J. Fluid Mech. (2014), *vol.* 745, R2, doi:10.1017/jfm.2014.128

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Creeping axisymmetric plumes with strongly temperature-dependent viscosity

Andrew Crosby† and John R. Lister

Institute of Theoretical Geophysics, Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK

(Received 1 January 2014; revised 17 February 2014; accepted 3 March 2014)

The structure of a steady axisymmetric thermal plume rising through a very viscous fluid with strongly temperature-dependent viscosity of the form $\mu \propto \exp(-\gamma T)$ is investigated. An analytic asymptotic solution is derived for the fast-flowing core of the plume, which predicts that the excess centreline temperature decays exponentially as $\exp\{-12\pi\kappa z/(\gamma A)\}$, where κ is the thermal diffusivity, z the height and A the vertical heat flux. This rate of decay, which is found to be in good agreement with numerical simulations of the boundary-layer equations, is three times faster than that predicted by the oft-quoted model of Olson, Schubert and Anderson (*J. Geophys. Res.*, vol. 98 (B4), 1993, pp. 6829–6844).

Key words: geophysical and geological flows, mantle convection, plumes/thermals

1. Introduction

Following on from the seminal work of Morton, Taylor & Turner (1956), there has been extensive research on the fluid-dynamical properties of plumes, which have been found to have diverse applications. One particular application of very viscous plumes is to the Earth's mantle, where the existence of plumes provides a possible explanation for the occurrence of hot spots (localised regions of volcanism, often away from plate boundaries) on the Earth's surface, as proposed by Morgan (1971).

Morgan's proposal motivated the study of very viscous thermal plumes with temperature-dependent viscosity, since such plumes are relevant to the Earth's mantle, where solid-state creep leads to fluid-like behaviour on sufficiently long time scales. The flow can occur by dislocation creep or by diffusion creep and when the latter mechanism is dominant, as is thought to be the case for the lower mantle (Schubert, Turcotte & Olson 2001), the resulting rheology is Newtonian. The associated viscosity varies strongly with temperature T, and somewhat with pressure, according to an Arrhenius viscosity law (Christensen 1989). When the pressure dependence can be neglected, the Arrhenius viscosity law associated with diffusion creep can be

† Email address for correspondence: a.crosby@damtp.cam.ac.uk

approximated, over a reasonable temperature interval, by $\mu(T) = \mu_0 \exp(-\gamma T)$, where μ_0 and γ are appropriately chosen constants.

Several theoretical models have been developed for creeping plumes with temperature-dependent viscosity. Yuen & Schubert (1976) considered the self-similar boundary layer along a vertical heated plate as a model for two-dimensional plumes with constant centreline temperature, while Hauri, Whitehead & Hart (1994) observed that self-similar solutions could also be found for three-dimensional axisymmetric plumes provided that the fluid viscosity law varied appropriately with height. Loper & Stacey (1983) developed a heuristic model of the flow of an axisymmetric compressible plume in a non-Newtonian fluid with an Arrhenius viscosity law and large viscosity contrast. This model was later simplified by Olson, Schubert & Anderson (1993) to focus on incompressible plumes within a Newtonian fluid.

The theoretical model of Olson *et al.* (1993) is of particular significance because it makes a number of simple predictions; notably, how the centreline temperature varies with height, and the relation between the width of the fast-flowing low-viscosity core and the wider slow-flowing thermal halo. The authors used these predictions to predict whether plumes could survive from the D'' layer at the core–mantle boundary up to the Earth's surface without being destroyed by thermal diffusion. Other authors have also made use of the predictions. For example, Steinberger & Antretter (2006) use the predicted relation for the width of the thermal halo in their modelling of hot-spot motion and of the shape of mantle plumes; and both Kerr & Mériaux (2004) and Jellinek & Manga (2004) quote the predicted variation of centreline temperature with height.

Despite its widespread use, we show here that the model of Olson *et al.* (1993) has some shortcomings: for example, it does not satisfy the energy equation throughout the fast-flowing core of the plume, which leads to a significant under-prediction for the rate of decay of the centreline temperature. The main aim of this paper is to present a simple improved asymptotic model that overcomes these shortcomings and thus gives much better agreement with numerical simulations. We derive this model in § 3 following a more detailed description of the problem setup in § 2. In § 4 we briefly relate our model back to that of Olson *et al.* (1993). In § 5, we compare the model against numerical simulations. Note that our analytic solution is not intended to capture the full complexities inherent in mantle convection.

2. Problem description

We consider a steady axisymmetric plume rising through a very viscous Newtonian fluid whose viscosity depends on temperature. Away from the plume, the fluid has a constant background temperature, a constant density ρ_0 and a constant viscosity μ_0. Within the plume, the density of the fluid varies linearly with temperature as $\rho(T) = \rho_0(1 - \beta T)$, where T is the excess temperature over the background temperature and β is the coefficient of thermal expansion, and the viscosity varies exponentially with the excess temperature as $\mu(T) = \mu_0 \exp(-\gamma T)$. We assume an infinite Prandtl number (in the Earth's mantle $Pr \sim 10^{23}$), and assume that density fluctuations are small ($\beta T \ll 1$) so that we can make a Boussinesq approximation. The equations of motion are then the variable-viscosity Stokes equations,

$$-\nabla p + \mu_0 \nabla \cdot \left(\mathrm{e}^{-\gamma T}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathsf{T}})\right) - \rho_0 \beta T \boldsymbol{g} = \boldsymbol{0}, \quad \nabla \cdot \boldsymbol{u} = 0, \qquad (2.1a)$$

and the evolution of the temperature is governed by

$$\boldsymbol{u} \cdot \nabla T = \kappa \nabla^2 T, \qquad (2.1b)$$

where κ is the thermal diffusivity. (The exponential variation of viscosity with temperature is typically much greater than any variation of β and κ with temperature, and so we take β and κ to be constants for simplicity.)

We assume that the scale of radial variation is much shorter than the scale of vertical variation, so that boundary-layer approximations can be applied to (2.1*a*) and (2.1*b*). We define dimensionless variables for temperature, radius, height, radial velocity and vertical velocity by

$$\tilde{T} \equiv \gamma T, \quad \tilde{r} \equiv \frac{r}{R}, \quad \tilde{z} \equiv \frac{z}{Ra\,R}, \quad \tilde{u} \equiv \frac{uR}{\kappa}, \quad \text{and} \quad \tilde{w} \equiv \frac{uR}{Ra\,\kappa}, \tag{2.2}$$

where R is a typical radial length scale (which could be the width of either the velocity or thermal profiles in the plume), and where we have defined a Rayleigh number

$$Ra \equiv \frac{\rho_0 \beta g R^3}{\gamma \kappa \mu_0}. \tag{2.3}$$

The assumption about the (dimensional) radial and vertical scales corresponds to $Ra \gg 1$. Using the boundary-layer approximations and dimensionless variables, (2.1) becomes

$$\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\mathrm{e}^{-\tilde{T}}\frac{\partial \tilde{w}}{\partial \tilde{r}}\right) + \tilde{T} = 0, \tag{2.4a}$$

$$\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}(\tilde{r}\tilde{u}) + \frac{\partial \tilde{w}}{\partial \tilde{z}} = 0, \tag{2.4b}$$

$$\tilde{u}\frac{\partial \tilde{T}}{\partial \tilde{r}} + \tilde{w}\frac{\partial \tilde{T}}{\partial \tilde{z}} = \frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial \tilde{T}}{\partial \tilde{r}}\right). \tag{2.4c}$$

We expect the boundary-layer approximation to be appropriate provided that the Rayleigh number is sufficiently large (although exactly how large will depend non-trivially on the choice of radial length scale and on the centreline viscosity contrast). To avoid notational clutter, we now drop the tildes from the dimensionless variables. Unless otherwise stated, all quantities from this point on should be considered dimensionless.

The use of a boundary-layer approximation means that the buoyancy of the plume cannot be supported by a lower boundary (as considered for an isoviscous plume by Whittaker & Lister 2006). Instead, we impose conditions $w(r_{max}) = T(r_{max}) = 0$ of zero vertical velocity and zero excess temperature at an effective outer boundary $r = r_{max}$. For the plumes of interest, the viscosity of the ambient fluid is exponentially larger than the viscosity within the plume, so we expect flow within the plume to be very weakly dependent on the choice of r_{max} provided it is in the region of exponentially large viscosity.

We denote the dimensionless vertical heat flux by A, which, under the boundary-layer approximation, is given by

$$A = 2\pi \int_0^{r_{max}} Twr\,\mathrm{d}r. \tag{2.5}$$

Assuming there is negligible heat flux through $r = r_{max}$, A is independent of height.

3. Asymptotic similarity solution for the fast-flowing core

We now derive an asymptotic similarity solution for the fast-flowing core (the region in which velocities are comparable to the centreline velocity) of a plume that has a large viscosity contrast between its centreline and the ambient fluid, i.e. a plume satisfying $T_0(z) \gg 1$, where $T_0(z)$ is the (dimensionless) centreline temperature.

The temperature decreases radially away from the centreline, and we define $\phi(r, z) \equiv T_0(z) - T(r, z)$ to be the deviation from the centreline temperature. As the temperature decreases, the viscosity of the fluid increases exponentially as $\exp(\phi)$. Consequently, we expect the fast-flowing core of the plume to be restricted to the region where $\phi = O(1)$. When the overall temperature contrast is large, $T_0 \gg 1$, the temperature in this region is still comparable to the centreline temperature, and thus the fast-flowing core $\phi = O(1)$ lies within the wider thermal halo where $\phi = O(T_0)$ (as noted by Loper & Stacey 1983).

At height z, we define $\delta(z)$ to be the radial length scale of the fast-flowing core and $W_0(z)$ to be the vertical velocity scale, both of which are to be determined below. We also define correspondingly scaled variables

$$\eta \equiv r/\delta(z) \quad \text{and} \quad W(r, z) \equiv w(r, z)/W_0(z). \tag{3.1a,b}$$

We rewrite the vertical-force balance (2.4*a*) in terms of η and W, and use $T_0 \gg 1$ to neglect the temperature deviation ϕ in the buoyancy force $T = T_0 - \phi$ (which leads to a small $O(\phi/T_0)$ relative error in the fast-flowing core). We thus obtain

$$\frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \mathrm{e}^{\phi}\frac{\partial W}{\partial \eta}\right) \sim -\frac{T_0 \mathrm{e}^{T_0}\delta^2}{W_0}. \tag{3.2}$$

We assume that the vertical heat flux is dominated by the fast-flowing core so that we can also neglect the effect of the temperature deviation ϕ in (2.5), which becomes

$$\int_0^\infty W\eta \, \mathrm{d}\eta \sim \frac{A}{2\pi W_0 \delta^2 T_0}. \tag{3.3}$$

(Recall that r_{max} is assumed to lie well into the region of exponentially larger viscosity so that the outer limit r_{max}/δ of the integral can be approximated by ∞.)

Motivated by the form of the right-hand sides of (3.2) and (3.3), we determine the radial and vertical-velocity scales to be defined by

$$\delta(z) \equiv \left(2\pi T_0^2 \mathrm{e}^{T_0}/A\right)^{-1/4} \quad \text{and} \quad W_0(z) \equiv \left(A\mathrm{e}^{T_0}/2\pi\right)^{1/2}. \tag{3.4a,b}$$

(So, as T_0 decreases with height, the core radius δ increases and the vertical velocity W_0 decreases.) The scaled velocity and temperature deviation can then be assumed to be self-similar functions, $W(\eta)$ and $\phi(\eta)$, of the scaled radial variable η, which satisfy

$$\frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta \mathrm{e}^{\phi}\frac{\mathrm{d}W}{\mathrm{d}\eta}\right) \sim -1 \quad \text{and} \quad \int_0^\infty W\eta \, \mathrm{d}\eta \sim 1. \tag{3.5a,b}$$

From (3.4) the vertical derivatives of $W_0(z)$ and $\delta(z)$ are related to that of the centreline temperature $T_0(z)$ by

$$\frac{1}{W_0}\frac{\mathrm{d}W_0}{\mathrm{d}z} = \frac{1}{2}\frac{\mathrm{d}T_0}{\mathrm{d}z} \quad \text{and} \quad \frac{1}{\delta}\frac{\mathrm{d}\delta}{\mathrm{d}z} = -\frac{1}{4}\frac{\mathrm{d}T_0}{\mathrm{d}z} - \frac{1}{2}\frac{1}{T_0}\frac{\mathrm{d}T_0}{\mathrm{d}z} \sim -\frac{1}{4}\frac{\mathrm{d}T_0}{\mathrm{d}z}; \tag{3.6a,b}$$

the asymptotic simplification of (3.6*b*) holds because $T_0 \gg 1$.

In order to determine the radial velocity, we first evaluate the vertical derivative of $w(r, z) = W_0(z)W(\eta = r/\delta)$, and use (3.6) to obtain

$$\frac{\partial w}{\partial z} = \frac{\mathrm{d}W_0}{\mathrm{d}z}W(\eta) - W_0(z)\frac{\eta}{\delta(z)}\frac{\mathrm{d}\delta}{\mathrm{d}z}\frac{\mathrm{d}W}{\mathrm{d}\eta} \sim -\frac{W_0}{\delta}\frac{\mathrm{d}\delta}{\mathrm{d}z}\frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta^2 W\right). \tag{3.7}$$

The continuity equation (2.4*b*) yields

$$\frac{\partial w}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}(ru) = -\frac{1}{\delta}\frac{1}{\eta}\frac{\partial}{\partial \eta}(\eta u), \tag{3.8}$$

which, in combination with (3.7), relates the radial velocity to the vertical velocity by

$$u \sim W_0(z)\frac{\mathrm{d}\delta}{\mathrm{d}z}\,\eta W(\eta). \tag{3.9}$$

(The radial velocity u is positive because the plume core decelerates as it rises and hence increases in width.)

In addition to (3.5), the temperature and velocity profiles must also satisfy the energy equation (2.4*c*). Rewritten in the scaled variables (3.1), (2.4*c*) becomes

$$\frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}\phi}{\mathrm{d}\eta}\right) = -W_0\delta^2\frac{\mathrm{d}T_0}{\mathrm{d}z}\,W + \left(\delta u - \delta W_0\frac{\mathrm{d}\delta}{\mathrm{d}z}\,\eta W\right)\frac{\mathrm{d}\phi}{\mathrm{d}\eta}. \tag{3.10}$$

The relationship (3.9) between the radial and vertical velocities is precisely that required for the last two terms, representing advection of the temperature deviation ϕ, to cancel. The cancellation occurs because the streamlines of (u, w) and the contours of constant ϕ are parallel to leading order. (Though $\phi \ll T_0$ in the fast-flowing core, the cancelling terms are, by (3.6*b*), individually of the same magnitude as the remaining term representing advection of T_0, and so the cancellation is a non-trivial simplification.)

By using the cancellation, and by substituting from (3.4), we can simplify (3.10) to

$$\frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}\phi}{\mathrm{d}\eta}\right) \sim -\frac{A}{2\pi T_0}\frac{\mathrm{d}T_0}{\mathrm{d}z}W. \tag{3.11}$$

Since W and ϕ are functions only of η, and T_0 only of z, (3.11) can hold only if

$$-\frac{A}{2\pi T_0}\frac{\mathrm{d}T_0}{\mathrm{d}z} \sim \alpha \quad \text{and} \quad \frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}\phi}{\mathrm{d}\eta}\right) \sim \alpha W \tag{3.12a,b}$$

where α is a constant.

Equations (3.5*a*) and (3.12*b*) are a fourth-order system of coupled ordinary differential equations for $W(\eta)$ and $\phi(\eta)$. Three of the necessary boundary conditions come from regularity at the origin and from the definition of ϕ, which imply that $\phi'(0) = W'(0) = \phi(0) = 0$. The final boundary condition is $W(\infty) \approx 0$, which can be thought of as matching an inner self-similar solution $W(\eta)$, which is valid where $\phi \ll T_0$ in the fast-flowing core, onto an outer solution where the vertical velocity is smaller due to the exponentially larger viscosity.

For each value of the constant α, there is a unique solution to (3.5*a*) and (3.12*b*) with the boundary conditions described above. However, there is exactly one value of

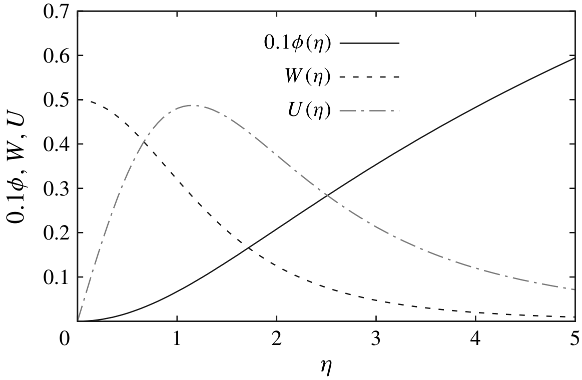

FIGURE 1. Structure of the self-similar velocity profiles (3.15*b*) and (3.13*a,b*) for U and W and the deviation $\phi = T_0 - T$ from the centreline temperature T_0 as functions of the radial similarity coordinate $\eta = r/\delta(z)$.

α for which the heat-flux constraint (3.5*b*) is also satisfied. This value is $\alpha = 6$, for which the solution is given analytically by

$$W(\eta) \sim \frac{1}{2}\left(1+\frac{\eta^2}{4}\right)^{-2} \quad \text{and} \quad \phi(\eta) \sim 3\ln\left(1+\frac{\eta^2}{4}\right). \tag{3.13a,b}$$

With $\alpha = 6$, (3.12*a*) implies that the centreline temperature decays exponentially as

$$T_0(z) \sim T_0(0)\exp\left(-\frac{12\pi}{A}z\right). \tag{3.14}$$

The characteristic width and velocity scales are then given in terms of T_0 by (3.4).

From (3.9) and (3.13*a*), the radial velocity is given by $u(r, z) = U_0(z)U(\eta)$, where

$$U_0(z) \sim (A/2\pi)^{-1/4}\mathrm{e}^{T_0/4}T_0^{1/2} \quad \text{and} \quad U(\eta) \sim \frac{3\eta}{4}\left(1+\frac{\eta^2}{4}\right)^{-2}. \tag{3.15a,b}$$

The radial velocity is outwards since the decrease in temperature and increase in viscosity with height means that the vertical velocity decreases with height (cf. (3.6*a*) and (3.14)), and thus fluid must flow outwards by mass conservation.

The asymptotic similarity solution for ϕ, W and U is plotted in figure 1. Though ϕ/T_0 is not small at sufficiently large radii, the velocities in this outer region are very small and not relevant to the inner solution for the fast-flowing core.‡

4. Relationship to the previous model of Olson *et al.* (1993)

In the model of Olson *et al.* (1993), instead of allowing the temperature deviation ϕ to be a general function of η that must be solved for, it is assumed to have the simple quadratic form $\phi = \lambda\eta^2$ for some constant λ. Given a quadratic profile for

‡This solution appears, with a slightly different derivation, in the PhD thesis of Professor Stephen Morris, University of California, Berkeley, (Morris 1980), but was not published elsewhere. We are grateful to a reviewer for alerting us to its existence, and to Professor Morris for providing us with a copy of his solution.

ϕ, the vertical force balance (3.5*a*) implies that the scaled vertical velocity has the Gaussian form

$$W \approx \frac{1}{4\lambda} \mathrm{e}^{-\lambda\eta^2}. \tag{4.1}$$

Since ϕ is no longer a general function of η, it is no longer possible to satisfy the energy equation (3.12*b*) at all points in the fast-flowing core. Instead, the energy equation is evaluated only on the centreline, to determine that, in our notation, $\alpha = 16\lambda^2$. The vertical heat-flux constraint (3.5*b*) gives $8\lambda^2 = 1$ for the Gaussian form (4.1) of W. This means that $\alpha = 2$, in contrast to $\alpha = 6$ in our asymptotic solution. Consequently the model of Olson *et al.* (1993) predicts that the centreline temperature of the plume decays three times more slowly than the asymptotic solution (3.14).

5. Numerical simulations

The asymptotic approximation in § 3 is based on the assumption $\phi/T_0 \ll 1$, which is valid in the fast-flowing plume core when $T_0 \gg 1$, but it may not be accurate at larger radii where $\phi = O(T_0)$ or at large heights where $T_0(z)$ has decreased towards one. In order to assess the accuracy of our asymptotic solution, we compared it against numerical solutions of the boundary-layer equations (2.4). The numerical method was similar to that of Albers & Christensen (1996): the temperature equation (2.4*c*) was marched upwards using a Crank–Nicolson scheme, and (2.4*a*) and (2.4*b*) were solved together to find (u, w) at each level. The accuracy of the results was checked by varying the spacing of the uniform numerical grid.

Figure 2(*a*) shows the structure of a numerically calculated plume with a base temperature profile $T(r, 0) = 12\exp(-r^2)$ and $r_{max} = 20$, which gives a vertical heat flux $A = 1.43 \times 10^5$. The initial viscosity contrast is $\mathrm{e}^{12} = 1.6 \times 10^5$. There is a short adjustment region at the base of the plume, roughly in $z \lesssim 150$, where there is, in comparison with greater heights, a relatively rapid spread of the temperature profile and a relatively strong inward radial velocity outside the fast-flowing core. The disappearance of this adjustment behaviour suggests convergence of the solution from the basal conditions onto a universal solution that is independent of the shape of the base temperature profile at greater heights. (See also figure 3.)

Above the adjustment region, the temperature profile spreads slowly outwards, and the centreline temperature decreases with height. As expected, the broad thermal halo contains a narrower fast-flowing core. The vertical velocity in the core decreases much more rapidly with height than the temperature, due to the exponential increase in viscosity with decreasing temperature (cf. (3.4*b*)). Within the fast-flowing core, the radial velocity is outwards as expected. At large radial distances, the radial velocity is weak but inwards, indicating that the plume as a whole is entraining ambient fluid.

Figure 2(*b*) shows the corresponding predictions of T, w and u from our asymptotic model. The model has the same parameters as the numerical solution: a centreline temperature $T_0 = 12$ at the base, and a vertical heat flux $A = 1.43 \times 10^5$.

The model temperature contours are in good agreement with those of the numerical solution in the regions where we expect them to be (i.e. where $T_0 \gg 1$ and $\phi/T_0 \ll 1$). The most notable difference is that, because there is no adjustment region at its base, the contours from the asymptotic model are shifted downwards slightly in comparison to those of the numerical solution.

The vertical-velocity contours are also in good agreement, as are those for the radial velocity in the fast-flowing core. Again there is a slight vertical offset between the model and the numerical solution due to the basal adjustment region in the numerical solution. The weak radial entrainment that occurs outside the fast-flowing core is not captured by the model as it occurs outside the region where $\phi/T_0 \ll 1$.

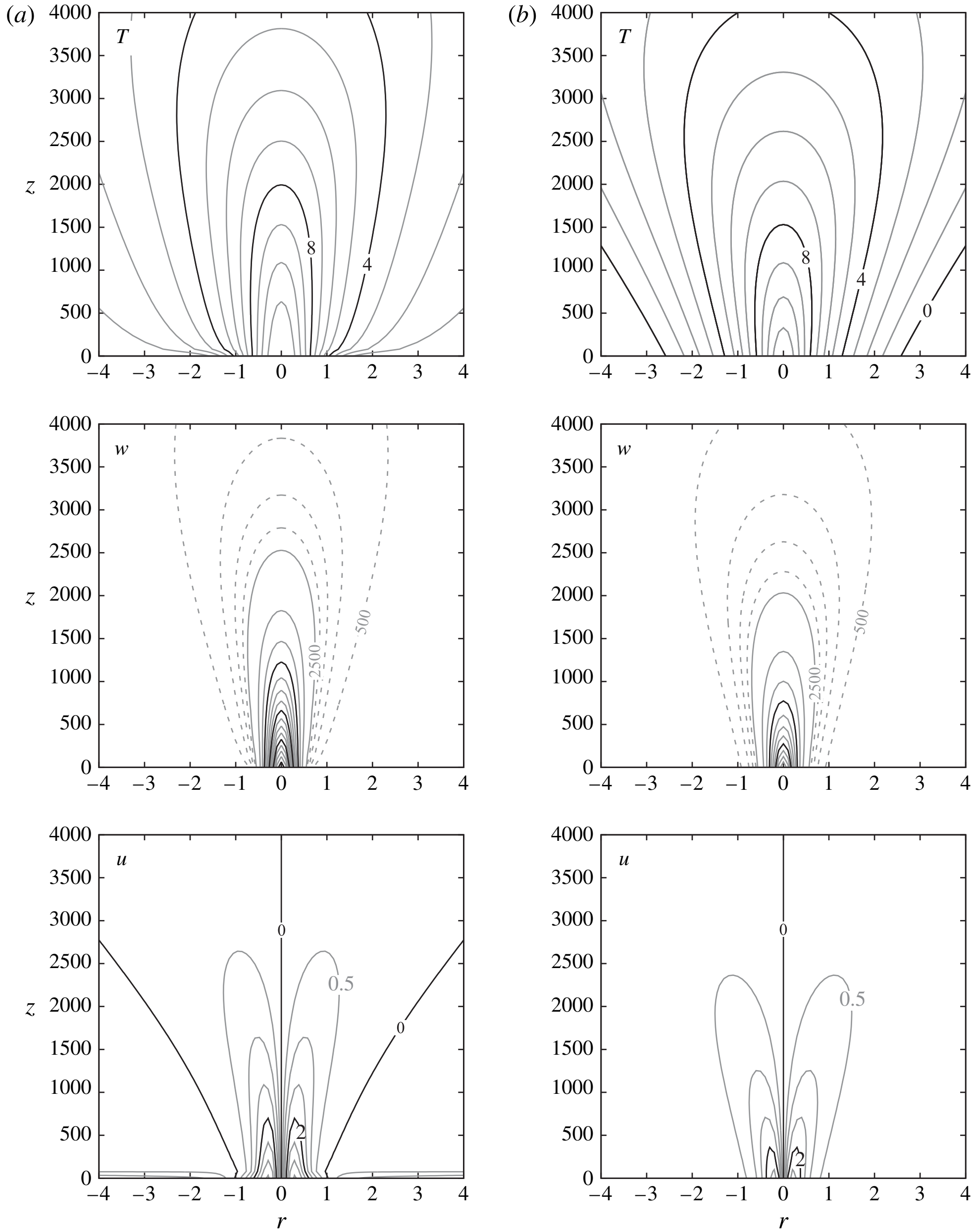

FIGURE 2. Dimensionless temperature T, vertical velocity w, and radial velocity u for a steady thermal plume in a fluid with viscosity law $\mu = \mathrm{e}^{-T}$. (*a*) A numerical solution to the boundary-layer equations (2.4) with a Gaussian temperature profile $\mathscr{T}\mathrm{e}^{-r^2}$ at the base with $\mathscr{T} = 12$. (*b*) The same quantities calculated from the asymptotic solution (3.13)–(3.15) with the same base centreline temperature and vertical heat flux. Contours of T are plotted at intervals of four (black) and one (grey), contours of w at intervals of 10 000 (black), 2500 (grey) and 500 (dashed), and contours of u at intervals of two (black) and 0.5 (grey). Some contour values are also labelled.

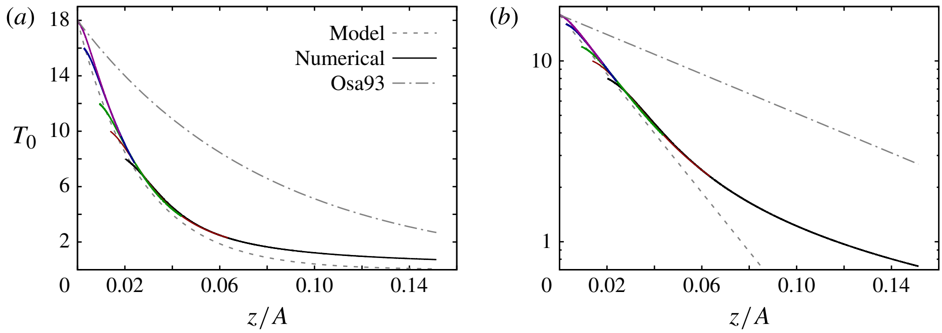

FIGURE 3. (*a*) Decay of the centreline temperature T_0 with scaled height z/A: comparison of numerical solutions and model predictions. The numerical solutions (solid lines) are for base temperature profiles $\mathscr{T} \exp(-r^2)$ with $\mathscr{T} = 8$, 10, 12, 16, 18; the z-offsets of the five solutions have been adjusted such that the curves overlap. After a short initial adjustment, the solutions collapse onto the same curve. Also shown are the prediction (3.14) from our model, and that from the model of Olson *et al.* (1993). (*b*) The same curves with T_0 on a logarithmic scale so that straight lines correspond to exponential decay. For comparison, plumes in the Earth's mantle are thought to have viscosity contrasts of around 100 (Schubert *et al.* 2001), which corresponds to a dimensionless centreline temperature of $T_0 \approx 4.5$.

5.1. *Centreline temperature*

The evolution of the centreline temperature with rescaled height z/A (as suggested by (3.14)) is shown in figure 3 for a number of numerical solutions with base temperature profiles of the form $\mathscr{T} \exp(-r^2)$ with different values of $\mathscr{T}$. (The values of r_{max} were in the range 20–80 and were chosen to ensure that r_{max} remained large compared to δ; the numerical solutions are then only weakly dependent on its value.) The vertical temperature profiles collapse onto a universal solution after an initial adjustment from the base temperature profile. Figure 3 also shows the prediction (3.14) of our model, which is in good agreement with the numerical solutions for large T_0.

5.2. *Radial structure*

A comparison between radial profiles from the numerical solution in figure 2 and those predicted by the asymptotic model is shown in figure 4. The profiles from the numerical solution are rescaled according to (3.4) and (3.15*a*). With the exception of the base profiles (red dot–dashed lines), which are below the basal adjustment region, there is good agreement between the model prediction and those numerical solutions for which $T_0 \gg 1$.

The model predicts a centreline velocity $0.5W_0(z)$ and that the velocity will fall to half its centreline value at a half-width $\delta_{w/2} = 1.29\,\delta(z)$. In figure 5(*a*), the values of these quantities in the numerical solutions are compared against the model predictions. In both cases, the numerical solutions exhibit an initial adjustment from the basal conditions followed by collapse onto a universal behaviour. The universal behaviour is in very good agreement with the model predictions when $T_0 \gg 1$.

Although we cannot use the asymptotic model, whose validity is limited to the region where $\phi/T_0 \ll 1$, to determine the thermal half-width (i.e. where $T = 0.5T_0$), we can investigate numerically the ratio between the thermal and velocity half-widths. This ratio is shown in figure 5(*b*). The solutions again collapse onto a universal

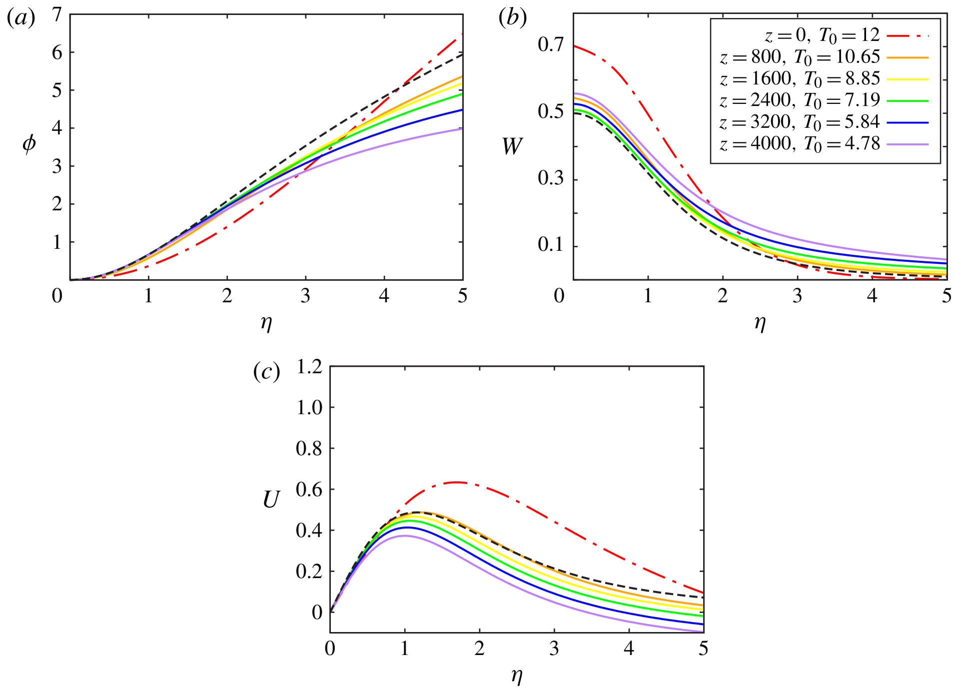

FIGURE 4. Scaled radial profiles of (*a*) the temperature deviation $\phi(\eta)$, (*b*) vertical velocity $W(\eta)$ and (*c*) radial velocity $U(\eta)$. The model predictions (3.13*b*), (3.13*a*) and (3.15*b*) (black dashed lines) are shown alongside profiles calculated from the numerical solution shown in figure 2 by scaling according to (3.4) and (3.15*a*).

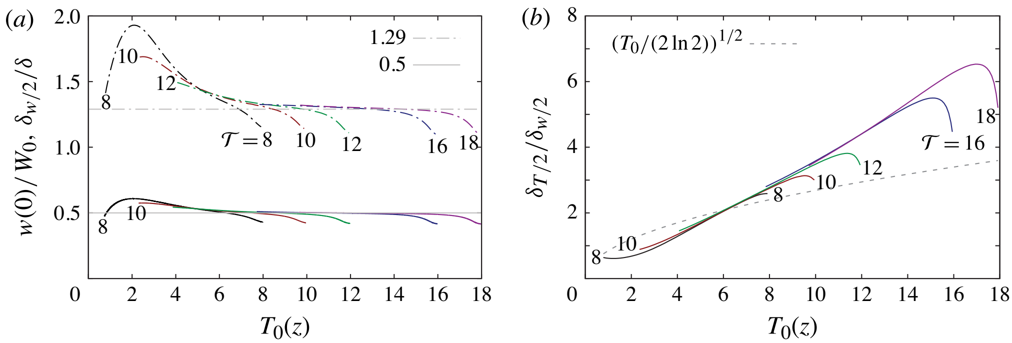

FIGURE 5. (*a*) Centreline velocity $w(0, z)$ scaled by the velocity scale $W_0(z)$ (solid lines), and the velocity half-width $\delta_{w/2}$ scaled by the radial scale δ (dot–dashed lines). The basal conditions are as in figure 3. The asymptotic model predicts values of 0.5 and 1.29 (grey lines). (*b*) Ratio of thermal half-width $\delta_{T/2}$ to velocity half-width $\delta_{w/2}$ in the numerical solutions (solid lines). Also shown is the prediction $(T_0/(2\ln 2))^{1/2}$ from the model of Olson *et al.* (1993).

behaviour after an initial adjustment. For $T_0 \gg 1$, the ratio is larger than one, indicating the existence of a wider thermal halo around the fast-flowing core, as expected. Furthermore, the ratio increases approximately linearly with T_0. The model of Olson *et al.* (1993) underestimates the variation of this ratio with T_0 (figure 5*b*).

6. Conclusion

We have derived an asymptotic similarity solution for the fast-flowing core of plumes with strongly temperature-dependent viscosity of an Arrhenius form $\mu \propto \exp(-\gamma T)$. This asymptotic approximation can be used as a simple analytic model for the plume structure. Comparison with numerical simulations confirms that the model is accurate when the dimensionless centreline temperature is large (implying a large viscosity contrast between the centreline and the ambient fluid), and that it provides a reasonable approximation for moderate centreline temperatures. In dimensional terms, our model predicts that the centreline temperature decays as $T_0(0)\exp\{-12\pi\kappa z/(\gamma A)\}$, which is three times faster than previous predictions. When applied to the Earth's mantle, the more rapid decay of the centreline temperature (see figure 3), and hence of the rise velocity, makes it harder for a weak plume to traverse the full depth of the mantle (e.g. Schubert *et al.* 2001, p. 532), increases the shearing of plumes by background mantle convection (e.g. Steinberger & Antretter 2006) and thus affects their stability (e.g. Lister *et al.* 2011).

References

ALBERS, M. & CHRISTENSEN, U. R. 1996 The excess temperature of plumes rising from the core–mantle boundary. *Geophys. Res. Lett.* **23**, 3567–3570.

CHRISTENSEN, U. R. 1989 Mantle rheology, constitution and convection. In *Mantle Convection: Plate Tectonics and Global Dynamics* (ed. W. R. Peltier), pp. 595–655. Gordon and Breach.

HAURI, E. H., WHITEHEAD, J. A. & HART, S. R. 1994 Fluid dynamic and geochemical aspects of entrainment in mantle plumes. *J. Geophys. Res.* **99** (B12), 24275–24300.

JELLINEK, A. M. & MANGA, M. 2004 Links between long-lived hot spots, mantle plumes, D'', and plate tectonics. *Rev. Geophys.* **42** (3), RG3002.

KERR, R. C. & MÉRIAUX, C. 2004 Structure and dynamics of sheared mantle plumes. *Geochem. Geophys. Geosyst.* **5** (12), Q12009.

LISTER, J. R., KERR, R. C., RUSSELL, N. J. & CROSBY, A. 2011 Rayleigh–Taylor instability of an inclined buoyant viscous cylinder. *J. Fluid Mech.* **671**, 313–338.

LOPER, D. E. & STACEY, F. D. 1983 The dynamical and thermal structure of deep mantle plumes. *Phys. Earth Planet. Inter.* **33**, 304–317.

MORGAN, W. J. 1971 Convection plumes in the lower mantle. *Nature* **230**, 42–43.

MORRIS, S. 1980 An asymptotic method for determining the transport of heat and matter by creeping flows with strongly variable viscosity. PhD thesis, Johns Hopkins University.

MORTON, B. R., TAYLOR, G. & TURNER, J. S. 1956 Turbulent gravitational convection from maintained and instantaneous sources. *Proc. R. Soc. Lond.* A **234** (1196), 1–23.

OLSON, P., SCHUBERT, G. & ANDERSON, C. 1993 Structure of axisymmetric mantle plumes. *J. Geophys. Res.* **98** (B4), 6829–6844.

SCHUBERT, G., TURCOTTE, D. L. & OLSON, P. 2001 *Mantle Convection in the Earth and Planets*. Cambridge University Press.

STEINBERGER, B. & ANTRETTER, M. 2006 Conduit diameter and buoyant rising speed of mantle plumes: implications for the motion of hot spots and shape of plume conduits. *Geochem. Geophys. Geosyst.* **7** (11), Q11018.

WHITTAKER, R. J. & LISTER, J. R. 2006 Steady axisymmetric creeping plumes above a planar boundary. Part 1. A point source. *J. Fluid Mech.* **567**, 361–378.

YUEN, D. A. & SCHUBERT, G. 1976 Mantle plumes: a boundary layer approach for Newtonian and non-Newtonian temperature-dependent rheologies. *J. Geophys. Res.* **81** (14), 2499–2510.