \frac{\kappa \delta_1 e^t}{a_1\,(1-\delta_1+\delta_1 e^t)} - \frac{\delta_2 \exp\left((1+\beta^2 a_1^2)t - \beta a_1 x\right)}{a_1\,(1-\delta_1+\delta_1 e^t)^\kappa}, \\ w &= \frac{\kappa\delta_1(1-a_4)e^t}{(1+a_1)\,(1-\delta_1+\delta_1 e^t)} \end{aligned} \tag{3.12}$$

of system (3.1) with $a_3 = 1$ is obtained.

It can be noted that the components of exact solutions of the form (3.12) are non-negative on the space interval $x \in (0, +\infty)$ provided the restrictions

$$\beta = \sqrt{\frac{1-a_4}{a_1(1+a_1 a_4)}}, \ a_4 < 1, \ \delta_1 > 1, \ \delta_2 < \kappa \tag{3.13}$$

hold. In this case, the solutions possess the asymptotical behaviour

$$u \to \delta_2 \delta_1^{-\kappa} e^{-\beta a_1 x}, \ v \to \frac{\kappa}{a_1} - \frac{\delta_2}{a_1}\,\delta_1^{-\kappa} e^{-\beta a_1 x}, \ w \to \frac{1-a_4}{1+a_1 a_4} \tag{3.14}$$

as $t \to \infty$.

Such behaviour predicts the scenario when the populations of initial farmers, converted farmers and hunter-gatherers co-exist in space-time; moreover, the distribution of two populations is inhomogeneous as $t \to \infty$. Notably, this scenario occurs at any semi-finite interval (instead of the fixed interval $(0, +\infty)$) because system (3.1) is invariant with respect to the space translations.

3.2 Case 5 of Table 1

The system and the most general linear combinations of the Lie symmetries from Case 5 of Table 1 have the forms

$$\begin{aligned} u_t &= u_{xx} + u(1-u), \\ v_t &= v_{xx} + v(1-u) + uw, \\ w_t &= d w_{xx} + a_3 w(1-w) - a_4 uw \end{aligned} \tag{3.15}$$

https://doi.org/10.1017/S0956792518000104 Published online by Cambridge University Press

and

$$X = \partial_t + \alpha\partial_x + \beta u\partial_v + \gamma e^t(u-1)\partial_v, \tag{3.16}$$

$$X = \partial_x + \beta u\partial_v + \gamma e^t(u-1)\partial_v. \tag{3.17}$$

As one can see, the first equation of system (3.15) is the famous Fisher equation [18] that is not integrable. There were many attempts to construct its exact solutions taking into account some reasonable initial and boundary conditions. In particular, the appropriate exact solution in the form of the TF

$$u \equiv U(\omega) = \frac{1}{4}\left(1 - \tanh\left[\frac{1}{2\sqrt{6}}\,\omega\right]\right)^2, \quad \omega = x - \frac{5}{\sqrt{6}}\,t \tag{3.18}$$

was found in [1]. We remind the reader that a plane wave solution of a partial differential equation (PDE), which is non-negative, bounded and satisfies the zero Neumann conditions at infinity, is usually called TF.

Ansatz corresponding to (3.16) and the reduced system for system (3.15) have the forms

$$\begin{aligned} &u = U(\omega), \ \omega = x - \alpha t,\\ &v = V(\omega) + \left(\beta t + \gamma e^t\right) U(\omega) - \gamma e^t,\\ &w = W(\omega) \end{aligned}$$

and

$$\begin{aligned} &U'' + \alpha U' + U(1-U) = 0,\\ &V'' + \alpha V' + V(1-U) + U\,(W-\beta) = 0,\\ &dW'' + \alpha W' + a_3 W(1-W) - a_4 UW = 0. \end{aligned} \tag{3.19}$$

It can be noted that the last equation of system (3.19) with the function U from (3.18) has the solutions

$$W = \frac{1-a_4}{4}\left(1 - \tanh\left[\frac{1}{2\sqrt{6}}\,\omega\right]\right)^2, \tag{3.20}$$

if $d = a_3 = 1$ and

$$W = 1 - \frac{1}{4}\left(1 - \tanh\left[\frac{1}{2\sqrt{6}}\,\omega\right]\right)^2, \tag{3.21}$$

if $d = 1, \ a_4 = 1 + a_3$.

Thus, we obtain the solutions of the HGF system (3.15) in the forms

$$\begin{aligned} &u = \tfrac{1}{4}\left(1 - \tanh\left[\tfrac{1}{2\sqrt{6}}\,\omega\right]\right)^2, \ \omega = x - \tfrac{5}{\sqrt{6}}\,t,\\ &v = V(\omega) + \tfrac{1}{4}\left(\beta t + \gamma e^t\right)\left(1 - \tanh\left[\tfrac{1}{2\sqrt{6}}\,\omega\right]\right)^2 - \gamma e^t,\\ &w = \tfrac{1-a_4}{4}\left(1 - \tanh\left[\tfrac{1}{2\sqrt{6}}\,\omega\right]\right)^2, \end{aligned} \tag{3.22}$$

if $d = a_3 = 1$ and

$$\begin{aligned} u &= \tfrac{1}{4}\left(1 - \tanh\left[\tfrac{1}{2\sqrt{6}}\,\omega\right]\right)^2, \ \omega = x - \tfrac{5}{\sqrt{6}}\,t, \\ v &= V(\omega) + \tfrac{1}{4}\left(\beta t + \gamma e^t\right)\left(1 - \tanh\left[\tfrac{1}{2\sqrt{6}}\,\omega\right]\right)^2 - \gamma e^t, \\ w &= 1 - \tfrac{1}{4}\left(1 - \tanh\left[\tfrac{1}{2\sqrt{6}}\,\omega\right]\right)^2 \end{aligned} \tag{3.23}$$

if $d = 1,\ a_4 = 1 + a_3$. In (3.22) and (3.23), the function V is an arbitrary solution of the linear ODE

$$V'' + \alpha V' + V(1 - U) + U\,(W - \beta) = 0 \tag{3.24}$$

with W from (3.20) and (3.21), respectively, while U is given by formula (3.18).

Remark 1 Although ODEs (3.6) and (3.24) are linear, we were unable to solve them exactly because they contain very complicated non-constant coefficients. Moreover, their solutions are not listed in the well-known handbooks like [22, 33].

Ansatz corresponding to operator (3.17) have the form

$$u = U(t), \quad v = V(t) + \left(\beta t + \gamma e^t\right) x\,U(t) - \gamma e^t x, \quad w = W(t). \tag{3.25}$$

Note that the exact solutions of the form (3.25) are not important from the applicability point of view because two components (u and w) depend only on the variable t.

3.3 Case 9 of Table 1

Finally, we examine the HGF system

$$\begin{aligned} u_t &= u_{xx} + u(1 - u - a_1 v), \ a_1 \neq 0, \\ v_t &= v_{xx} + v(1 - u - a_1 v) + uw + a_1 vw, \\ w_t &= w_{xx} - a_4 uw - a_1 a_4 vw, \end{aligned} \tag{3.26}$$

corresponding to Case 9 of Table 1. The most general linear combinations of its Lie symmetries

$$\begin{aligned} X = \partial_t + \alpha\partial_x + \beta(-a_1 u\partial_u + u\partial_v) + \\ \gamma e^t\left(\tfrac{a_4 - 1}{a_1}\,u + (a_4 - 1)v + w + \tfrac{1 - a_4}{a_1}\right)\left(\partial_u - \tfrac{1}{a_1}\partial_v\right) \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} X = \partial_x + \beta(-a_1 u\partial_u + u\partial_v) + \\ \gamma e^t\left(\tfrac{a_4 - 1}{a_1}\,u + (a_4 - 1)v + w + \tfrac{1 - a_4}{a_1}\right)\left(\partial_u - \tfrac{1}{a_1}\partial_v\right) \end{aligned} \tag{3.28}$$

lead to the ansätze and the reduced systems for system (3.26) presented in Table 2.

4 Travelling wave solutions and their interpretation

In this section, we look for TFs (a special sub-class of the plane wave solutions) of the HGF system (1.3). TFs are the most common in theoretical and applied studies of

https://doi.org/10.1017/S0956792518000104 Published online by Cambridge University Press

Table 2. *Reductions of the HGF system (3.26)*

Operator	Ansatz	Reduced system
(3.27) with $1+\beta a_1 \neq 0$	$u = e^{-\beta a_1 t}U(\omega) + \frac{\gamma e^t}{1+\beta a_1}\left((a_4-1)V(\omega)+ W(\omega) + (1-a_4)/a_1\right),\ \omega = x - \alpha t,$ $v = V(\omega) - \frac{u}{a_1},$ $w = W(\omega)$	$U'' + \alpha U' + U(1 + a_1\beta - a_1 V) = 0,$ $V'' + \alpha V' + V(1 - a_1 V + a_1 W) = 0,$ $W'' + \alpha W' - a_1 a_4 V W = 0$
(3.27) with $1+\beta a_1 = 0$	$u = e^t\left(U(\omega) + \gamma\left((a_4-1)V(\omega)+ W(\omega) + (1-a_4)/a_1\right)t\right),\ \omega = x - \alpha t,$ $v = V(\omega) - \frac{u}{a_1},$ $w = W(\omega)$	$U'' + \alpha U' - a_1 U V -$ $\gamma\left((a_4-1)V + W + (1-a_4)/a_1\right) = 0,$ $V'' + \alpha V' + V(1 - a_1 V + a_1 W) = 0,$ $W'' + \alpha W' - a_1 a_4 V W = 0,$
(3.28) with $\beta \neq 0$	$u = e^{-\beta a_1 x}U(t) + \frac{\gamma e^t}{\beta a_1}((a_4-1)V(t)+ W(t) + (1-a_4)/a_1),$ $v = V(t) - \frac{u}{a_1},$ $w = W(t)$	$U' + U\left(a_1 V - 1 - a_1^2\beta^2\right) = 0,$ $V' + V(a_1 V - a_1 W - 1) = 0,$ $W' + a_1 a_4 V W = 0,$
(3.28) with $\beta = 0$	$u = U(t) + \gamma e^t((a_4-1)V(t)+ W(t) + (1-a_4)/a_1)x,$ $v = V(t) - \frac{u}{a_1},$ $w = W(t)$	$U' + U(a_1 V - 1) = 0,$ $V' + V(a_1 V - a_1 W - 1) = 0,$ $W' + a_1 a_4 V W = 0,$

non-linear real world models (see, e.g., [7, 27, 28]). In the case of a single RD equation, a substantial number of such solutions are presented in [20]. Although paper [3] devoted to study TFs of system (1.1), such solutions are not explicitly presented therein. Here, we construct several TFs of the HGF system (1.3) and present their interpretation.

As we noted above, one diffusivity can be set 1 in (1.3) without losing a generality; hence, we consider system

$$\begin{aligned} u_t &= u_{xx} + u(1 - u - a_1 v),\\ v_t &= d_2 v_{xx} + a_2 v(1 - u - a_1 v) + uw + a_1 vw,\\ w_t &= d_3 w_{xx} + a_3 w(1 - w) - a_4 uw - a_5 vw \end{aligned} \tag{4.1}$$

in what follows. Because system (4.1) with arbitrary coefficients admits only the trivial algebra (2.2), the plane wave ansatz

$$u = U(\omega),\ \omega = x - \alpha t,\ v = V(\omega),\ w = W(\omega)$$

can be easily derived, which reduces (4.1) to the non-linear ODE system

$$\begin{aligned} &U'' + \alpha U' + U(1 - U - a_1 V) = 0,\\ &d_2 V'' + \alpha V' + a_2 V(1 - U - a_1 V) + UW + a_1 VW = 0,\\ &d_3 W'' + \alpha W' + a_3 W(1 - W) - a_4 UW - a_5 VW = 0. \end{aligned} \tag{4.2}$$

https://doi.org/10.1017/S0956792518000104 Published online by Cambridge University Press

Obviously, the ODE system (4.2) with arbitrary coefficients is not integrable, hence, we seek for its particular solutions. Our aim is to find TFs, i.e., such plane wave solutions, which are positive and bounded for arbitrary x and $t > 0$. Moreover, in order to provide a biological interpretation of determined solutions, we assume that the solutions to-be-determined connect the steady-state points of system (4.1). Taking into account the arguments presented above, we consider the *ad hoc* ansatz

$$\begin{aligned} U &= \sigma_1 (1 - \tanh \mu\,\omega)^{k_1}, \\ V &= \sigma_2 (1 - \tanh \mu\,\omega)^{k_2}, \\ W &= 1 - \sigma_3 (1 - \tanh \mu\,\omega)^{k_3}. \end{aligned} \tag{4.3}$$

Notably, ansätze of such form are often used and the corresponding technique is often called the tanh method (see, e.g., [26, 39]).

We assume that the exact solution of the form (4.3) connects steady-state points of (4.2), namely $(U_0, V_0, 0)$ (as $\omega \to -\infty$) and $(0, 0, 1)$ (as $\omega \to +\infty$). Having such assumption, one immediately obtains the restrictions

$$1 - 2^{k_1}\sigma_1 - a_1 2^{k_2}\sigma_2 = 0, \;\; 1 - 2^{k_3}\sigma_3 = 0.$$

Substituting ansatz (4.3) into (4.2) and making the corresponding calculations, the exact solution

$$\begin{aligned} U &= \tfrac{1}{4}(1 - 2a_1\delta)\left(1 - \tanh\left[\tfrac{\sqrt{1-2a_1\delta}}{2\sqrt{6}}\,\omega\right]\right)^2, \\ V &= \delta - \delta \tanh\left[\tfrac{\sqrt{1-2a_1\delta}}{2\sqrt{6}}\,\omega\right], \\ W &= \tfrac{1}{2} + \tfrac{1}{2}\tanh\left[\tfrac{\sqrt{1-2a_1\delta}}{2\sqrt{6}}\,\omega\right] \end{aligned} \tag{4.4}$$

of system (4.2) was constructed. Here, $a_1 \leqslant \frac{1}{2\delta}$ (otherwise, the solution is complex), $\delta > 0$ (otherwise, V is negative) and the additional restrictions

$$\begin{gathered} \alpha = \tfrac{5-4a_1\delta}{\sqrt{6-12a_1\delta}}, \;\; d_2 = \tfrac{-3-5\delta+6a_1\delta+4a_1\delta^2}{\delta(-3+2a_1\delta)}, \;\; a_2 = \tfrac{3-10\delta+6a_1\delta+8a_1\delta^2}{6\delta(-3+2a_1\delta)}, \\ a_4 = \tfrac{d_3}{3}, \;\; a_5 = \tfrac{5-d_3+6a_3-4a_1\delta+2a_1d_3\delta}{12\delta} \end{gathered} \tag{4.5}$$

must take place. Because $d_2 > 0$, $a_2 \geqslant 0$ and $a_5 \geqslant 0$, the further restrictions

$$\begin{aligned} & a_3 \leqslant \tfrac{1}{6}(-5 + 4\delta a_1 + d_3 - 2\delta a_1 d_3), \;\; d_3 \geqslant \tfrac{5-4\delta a_1}{1-2\delta a_1}, \\ & a_1 \leqslant \begin{cases} \frac{1}{2\delta}, \text{ if } \delta > 1, \\ \frac{-3+10\delta}{2\delta(3+4\delta)}, \text{ if } \frac{3}{10} \leqslant \delta \leqslant 1 \end{cases} \end{aligned} \tag{4.6}$$

are obtain from (4.5).

As one can see, the exact solution (4.4) is nothing else but the exact solution connecting the steady-state points $(1 - 2a_1\delta, 2\delta, 0)$ and $(0, 0, 1)$ of system (4.2), because

$$(U, V, W) \to (1 - 2a_1\delta, 2\delta, 0) \text{ if } \omega \to -\infty,$$
$$(U, V, W) \to (0, 0, 1) \text{ if } \omega \to +\infty.$$

An example of the exact solution (4.4) is presented in Figure 1.

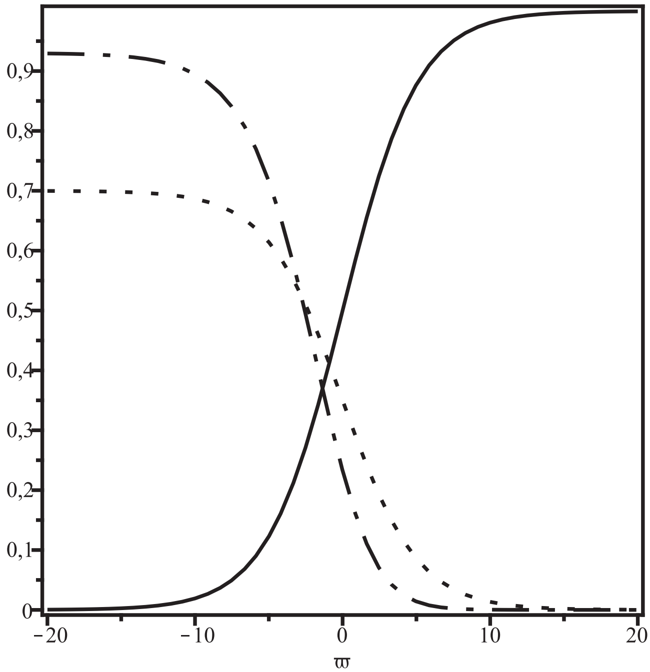

FIGURE 1. Curves representing the functions $U(\omega)$ (dash-dot), $V(\omega)$ (dot) and $W(\omega)$ (solid) from (4.4) for parameters $a_1 = 1/10$ and $\delta = 7/20$.

Thus, the one-parameter family of TFs

$$\begin{aligned} u &= \tfrac{1}{4}(1-2a_1\delta)\left(1-\tanh\left[\tfrac{\sqrt{1-2a_1\delta}}{2\sqrt{6}}(x-\alpha t)\right]\right)^2, \\ v &= \delta - \delta\tanh\left[\tfrac{\sqrt{1-2a_1\delta}}{2\sqrt{6}}(x-\alpha t)\right], \\ w &= \tfrac{1}{2}+\tfrac{1}{2}\tanh\left[\tfrac{\sqrt{1-2a_1\delta}}{2\sqrt{6}}(x-\alpha t)\right] \end{aligned} \tag{4.7}$$

of the HGF system (4.1) with restrictions (4.5) and (4.6) is derived. This solution has a clear biological interpretation and describes such interaction between farmers and hunter-gatherers that hunter-gatherers die, while the initial and converted farmers co-exist (see Figure 2). Actually, one may say that extinction of hunter-gatherers takes place because all of them are converted into farmers.

Now, we turn to system (4.1) with $a_1 = 0$:

$$\begin{aligned} u_t &= u_{xx} + u(1-u), \\ v_t &= d_2 v_{xx} + a_2 v(1-u) + uw, \\ w_t &= d_3 w_{xx} + a_3 w(1-w) - a_4 uw - a_5 vw. \end{aligned} \tag{4.8}$$

It follows from Theorem 2.1 that system (4.8) with $a_2 \neq 0$ and $a_3 + a_5 > 0$ admits only the trivial algebra (2.2).

https://doi.org/10.1017/S0956792518000104 Published online by Cambridge University Press

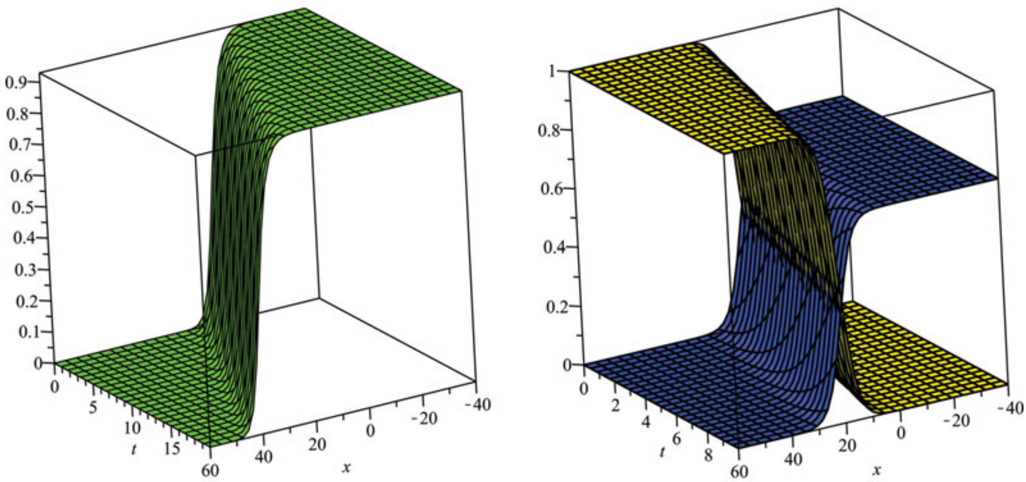

FIGURE 2. Surfaces representing the components u (green), v (blue) and w (yellow) of TF (4.7) with $\alpha = 81/5\sqrt{62}$ and $\delta = 7/20$ of the HGF system (4.1) with the parameters $a_1 = 1/10$, $a_2 = 64/2051$, $a_4 = d_3/3$, $a_5 = (162 + 200a_3 - 31d_3)/140$, $d_2 = 8982/2051$.

In order to construct exact solution of system (4.8), an analog of *ad hoc* ansatz (4.3) has been again used. So, the coefficients of (4.8) were specified as follows:

$$d_2 = \frac{1}{2},\ a_2 = 1,\ a_3 = \frac{5 - d_3}{6},\ a_4 = \frac{5}{3},\ a_5 = 0.$$

As a result, TF

$$\begin{aligned} u(t,x) &\equiv U(\omega) = \tfrac{1}{4}\left(1 - \tanh\left[\tfrac{1}{2\sqrt{6}}\,\omega\right]\right)^2, \quad \omega = x - \tfrac{5}{\sqrt{6}}\,t,\\ v(t,x) &\equiv V(\omega) = \tfrac{3d-5}{3(d-5)}\left(1 - \tanh\left[\tfrac{1}{2\sqrt{6}}\,\omega\right]\right)^3,\\ w(t,x) &\equiv W(\omega) = \tfrac{3d-5}{2(d-5)}\left(1 - \tanh^2\left[\tfrac{1}{2\sqrt{6}}\,\omega\right]\right) \end{aligned} \tag{4.9}$$

of the system

$$\begin{aligned} u_t &= u_{xx} + u(1-u),\\ v_t &= \frac{1}{2}\,v_{xx} + v(1-u) + uw,\\ w_t &= dw_{xx} + \frac{5-d}{6}\,w(1-w) - \frac{5}{3}\,uw \end{aligned}$$

was constructed (here, $d_3 \equiv d \leqslant 5/3$).

TF (4.9) connects the steady-state points $\left(1, \frac{8(3d-5)}{3(d-5)}, 0\right)$ and $(0, 0, 0)$ because

$$\begin{aligned} (U, V, W) &\to \left(1, \frac{8(3d-5)}{3(d-5)}, 0\right) \text{ if } \omega \to -\infty,\\ (U, V, W) &\to (0, 0, 0) \text{ if } \omega \to +\infty. \end{aligned}$$

Thus, the biological interpretation of solution (4.9) is similar to that for solution (4.7). Notably, TF (4.9) in contrast to that (4.7) has the fixed wave velocity $\alpha = \frac{5}{\sqrt{6}}$, which is exactly the same as for TF (3.18) of the Fisher equation.

Remark 2 Because the HGF system (1.3) is invariant with respect to the discrete transformation $x \to -x$, all the solutions obtained above can be transformed to another solutions using this transformation.

5 Conclusions

In this paper, the three-component non-linear system of PDEs (1.1) introduced in [3] for describing the spread of an initially localized population of farmers into a region occupied by hunter-gatherers was studied by the classical Lie method. First of all, the system was transformed to the non-dimensional form (1.2) in order to reduce the number of parameters. All possible Lie symmetries of system (1.2) were identified (Theorem 2.1), inequivalent symmetry reductions to the ODE systems in the most interesting case (from applicability point of view) were conducted (Section 3), several families of exact solutions (including the travelling fronts) were found and a possible biological interpretation for some of them was provided (Section 4).

It is worth noting that system (1.1) was studied under the restriction $e_1 \neq 0$, otherwise, one reduces to the three-component diffusive Lotka–Volterra system. Lie symmetries of the three-component diffusive Lotka–Volterra system are completely described in [9], while its exact solutions are constructed in [8, 9, 21].

To the best of our knowledge, this paper is the first study of the HGF model by symmetry-based methods. In [3], the authors studied the existence and behaviour of TFs of the model; however, any exact solutions are not presented therein. In particular, it is stated that there are TFs connecting the stable and unstable steady-state points of the model (see [3, P.10]). Interestingly that TF (4.7) corresponds exactly to such case provided restrictions (4.5) and (4.6) hold. Moreover, we constructed the exact solution (3.12), which predicts co-existence of all the populations at any semi-final space interval (see formulae (3.14)) provided the coefficients of the HGF system (1.3) satisfy the restrictions (3.13). Such type of behaviour was not identified in [3].

The results obtained in this paper can be useful for investigation of some other three-component non-linear system of PDEs with similar structure arising in applications (in particular for the models describing language competition [23, 35]). Such systems can possess travelling wave solutions with similar structure to those obtained above in Section 4. On the other hand, it is unlikely that modified systems will possess Lie symmetries presented in Table 1 because even a modest modification of a given system may drastically change its Lie symmetry.

A natural continuation of this research is searching for non-Lie (non-classical, conditional, etc.) symmetries of the non-linear system (1.1) and their application for constructing exact solutions. We have achieved some progress in this direction and an interesting particular result is presented in Section 3.4 [10]. We plan to report more new results in a forthcoming paper.

References

[1] Ablowitz, M. & Zeppetella, A. (1979) Explicit solutions of Fisher's equation for a special wave speed. *Bull. Math. Biol.* **41**, 835–840.

[2] Ames, W.F. (1972) *Nonlinear Partial Differential Equations in Engineering*, Academic Press, New York.

[3] Aoki, K., Shida, M. & Shigesada, N. (1996) Travelling wave solutions for the spread of farmers into a region occupied by hunter-gatherers. *Theor. Popul. Biol.* **50**, 1–17.

[4] Aris, R. (1975) *The Mathematical Theory of the Diffusion and Reaction in Permeable Catalysts*, Vol. 1–2, Oxford University Press, Oxford.

[5] Arrigo, D. J. (2015) *Symmetry Analysis of Differential Equations*, John Wiley & Sons, Inc., Hoboken, NJ.

[6] Bluman, G. W., Cheviakov, A. F. & Anco, S. C. (2010) *Applications of Symmetry Methods to Partial Differential Equations*, Springer, New York.

[7] Britton, N. F. (2003) *Essential Mathematical Biology*, Springer, Berlin.

[8] Chen, C.-C. & Hung, L.-C. (2012) Exact travelling wave solutions of three-species competition-diffusion systems. *Discrete Contin. Dyn. Syst. Ser. B* **17**, 2653–2669.

[9] Cherniha, R. & Davydovych, V. (2013) Lie and conditional symmetries of the three-component diffusive Lotka–Volterra system. *J. Phys. A: Math. Theor.* **46**, 185204 (14 pp).

[10] Cherniha, R. & Davydovych, V. (2017) *Nonlinear Reaction-Diffusion Systems – Conditional Symmetry, Exact Solutions and Their Applications in Biology,* Lecture Notes in Mathematics, Vol. 2196, Springer, Cham, Switzerland.

[11] Cherniha, R., Davydovych, V. & Muzyka, L. (2017) Lie symmetries of the Shigesada–Kawasaki–Teramoto system. *Comm. Nonlinear Sci. Numer. Simul.* **45**, 81–92.

[12] Cherniha, R. & King, J. R. (2000) Lie symmetries of nonlinear multidimensional reaction-diffusion systems: I. *J. Phys. A: Math. Gen.* **33**, 267–282.

[13] Cherniha, R. & King, J. R. (2000) Addendum: "Lie symmetries of nonlinear multidimensional reaction-diffusion systems: I". *J. Phys. A: Math. Gen.* **33**, 7839–7841.

[14] Cherniha, R. & King, J. R. (2003) Lie symmetries of nonlinear multidimensional reaction-diffusion systems: II. *J. Phys. A: Math. Gen.* **36**, 405–425.

[15] Cherniha, R. & King, J. R. (2005) Nonlinear reaction-diffusion systems with variable diffusivities: Lie symmetries, ansätze and exact solutions. *J. Math. Anal. Appl.* **308**, 11–35.

[16] Cherniha, R. & King, J. R. (2006) Lie symmetries and conservation laws of nonlinear multidimensional reaction-diffusion systems with variable diffusivities. *IMA J. Appl. Math.* **71**, 391–408.

[17] Cherniha, R. M. & Wilhelmsson, H. (1996) Symmetry and exact solution of heat-mass transfer equations in thermonuclear plasma. *Ukr. Math. J.* **48**, 1434–1449.

[18] Fisher, R. A. (1937) The wave of advance of advantageous genes. *Ann. Eugen.* **7**, 353–369.

[19] Fushchych, W. I., Shtelen, W. M. & Serov, M. I. (1993) *Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics*, Kluwer, Dordrecht.

[20] Gilding, B. H. & Kersner, R. (2004) *Travelling Waves in Nonlinear Diffusion-Convection Reaction,* Birkhauser Verlag, Basel.

[21] Hung, L.-C. (2011) Traveling wave solutions of competitive-cooperative Lotka–Volterra systems of three species. *Nonlinear Anal. Real World Appl.* **12**, 3691–3700.

[22] Kamke, E. (1977) *Differentialgleichungen. Lösungsmethoden und Lösungen. I: Gewöhnliche* (in German), Teubner Stuttgart.

[23] Kandler, A., Unger, R. & Steele, J. (2010) Language shift, bilingualism and the future of Britain's Celtic languages. *Philos. Trans. R. Soc. B* **365**, 3855–3864.

[24] Knyazeva, I. V. & Popov, M. D. (1994) A system of two diffusion equations. In: N. H. Ibragimov (editor), *CRC Handbook of Lie Group Analysis of Differential Equations*, Vol. 1, CRC Press, Boca Raton, pp. 171–176.

[25] Kuang, Y., Nagy, J. D. & Eikenberry, S. E. (2016) *Introduction to Mathematical Oncology,* CRC Press, Boca Raton.

[26] Malfliet, W. (2004) The tanh method: A tool for solving certain classes of nonlinear evolution and wave equations. *J. Comp. Appl. Math.* **164**, 529–541.
[27] Murray, J. D. (2003) *An Introduction I: Models and Biomedical Applications*, Springer, Berlin.
[28] Murray, J. D. (2003) *Mathematical Biology II: Spatial Models and Biomedical Applications*, Springer, Berlin.
[29] Nikitin, A. G (2005) Group classification of systems of nonlinear reaction-diffusion equations. *Ukr. Math. Bull.* **2**, 153–204.
[30] Okubo, A. & Levin, S. A. (2001) *Diffusion and Ecological Problems. Modern Perspectives*, 2nd ed., Springer, Berlin.
[31] Olver, P. (1986) *Applications of Lie Groups to Differential Equations*, Springer, Berlin.
[32] Ovsiannikov, L. V. (1980) *The Group Analysis of Differential Equations*, Academic Press, New York.
[33] Polyanin, A. D. & Zaitsev, V. F. (2003) *Handbook of Exact Solutions for Ordinary Differential Equations*, CRC Press, London.
[34] Samarskii, A. A., Galaktionov, V. A., Kurdyumov, S. P. & Mikhailov, A. P. (1995) *Blow-up in Quasilinear Parabolic Equations,* Walter de Gruyter, Berlin.
[35] Straughan, B. (2014) Shocks and acceleration waves in modern continuum mechanics and in social systems. *Evol. Equat. Contr. Theory* **3**, 541–555.
[36] Serov, M. & Omelian, O. (2008) Classification of the symmetry properties of a system of chemotaxis equations. *Ukr. Math. Bull.* **5**, 529–557.
[37] Torrisi, M., Tracina, R. & Valenti, A. (1996) A group analysis approach for a non linear differential system arising in diffusion phenomena. *J. Math. Phys.* **37**, 4758–4767.
[38] Turing, A. M. (1952) The chemical basis of morphogenesis. *Philos. Trans. Roy. Soc. Lond.* **237**, 37–72.
[39] Wazwaz, A. M. (2008) The extended tanh method for the Zakharo–Kuznetsov (ZK) equation, the modified ZK equation, and its generalized forms. *Commun. Nonlinear Sci. Numer. Simul.* **13**, 1039–1047.
[40] Zulehner, W. & Ames, W. F. (1983) Group analysis of a semilinear vector diffusion equation. *Nonlinear Anal.* **7**, 945–969.