J. Inst. Math. Jussieu (2017) **16**(4), 887–898 887
doi:10.1017/S1474748015000274

THE LOEWY STRUCTURE OF G_1T-VERMA MODULES OF SINGULAR HIGHEST WEIGHTS

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(Received 7 May 2015; revised 4 August 2015; accepted 4 August 2015;
first published online 2 October 2015)

Abstract Let G be a reductive algebraic group over an algebraically closed field of positive characteristic, G_1 the Frobenius kernel of G, and T a maximal torus of G. We show that the parabolically induced G_1T-Verma modules of singular highest weights are all rigid, determine their Loewy length, and describe their Loewy structure using the periodic Kazhdan–Lusztig P- and Q-polynomials. We assume that the characteristic of the field is sufficiently large that, in particular, Lusztig's conjecture for the irreducible G_1T-characters holds.

Keywords: Frobenius kernel; singular block; Loewy series

2010 *Mathematics subject classification*: Primary 20G05

Let G be a reductive algebraic group over an algebraically closed field $\Bbbk$ of positive characteristic p. The Frobenius kernel G_1 of G is an analog of the Lie algebra of G in characteristic 0. To keep track of weights, we consider representations of G_1T with T a maximal torus of G. In this paper we study G_1T-Verma modules, standard objects of the theory.

Many years ago, Andersen and the second author of the present paper showed that the G_1T-Verma modules of p-regular highest weights are all rigid of Loewy length 1 plus the dimension of the flag variety of G, and they described their Loewy structure using the periodic Kazhdan–Lusztig Q-polynomials [3]. For that they assumed the validity of Lusztig's conjecture on the irreducible characters for G_1T-modules, or rather Vogan's equivalent version on the semisimplicity of certain G_1T-modules, modeling after Irving's method [9, 10]. Lusztig's conjecture is now a theorem for large p as established by Andersen *et al.* [2]. Pushing their graded representation theory, with a machinery of Beilinson *et al.* [4], we showed in [1] that the parabolic induction is graded on p-regular blocks, and determined the Loewy structure of parabolically induced G_1T-Verma modules of p-regular highest weights. In this paper we use Riche's Koszulity of the G_1-block

Supported in part by JSPS grants in aid for Scientific Research 26707001 AN and 15K04789 KM.

algebras [16] to uncover the structure of the parabolically induced G_1T-Verma modules of p-singular highest weights, to complete the entire picture.

To describe our results precisely, let us introduce some notation. For simplicity we will assume throughout the paper that G is simply connected and simple. Fix a Borel subgroup B of G containing T, and choose a positive system R^+ of R such that the roots of B are $-R^+$. Let R^{s} denote the set of simple roots of R^+. Let Λ denote the weight lattice of T equipped with a partial order such that $\lambda \geqslant \mu$ iff $\lambda - \mu \in \sum_{\alpha\in R^+} \mathbb{N}\alpha$. Put $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$. Let W denote the Weyl group of G relative to T, and let $W_a = W \ltimes \mathbb{Z}R$ be the affine Weyl group with elements of $\mathbb{Z}R$ in W_a acting on Λ by translations. We let W_a act on Λ also via $x \bullet \lambda = px\frac{1}{p}(\lambda+\rho)-\rho$, $x \in W_a$, $\lambda \in \Lambda$. In particular, for $x = \gamma \in \mathbb{Z}R$, $x \bullet \lambda = \lambda + p\gamma$. Let $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$ denote the set of coroots of R, and put $H_{\alpha,n} = \{v \in \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle v+\rho, \alpha^\vee\rangle = pn\}$, $\alpha \in R$ and $n \in \mathbb{Z}$. We call a connected component of $(\Lambda \otimes_{\mathbb{Z}} \mathbb{R}) \setminus \bigcup_{\alpha\in R, n\in\mathbb{Z}} H_{\alpha,n}$ an alcove. We say that $\lambda \in \Lambda$ is p-regular iff it belongs to an alcove; otherwise, λ is p-singular. Let also $\Lambda^+ = \{\lambda \in \Lambda \mid \langle \lambda, \alpha^\vee\rangle \geqslant 0\ \forall \alpha \in R^+\}$, the set of dominant weights. We let $A^+ = \{v \mid \langle v+\rho, \alpha^\vee\rangle \in]0, p[\ \forall \alpha \in R^+\}$ denote the bottom dominant alcove. For a closed subgroup H of G we let H_1 denote its Frobenius kernel. Let $\hat{\nabla} = \mathrm{ind}_{B_1T}^{G_1T}$ denote the induction functor [11, I.3] from the category of B_1T-modules to the category of G_1T-modules. The G_1T-simple modules are parameterized by their highest weights in Λ. We let $\hat{L}(\nu)$ denote the simple G_1T-module of highest weight $\nu \in \Lambda$. If M is a finite-dimensional G_1T-module, we will write $[M : \hat{L}(\nu)]$ for the composition factor multiplicity of $\hat{L}(\nu)$ in M.

A Loewy filtration of a finite-dimensional G_1T-module M is a filtration of M of minimal length such that each of its subquotients is semisimple. The length of a Loewy filtration is uniform, and is called the Loewy length of M, denoted $\ell\ell(M)$. Among the Loewy filtrations, the socle series of M is defined by $0 < \mathrm{soc}^1 M < \mathrm{soc}^2 M < \cdots < \mathrm{soc}^{\ell\ell(M)} M = M$ with $\mathrm{soc}^1 M = \mathrm{soc} M$, called the socle of M, which is the sum of simple submodules of M, and $\mathrm{soc}^i M/\mathrm{soc}^{i-1} M = \mathrm{soc}(M/\mathrm{soc}^{i-1} M)$ for $i > 1$. Also the radical series of M is defined by $0 = \mathrm{rad}^{\ell\ell(M)} M < \cdots < \mathrm{rad}^2 M < \mathrm{rad}^1 M < M$ with $\mathrm{rad}^1 M = \mathrm{rad} M$, called the radical of M, which is the intersection of the maximal submodules of M, and $\mathrm{rad}^i M = \mathrm{rad}(\mathrm{rad}^{i-1} M)$ for $i > 1$. If $0 < M^1 < \cdots < M^{\ell\ell(M)} = M$ is any Loewy filtration of M, $\mathrm{rad}^{\ell\ell(M)-i} M \leqslant M^i \leqslant \mathrm{soc}^i M$ for each i. We say that M is rigid iff the socle and the radical series of M coincide. We put $\mathrm{soc}_i M = \mathrm{soc}^i M/\mathrm{soc}^{i-1} M$ and $\mathrm{rad}_j M = \mathrm{rad}^j M/\mathrm{rad}^{j+1} M$.

In this paper we show the following.

Theorem. *Assume that $p \gg 0$. Let $\nu \in \Lambda$, and let $N(\nu)$ denote the number of hyperplanes $H_{\alpha,n}$ on which ν lies. The G_1T-Verma module $\hat{\nabla}(\nu)$ of highest weight ν is rigid of Loewy length $1 + \dim G/B - N(\nu)$. If $x \in W_a$ is such that ν belongs to the upper closure of $x \bullet A^+$, and, if $\nu_0 = x^{-1} \bullet \nu$, the Loewy structure of $\hat{\nabla}(\nu)$ is given by*

$$\sum_{i\in\mathbb{N}} q^{\frac{\mathrm{d}(y,x)-i}{2}} [\mathrm{soc}_{i+1} \hat{\nabla}(\nu) : \hat{L}(y \bullet \nu_0)] = \begin{cases} Q^{y\bullet A^+, x\bullet A^+} & \text{if } y \in W_a \text{ with } y \bullet \nu_0 \text{ belonging to the upper closure of } y \bullet A^+, \\ 0 & \text{otherwise,} \end{cases}$$

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where $\mathrm{d}(y, x)$ *is the distance from the alcove* $y \bullet A^+$ *to the alcove* $x \bullet A^+$ *[15, 1.4] and* $Q^{y\bullet A^+, x\bullet A^+}$ *is a polynomial from [15, 1.8].*

For this theorem to hold, we assume that $p \gg 0$ so that Lusztig's conjecture for the irreducible characters of G_1T-modules and also that the conditions [16, (10.1.1) and (10.2.1)] from [5] hold. While Fiebig [8] gives an explicit lower bound of p, crude as it may be, for Lusztig's conjecture to hold, a recent work of Williamson [17] reveals that p has, in general, to be much bigger than h, the Coxeter number of G, which was the original bound for the conjecture to hold. Compared to the restriction required for Lusztig's conjecture to hold, the other conditions in [16] are innocent.

We actually obtain, more generally, analogous results for the parabolically induced module $\hat{\nabla}_P(\hat{L}^P(\nu)) = \mathrm{ind}_{P_1T}^{G_1T}(\hat{L}^P(\nu))$ with $\hat{L}^P(\nu)$ denoting a simple P_1T-module of highest weight ν for a parabolic subgroup P of G.

For a category $\mathcal{C}$ we will denote the set of morphisms from object X to Y in $\mathcal{C}$ by $\mathcal{C}(X, Y)$.

1. Koszulity of the G_1-block algebras

Throughout the paper we will assume that $p > h$, the Coxeter number of G, unless otherwise specified. In particular, $p\Lambda \cap \mathbb{Z}R = p\mathbb{Z}R$. All modules we consider are finite dimensional over $\Bbbk$. Our basic strategy is to transport the known structure of a G_1T-block $\mathcal{C}(\lambda)$ of p-regular $\lambda \in \Lambda$ to an arbitrary block $\mathcal{C}(\mu)$ by the translation functor. For $p \gg 0$, thanks to [16], the corresponding translation functor for the G_1-blocks is graded, and the G_1-block algebras are all Koszul.

1.1.

For $\nu \in \Lambda$, let $\hat{L}(\nu)$ denote the simple G_1T-module of highest weight ν, and let $\hat{P}(\nu)$ denote the G_1T-projective cover of $\hat{L}(\nu)$. Let Ω be a p-regular orbit of W_a in Λ, and let $\mathcal{C}(\Omega)$ denote the corresponding G_1T-block. Thus $\mathcal{C}(\Omega) = \mathcal{C}(\nu)$, $\nu \in \Omega$, consists of G_1T-modules whose composition factors all have highest weights in Ω. Let Ω' be a system of representatives of Ω under the translations by $p\mathbb{Z}R$, and let $\hat{P}(\Omega) = \coprod_{\nu\in\Omega'} \hat{P}(\nu)$. Then $\coprod_{\gamma\in p\mathbb{Z}R} \mathcal{C}(\Omega)(\hat{P}(\Omega) \otimes \gamma, \hat{P}(\Omega))$ forms a $p\mathbb{Z}R$-graded $\Bbbk$-algebra under the composition using the auto-functor $? \otimes \gamma$, $\gamma \in p\mathbb{Z}R$, on $\mathcal{C}(\Omega)$. If we let $\hat{\mathbb{E}}(\Omega)$ denote its opposite algebra, $\coprod_{\gamma\in p\mathbb{Z}R} \mathcal{C}(\Omega)(\hat{P}(\Omega) \otimes \gamma, ?)$ gives an equivalence of categories from $\mathcal{C}(\Omega)$ to the category of finite-dimensional $p\mathbb{Z}R$-graded $\hat{\mathbb{E}}(\Omega)$-modules. Moreover, $\hat{\mathbb{E}}(\Omega)$ admits a $\mathbb{Z}$-grading compatible with its $p\mathbb{Z}R$-gradation [2, 18.17.1]. For p sufficiently large that Lusztig's conjecture holds, [2, 18.17] has proved that $\hat{\mathbb{E}}(\Omega)$ is Koszul with respect to its $\mathbb{Z}$-gradation. Let us state Lusztig's conjecture in an equivalent form, which is the inverted version of the conjecture for G_1T (cf. [13, 3.3], [7, 3.4]), as follows: $\forall x, y \in W_a$,

$$[\hat{\nabla}(x \bullet 0) : \hat{L}(y \bullet 0)] = Q^{y\bullet A^+, x\bullet A^+}(1), \tag{L}$$

where $Q^{y\bullet A^+, x\bullet A^+}$ is a polynomial from [15, 1.8].

Assuming (L), let $\tilde{\mathcal{C}}(\Omega)$ denote the category of finite-dimensional $(p\mathbb{Z}R \times \mathbb{Z})$-graded $\hat{\mathbb{E}}(\Omega)$-modules. For each $\nu \in \Omega'$ let $\tilde{L}(\nu) = \coprod_{\gamma\in p\mathbb{Z}R} \mathcal{C}(\Omega)(\hat{P}(\Omega) \otimes \gamma, \hat{L}(\nu))$ be the lift of

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$\hat{L}(\nu)$ in $\tilde{\mathcal{C}}(\Omega)$. This is a simple quotient of $\hat{\mathbb{E}}(\Omega)$, and hence is a direct summand of the degree-0 part of $\hat{\mathbb{E}}(\Omega)$ in its Koszul $\mathbb{Z}$-grading [2, F.2]. If we denote the degree shift of objects in $\tilde{\mathcal{C}}(\Omega)$ by $[\gamma]$, $\gamma \in p\mathbb{Z}R$, and by $\langle i\rangle$, $i \in \mathbb{Z}$, any simple of $\tilde{\mathcal{C}}(\Omega)$ may be written $\tilde{L}(\nu)[\gamma]\langle i\rangle$, $\nu \in \Omega'$, $\gamma \in p\mathbb{Z}R$, and $i \in \mathbb{Z}$, in a unique way up to isomorphism. As $\tilde{L}(\nu)[\gamma]$ is a lift of $\hat{L}(\nu+\gamma) = \hat{L}(\nu)\otimes\gamma$, we will also write $\tilde{L}(\nu+\gamma)$ for $\tilde{L}(\nu)[\gamma]$. For each $\nu \in \Omega$ the G_1T-Verma module $\hat{\nabla}(\nu)$ of highest weight ν admits a lift $\tilde{\nabla}(\nu)$ in $\tilde{\mathcal{C}}(\Omega)$ such that its socle is $\tilde{L}(\nu)$. Likewise each projective $\hat{P}(\nu)$ admits a lift $\tilde{P}(\nu)$ which is the projective cover of $\tilde{L}(\nu)$.

1.2.

Let $\Lambda_p = \{\nu \in \Lambda^+ \mid \langle \nu, \alpha^\vee\rangle < p \ \forall\alpha \in R^{\mathrm{s}}\}$. For $\nu \in \Lambda$ we write $\nu = \nu^0 + p\nu^1$ with $\nu^0 \in \Lambda_p$ and $\nu^1 \in \Lambda$. We let $L(\nu^0)$ denote the simple G-module of highest weight ν^0, which remains simple regarded as a G_1-module. All simple G_1-modules are obtained thus from simple G-modules of highest weights in Λ_p. One has, as G_1T-modules, $\hat{L}(\nu) = L(\nu^0)\otimes p\nu^1$ and $\hat{P}(\nu) = \hat{P}(\nu^0)\otimes p\nu^1$, with $\hat{P}(\nu^0)$ providing the G_1-projective cover of $L(\nu^0)$, which we will denote by $P(\nu^0)$.

Let now $\mathfrak{g}$ denote the Lie algebra of G, $\mathbf{U}\mathfrak{g}$ the universal enveloping algebra of $\mathfrak{g}$, and $(\mathbf{U}\mathfrak{g})_0$ the central reduction of $\mathbf{U}\mathfrak{g}$ with respect to the Frobenius central character 0. As $(\mathbf{U}\mathfrak{g})_0$ coincides with the algebra of distributions of G_1, the representation theory of G_1 is equivalent to that of $(\mathbf{U}\mathfrak{g})_0$. For each $\nu \in \Lambda$ let $(\mathbf{U}\mathfrak{g})_0^{\hat{\nu}}$ be the central reduction of $(\mathbf{U}\mathfrak{g})_0$ with respect to the Harish-Chandra generalized character $\hat{\nu}$. This is the G_1-block component of $(\mathbf{U}\mathfrak{g})_0$ associated to ν. Let $\mathcal{B}(\nu)$ denote the category of finite-dimensional $(\mathbf{U}\mathfrak{g})_0^{\hat{\nu}}$-modules. The algebra $(\mathbf{U}\mathfrak{g})_0^{\hat{\nu}}$ is equipped with a $\mathbb{Z}$-grading [16, 6.3 and 10.2 line 16, p. 126]. We let $\mathcal{B}^{\mathrm{gr}}(\nu)$ denote the category of finite-dimensional graded $(\mathbf{U}\mathfrak{g})_0^{\hat{\nu}}$-modules. Let $\Lambda(\nu) = \{(w\bullet\nu)^0 \mid w \in W\}$. Each $P(\eta)$, $\eta \in \Lambda(\nu)$, admits a lift $P^{\mathrm{gr}}(\eta)$ in $\mathcal{B}^{\mathrm{gr}}(\nu)$. Let $P^\nu = \coprod_{\eta\in\Lambda(\nu)} P^{\mathrm{gr}}(\eta)$, and set $\mathbb{E}(\nu) = \mathcal{B}(\nu)(P^\nu, P^\nu)^{\mathrm{op}}$. As P^ν is a projective generator of $\mathcal{B}(\nu)$, and as $\mathbb{E}(\nu) = \coprod_{i\in\mathbb{Z}} \mathcal{B}^{\mathrm{gr}}(\nu)(P^\nu\langle i\rangle, P^\nu)$ is equipped with a $\mathbb{Z}$-gradation, $\langle i\rangle$ denoting the degree shift, $\mathcal{B}(\nu)(P^\nu, ?)$ induces an equivalence from $\mathcal{B}^{\mathrm{gr}}(\nu)$ to the category of finite-dimensional $\mathbb{Z}$-graded $\mathbb{E}(\nu)$-modules, which we will denote by $\tilde{\mathcal{B}}(\nu)$. For $p \gg 0$, thanks to [16, 10.3], all $\mathbb{E}(\nu)$ are Koszul by a careful choice of graded lift $P^{\mathrm{gr}}(\eta)$, $\eta \in \Lambda(\nu)$.

To be precise, let $I \subseteq R^{\mathrm{s}}$, and let P denote the corresponding standard parabolic subgroup of G with the Weyl group $W_I = \langle s_\alpha \mid \alpha \in I\rangle$, where s_α is the reflection associated to α. We take and fix $\lambda \in \Lambda$ belonging to the alcove A^+ and $\mu \in \Lambda$ lying in its closure $\overline{A^+}$ in the rest of the paper as follows. Take μ such that $\mathrm{C}_{W_a\bullet}(y\bullet\mu) := \{x \in W_a \mid xy\bullet\mu = y\bullet\mu\} = W_I$ for some $y \in W_a$, and for this $y \in W_a$ take λ to satisfy $\langle y\bullet\lambda, \alpha^\vee\rangle = 0 \ \forall\alpha \in I$. If $p \gg 0$ so that condition (L) holds, one can take each $P^{\mathrm{gr}}((w\bullet\lambda)^0)$ to satisfy a certain condition [16, 8.1(‡)]. With this choice [16, Theorem 9.5.1] shows that the graded algebra $\mathbb{E}(\lambda)$ is Koszul. For μ, assume in addition to (L) two more conditions, which go as follows. The first one [16, 10.1.1], coming from [5, Lemma 1.10.9(ii)], reads, with $\mathcal{D}^\lambda_{G/P}$ denoting the sheaf of PD-differential operators on G/P twisted by the invertible sheaf $\mathcal{L}_{G/P}(\lambda)$,

$$R^i\Gamma(G/P, \mathcal{D}^\lambda_{G/P}) = 0 \quad \forall i > 0. \tag{R1}$$

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The center $\mathfrak{Z}$ of $\mathbf{U}\mathfrak{g}$ consists of two parts, the Harish-Chandra center $\mathfrak{Z}_{\mathrm{HC}} = (\mathbf{U}\mathfrak{g})^G$, the set of invariants under the adjoint G-action, and the Frobenius center $\mathfrak{Z}_{\mathrm{Fr}}$ generated by $X^p - X^{[p]}$, $X \in \mathfrak{g}$, with $X^{[p]}$ denoting the pth power of X in the algebra of distributions of G_1 [12, § 9]. Letting $\mathfrak{t}$ denote the Lie algebra of T, the Harish-Chandra center is isomorphic as $\Bbbk$-algebras to the set of $(W\bullet)$-invariants of the symmetric algebra of $\mathfrak{t}$. Then λ defines a structure of $\mathfrak{Z}_{\mathrm{HC}}$-algebra on $\Bbbk$, denoted $\Bbbk_\lambda$ and called a Harish-Chandra central character afforded by λ. With $(\mathbf{U}\mathfrak{g})^\lambda = \mathbf{U}\mathfrak{g} \otimes_{\mathfrak{Z}_{\mathrm{HC}}} \Bbbk_\lambda$ denoting the central reduction of $\mathbf{U}\mathfrak{g}$ by the Harish-Chandra central character λ, the second condition [16, 10.2.1], coming also from [5, Lemma 1.10.9], reads that

$$\text{the natural morphism } (\mathbf{U}\mathfrak{g})^\lambda \to \Gamma(G/P, \mathcal{D}^\lambda_{G/P}) \text{ be surjective.} \tag{R2}$$

If $p \gg 0$ so that (L), (R1), and (R2) all hold, one can take each $P^{\mathrm{gr}}(\eta)$, $\eta \in \Lambda(\mu)$, to satisfy [16, Theorem 10.2.4], which makes $\mathbb{E}(\mu)$ also Koszul [16, Theorem 10.3.1]. For any $\nu \in \Lambda$ there is by [5, Lemma 1.5.2] some $\xi \in \Lambda$ such that $\nu + p\xi \in W_a \bullet \mu$ with μ as above. Thus under conditions (L), (R1), and (R2) we may assume that all G_1-block algebras $\mathbb{E}(\nu)$ are Koszul. For each $\eta \in \Lambda(\nu)$ we denote by $\tilde{L}(\eta)$ the graded lift in $\tilde{\mathcal{B}}(\nu)$ of the G_1-simple $L(\eta)$ as a direct summand of $\mathbb{E}(\nu)_0$. Let also $\tilde{P}(\eta) = \coprod_{i\in\mathbb{Z}} \mathcal{B}^{\mathrm{gr}}(\nu)(P^\nu\langle i\rangle, P^{\mathrm{gr}}(\eta))$ be a graded lift in $\tilde{\mathcal{B}}(\nu)$ of $P(\eta)$ to form the projective cover of $\tilde{L}(\eta)$.

1.3.

Assume from now on throughout the rest of the paper that $p \gg 0$ so that all the conditions (L), (R1), and (R2) from §§ 1.1 and 1.2 hold, unless otherwise specified. Fix also λ and μ as specified in § 1.2.

For our purposes, as tensoring with $p\eta$, $\eta \in \Lambda$, is an equivalence from the G_1T-block $\mathcal{C}(\Gamma)$ of a W_a-orbit Γ to the G_1T-block $\mathcal{C}(\Gamma + p\eta)$, we have only to determine the structure of parabolically induced G_1T-Verma modules of highest weight $x \bullet \mu$ with μ as above and $x \in W_a$.

If $\Omega = W_a \bullet \lambda$, as $p > h$ by the standing hypothesis, $\mathbb{E}(\lambda)$ is isomorphic by the linkage principle to $\hat{\mathbb{E}}(\Omega)$ from § 1.1 as $\Bbbk$-algebras. As two $\mathbb{Z}$-gradations on the algebra must agree by their Koszulity [2, F.2], there is no ambiguity about the functor from $\tilde{\mathcal{C}}(\Omega)$ to $\tilde{\mathcal{B}}(\lambda)$ forgetting the $p\mathbb{Z}R$-gradation, which is compatible with the forgetful functor from the category of G_1T-modules to that of G_1-modules. Thus one has a commutative diagram of forgetful functors

$$\begin{array}{ccc} \mathcal{C}(\Omega) & \longleftarrow & \tilde{\mathcal{C}}(\Omega) \\ \downarrow & & \downarrow \\ \mathcal{B}(\lambda) & \longleftarrow & \tilde{\mathcal{B}}(\lambda) \end{array}$$

1.4.

For each $\nu \in \Lambda$ let $\overline{\mathrm{pr}}_\nu$ denote the projection from the category of finite-dimensional G_1-modules to its ν-block $\mathcal{B}(\nu)$. For $\nu, \eta \in \overline{A^+}$ recall from [6] the translation functor $T_\nu^\eta = \overline{\mathrm{pr}}_\eta(L((\eta - \nu)^+) \otimes\ ?) : \mathcal{B}(\nu) \to \mathcal{B}(\eta)$ with $(\eta - \nu)^+ \in W(\eta - \nu) \cap \Lambda^+$.

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By [16, Proposition 5.4.3 and Theorem 6.3.4] the adjoint translation functors T_λ^μ and T_μ^λ are graded to form a pair of functors $\mathcal{B}^{\mathrm{gr}}(\lambda) \rightleftarrows \mathcal{B}^{\mathrm{gr}}(\mu)$ such that graded T_λ^μ is right adjoint to graded T_μ^λ. In turn, they induce a pair of graded functors, which we will denote by $\tilde{T}_\lambda^\mu$ and $\tilde{T}_\mu^\lambda$:

$$\tilde{T}_\mu^\lambda = \coprod_{i\in\mathbb{N}} \mathcal{B}^{\mathrm{gr}}(\lambda)(P^\lambda\langle i\rangle, T_\mu^\lambda\,?) \circ (P^\mu \otimes_{\mathbb{E}(\mu)} ?) : \tilde{\mathcal{B}}(\mu) \to \tilde{\mathcal{B}}(\lambda),$$
$$\tilde{T}_\lambda^\mu = \coprod_{i\in\mathbb{N}} \mathcal{B}^{\mathrm{gr}}(\mu)(P^\mu\langle i\rangle, T_\lambda^\mu\,?) \circ (P^\lambda \otimes_{\mathbb{E}(\lambda)} ?) : \tilde{\mathcal{B}}(\lambda) \to \tilde{\mathcal{B}}(\mu).$$

Thus $\tilde{T}_\lambda^\mu$ is right adjoint to $\tilde{T}_\mu^\lambda$.

Put $N = \dim G/B$ and $N_P = \dim G/P$. A crucial fact to our results is Riche's [16, 10.2.8], which asserts that, for each $w \in W$ with $w' \in W$ such that $(w \bullet \mu)^0$ belongs to the upper closure of an alcove containing $(w' \bullet \lambda)^0$, $T_\mu^\lambda P^{\mathrm{gr}}((w\bullet\mu)^0) = P^{\mathrm{gr}}((w'\bullet\lambda)^0)\langle N - N_P\rangle$, and hence

$$\tilde{T}_\mu^\lambda \tilde{P}((w\bullet\mu)^0) = \tilde{P}((w'\bullet\lambda)^0)\langle N - N_P\rangle. \tag{1}$$

1.5.

For each $\nu \in \Lambda$ let $\widehat{\mathrm{pr}}_\nu$ denote the projection from the category of finite-dimensional G_1T-modules to the block $\mathcal{C}(\nu)$. For $\nu, \eta \in \overline{A^+}$ one has as in § 1.4 the translation functor $\hat{T}_\nu^\eta = \widehat{\mathrm{pr}}_\eta(L((\eta-\nu)^+) \otimes ?) : \mathcal{C}(\nu) \to \mathcal{C}(\eta)$ [11, II.9.22]. Under the assumption that $p > h$, the functors $\hat{T}_\lambda^\mu$ and T_λ^μ commute with the forgetful functors as in [16, Lemma 4.3.1]:

$$\begin{array}{ccc} \mathcal{C}(\lambda) & \xrightarrow{\hat{T}_\lambda^\mu} & \mathcal{C}(\mu) \\ \downarrow & \circlearrowleft & \downarrow \\ \mathcal{B}(\lambda) & \xrightarrow[T_\lambda^\mu]{} & \mathcal{B}(\mu). \end{array}$$

1.6.

Under the forgetful functors, $\hat{\nabla} = \mathrm{ind}_{B_1T}^{G_1T}$ yields an induction functor $\bar{\nabla} = \mathrm{ind}_{B_1}^{G_1}$ from the category of B_1-modules to the category of G_1-modules. If M is a G_1T-module, it is semisimple iff it is semisimple as a G_1-module [11, I.6.15]. Thus, in order to show that $\hat{\nabla}(x\bullet\mu)$, $x \in W_a$, is rigid, we have only to show that $\bar{\nabla}(x\bullet\mu)$ is rigid.

For a facet F in $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ with respect to W_a let $\hat{F}$ denote its upper closure. As $\hat{\nabla}(x\bullet\mu)$ $= \hat{T}_\lambda^\mu \hat{\nabla}(x\bullet\lambda)$, $\tilde{T}_\lambda^\mu \tilde{\nabla}(x\bullet\lambda) \in \tilde{\mathcal{B}}(\mu)$ is a graded lift of $\bar{\nabla}(x\bullet\mu)$, which we will denote by $\tilde{\nabla}(x\bullet\mu)\langle i + N_P - N\rangle$ if $x\bullet\mu \in \widehat{x'\bullet A^+}$, $x' \in W_a$, and if $[\mathrm{soc}_{i+1}\hat{\nabla}(x\bullet\lambda) : \hat{L}(x'\bullet\lambda)] =$ $[\tilde{\nabla}(x\bullet\lambda) : \tilde{L}(x'\bullet\lambda)\langle i\rangle] \neq 0$. Such i is uniquely determined as $[\hat{\nabla}(x\bullet\lambda) : \hat{L}(x'\bullet\lambda)] = 1$ by the translation principle. As $\bar{\nabla}(x\bullet\mu)$ has a simple socle and a simple head, so does its lift, and hence the lift is rigid by [4, Proposition 2.4.1]. There now follows the rigidity of $\hat{\nabla}(x\bullet\mu)$.

Proposition. *All G_1T-Verma modules $\hat{\nabla}(\nu)$, $\nu \in \Lambda$, are rigid.*

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1.7.

Let $w \in W$, and put ${}^wB = wBw^{-1}$, $\hat\nabla_w = \mathrm{ind}_{{}^wB_1T}^{G_1T}$. If M is a G_1T-module, let ${}^wM$ denote the G_1T-module M with the G_1T-action twisted by w, i.e., we let $g \in G_1T$ act on $m \in M$ by $w^{-1}gw$. For each $\nu \in \Lambda$ one has an isomorphism ${}^w\hat\nabla(\nu) \simeq \hat\nabla_w(w\nu)$ [11, II.9.3]. Thus we obtain the following.

Corollary. *All $\hat\nabla_w(\nu)$, $w \in W$, $\nu \in \Lambda$, are rigid.*

1.8.

Let $J \subseteq R^s$, let Q be the standard parabolic subgroup of G associated to J with the Weyl group denoted W_J, and let $\hat\nabla_J = \mathrm{ind}_{Q_1T}^{G_1T}$ denote the induction functor from the category of Q_1T-modules to the category of G_1T-modules. Let $\nu \in \Lambda$, and let $\hat L^J(\nu)$ denote the simple Q_1T-module of highest weight ν. Choose a p-regular $\eta \in \Lambda$ such that ν belongs to the upper closure of the $W_{J,a}$-alcove containing η. Under the Lusztig conjecture (L) we have shown in [1, 3.9] that $\hat\nabla_J(\hat L^J(\eta))$ is graded, and in [1, 2.3] that $\hat T_\eta^\nu(\hat\nabla_J(\hat L^J(\eta))) \simeq \hat\nabla_J(\hat L^J(\nu))$. As $\hat\nabla_J(\hat L^J(\nu))$ has a simple head and socle [1, 1.4]. Again, from [4, Proposition 2.4.1], we obtain the following proposition.

Proposition. *All parabolically induced G_1T-Verma modules $\hat\nabla_J(\hat L^J(\nu))$, $\nu \in \Lambda$, are rigid.*

2. The Loewy structure

We keep the notation from § 1.

2.1.

For each $\nu \in \Lambda$ we will denote $\hat L(\nu)$ by $\bar L(\nu)$ when regarded as a G_1-module. Thus $\bar L(\nu) = L(\nu^0)$.

Lemma. *Let $x \in W_a$.*

(i) *One has*

$$\tilde T_\lambda^\mu \tilde L((x\bullet\lambda)^0) = \begin{cases} \tilde L((x\bullet\mu)^0)\langle N_P - N\rangle & \text{if } x\bullet\mu \in \widehat{x\bullet A^+},\\ 0 & \text{else.}\end{cases}$$

(ii) *If $x\bullet\mu \in \widehat{x\bullet A^+}$, one has for each $i \in \mathbb{N}$*

$$\hat T_\lambda^\mu \operatorname{soc}^i \hat\nabla(x\bullet\lambda) = \operatorname{soc}^i \hat\nabla(x\bullet\mu).$$

Proof. (i) We may by § 1.5 assume that $x\bullet\mu \in \widehat{x\bullet A^+}$ [11, II.7.15, 9.22.4], which occurs iff $(x\bullet\mu)^0$ lies in the upper closure of the alcove $(x\bullet\lambda)^0$ belongs to. Thus we are to show in that case that $\tilde T_\lambda^\mu \tilde L((x\bullet\lambda)^0) = \tilde L((x\bullet\mu)^0)\langle N_P - N\rangle$.

As $\tilde{P}((x\bullet\mu)^0)$ (respectively, $\tilde{P}((x\bullet\lambda)^0)$) is a projective cover of $\tilde{L}((x\bullet\mu)^0)$ (respectively, $\tilde{L}((x\bullet\lambda)^0)$), we have for each $n\in\mathbb{Z}$

$$\begin{aligned}\tilde{\mathcal{B}}(\mu)(\tilde{P}((x\bullet\mu)^0)\langle n\rangle, \tilde{T}_\lambda^\mu\tilde{L}((x\bullet\lambda)^0)) &\simeq \tilde{\mathcal{B}}(\lambda)(\tilde{T}_\mu^\lambda\tilde{P}((x\bullet\mu)^0)\langle n\rangle, \tilde{L}((x\bullet\lambda)^0))\\ &\simeq \tilde{\mathcal{B}}(\lambda)(\tilde{P}((x\bullet\lambda)^0)\langle n+N-N_P\rangle, \tilde{L}((x\bullet\lambda)^0)) \quad \text{by (1.4.1)},\end{aligned}$$

which is nonzero iff $n+N-N_P=0$, and hence the assertion follows.

(ii) Let $\mathrm{soc}^i_{G_1}\bar{\nabla}((x\bullet\lambda)^0)$, $x\in W_a$, denote the ith term of the G_1-socle series of $\bar{\nabla}((x\bullet\lambda)^0)$, which is just $\mathrm{soc}^i\hat{\nabla}(x\bullet\lambda)$ regarded as a G_1-module. As the socle series and the gradation over $\mathbb{E}(\lambda)$ (respectively, $\mathbb{E}(\mu)$) coincide on $\tilde{\nabla}((x\bullet\lambda)^0)$ (respectively, $\tilde{\nabla}((x\bullet\mu)^0)$) up to shift by [4, Proposition 2.4.1], and as the socle of $\tilde{\nabla}((x\bullet\lambda)^0)$ is sent onto the socle of $\tilde{\nabla}((x\bullet\mu)^0)$, it follows from (i) that $\tilde{T}_\lambda^\mu\mathrm{soc}^i\tilde{\nabla}((x\bullet\lambda)^0)=\mathrm{soc}^i\tilde{\nabla}((x\bullet\mu)^0)$, and hence also $T_\lambda^\mu\mathrm{soc}^i_{G_1}\bar{\nabla}((x\bullet\lambda)^0)=\mathrm{soc}^i_{G_1}\bar{\nabla}((x\bullet\mu)^0)$. As the G_1T-socle series and the G_1-socle series on G_1T-modules coincide, the assertion holds. □

2.2.

$\forall x,y\in W_a$, let $Q^{y\bullet A^+,x\bullet A^+}(q)=\sum_j Q_j^{y,x}q^{\frac{j}{2}}\in\mathbb{Z}[q]$ be the periodic Kazhdan–Lusztig Q-polynomial from [15]. Put $Q^{y,x}=Q^{y\bullet A^+,x\bullet A^+}(q)$ for simplicity. Recall from [3], [2, 18.19]/[1, 5.1, 2] that

$$\sum_{i\in\mathbb{N}} q^{\frac{\mathrm{d}(y,x)-i}{2}}[\mathrm{soc}_{i+1}\hat{\nabla}(x\bullet\lambda):\hat{L}(y\bullet\lambda)] = \sum_{i\in\mathbb{N}} q^{\frac{\mathrm{d}(y,x)-i}{2}}[\tilde{\nabla}(x\bullet\lambda):\tilde{L}(y\bullet\lambda)\langle -i\rangle] = Q^{y,x},$$

where $\mathrm{d}(y,x)=\mathrm{d}(y\bullet A^+,x\bullet A^+)$ is the distance from the alcove $y\bullet A^+$ to the alcove $x\bullet A^+$ [15]. Let $W_a(\mu)=\{x\in W_a \mid x\bullet\mu\in\widehat{x\bullet A^+}\}$. For each $x\in W_a(\mu)$, Proposition 2.1(ii) shows that

$$\sum_{i\in\mathbb{N}} q^{\frac{\mathrm{d}(y,x)-i}{2}}[\mathrm{soc}_{i+1}\hat{\nabla}(x\bullet\mu):\hat{L}(y\bullet\mu)] = \begin{cases} Q^{y,x} & \text{if } y\in W_a(\mu),\\ 0 & \text{else.}\end{cases}$$

2.3.

One can likewise determine the Loewy series of parabolically induced G_1T-Verma modules $\hat{\nabla}_J(\hat{L}^J(\nu))$, $J\subseteq R^s$, $\nu\in\Lambda$, from §1.8, using [1, 2.3]. Let $W_{J,a}=W_J\ltimes\mathbb{Z}R_J$ denote the affine Weyl group for P_J.

Theorem. *Let $\nu\in\Lambda$, $x\in W_a$ such that $\nu\in\widehat{x\bullet A^+}$, and put $\nu_0=x^{-1}\bullet\nu$. Then*

$$\sum_{i\in\mathbb{N}} q^{\frac{\mathrm{d}(y,x)-i}{2}}[\mathrm{soc}_{i+1}\hat{\nabla}_J(\hat{L}^J(\nu)):\hat{L}(y\bullet\nu_0)] = \begin{cases} \displaystyle\sum_{z\in W_{J,a}} Q^{y\bullet A^+,z\bullet A^+}(-1)^{\mathrm{d}_J(z,x)}\hat{P}^J_{z\bullet A^+,x\bullet A^+} & \textit{if } y\in W_a(\mu),\\ 0 & \textit{else,}\end{cases}$$

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where $\hat{P}^J_{z\bullet A^+,x\bullet A^+}$ is a $\hat{P}$-polynomial from [14] for $W_{J,a}$ and $\mathrm{d}_J(z,x)$ is the distance from $z\bullet A^+$ to $x\bullet A^+$ with respect to $W_{J,a}$.

2.4.

Finally we determine the Loewy length of all parabolically induced G_1T-Verma modules. We first need analogs of [9, Propositions 3.2 and 3.3].

For an arbitrary $\nu\in\Lambda$ let $\hat{\Delta}(\nu)=\mathrm{Dist}(G_1)\otimes_{\mathrm{Dist}(B_1)}\nu=\hat{\nabla}(\nu)^\tau$, the $\Bbbk$-linear dual of $\hat{\nabla}(\nu)$ twisted by the Chevalley anti-involution τ of G, which is denoted ${}^\tau\hat{\nabla}(\nu)$ in [11, II.2.12]. We say that a G_1T-module M admits a $\hat{\nabla}$-filtration iff there is a filtration $0=M^0<M^1<\cdots<M^r=M$ of G_1T-modules with each $M^i/M^{i-1}\simeq\hat{\nabla}(\nu_i)$ for some $\nu_i\in\Lambda$, in which case one can arrange the filtration such that $\nu_i\not<\nu_j$ if $i>j$ [11, II.9.8]. Whenever M admits a $\hat{\nabla}$-filtration, we will assume that such a rearrangement has been done.

Let $W_\nu=\{x\in W_a\mid x\bullet\nu=\nu\}$, and take an alcove A in the closure of which ν lies. Choose $\eta\in\Lambda$ in A. Let η^+ (respectively, η^-) denote the highest (respectively, lowest) weight in $W_\nu\bullet\eta$. Let us also denote by $\hat{T}^\eta_\nu:\mathcal{C}(W_a\bullet\nu)\to\mathcal{C}(W_a\bullet\eta)$ and $\hat{T}^\nu_\eta:\mathcal{C}(W_a\bullet\eta)\to\mathcal{C}(W_a\bullet\nu)$ the associated translation functors. Let $N(\nu)$ denote the number of hyperplanes $H_{\alpha,n}$ on which $\nu\in\Lambda$ lies.

Lemma. *Assume that $p\gg0$ so that (L) holds.*

(i) $\hat{\Delta}(\eta^+)\leqslant\mathrm{rad}^{N(\nu)}\hat{T}^\eta_\nu\hat{\Delta}(\nu)$.

(ii) $\hat{L}(\eta^-)\leqslant\mathrm{soc}_{N(\nu)+1}\hat{\nabla}(\eta^+)$.

(iii) $\ell\ell(\hat{T}^\eta_\nu\hat{L}(\nu))\geqslant2N(\nu)+1$.

(iv) $\forall M\in\mathcal{C}(\nu)$, $\ell\ell(\hat{T}^\eta_\nu M)\geqslant2N(\nu)+\ell\ell(M)$.

Proof. (i) Recall from [2, 18.13] that the translation functors $\hat{T}^\nu_\eta$ and $\hat{T}^\eta_\nu$ admit graded versions, denoted $T_!$ and T^*, respectively. If we let $\tilde{\Delta}(\eta)$ denote the graded version of $\hat{\Delta}(\eta)$, $T^*T_!\tilde{\Delta}(\eta^-)$ admits by [2, 18.15] a filtration with the subquotients $\tilde{\Delta}(w\bullet\eta^-)\langle o(w\bullet\eta^-)\rangle$, $w\in W_\nu$, where $o(w\bullet\eta^-)$ denotes the number of hyperplanes $H_{\alpha,n},\alpha\in R^+,n\in\mathbb{Z}$, on which ν lies and such that $w\bullet\eta^-$ belongs to their positive sides [2, 15.13]. Thus the graded version of $\hat{L}(\eta^+)=\mathrm{hd}\hat{\Delta}(\eta^+)$ appears in $T^*T_!\tilde{\Delta}(\eta^-)$ as $\tilde{L}(\eta^+)\langle N(\nu)\rangle$ while that of $\hat{L}(\eta^-)=\mathrm{hd}\hat{\Delta}(\eta^-)=\mathrm{hd}\hat{T}^\eta_\nu\hat{\Delta}(\nu)$ appears as $\tilde{L}(\eta^-)$. Under assumption (L), $\tilde{\Delta}(\eta^-)$ is graded over the Koszul algebra $\hat{\mathbb{E}}(W_a\bullet\eta)$ from § 1.1, and so therefore is $T^*T_!\tilde{\Delta}(\eta^-)$. As $\hat{T}^\eta_\nu\hat{\Delta}(\nu)$ has a simple socle and a simple head, its Loewy series coincides with the grading filtration up to degree shift by [4]. It follows that $\hat{L}(\eta^+)$ appears in $\mathrm{rad}_{N(\nu)}\hat{T}^\eta_\nu\hat{\Delta}(\nu)$, and hence $\hat{\Delta}(\eta^+)\leqslant\mathrm{rad}^{N(\nu)}\hat{T}^\eta_\nu\hat{\Delta}(\nu)$.

(ii) Note first that the number $(\hat{P}(\eta^-):\hat{\nabla}(\eta^+))$ of times $\hat{\nabla}(\eta^+)$ appears in the $\hat{\nabla}$-filtration of $\hat{P}(\eta^-)$ is by Brauer–Humphreys reciprocity [11, II.11.4] equal to $[\hat{\nabla}(\eta^+):\hat{L}(\eta^-)]$, which is by the translation principle equal to 1:

$$(\hat{P}(\eta^-):\hat{\nabla}(\eta^+))=[\hat{\nabla}(\eta^+):\hat{L}(\eta^-)]=1. \tag{1}$$

Thus, if $r=\max\{i\in\mathbb{N}\mid\eta_i=\eta^+\}$ in the $\hat{\nabla}$-filtration $M^\bullet$ of $\hat{P}(\eta^-)$ with the subquotients $M^i/M^{i-1}\simeq\hat{\nabla}(\eta_i)$, one has $\hat{T}^\eta_\nu\hat{\nabla}(\nu)\leqslant M^r$. By (1) and by [3, 3.5] there is unique $j\in\mathbb{N}$

such that $[\mathrm{soc}_{j+1}M^r : \hat{L}(\eta^+)] = [\mathrm{soc}_{j+1}\hat{\nabla}(\eta^+) : \hat{L}(\eta^-)] = 1$. As $[\hat{T}_\nu^\eta\hat{\nabla}(\nu) : \hat{L}(\eta^+)] \neq 0$, we must have $[\mathrm{soc}_{j+1}\hat{T}_\nu^\eta\hat{\nabla}(\nu) : \hat{L}(\eta^+)] = 1$. Then taking the τ-dual yields that $[\mathrm{rad}_j\hat{T}_\nu^\eta\hat{\Delta}(\nu) : \hat{L}(\eta^+)] = 1$, and hence $j = N(\nu)$ by (i).

(iii) Consider a filtration of $\hat{T}_\nu^\eta\hat{\Delta}(\nu)$ with the subquotients $\hat{\Delta}(w \bullet \eta)$, $w \in W_\nu$. By weight considerations $\hat{T}_\nu^\eta\hat{L}(\nu)$ must contain all the composition factors of $\hat{T}_\nu^\eta\hat{\Delta}(\nu)$ isomorphic to $\hat{L}(\eta^-)$.

On the other hand, $[\hat{\Delta}(w \bullet \eta^+) : \hat{L}(\eta^-)] = 1\ \forall w \in W_\nu$ as in (1). Thus $\hat{T}_\nu^\eta\hat{L}(\nu)$ contains a composition factor $\hat{L}(\eta^-)$ corresponding to one in each of $\hat{\Delta}(w \bullet \eta^+)$, $w \in W_\nu$. Consider the factor corresponding to the one in $\hat{\Delta}(\eta^+)$. Let $\theta \in G_1T\mathbf{Mod}(\hat{\Delta}(\eta^+), \hat{T}_\nu^\eta\hat{L}(\nu))$ be the restriction to $\hat{\Delta}(\eta^+)$ of the quotient $\hat{T}_\nu^\eta\hat{\Delta}(\nu) \to \hat{T}_\nu^\eta\hat{L}(\nu)$. Then $\mathrm{im}\,\theta \leqslant \mathrm{rad}^{N(\nu)}\hat{T}_\nu^\eta\hat{L}(\nu)$ by (i). As the composition factor $\hat{L}(\eta^-)$ comes from the one in $\mathrm{rad}_{N(\nu)}\hat{\Delta}(\eta^+)$ by (ii), it lies in $\mathrm{rad}_{N(\nu)}(\mathrm{im}\,\theta)$. It follows that $2N(\nu)+1 \leqslant \ell\ell(\mathrm{im}\,\theta) + N(\nu) \leqslant \ell\ell(\hat{T}_\nu^\eta\hat{L}(\nu))$.

(iv) Consider a nonsplit exact sequence $0 \to \hat{L}(y \bullet \nu) \to K \to \hat{L}(x \bullet \nu) \to 0$, $x, y \in W_a$, with $x \bullet \nu > y \bullet \nu$. There is an epimorphism $\hat{\Delta}(x \bullet \nu) \twoheadrightarrow K$. As $\hat{T}_\nu^\eta\hat{\Delta}(x \bullet \nu)$ has a simple head, so does $\hat{T}_\nu^\eta K$. In particular, $\hat{T}_\nu^\eta K$ is indecomposable, and so therefore is $(\hat{T}_\nu^\eta K)^\tau \simeq \hat{T}_\nu^\eta(K^\tau)$.

We now argue by induction on $\ell\ell(M)$. We may assume that M has a simple head. Let $\hat{L}(x \bullet \nu) = \mathrm{hd}M$, $x \in W_a$. Take a quotient M/M' with $\mathrm{rad}M > M' > \mathrm{rad}^2 M$ which fits in a short exact sequence $0 \to \hat{L}(y \bullet \nu) \to M/M' \to \hat{L}(x \bullet \nu) \to 0$ for some $y \in W_a$. As $\hat{T}_\nu^\eta(M/M')$ is indecomposable, the exact sequence $0 \to \hat{T}_\nu^\eta(\mathrm{rad}M) \to \hat{T}_\nu^\eta M \to \hat{T}_\nu^\eta\hat{L}(x \bullet \nu) \to 0$ cannot split. Thus $\ell\ell(\hat{T}_\nu^\eta M) \geqslant \ell\ell(\hat{T}_\nu^\eta(\mathrm{rad}M)) + 1$, as desired. □

2.5.

Keep the notation of § 2.4. Let w_0 denote the longest element of W. $\forall x \in W_a$, recall from [3, 3.4.2] that

$$\ell\ell\hat{P}(\nu) \geqslant 2\ell\ell(\hat{\nabla}(w_0 \bullet \nu)) - 1 \geqslant 2N - 2N(\nu) + 1, \tag{1}$$

and from [3, 2.3] that $\ell\ell\hat{\nabla}(w_0 \bullet \nu) \geqslant N - N(\nu) + 1$. Thus

$$\begin{aligned} 2N+1 &= \ell\ell\hat{P}(\eta^-) \quad \text{by [3, 5.4]} \\ &= \ell\ell(\hat{T}_\nu^\eta\hat{P}(\nu)) \geqslant \ell\ell(\hat{P}(\nu)) + 2N(\nu) \quad \text{by § 2.4} \\ &\geqslant 2N - 2N(\nu) + 1 + 2N(\nu) = 2N+1. \end{aligned}$$

It follows that $\ell\ell\hat{P}(\nu) = 2N - 2N(\nu) + 1$, and then $\ell\ell\hat{\nabla}(w_0 \bullet \nu) = N - N(\nu) + 1$ by (1). As $\hat{\nabla}_w((w \bullet \nu)\langle w\rangle) \simeq {}^w\hat{\nabla}(\nu) \otimes p(w \bullet 0)\ \forall w \in W$ by § 1.5, we have

$$\ell\ell\hat{\nabla}_w((w \bullet \nu)\langle w\rangle) = 1 + N - N(\nu) = 1 + \dim G/B - N(\nu),$$

determining the Loewy length of $\hat{\nabla}_w((w \bullet \nu)\langle w\rangle)$. Let us also record the following.

Theorem. *Assume that $p \gg 0$ so that (L) holds.* $\forall \nu \in \Lambda$, $\ell\ell(\hat{P}(\nu)) = 2N - 2N(\nu) + 1$.

2.6.

Recall the notation of § 1.8. To find the Loewy length of $\hat{\nabla}_J(\hat{L}^J(\nu))$, we first recall some identities from [1]. These hold without restrictions on p. Let w_J denote the

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longest element of W_J, and put $w^J = w_0 w_J$. Let $\nu \in \Lambda$. We will write $\nu\langle w\rangle$, $w \in W$, for $\nu + (p-1)(w \bullet 0)$.

One can reformulate [1, 1.4] as an isomorphism $\mathrm{hd}\hat{\nabla}_J(\hat{L}^J(\nu)) \simeq L((w^J \bullet \nu)^0) \otimes_{\Bbbk} p\{(w^J)^{-1} \bullet (w^J \bullet \nu)^1\}$. Also, [1, 4.5] carries over to arbitrary $\nu \in \Lambda$: $\mathrm{hd}^{w^J}\hat{\nabla}_J(\hat{L}^J(\nu)) \simeq \hat{L}(w^J \bullet \nu) \otimes_{\Bbbk} \{-p(w^J \bullet 0)\}$. We then have a commutative diagram from [1, 4.6.1],

$$
\begin{array}{ccc}
\hat{\nabla}_{w^J}((w^J \bullet \nu)\langle w^J\rangle) \otimes \{-p(w^J \bullet 0)\} & \xrightarrow{\phi_{w^J} \otimes \{-p(w^J \bullet 0)\}} & \hat{\nabla}(w^J \bullet \nu) \otimes \{-p(w^J \bullet 0)\} \\
\uparrow & & \uparrow \\
{}^{w^J}\hat{\nabla}_J(\hat{L}^J(\nu)) & \twoheadrightarrow & \hat{L}(w^J \bullet \nu) \otimes \{-p(w^J \bullet 0)\}
\end{array}
$$

and another from [1, 4.6.3],

$$
\begin{array}{ccc}
\hat{\nabla}_{w_0}((w^J \bullet \nu)\langle w_0\rangle) \otimes \{-p(w^J \bullet 0)\} & & \\
\downarrow {\scriptstyle \phi'_{w^J} \otimes \{-p(w^J \bullet 0)\}} & \searrow\!\!\!\!\searrow & \\
 & & {}^{w^J}\hat{\nabla}_J(\hat{L}^J(\nu)) \\
 & \swarrow & \\
\hat{\nabla}_{w^J}((w^J \bullet \nu)\langle w^J\rangle) \otimes \{-p(w^J \bullet 0)\}. & &
\end{array}
$$

If we write $w^J = s_{i_1} \ldots s_{i_n}$ in a reduced expression with s_i denoting the reflection associated to the simple root α_i, the homomorphism $\phi_{w^J} : \hat{\nabla}_{w^J}((w^J \bullet \nu)\langle w^J\rangle) \to \hat{\nabla}(w^J \bullet \nu)$ is the composite

$$
\begin{aligned}
\hat{\nabla}_{s_{i_1}\cdots s_{i_n}}((w^J \bullet \nu)\langle s_{i_1}\cdots s_{i_n}\rangle) &\to \hat{\nabla}_{s_{i_1}\cdots s_{i_{n-1}}}((w^J \bullet \nu)\langle s_{i_1}\cdots s_{i_{n-1}}\rangle) \to \cdots \\
&\to \hat{\nabla}_{s_{i_1}s_{i_2}}((w^J \bullet \nu)\langle s_{i_1}s_{i_2}\rangle) \to \hat{\nabla}_{s_{i_1}}((w^J \bullet \nu)\langle s_{i_1}\rangle) \to \hat{\nabla}((w^J \bullet \nu))
\end{aligned}
$$

with each $\hat{\nabla}_{s_{i_1}\cdots s_{i_r}}((w^J \bullet \nu)\langle s_{i_1}\cdots s_{i_r}\rangle) \to \hat{\nabla}_{s_{i_1}\cdots s_{i_{r-1}}}((w^J \bullet \nu)\langle s_{i_1}\cdots s_{i_{r-1}}\rangle)$ bijective iff $\langle w^J \bullet \nu + \rho, s_{i_1}\cdots s_{i_{r-1}}\alpha_{i_r}^\vee\rangle \equiv 0 \mod p$ [3, 2.2]. Thus, if we put $R^+(w) = \{\alpha \in R^+ \mid w\alpha < 0\}$, $w \in W$, and $R^+_\nu = \{\alpha \in R^+ \mid \langle \nu + \rho, \alpha^\vee\rangle \equiv 0 \mod p\}$, then $\phi_{w^J} \otimes \{-p(w^J \bullet 0)\}$ annihilates $\mathrm{soc}^{\ell(w^J) - |R^+(w^J) \cap R^+_\nu|}\hat{\nabla}_{w^J}((w^J \bullet \nu)\langle w^J\rangle) \otimes \{-p(w^J \bullet 0)\}$, and hence

$$\ell\ell^{w^J}\hat{\nabla}_J(\hat{L}^J(\nu)) \geqslant \ell(w^J) - |R^+(w^J) \cap R^+_\nu| + 1. \tag{1}$$

Likewise, $\phi'_{w^J} \otimes \{-p(w^J \bullet 0)\}$ annihilates $\mathrm{soc}^{\ell(w_J) - |(R^+ \setminus R^+(w^J)) \cap R^+_\nu|}\hat{\nabla}_{w_0}((w^J \bullet \nu)\langle w_0\rangle) \otimes \{-p(w^J \bullet 0)\}$.

We now assume (L) again. As $\ell\ell\hat{\nabla}_{w_0}((w^J \bullet \nu)\langle w_0\rangle) = N - N(\nu) + 1$ by § 2.5,

$$\ell\ell^{w^J}\hat{\nabla}_J(\hat{L}^J(\nu)) \leqslant N - N(\nu) + 1 - \{\ell(w_J) - |(R^+ \setminus R^+(w^J)) \cap R^+_\nu|\}. \tag{2}$$

As $N(\nu) = |R^+(w^J) \cap R^+_\nu| + |(R^+ \setminus R^+(w^J)) \cap R^+_\nu|$, it now follows from (1) and (2) that

$$
\begin{aligned}
\ell\ell\hat{\nabla}_J(\hat{L}^J(\nu)) &= \ell(w^J) - |R^+(w^J) \cap R^+_\nu| + 1 = 1 + \dim G/Q - |R^+(w^J) \cap R^+_\nu| \\
&= |R^+(w^J)| - |R^+(w^J) \cap R^+_\nu| + 1 = 1 + |R^+(w^J) \setminus R^+_\nu|.
\end{aligned}
$$

Thus

Theorem. *Assume that $p \gg 0$ so that (L) holds. All $\hat{\nabla}_J(\hat{L}^J(\nu))$, $J \subseteq R^s$, $\nu \in \Lambda$, have Loewy length $1 + |R^+(w^J) \setminus R_\nu^+|$.*

Acknowledgements. We are grateful to Wolfgang Soergel for suggesting the problem to us and for referring us [16]. Thanks are also due to Simon Riche for helpful comments to clarify the presentation of the paper. The first author of the paper acknowledges the hospitality of both the Institute of Mathematics of Jussieu and the Mittag-Leffler Institute during visits of which the work was done. Finally, we are grateful to the referee for the careful reading of the manuscript.

References

1. N. Abe and M. Kaneda, On the structure of parabolically induced G_1T-Verma modules, *J. Inst. Math. Jussieu* **14**(01) (2015), 185–220.
2. H. H. Andersen, J. C. Jantzen and W. Soergel, Representations of quantum groups at a p-th root of unity and of semisimple groups in characteristic p: independence of p, *Astérisque* **220** (1994), 1–321 (SMF).
3. H. H. Andersen and M. Kaneda, Loewy series of modules for the first Frobenius kernel in a reductive algebraic group, *Proc. Lond. Math. Soc.* (3) **59** (1989), 74–98.
4. A. Beilinson, D. Ginzburg and W. Soergel, Koszul duality patterns in representation theory, *J. Amer. Math. Soc.* **9**(2) (1996), 473–527.
5. R. Bezrukavnikov, I. Mirkovic and D. Rumynin, Singular localization and intertwining functors for reductive Lie algebras in prime characteristic, *Nagoya Math. J.* **184** (2006), 1–55.
6. R. Bezrukavnikov, I. Mirkovic and D. Rumynin, Localization of modules for a semisimple Lie algebra in prime characteristic, *Ann. of Math.* (2) **167** (2008), 945–991.
7. P. Fiebig, Lusztig's conjecture as a moment graph problem, *Bull. Lond. Math. Soc.* **42**(6) (2010), 957–972.
8. P. Fiebig, An upper bound on the exceptional characteristics for Lusztig's character formula, *J. reine angew. Math.* **673** (2012), 1–31.
9. R. S. Irving, Projective modules in the category $\mathcal{O}_S$: Loewy series, *Trans. Amer. Math. Soc.* **291**(2) (1985), 733–754.
10. R. S. Irving, The socle filtration of a Verma module, *Ann. Sci. Éc. Norm. Supér.* (4) **21** (1988), 47–65.
11. J. C. Jantzen, *Representations of Algebraic Groups* (American Mathematical Society, 2003).
12. J. C. Jantzen, Representations of Lie algebras in prime characteristic, in *Representation Theories and Algebraic Geometry* (ed. A. Broer), pp. 185–235 (Kluwer, 1998).
13. M. Kaneda, The Kazhdan–Lusztig polynomials arising in the modular representation theory of reductive algebraic groups, *RIMS Kōkyūroku* **670** (1988), 129–162.
14. S. Kato, On the Kazhdan–Lusztig polynomials for affine Weyl groups, *Adv. Math.* **55** (1995), 103–130.
15. G. Lusztig, Hecke algebras and Jantzen's generic decomposition patterns, *Adv. Math.* **37** (1980), 121–164.
16. S. Riche, Koszul duality and modular representations of semisimple Lie algebras, *Duke Math. J.* **154** (2010), 31–134.
17. G. Williamson, A. Kontorovich and P. McNamara, Schubert calculus and torsion explosion, arXiv:1309.5055v2.