Euro. Jnl of Applied Mathematics (2014), *vol.* 25, *pp.* 579–594.
doi:10.1017/S0956792514000151

Solutions of a free boundary problem in a doubly connected domain via a circular-arc polygon

J. S. MARSHALL

Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2AZ, UK
email: jonathan.marshall1@imperial.ac.uk

(*Received 18 November 2013; revised 29 April 2014; accepted 12 May 2014; first published online 6 June 2014*)

This paper addresses a free boundary problem for a steady, uniform patch of vorticity surrounding a single flat plate of zero thickness and finite length. Exact solutions to this problem have previously been found in terms of conformal maps represented by Cauchy-type integrals. Here, however, it is demonstrated how, by considering an associated circular-arc polygon and using ideas from automorphic function theory, these maps can be expressed in a simple non-integral form.

Key words: Free boundary problem; Multiply connected; Conformal mapping; Automorphic functions

1 Introduction

In this paper, we consider a two-dimensional free boundary problem which arises in numerous diverse physical applications, describing a layer surrounding a given fixed, solid object, where the outer boundary of this layer is free and to be determined. The problem itself is stated in terms of some function Ψ, which satisfies Poisson's equation with constant forcing term (that is, $\nabla^2\Psi = \text{constant}, -\Omega$) in the layer, with the constraints that Ψ is constant along both the fixed boundary of the object and the free boundary of the layer, while the first derivatives of Ψ vanish on the free boundary.

One situation in which this problem arises is in a model for an industrial process for coating objects with viscous fluids originally formulated by Tuck *et al.* [19]. Another, identified by Crowdy [5] and the context in which we shall primarily present our discussion, is in two-dimensional flows of ideal inviscid fluids featuring vortical layers around solid boundaries. Such flows are of geophysical interest, as bands of vorticity are often observed in the ocean along coastlines. In this case, Ψ represents the stream function of the flow, and the above conditions imply that the vortical layer has uniform vorticity Ω, while the fixed and free boundaries are streamlines of the flow and the flow velocity vanishes everywhere on the free boundary.

Exact solutions to this problem have been found for the case of a semi-infinite plate by Howison [11] using a mapping to a potential plane, and for wedges extending to infinity by Howison and King [12] and more generally by Craster [2], using the differential equation method originally devised by Polubarinova-Kochina [1, 18].

In this paper, we consider the case of a single plate of finite length. Exact solutions for this were in fact found by Johnson and McDonald in [14]. These were derived from consideration of the Schwarz function [8] of the free boundary of the layer, extending ideas described by Crowdy in [5] and earlier in [4], and which have subsequently been used to construct a wide variety of vortical equilibria. The solutions found in [14] are stated in terms of conformal maps from a parametric domain. These maps were constructed by first determining their imaginary part on the domain boundary, and then using standard results to write them as Cauchy-type integrals. Subsequently, following a similar approach, although using a different parametric variable and employing automorphic function theory, Marshall [16] was able to derive solutions for an arbitrary array of finitely many plates of finite length aligned in a parallel direction (not necessarily collinear), retrieving the results of [14] as a special case.

In this paper, however, we derive a representation for these maps in a simple non-integral form. We do so by following a similar approach to that in [14, 16], but with the key difference of constructing an auxiliary function not considered in either of these two earlier publications. As in [16], our derivation rests on the theory of automorphic functions.

We report these results for the reasons that, not only are the formulae for the maps found here arguably simpler than those presented previously, but the method used to derive them is of theoretical interest and may be applicable to other similar free boundary problems involving a single flat plate, or possibly arrays of multiple flat plates, or even plates which are only piecewise flat. We mention here that, in the context of the vortical flow problem, in addition to the cases referred to above, results are known for a periodic, collinear array of infinitely many plates of finite length [15], and also for layers along walls driven by sources and sinks [13], including walls with gaps [17].

Finally, we summarise the contents of the paper as follows. In Section 2, we present the formulation of the problem. In Section 3, we describe a parameterisation for its solutions in terms of conformal maps from a parametric domain. Next, in Section 4, we present an auxiliary function previously used in [14, 16]. The remaining sections contain this paper's new contributions. In Section 5, we introduce and construct a second auxiliary function not considered in either [14] or [16]. Finally, in Section 6, we combine our two auxiliary functions to obtain a formula for the parameterising conformal map in a non-integral form.

2 Problem formulation

We consider the two-dimensional flow in a z-plane (where $z = x + \mathrm{i}y$) of an ideal inviscid fluid in the region exterior to a single flat plate, Γ_1, of zero thickness and finite length l, lying along the real z-axis, centred on the origin, with endpoints $z_1 = -l/2$ and $z_2 = l/2$. We suppose that Γ_1 is surrounded by a bounded patch $\mathscr{P}$ of uniform vorticity Ω, with free outer boundary Γ_0. We also assume that the flow velocity is zero on Γ_0 and everywhere outside $\mathscr{P}$. Furthermore, we assume Γ_0 to be an analytic curve. Clearly, this represents a very particular subclass of all possible vortical flows around Γ_1. Nevertheless, given the lack of exact solutions known for vortical flows around solid boundaries in general, any such results are of interest. A schematic illustrating $\mathscr{P}$ is shown in Figure 1. Note that we label the points where Γ_0 intersects the positive real and imaginary z-axes by x_0 and $\mathrm{i}y_0$ respectively.

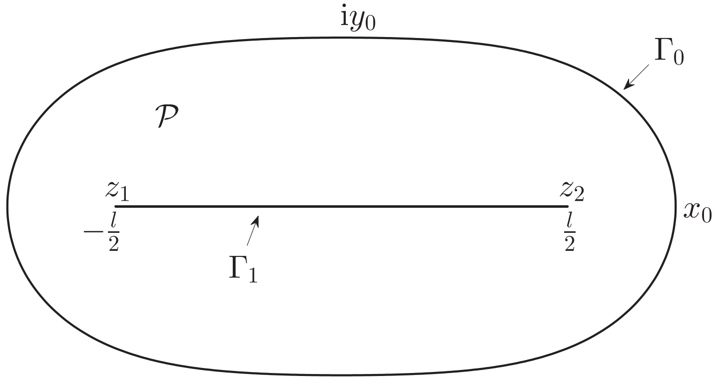

FIGURE 1. A patch $\mathscr{P}$ of uniform vorticity Ω, with outer boundary Γ_0, surrounding a plate Γ_1 with endpoints $z_1 = -l/2$ and $z_2 = l/2$. Γ_0 intersects the positive real and imaginary z-axes at the points x_0 and iy_0 respectively. The flow velocity is zero on Γ_0 and everywhere outside $\mathscr{P}$.

Denoting the streamfunction of the flow by Ψ, we have, for $z \in \mathscr{P}$, $\nabla^2\Psi = -\Omega$. Then, denoting the components of the fluid velocity in the x- and y-directions by u and v respectively, by integrating this last equation one arrives at

$$u - iv = -\frac{i\Omega}{2}(\overline{z} - S(z)), \quad \text{for } z \in \mathscr{P}, \tag{2.1}$$

where $S(z)$ is a function of z which is analytic for $z \in \mathscr{P}$ except possibly for isolated singularities. One may identify $S(z)$ as follows.

Since we require the flow velocity to vanish on Γ_0, (2.1) implies that we must have

$$S(z) = \overline{z}, \quad \text{for } z \in \Gamma_0. \tag{2.2}$$

Then assuming Γ_0 to be an analytic curve, (2.2) implies that $S(z)$ must be the (unique) Schwarz function of Γ_0 [8].

The Schwarz function of a general analytic curve is analytic only in an annular neighbourhood of the curve. Γ_0 is thus special in that we require its Schwarz function $S(z)$ to be analytic everywhere in $\mathscr{P}$ except for the following singularities.

Consider the flow in the neighbourhood of the plate Γ_1. Along the plate itself the fluid velocity is purely tangential, directed along the upper side one way and along the lower side the other. In fact, local to the plate, we expect the flow round it to be clockwise if $\Omega > 0$ and anticlockwise if $\Omega < 0$. This follows from the fact that, since we require the fluid velocity to vanish on Γ_0, the net vorticity of the patch and plate must be zero. One may show that the flow will possess these properties if we impose the following singularities at the plate endpoints z_1 and z_2:

$$u - iv \sim \begin{cases} \lambda(z - z_1)^{-1/2} & \text{for } z \text{ local to } z_1 \\ i\lambda(z - z_2)^{-1/2} & \text{for } z \text{ local to } z_2 \end{cases}, \tag{2.3}$$

where λ is a real constant of the same sign as Ω. Here we pick branches of the square roots so that, in terms of local coordinates at each endpoint, we have $0 < \arg(z - z_1) < 2\pi$ and $-\pi < \arg(z - z_2) < \pi$, with a branch cut along the plate so that the velocity is continuous

https://doi.org/10.1017/S0956792514000151 Published online by Cambridge University Press

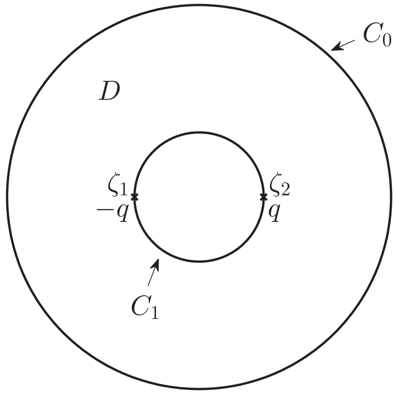

FIGURE 2. The pre-image annulus D of $\mathscr{P}$. D is bounded by the concentric circles C_0 and C_1. The points $\zeta_1 = -q$ and $\zeta_2 = q$ are the pre-images of the plate endpoints z_1 and z_2 respectively.

in the interior of $\mathscr{P}$. Apart from these singularities at z_1 and z_2, the flow velocity must be non-singular everywhere else in $\mathscr{P}$. It then follows from (2.1) and (2.3) that $S(z)$ must be analytic everywhere in $\mathscr{P}$ except for singularities at z_1 and z_2 of the form

$$S(z) \sim \begin{cases} -2\mathrm{i}\lambda\Omega^{-1}(z-z_1)^{-1/2} & \text{for } z \text{ local to } z_1, \\ 2\lambda\Omega^{-1}(z-z_2)^{-1/2} & \text{for } z \text{ local to } z_2. \end{cases} \tag{2.4}$$

3 Parameterising the patch

Both Johnson and McDonald [14] and Marshall [16] seek $\mathscr{P}$ in terms of parameterisations by one-to-one conformal maps. Each, however, uses a different parametric domain. In this paper, we shall work with that used in [16], namely a concentric circular annulus D in a ζ-plane. (Note that, $\mathscr{P}$ is doubly connected, and any doubly connected domain is the image under a one-to-one conformal map of a concentric annulus [10].) We label the map from D onto $\mathscr{P}$ by $z(\zeta)$. We denote the outer and inner boundary circles of D by C_0 and C_1 respectively, and for $k = 0, 1$, take C_k to correspond Γ_k under $z(\zeta)$. We are free to take C_0 to be the unit circle $\{\zeta : |\zeta| = 1\}$, and label the radius of C_1 by q, where $0 < q < 1$.

On the grounds of symmetry, we assume that

$$\overline{z(\zeta)} \equiv z(\overline{\zeta}) \quad \text{and} \quad z(-\zeta) \equiv -z(\zeta). \tag{3.1}$$

We also assume the normalisation that $\zeta = 1$ maps to the point x_0. Thus, for $k = 1, 2$, denoting by ζ_k the point on C_1 which corresponds to z_k under $z(\zeta)$, we must have $\zeta_1 = -q$ and $\zeta_2 = q$. Furthermore, $z(\pm 1) = \pm x_0$, $z(\pm\mathrm{i}) = \pm\mathrm{i}y_0$, and $z(\pm\mathrm{i}q) = 0$. A schematic illustrating D is shown in Figure 2.

We are ultimately going to derive an explicit formula for $z(\zeta)$. We begin this derivation as follows.

Firstly, it will be helpful to introduce the region D', which is the image of D under reflection in C_0, by which we mean the transformation $\zeta \mapsto 1/\overline{\zeta}$. D' is bounded by the circles C_0 and C_1', where the latter denotes the reflection of C_1 in C_0, and is centred on the origin with radius q^{-1}. We also introduce the region F, which is the union of D, D', C_0 and C_1, i.e. the region $\{\zeta : q \leqslant |\zeta| < q^{-1}\}$. Note that C_1' is not included in F.

Now, let us consider $S(z)$ in terms of ζ, that is, $S(z(\zeta))$. We shall denote this as $\tilde{S}(\zeta)$. For $\zeta \in C_0$, since $\overline{\zeta} = \zeta^{-1}$, it follows from (2.2) and the first relation of (3.1) that

$$\tilde{S}(\zeta) = z(\zeta^{-1}). \tag{3.2}$$

But $z(\zeta)$ must be analytic for all ζ in the closure of D. Furthermore, it follows from the properties of $S(z)$ described in Section 2 that $\tilde{S}(\zeta)$ must also be analytic for all ζ in the closure of D except for singularities at $\zeta = \pm q$. Hence, by analytic continuation, one may deduce that (3.2) must in fact hold for all ζ in the closure of F. An immediate consequence of this is that both $\tilde{S}(\zeta)$ and $z(\zeta)$ must be analytic everywhere in the closure of F except for singularities at $\zeta = \pm q$ and $\zeta = \pm q^{-1}$ respectively.

Let us now analyse the singularities of $\tilde{S}(\zeta)$ at $\zeta = \pm q$ in more detail. Firstly note that, for $k = 1, 2$, since ζ_k maps to an endpoint of the plate, we must have $z'(\zeta_k) = 0$ (where we use the notation $'$ to denote differentiation) and, for ζ local to ζ_k,

$$z(\zeta) \sim z_k + \frac{z''(\zeta_k)}{2}(\zeta - \zeta_k)^2. \tag{3.3}$$

One may deduce from (3.3) that for $k = 1, 2$, $z''(\zeta_k)$ must be real and, furthermore, negative for $k = 1$ and positive for $k = 2$. In fact, it follows from the second relation of (3.1) that we must have $z''(\zeta_2) = -z''(\zeta_1) = \sigma$, for some $\sigma \in \mathbb{R}, > 0$.

We now take the square root of (3.3). In order to remain consistent with the behaviour local to z_k resulting from the choice of branches already made implicitly in (2.3), one may show that we must have

$$\begin{aligned} (z - z_1)^{-1/2} &\sim \mathrm{i}\sqrt{\frac{2}{\sigma}}(\zeta - \zeta_1)^{-1} \quad \text{for } \zeta \text{ local to } \zeta_1, \\ (z - z_2)^{-1/2} &\sim \sqrt{\frac{2}{\sigma}}(\zeta - \zeta_2)^{-1} \quad \text{for } \zeta \text{ local to } \zeta_2, \end{aligned} \tag{3.4}$$

where we take $\sqrt{2/\sigma}$ to denote the positive square root.

It then follows from (2.4) and (3.4) that for $k = 1, 2$, for ζ local to ζ_k, we require

$$\tilde{S}(\zeta) \sim \frac{\chi}{\zeta - \zeta_k}, \tag{3.5}$$

where

$$\chi = 2\sqrt{\frac{2}{\sigma}}\lambda\Omega^{-1}, \tag{3.6}$$

i.e. $\tilde{S}(\zeta)$ must have simple poles at $\zeta = \pm q$, both with the same residue χ given by (3.6).

Having identified these properties of $z(\zeta)$ and $\tilde{S}(\zeta)$, we shall now introduce, in the following two sections, two auxiliary functions which we shall use to construct $z(\zeta)$.

https://doi.org/10.1017/S0956792514000151 Published online by Cambridge University Press

4 An auxiliary function, $h(\zeta)$

In this section, we present the first of our two auxiliary functions. This was introduced by Johnson and McDonald in [14] as a function of z, and subsequently used by Marshall in [16], parameterised in terms of ζ. For completeness, we give details of its construction as follows.

In terms of z, it is defined as

$$g(z) = -\frac{\mathrm{i}\Omega}{2}(z - S(z)). \tag{4.1}$$

Note that it follows from (2.1) that, for z in the closure of $\mathscr{P}$, $g(z) = \Omega y + u - \mathrm{i}v$. Thus, since $v = 0$ on both Γ_0 and Γ_1,

$$\mathrm{Im}\{g(z)\} = 0 \quad \text{for } z \in \Gamma_0, \Gamma_1. \tag{4.2}$$

Let us now consider $g(z)$ in terms of ζ. We define

$$h(\zeta) = g(z(\zeta)) = -\frac{\mathrm{i}\Omega}{2}(z(\zeta) - \tilde{S}(\zeta)). \tag{4.3}$$

One may deduce from the properties of $z(\zeta)$ and $\tilde{S}(\zeta)$ described in Section 3 that $h(\zeta)$ must be analytic for all ζ in the closure of F except for singularities at $\zeta = \pm q, \pm q^{-1}$.

Next, it follows from (4.2) that

$$\begin{aligned} h(\zeta) + h(\zeta^{-1}) = 0 \quad &\text{for } \zeta \in C_0, \\ h(\zeta) + h(q^2\zeta^{-1}) = 0 \quad &\text{for } \zeta \in C_1, \end{aligned} \tag{4.4}$$

where we have used (3.2) and the first relation of (3.1), together with the fact that $\overline{\zeta}$ equals ζ^{-1} and $q^2\zeta^{-1}$ for ζ on C_0 and C_1 respectively. But then, by the analytic continuation of the two relations in (4.4), one may deduce that

$$h(q^2\zeta) = h(\zeta) \quad \text{for all } \zeta. \tag{4.5}$$

Equation (4.5) will be crucial to our construction of $h(\zeta)$, as we now show.

Let us define $\theta(\zeta) = q^2\zeta$. Furthermore, for $n \in \mathbb{Z}$, let us denote $\theta^n(\zeta) = q^{2n}\zeta$, so that, for example, $\theta^0(\zeta)$ is simply the identity, and $\theta^{-1}(\zeta)$ is the inverse of $\theta(\zeta)$. Now define $\Theta = \{\theta^n(\zeta) : n \in \mathbb{Z}\}$. Θ is in fact an example of a *Schottky group* [9, 16]. Furthermore, F is a *fundamental region* of Θ. In fact, any concentric annulus centred on the origin whose two boundary circles differ in radii by a multiplicative factor of q^2, forms a fundamental region of Θ. It follows from (4.5) that $h(\zeta)$ is invariant under each element of Θ, or, in other words, is *automorphic* with respect to Θ [9]. The identification of this property will enable us to exploit the theory of automorphic functions to construct $h(\zeta)$ as follows.

Let us first examine the singularities of $h(\zeta)$ in F. Recall that this region contains C_1 but not C_1'. It then follows from the properties of $z(\zeta)$ and $\tilde{S}(\zeta)$ described in Section 3 that $h(\zeta)$ must be analytic everywhere in F except for simple poles at $\zeta = \pm q$, both with residue $\mathrm{i}\lambda\sqrt{2/\sigma}$. Note that this residue is purely imaginary, and the sign of its imaginary part is the same as that of Ω.

Let us now demonstrate that (4.5) together with the singularities in F just described can be used to define $h(\zeta)$ uniquely. For, suppose that some function $\tilde{h}(\zeta)$ also possesses these properties and consider $H(\zeta) = h(\zeta) - \tilde{h}(\zeta)$. Since $h(\zeta)$ and $\tilde{h}(\zeta)$ have the same singularities in F, $H(\zeta)$ must be analytic everywhere in this region. But since both $h(\zeta)$ and $\tilde{h}(\zeta)$ satisfy (4.5), so must $H(\zeta)$. It then follows that $H(\zeta)$ must be analytic everywhere in the ζ-plane. Thus, by Liouville's theorem, $H(\zeta)$ must be a constant. But now note that, by our assumption that $z(\zeta)$ maps $\zeta = 1$ to the point x_0, it follows from (4.3) that $h(1) = 0$. Then assuming also that $\tilde{h}(1) = 0$, it follows that $H(1) = 0$. Hence, $H(\zeta) \equiv 0$ and $\tilde{h}(\zeta) \equiv h(\zeta)$. Thus, if one can construct a function with the properties just described, one may identify it as $h(\zeta)$. We demonstrate how this may be done as follows.

4.1 Construction of $h(\zeta)$

Let us first introduce the function $P(\zeta, q)$ defined by

$$P(\zeta, q) = (1-\zeta)\prod_{n=1}^{\infty}(1-q^{2n}\zeta)(1-q^{2n}\zeta^{-1}). \tag{4.6}$$

Note that it can be shown that the infinite product in (4.6) converges for all q, $0 < q < 1$.

One can check that $P(\zeta, q)$ is analytic everywhere in the ζ-plane except at 0 and ∞. It has a simple zero in F at $\zeta = 1$, and additional simple zeros at all images of 1 under the non-identity maps in Θ. Furthermore, two useful properties which can be deduced directly from (4.6) are

$$P(q^2\zeta, q) = -\zeta^{-1}P(\zeta, q) \quad \text{and} \quad P(\zeta^{-1}, q) = -\zeta^{-1}P(\zeta, q). \tag{4.7}$$

Next, we define

$$K(\zeta, q) = \zeta \frac{d}{d\zeta} \log P(\zeta, q). \tag{4.8}$$

It follows from (4.6) that

$$K(\zeta, q) = 1 + \frac{1}{\zeta - 1} + \sum_{n=1}^{\infty} q^{2n}\left(\frac{1}{\zeta - q^{2n}} - \frac{1}{\zeta^{-1} - q^{2n}}\right). \tag{4.9}$$

Note that $K(\zeta, q)$ is analytic everywhere in F except for a simple pole at $\zeta = 1$ of residue 1. Also, two useful properties that follow from (4.7) are

$$K(q^2\zeta, q) = K(\zeta, q) - 1 \quad \text{and} \quad K(\zeta^{-1}, q) = 1 - K(\zeta, q). \tag{4.10}$$

Then $h(\zeta)$ can, in fact, be constructed in terms of $K(\zeta, q)$ as

$$h(\zeta) = \mathrm{i}\mu(K(\zeta/q, q) - K(-\zeta/q, q)), \tag{4.11}$$

where μ is some real constant with the same sign as Ω. To verify this, let us check that the function on the right-hand side of (4.11) possesses the properties identifying $h(\zeta)$ described above. To do so, we make use of the properties of $K(\zeta, q)$ just stated. Firstly, one may deduce from the first relation of (4.10) that this function satisfies (4.5). Next, one can check that it is analytic everywhere in F except for simple poles at $\zeta = \pm q$, both with residue $\mathrm{i}\mu q$, which is purely imaginary and has imaginary part of the same sign as Ω. Finally, one can show, using (4.9), that it vanishes at $\zeta = 1$. This completes our check on (4.11). Before continuing, we also point out that the values of μ and q in (4.11) may be determined by the length l of the plate and the size of the patch, as will be discussed in greater detail in Section 6.

Now note that, in [16], having constructed $h(\zeta)$, a representation for the map $z(\zeta)$ is derived as follows. One may deduce from (4.3) that, for $\zeta \in C_0$, $h(\zeta) = \Omega y$. But, since y is simply zero along Γ_1, we thus know the imaginary part of $z(\zeta)$ on both of the boundaries of D. Given this information, retrieving $z(\zeta)$ in full is a modified Schwarz problem, and the solution to this is known [6]. As shown by Marshall in [16], this gives $z(\zeta)$ as the following Cauchy-type integral,

$$z(\zeta) = \frac{-1}{\pi\Omega} \oint_{C_0} \frac{h(\xi)K(\zeta/\xi, q)}{\xi} d\xi, \tag{4.12}$$

where we integrate around C_0 in the anticlockwise direction. Note that $K(\zeta/\xi, q)$ has a pole at $\xi = \zeta$, and this gives rise to a singularity of the integrand in (4.12).

We point out that this approach is similar to that taken by Johnson and McDonald in [14], who there also construct an integral representation for z, albeit in terms of a different parametric variable. In fact, in [16], Marshall demonstrated that (4.12) may be rearranged into this alternative form.

In this paper, however, we shall derive a non-integral representation for $z(\zeta)$. To do this, we introduce a second auxiliary function, not considered by either Johnson and McDonald in [14] or Marshall in [16]. This is described in the following section.

5 A second auxiliary function, $h_2(\zeta)$

Along each of Γ_0 and Γ_1 we have two real boundary conditions, which we may express as

$$\begin{aligned} \mathrm{Im}\{z + S(z)\} = 0, \quad \mathrm{Im}\{\mathrm{i}z - \mathrm{i}S(z)\} = 0, \quad \text{for } z \in \Gamma_0, \\ \mathrm{Im}\{z\} = 0, \quad \mathrm{Im}\{\mathrm{i}z - \mathrm{i}S(z)\} = 0, \quad \text{for } z \in \Gamma_1. \end{aligned} \tag{5.1}$$

Then using arguments described, for example, by Polubarinova-Kochina [18, Ch. VII, Sec. 2], one can show that the function $w(z)$ defined as

$$w(z) = \frac{z}{S(z)} \tag{5.2}$$

must map each of Γ_0 and Γ_1 onto sections of circular arcs. Indeed, by (2.2), at a point z on Γ_0, $w(z) = e^{2\mathrm{i}\arg(z)}$. Thus, $w(z)$ maps Γ_0 onto the unit circle, in fact covering it twice. Furthermore, taking the ratio of the latter two conditions in (5.1), one finds that

$$\mathrm{Re}\left\{\frac{1}{w(z) - 1}\right\} = -1, \quad \text{for } z \in \Gamma_1. \tag{5.3}$$

It is then straightforward to show from (5.3) that $w(z)$ maps Γ_1 onto a section of the circle centred on $w = \frac{1}{2}$ of radius $\frac{1}{2}$. One may deduce that this image of Γ_1 does not cover the

https://doi.org/10.1017/S0956792514000151 Published online by Cambridge University Press

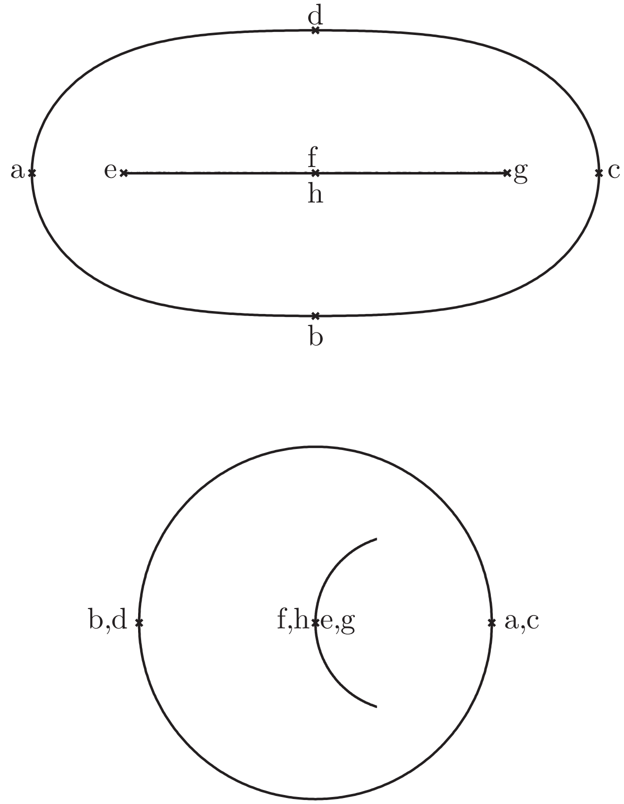

FIGURE 3. Under $w(z)$, $\mathscr{P}$ (top) maps onto (a double covering of) the interior of a circular-arc polygon (bottom), bounded by the unit circle $\{w : |w| = 1\}$ and a section of the circle $\{w : |w - \frac{1}{2}| = \frac{1}{2}\}$. Points labelled by the same letter (a, b, ..., h) correspond to one another under this transformation.

whole of this circle, and in particular does not contain the point $w = 1$, as follows. Since $h(\zeta)$ is automorphic with respect to the Schottky group Θ, following from the general theory of automorphic functions [9], $h(\zeta)$ must have the same number of zeros as poles in the fundamental region F, where we count these according to their multiplicity. But, as discussed in Section 4, $h(\zeta)$ has precisely two simple poles in F (at $\zeta = \pm q$). Furthermore, as one may deduce from (4.3), $h(\zeta)$ must vanish at $\zeta = \pm 1$. It thus follows that these must be the only zeros of $h(\zeta)$ in F (and, in fact, both simple). Hence, one may deduce that $h(\zeta)$ does not vanish on C_1, and thus that nowhere on Γ_1 does $S(z)$ equal z, or $w(z)$ equal 1. We remark that this makes sense on physical grounds, since one would not expect there to be a stagnation point of the flow on Γ_1, and hence, from (2.1), that $S(z) = z$ at any point along this boundary. A schematic illustrating the image of $\mathscr{P}$ in the w-plane is given in Figure 3.

However, now note that, inverting in $w = 1$, it follows from the basic properties of Möbius transformations that the circular-arc images of both Γ_0 and Γ_1 in the w-plane map onto sections of straight lines, of infinite and finite lengths, respectively. In fact, let

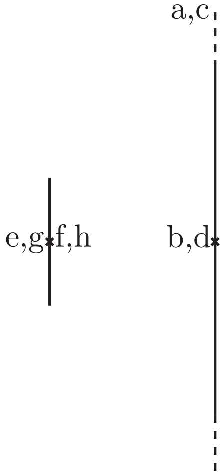

FIGURE 4. Under $g_2(z)$, $\mathscr{P}$ maps onto (a double covering of) the region bounded by the line $\mathrm{Re}\{g_2(z)\} = \frac{1}{2}$ and a section of the line $\mathrm{Re}\{g_2(z)\} = 0$, shown here. Again, points labelled by the same letter in this and Figure 3 correspond to one another.

us consider the function

$$g_2(z) = \frac{1}{w(z) - 1} + 1. \tag{5.4}$$

Elementary rearrangements lead to

$$g_2(z) = \frac{z}{z - S(z)}. \tag{5.5}$$

Then, from (2.2) and (5.3), it follows that

$$\mathrm{Re}\{g_2(z)\} = \begin{cases} \frac{1}{2} & \text{for } z \in \Gamma_0, \\ 0 & \text{for } z \in \Gamma_1. \end{cases} \tag{5.6}$$

A schematic illustrating the image of $\mathscr{P}$ under $g_2(z)$ is given in Figure 4.

Let us now consider $g_2(z)$ in terms of ζ. We define

$$h_2(\zeta) = g_2(z(\zeta)) = \frac{z(\zeta)}{z(\zeta) - \tilde{S}(\zeta)}. \tag{5.7}$$

One may deduce from the properties of $z(\zeta)$ and $\tilde{S}(\zeta)$ discussed in Section 3, and the zeros of $h(\zeta)$ just described, that $h_2(\zeta)$ must be analytic for all ζ in the closure of F except for singularities at $\zeta = \pm 1$.

Next, it follows from (5.6) that

$$\begin{aligned} h_2(\zeta) + h_2(\zeta^{-1}) &= 1 \quad \text{for } \zeta \in C_0, \\ h_2(\zeta) + h_2(q^2\zeta^{-1}) &= 0 \quad \text{for } \zeta \in C_1, \end{aligned} \tag{5.8}$$

where we have used (3.2) and the first relation of (3.1). But then, by the analytic continuation of the two relations in (5.8), one may deduce that

$$h_2(q^2\zeta) = h_2(\zeta) - 1 \quad \text{for all } \zeta. \tag{5.9}$$

Thus, unlike $h(\zeta)$, $h_2(\zeta)$ is only *quasi*-automorphic with respect to Θ. However, using arguments similar to those applied to $h(\zeta)$ in Section 4, we can still identify it as follows.

Firstly, let us examine the singularities of $h_2(\zeta)$ in F in more detail. As stated above, these are located at $\zeta = \pm 1$. Let us first consider that at $\zeta = 1$. Local to this point we may expand $z(\zeta)$ and $\tilde{S}(\zeta)$ as Taylor series in ζ as follows. We have

$$z(\zeta) = z(1) + z'(1)(\zeta - 1) + \ldots, \tag{5.10}$$

Next, from (2.2) one may deduce that $\tilde{S}(1) = z(1)$, and furthermore that $\tilde{S}'(1) = -z'(1)$. Hence, we also have

$$\tilde{S}(\zeta) = z(1) - z'(1)(\zeta - 1) + \ldots. \tag{5.11}$$

Straightforward manipulations then lead one to find that, local to $\zeta = 1$,

$$h_2(\zeta) = \frac{z(1)}{2z'(1)(\zeta - 1)} + \mathcal{O}(1). \tag{5.12}$$

Using similar arguments, one can also show that, local to $\zeta = -1$,

$$h_2(\zeta) = \frac{z(-1)}{2z'(-1)(\zeta + 1)} + \mathcal{O}(1). \tag{5.13}$$

But it follows from the second relation of (3.1) that $z(1) = -z(-1)$ and, furthermore, that $z'(1) = z'(-1)$. Hence, we deduce that $h_2(\zeta)$ must have simple poles at $\zeta = \pm 1$, with residues $\pm\tau$ respectively, where $\tau = z(1)/(2z'(1))$.

We may in fact determine τ as follows. Consider the integral, I, of $h_2(\zeta)/\zeta$ around the boundary of F. $h_2(\zeta)/\zeta$ is analytic everywhere in F except for simple poles at $\zeta = \pm 1$, where the residues of these are *both* τ. Hence, by the residue theorem, I must equal $4\pi i\tau$. However, we also have

$$I = \oint_{C_1'} \frac{h_2(\zeta)}{\zeta} d\zeta - \oint_{C_1} \frac{h_2(\zeta)}{\zeta} d\zeta = \mathrm{i}\int_{\theta=0}^{2\pi} (h_2(q^{-1}e^{\mathrm{i}\theta}) - h_2(qe^{\mathrm{i}\theta}))d\theta = 2\pi\mathrm{i}, \tag{5.14}$$

where the last equality follows from the quasi-automorphicity property (5.9). Hence, combining the above, one may deduce that $\tau = \frac{1}{2}$.

Finally, since we assume $\zeta = q$ maps to one of the plate endpoints, where z is finite and non-zero but $S(z)$ is singular, we must have $h_2(q) = 0$. This fact, together with (5.9) and the singularities in F just described, are in fact enough to identify $h_2(\zeta)$ uniquely. One can show this using arguments similar to those applied to $h(\zeta)$ in Section 4 by supposing that some function $\tilde{h}_2(\zeta)$ also possesses these properties and then considering $H_2(\zeta) = h_2(\zeta) - \tilde{h}_2(\zeta)$. Note that, even though $h_2(\zeta)$ and $\tilde{h}_2(\zeta)$ are quasi-automorphic with respect to Θ, $H_2(\zeta)$ will itself be automorphic.

Thus, if we can construct a function with these properties, we may identify it as $h_2(\zeta)$. We show how one may do this as follows.

5.1 Construction of $h_2(\zeta)$

Given the properties of $K(\zeta, q)$ described in Section 4.1, one may, at first glance, expect that some simple combination of this and $K(-\zeta, q)$ could be used to construct $h_2(\zeta)$. However, on closer inspection, it becomes apparent that a construction in these terms is not so obvious. Instead, $h_2(\zeta)$ is given by

$$h_2(\zeta) = K(\zeta^2, q^2). \tag{5.15}$$

We may verify (5.15) as follows. Firstly, one may show from the first relation of (4.10) that $K(q^4\zeta^2, q^2) = K(\zeta^2, q^2) - 1$. Next, it follows from (4.9) that for ζ^2 close to 1,

$$K(\zeta^2, q^2) = \frac{1}{\zeta^2 - 1} + \mathcal{O}(1) = \frac{1}{2(\zeta - 1)} - \frac{1}{2(\zeta + 1)} + \mathcal{O}(1). \tag{5.16}$$

Hence, $K(\zeta^2, q^2)$ has simple poles at $\zeta = \pm 1$ with residues $\pm\frac{1}{2}$ respectively. Finally, using (4.9), one may show that $K(q^2, q^2) = 0$. Thus, $K(\zeta^2, q^2)$ possesses the properties identifying $h_2(\zeta)$. This completes our check on (5.15).

Finally, we make the following remark. That $h_2(\zeta)$ is given by (5.15) may also be understood from the point of view of conformal mappings as follows. By considering the effects of $w(z)$ and $g_2(z)$, one may deduce that $h_2(\zeta)$ must map the annulus D onto a *double* covering of the slit domain illustrated in Figure 4, with each of the upper and lower halves of D mapping onto a separate copy. Now note that, as follows from its properties described in Section 4.1, and as is discussed more generally by Crowdy and Marshall in [7], one can show that $K(\zeta, q)$ maps D onto a slit domain of this type, but in a one-to-one manner. One may then deduce that $K(\zeta^2, q^2)$ produces the double-covering required of $h_2(\zeta)$. This confirms (5.15).

6 Final formula for $z(\zeta)$

Finally, from (4.3) and (5.7), together with (4.11) and (5.15), we obtain the following expression for $z(\zeta)$, for a patch $\mathscr{P}$ of vorticity Ω:

$$z(\zeta) = \frac{-2\mu}{\Omega}(K(\zeta/q, q) - K(-\zeta/q, q))K(\zeta^2, q^2), \tag{6.1}$$

where μ is some real constant with the same sign as Ω. This is the principal new result of this paper.

6.1 Determining the mapping parameters

As mentioned in Section 4.1, the values of q and μ in this parameterisation may be determined by the plate length l and the size of the patch. One may find relations between these quantities as follows.

Firstly, we have $l = 2z(q)$. Note that $K(\zeta/q, q)$ has a simple pole at $\zeta = q$, but, as stated in Section 5.1, $K(\zeta^2, q^2)$ vanishes there. In fact, a local analysis of (6.1) about $\zeta = q$

https://doi.org/10.1017/S0956792514000151 Published online by Cambridge University Press

reveals that

$$l = -\frac{8\mu}{\Omega} q^2 K'(q^2, q^2). \tag{6.2}$$

Then, using (4.9), one may obtain an expression for $K'(q^2, q^2)$ in terms of q. Substituting this into (6.2), one finds

$$l = \frac{16\mu}{\Omega} \sum_{n=0}^{\infty} \frac{q^{2(2n+1)}}{(1 - q^{2(2n+1)})^2}. \tag{6.3}$$

Next, as a measure of the size of the patch, let us consider y_0. We have $y_0 = -\mathrm{i}z(\mathrm{i})$. Alternatively, as one may deduce from (4.3), $y_0 = h(\mathrm{i})/\Omega$. It then follows from (4.11) and (4.9) that

$$y_0 = \frac{4\mu}{\Omega} \sum_{n=0}^{\infty} \frac{q^{2n+1}}{1 + q^{2(2n+1)}}. \tag{6.4}$$

Equations (6.3) and (6.4) constitute two relations which may be used to determine q and μ for a given plate length l and patch thickness y_0. Furthermore, one may also derive an expression for the "aspect ratio" x_0/y_0 of the patch. In fact, as for y_0 above, one may do this without requiring the full form for $z(\zeta)$. Details are given in the appendix. One may also derive expressions for these quantities using the parameterisations presented in [14, 16].

An example of a patch constructed using this parameterisation is given in Figure 1. It should be pointed out that the author has so far been unable to show that (6.1) may be rearranged into the form (4.12) for $z(\zeta)$ previously derived in [16], or that presented in [14]. However, numerical checks demonstrate agreement between these results.

Finally, we point out that, using (6.1) and (4.11), one may easily compute the flow velocity at points in the interior of the patch from (2.1).

7 Conclusions

In this paper, we have constructed solutions to a free boundary problem in a doubly connected domain, describing a vortex patch around a single flat plate of zero thickness and finite length. These solutions are stated in terms of conformal maps from a parametric domain. Formulae for these maps have been constructed in earlier publications in terms of Cauchy-type integrals. Here, however, we derive a representation for them in a simple non-integral form.

Key steps in the derivation of this new formulation are the introduction of the function $w(z) = z/S(z)$ which maps the patch onto the interior of a circular-arc polygon, and the observation that, by a simple inversion, this can then be transformed into a region with straight boundaries. The use of such techniques in free boundary problems set in domains whose fixed boundaries are straight is well known [1, 18]. However, the majority of such problems to which they are applied in the existing literature are ones set in domains of just simple connectivity.

Another vital component in the derivation of these solutions is the use of the theory of automorphic functions, which facilitates the construction of two auxiliary functions $h(\zeta)$ and $h_2(\zeta)$.

It is possible that the ideas and methods presented in this paper could be applied to other similar problems for the free boundary of a layer surrounding a single flat plate, or possibly arrays of multiple flat plates, or even plates which are only piecewise flat. We point out, however, that there appear to be certain restrictions on their use. Firstly, as mentioned by Marshall in [16], it seems that the auxiliary function $h(\zeta)$ is only suited to cases with plates aligned in a parallel direction. Secondly, while the function $w(z)$ will always map the patch onto a circular-arc domain, typically this will not be as simple as the one encountered here. Nevertheless, further examples are currently being investigated.

As yet another possible method of solving this problem, Howison and King [12] suggest that based on differential equations originally devised by Polubarinova-Kochina [1, 18]. Again, this is a technique more commonly associated with free boundary problems set in simply connected domains. One could reduce this problem to one in a simply connected region by exploiting the symmetry of the patch and focusing on just one half or one quarter of it. Doing so would lead to a differential equation with rational coefficients. Alternatively, retaining the double connectivity, one could consider a differential equation with independent variable ζ satisfied by $z(\zeta)$ and $\tilde{S}(\zeta)$. Such an equation would have coefficients that are not rational, but rather expressible in terms of the function $P(\zeta, q)$ and its derivatives. From our above analysis, it should have singular points in F at $\zeta = \pm q$, corresponding to the singularities of $\tilde{S}(\zeta)$ there, but also at four other points on C_1. These additional singular points are not singularities of either $z(\zeta)$ or $\tilde{S}(\zeta)$, but rather correspond, under $w(z(\zeta))$, to the end points of the circular-arc image of C_1 in the w-plane. There are four of them since the image of D in this plane is a double-covering. At each, $dw(z(\zeta))/d\zeta = 0$, or equivalently, the Wronskian $z'(\zeta)\tilde{S}(\zeta) - z(\zeta)\tilde{S}'(\zeta)$ vanishes. Under $z(\zeta)$, these correspond to points on the plate Γ_1. Such singularities are commonly observed in the analysis of free boundary problems, using this or related methods, often corresponding to points of inflection of the free surface (see, for example, [3]). In this particular case, however, their physical significance is not obvious. We also point out that, in addition to these singular points in F, there will be others at their images under all the non-identity maps in Θ.

Finally, we mention that other issues relating to this particular problem are discussed by Johnson and McDonald in [14] and Marshall in [16]. The main purpose of this paper has been to present and discuss the new formulation for its parameterisation.

Appendix Aspect ratio of the patch

As mentioned in Section 6, here we derive an expression for the 'aspect ratio' x_0/y_0 of the patch. We do so without requiring the full form for the map $z(\zeta)$, using, rather, properties of the auxiliary functions $h(\zeta)$ and $h_2(\zeta)$.

We point out that, while one could conduct the following analysis in terms of functions associated with the Schottky group Θ, as in the rest of this paper, here we shall in fact use the Jacobi elliptic functions. This is simply for the reason that it is convenient to make use of the well-documented properties of the latter.

Firstly, as follows from identifications made by Marshall in [16], one can show that (4.11) can be written as

$$h(\zeta) = \frac{2\rho\mathscr{K}\mu}{\pi}\mathrm{sn}(U(\zeta), \rho), \tag{7.1}$$

https://doi.org/10.1017/S0956792514000151 Published online by Cambridge University Press

where

$$U(\zeta) = \frac{-2\mathscr{K}\mathrm{i}}{\pi}\log\zeta, \tag{7.2}$$

sn denotes one of the Jacobi elliptic functions, $\mathscr{K} = \mathscr{K}(\rho)$ is the complete elliptic integral of the first kind defined by

$$\mathscr{K}(\rho) = \int_0^1 \frac{dt}{((1-t^2)(1-\rho^2t^2))^{1/2}}, \tag{7.3}$$

and the modulus ρ is related to q by

$$q = \exp\left(-\frac{\pi}{2}\frac{\mathscr{K}'}{\mathscr{K}}\right), \tag{7.4}$$

where

$$\mathscr{K}' = \mathscr{K}'(\rho) = \mathscr{K}((1-\rho^2)^{1/2}). \tag{7.5}$$

Now, as stated in Section 6.1, $y_0 = h(\mathrm{i})/\Omega$. Using standard properties of sn, it then follows from (7.1) that $\mu = \pi\Omega y_0/(2\rho\mathscr{K})$. Hence, (7.1) may be written as

$$h(\zeta) = \Omega y_0 \mathrm{sn}(U(\zeta), \rho). \tag{7.6}$$

Now, recalling $x_0 = z(1)$ and the fact, following (2.2), that $\tilde{S}'(1) = -z'(1)$, one may deduce from (4.3) that

$$z'(1) = \frac{\mathrm{i}}{\Omega}h'(1). \tag{7.7}$$

But, with cn and dn denoting two more of the Jacobi elliptic functions, by differentiating (7.6), standard results give

$$h'(\zeta) = \frac{-2\mathscr{K}\Omega y_0\mathrm{i}}{\pi\zeta}\mathrm{cn}(U(\zeta),\rho)\mathrm{dn}(U(\zeta),\rho), \tag{7.8}$$

and then

$$h'(1) = \frac{-2\mathscr{K}\Omega y_0\mathrm{i}}{\pi}. \tag{7.9}$$

It then follows from (7.7) and (7.9) that

$$z'(1) = \frac{2\mathscr{K}}{\pi}y_0. \tag{7.10}$$

But, as follows from the fact, shown in Section 5, that $\tau = z(1)/(2z'(1)) = \frac{1}{2}$, we have $z(1) = z'(1)$. Hence, from (7.10), one may finally deduce that

$$\frac{x_0}{y_0} = \frac{2\mathscr{K}}{\pi}. \tag{7.11}$$

Lastly, let us provide a quick check on (7.11). Following steps taken by Marshall in [16] (see equation (4.34) thereof), one can rewrite (4.12) as

$$z(\zeta) = \frac{-2\mathscr{K}y_0\mathrm{i}}{\pi^2}\int_{C_0^+}\frac{h(\xi)}{h(\xi/\zeta)}\frac{d\xi}{\xi}, \tag{7.12}$$

where C_0^+ denotes the section of C_0 in the upper half plane, and we integrate along this in the anticlockwise direction from 1 to -1. Equation (7.11) then follows from (7.12) with $\zeta = 1$.

References

[1] Crank, J. (1984) *Free and Moving Boundary Problems*, Oxford University Press, Oxford, UK.
[2] Craster, R. V. (1994) Two related free boundary problems. *IMA J. Appl. Math.* **52**, 253–270.
[3] Craster, R. V. (1997) The solution of a class of free boundary problems. *Proc. R. Soc. A* **453**, 607–630.
[4] Crowdy, D. G. (1999) A class of exact multipolar vortices. *Phys. Fluids* **11**, 2556–2564.
[5] Crowdy, D. G. (2004) Exact solutions for uniform vortex layers attached to corners and wedges. *Euro. J. Appl. Math.* **15**, 643–650.
[6] Crowdy, D. G. (2008) The Schwarz problem in multiply connected domains and the Schottky–Klein prime function. *Complex Var. Elliptic Equ.* **53**(3), 221–236.
[7] Crowdy, D. G. & Marshall, J. S. (2006) Conformal mappings between canonical multiply connected domains. *Comput. Methods Funct. Theory* **6(1)**, 59–76.
[8] Davis, P. J. (1974) *The Schwarz Function and its Applications*, The Mathematical Association of America, Washington, DC.
[9] Ford, L. R. (1972) *Automorphic Functions*, Chelsea, New York, NY.
[10] Goluzin, G. M. (1969) *Geometric Theory of Functions of a Complex Variable*, American Mathematical Society, Providence, RH.
[11] Howison, S. D. (1987) Complex variables in industrial mathematics. In: H. Neunzert (editor), *Proceedings of the 2nd European Symposium on Mathematics in Industry*, Oberwolfach, Germany, Kluwer Academic, Stuttgart, Germany, pp. 155–166.
[12] Howison, S. D. & King, J. R. (1989) Explicit solution to six free boundary problems in fluid flow and diffusion. *IMA J. Appl. Math.* **42**, 155–175.
[13] Johnson, E. R. & McDonald, N. R. (2006) Vortical source-sink flow against a wall: The initial value problem and exact steady states. *Phys. Fluids* **18**, 076601.
[14] Johnson, E. R. & McDonald, N. R. (2007) Steady vortical flow around a finite plate. *Q. J. Mech. Appl. Math.* **60**(1), 65–72.
[15] Johnson, E. R. & McDonald, N. R. (2009) Necking in coating flow over periodic substrates. *J. Eng. Math.* **65**, 171–178.
[16] Marshall, J. S. (2012) Steady uniform vortex patches around an assembly of walls or flat plates. *Q. J. Mech. Appl. Math.* **65**(1), 27–60.
[17] McDonald, N. R. & Johnson, E. R. (2009) Gap-leaping vortical currents. *J. Phys. Ocean.* **39**, 2665–2674.
[18] Polubarinova-Kochina, P. Ya. (1962) *Theory of Groundwater Movement*, Princeton University Press, Princeton, NJ.
[19] Tuck, E. O., Bentwick, M. & van der Hoek, J. (1983) The free boundary problem for gravity-driven unidirectional viscous flows. *IMA J. Appl. Math.* **30**, 191–208.

https://doi.org/10.1017/S0956792514000151 Published online by Cambridge University Press