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PAPER

Quantifiers on languages and codensity monads*

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Abstract
This paper contributes to the techniques of topo-algebraic recognition for languages beyond the regular setting as they relate to logic on words. In particular, we provide a general construction on recognisers corresponding to adding one layer of various kinds of quantifiers and prove a corresponding Reutenauer-type theorem. Our main tools are codensity monads and duality theory. Our construction hinges on a measure-theoretic characterisation of the profinite monad of the free S-semimodule monad for finite and commutative semirings S, which generalises our earlier insight that the Vietoris monad on Boolean spaces is the codensity monad of the finite powerset functor.

Keywords: Formal languages; logic on words; semiring quantifiers; stone duality; codensity monads; semiring-valued measures

1. Introduction

It is well known that the combinatorial property of a language of being given by a star-free regular expression can be described both by algebraic and by logical means. Indeed, on the algebraic side, the star-free languages are exactly those languages whose syntactic monoids do not contain any non-trivial groups as subsemigroups. On the logical side, properties of words can be expressed in predicate logic by considering variables as positions in the word, relation symbols asserting that a position in a word has a certain letter of the alphabet, and possibly additional predicates on positions known as numerical predicates. As shown in McNaughton and Papert (1971), the class of languages definable by first-order sentences over the numerical predicate $<$ consists precisely of the star-free ones.

The theory of formal languages abounds with such results showing the strong interplay between logic and algebra. For instance, Straubing et al. introduced in (1995) a class of additional quantifiers, the so-called modular quantifiers $\exists_{p \bmod q}$. (Recall that a word satisfies a formula $\exists_{p \bmod q} x.\varphi(x)$ provided the number of positions x for which $\varphi(x)$ holds is congruent to p modulo q.) There it is shown, for example, that the languages definable using modular quantifiers of modulus q are exactly the languages whose syntactic monoids are solvable groups of cardinality dividing a power of q.

Studying modular quantifiers is relevant for tackling open problems in Boolean circuit complexity, see, for example, Straubing and Thérien (2008) for a discussion. Since Boolean circuit

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classes contain non-regular languages, expanding the automata theoretic techniques beyond the regular setting is also relevant for addressing these problems.

A fundamental tool in studying the connection between algebra and logic in this setting is the availability of constructions on monoids which mirror the action of quantifiers. That is, given the syntactic monoid for a language with a free variable, one wants to construct a monoid that recognises the quantified language. Constructions of this type abound, and are all versions of semidirect products, with the block product playing a central rôle as it allows one to construct recognisers for many different quantifiers (Tesson and Thérien, 2007).

The present article is an expanded and improved version of the publication (Gehrke et al., 2017), where the main results were first announced. Its purpose is to expand the techniques available for monoids and provide the topo-algebraic characterisation of adding one layer of various kinds of quantifiers, beyond the regular setting. The first step was made in Gehrke et al. (2016), where (a) we introduced a topological notion of recogniser, which will be motivated in the next subsection and (b) we gave a notion of unary Schützenberger product that corresponds, on the recogniser side, to adding one layer of the existential quantifier for arbitrary languages of words.

In Section 1.1, we provide a gentle introduction and motivate the duality-theoretic approach to language recognition. In Section 1.2, we present codensity monads, our tool of choice for systematically obtaining the relevant topological constructions and briefly discuss related work. Finally, in Section 1.3, we present the main contributions of this paper and provide an overview of the remainder of the paper.

1.1 Duality for language recognition

Stone duality plays an important rôle and has a long tradition in many areas of semantics, for example, in domain theory and modal logic. In Pippenger (1997), Pippenger made the link between Stone duality and regular languages explicit, by proving that the Boolean algebra of regular languages over a finite alphabet A is the dual of the free profinite monoid on A. Yet, only recently, starting with the papers (Gehrke et al., 2008, 2010), the deep connection between this field and formal language theory started to emerge. In these papers a new notion of language recognition, based on topological methods, was proposed for the setting of non-regular languages. Moreover, the scene was set for a new duality-theoretic understanding of the celebrated Eilenberg–Reiterman theorems, establishing a connection between varieties of languages, pseudo-varieties of finite algebras and profinite equations. This led to an active research area, where categorical and duality-theoretic methods are used to encompass notions of language recognition for various automata models. See, for example, the monadic approach to language recognition put forward by Bojańczyk (2015), or the series of papers on a category-theoretic approach to Eilenberg–Reiterman theory (cf. Adámek et al., 2015 and references herein).

Let us illustrate the interplay between duality theory and the theory of regular languages by explaining the duality between the syntactic monoid of a regular language L on a finite alphabet A, and the Boolean subalgebra $\mathcal{B} \hookrightarrow \mathcal{P}(A^*)$ generated by the quotients of L, that is by the sets

$$w^{-1}Lv^{-1} = \{u \in A^* \mid wuv \in L\}$$

for $w, v \in A^*$. In this setting, one makes use only of the finite duality between the category of finite Boolean algebras and the category of finite sets which, at the level of objects, asserts that each finite Boolean algebra is isomorphic to the powerset of its atoms.

Since the language L is regular, it has only finitely many quotients, say

$$\{w_1^{-1}Lv_1^{-1}, \dots, w_n^{-1}Lv_n^{-1}\}.$$

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The finite Boolean algebra generated by this set has as atoms the non-empty subsets of A^* of the form

$$\bigcap_{i\in I} w_i^{-1}Lv_i^{-1} \cap \bigcap_{j\in J} (w_j^{-1}Lv_j^{-1})^c$$

for some partition $I \cup J$ of $\{1, \ldots, n\}$. We clearly see that such atoms are in one-to-one correspondence with the equivalence classes of the Myhill syntactic congruence $\sim_L$, and thus with the elements of the syntactic monoid $A^*/\sim_L$ of L.

However, the more interesting aspect of this approach is that one can also explain the monoid structure of $A^*/\sim_L$ and the syntactic morphism in duality-theoretic terms. For this, we have to recall first the duality between the category of sets and the category of complete atomic Boolean algebras. At the level of objects, every complete atomic Boolean algebra is isomorphic to the powerset of its atoms. So the dual of A^* is $\mathcal{P}(A^*)$, but the duality also tells us that quotients on one side are turned into embeddings on the other. Thus, we have the duality between the following morphisms:

$$\mathcal{B} \hookrightarrow \mathcal{P}(A^*) \qquad \Big| \qquad A^* \twoheadrightarrow A^*/\sim_L.$$

Also, the left action of A^* on itself given by appending a word w on the left corresponds, on the dual side, to a left quotient operation (which is a right action):

$$\begin{array}{ccc} \mathcal{P}(A^*) & \xrightarrow{\Lambda_w} & \mathcal{P}(A^*) \\ U & \longmapsto & w^{-1}U \end{array} \qquad \Bigg| \qquad \begin{array}{ccc} A^* & \xrightarrow{l_w} & A^* \\ v & \longmapsto & wv \end{array}$$

Since the Boolean algebra $\mathcal{B}$ is closed under quotients, the homomorphisms $\Lambda_w \colon \mathcal{P}(A^*) \to \mathcal{P}(A^*)$ restrict to homomorphisms $\mathcal{B} \to \mathcal{B}$, that is we have commuting squares as the left one in diagram (1) below. By duality, we obtain a left action of A^* on $A^*/\sim_L$, whose component at w is as in the right-hand diagram in (1). Reasoning in a similar manner, one also obtains a right action of A^* on $A^*/\sim_L$, and the two actions commute. It is a simple lemma, see Gehrke et al. (2016), that since $A^*/\sim_L$ is a quotient of A^* and it is equipped with commuting left and right A^*-actions (called in loc. cit. an A^*-*biaction*), then one can uniquely define a monoid multiplication on $A^*/\sim_L$ so that the quotient $A^* \twoheadrightarrow A^*/\sim_L$ is a monoid morphism.

$$\begin{array}{ccc} \mathcal{B} & \hookrightarrow & \mathcal{P}(A^*) \\ {\scriptstyle \Lambda_w} \downarrow & & \downarrow {\scriptstyle \Lambda_w} \\ \mathcal{B} & \hookrightarrow & \mathcal{P}(A^*) \end{array} \qquad \Bigg| \qquad \begin{array}{ccc} A^* & \twoheadrightarrow & A^*/\sim_L \\ {\scriptstyle l_w} \downarrow & & \downarrow {\scriptstyle l_w} \\ A^* & \twoheadrightarrow & A^*/\sim_L \end{array} \tag{1}$$

This approach paves the way to a notion of recogniser and syntactic object pertinent for non-regular languages. In the case of a non-regular language L, the Boolean algebra $\mathcal{B}$ spanned by the quotients of L is no longer finite, so the finite or discrete duality theorems we have employed previously are no longer applicable. Instead, we use the full Stone duality, which establishes a dual equivalence between the category of Boolean algebras and the category BStone of Boolean (Stone) spaces, that is, compact Hausdorff spaces whose clopen subsets form a basis for the topology. In this setting, the dual of $\mathcal{P}(A^*)$ is the Stone–Čech compactification $\beta(A^*)$ of the discrete space A^*. The embedding of $\mathcal{B}$ into $\mathcal{P}(A^*)$ is turned by the duality theorem into a quotient of topological spaces as displayed below, where we denote the dual of $\mathcal{B}$ by X.

$$\mathcal{B} \hookrightarrow \mathcal{P}(A^*) \qquad \Big| \qquad \beta(A^*) \twoheadrightarrow X$$

The syntactic monoid of the language L, which is in general infinite, can be seen as a dense subset of X, and is indeed the image of the composite map $A^* \hookrightarrow \beta(A^*) \twoheadrightarrow X$ where the first arrow is the embedding of A^* in its Stone–Čech compactification. We thus obtain a commuting diagram as follows:

$$\begin{array}{ccc} \beta(A^*) & \twoheadrightarrow & X \\ \uparrow & & \uparrow \\ A^* & \twoheadrightarrow & A^*/{\sim_L} \end{array}$$

Furthermore, one can show that the syntactic monoid acts (continuously) on X both on the left and on the right, and these actions commute. This led us, in Gehrke et al. (2016), to the definition of a *Boolean space with an internal monoid* (BiM) as a suitable notion for language recognition beyond the regular setting. We recall this (in fact a small variation of it) in Definition 2.2 below.

1.2 Profinite monads

Profinite methods have a long tradition in language theory, see, for example, Almeida and Weil (1998). To accommodate these tools in his monadic approach to language recognition, Bojańczyk (2015) has recently introduced a construction transforming a monad T on Set (the category of sets and functions) into a so-called *profinite monad*, again on the category of sets. The latter monad allowed him to study in this generic framework the profinite version of the objects modelled by T such as profinite words, profinite countable chains and profinite trees.

A very much related construction of a *profinite monad of* T was introduced in Adámek et al. (2016*b*), this time as a monad on the category of Boolean spaces, obtained as a so-called *codensity monad* for a functor from the category of finitely carried T-algebras to Boolean spaces, that we have described in the next section.

The codensity monad is a standard construction in category theory, which goes back to the work of Kock in the 1960s. It is well known that any right adjoint functor G induces a monad obtained by composition with its left adjoint, and this is exactly the codensity monad of G. In general, the codensity monad of a functor that is not necessarily right adjoint, provided it exists, is the best approximation to this phenomenon. For example, the codensity monad of the forgetful functor $|-|\colon \mathsf{BStone} \to \mathsf{Set}$ on Boolean spaces is the ultrafilter monad on Set obtained by composition with its left adjoint $\beta\colon \mathsf{Set} \to \mathsf{BStone}$. The same monad has yet another description as a codensity monad, this time for the inclusion of the category Set_f of finite sets into Set, a fact proved in Kennison and Gildenhuys (1971) and recently revisited in the elegant paper (Leinster, 2013).

The starting point of the present paper is the observation that the unary Schützenberger product $(\Diamond X, \Diamond M)$ of a BiM (X, M) from our paper (Gehrke et al., 2016) hinges, at a deeper level, on the fact that the *Vietoris monad* $\mathcal{V}$ on the category of Boolean spaces (which is heavily featured in that construction) is the profinite monad of the finite powerset monad $\mathcal{P}_f$ on Set. Recall that any Boolean space X is the cofiltered (or inverse) limit of its finite quotients X_i, see, for example, Engelking (1989, 6.2.C.(a)). Then, one can check that the Vietoris space $\mathcal{V}X$ can be obtained as the cofiltered limit of the finite discrete spaces $\mathcal{P}_f X_i$.

In order to find suitable recognisers for languages quantified by, for example, modular existential quantifiers, we need a slightly different construction than $(\Diamond X, \Diamond M)$ of Gehrke et al. (2016). Specifically, we observe that the semantics of these quantifiers can be modelled, at least at the level of finite monoids, by the free S-semimodule monad $\mathcal{S}$, for a suitable choice of the semiring S. It should be noted that $\mathcal{P}_f$ is also an instance of the free S-semimodule monad, for the Boolean semiring $\mathbf{2}$. To obtain corresponding constructions at the level of Boolean spaces with internal monoids, one needs to understand the analogue of the Vietoris construction for the monad $\mathcal{S}$. A prime candidate, from a category-theoretic perspective, is the codensity monad of $\mathcal{S}$.

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1.3 Contributions

This paper contributes to the connection between the topological approach to language recognition and logical formalisms beyond the setting of regular languages, and furthers, along the way, the study of profinite monads in formal language theory.

The main result of Section 3 allows one to extend finitary commutative Set-monads to the category of Boolean spaces with internal monoids. A particular instance of this result is presented in Section 4, where duality-theoretic insights are used to provide a concrete and useful description of the constructions involved in terms of measures. In Section 5, we develop a generic approach for mirroring operations on languages, such as modular quantifiers, associating to a BiM (X, M) a new BiM $(\Diamond_S X, \Diamond_S M)$. Finally, in Section 6, we explain how these constructions are indeed canonical, and provide a Reutenauer-type result characterising the Boolean algebra generated by the languages recognised by $(\Diamond_S X, \Diamond_S M)$.

Let us note that all the operations on languages that we consider in this paper preserve regular languages, that is they transform regular languages into regular languages. This is the case, for example, for modular quantifiers. However, in our setting, the main source of non-regularity does not reside in the application of logical quantifiers, but rather in the fact that we allow *arbitrary* numerical predicates in logic on words. In this sense, we focus on the understanding of regular quantifiers in the setting of non-regular languages.

2. Preliminaries

2.1 Logic on words

Fix an arbitrary finite set A, and write A^* for the free monoid over A. A word over the alphabet A (an A-*word*, for short) is an element $w \in A^*$. In the logical approach to language theory, the word w is regarded as a (relational) structure on the set $\{1, \ldots, |w|\}$, where $|w|$ denotes the length of the word, equipped with a unary relation P_a for each $a \in A$, which singles out the positions in the word where the letter a appears. If φ is a sentence (i.e. a formula in which every variable is in the scope of a quantifier) in a language interpretable over words, we denote by L_φ the set of A-words satisfying φ.

Assume now that $\varphi(x)$ is a formula with a free first-order variable x (intuitively, this means that $\varphi(x)$ can talk about positions in the words). In the classical model-theoretic approach, in order to interpret a free variable x, one considers structures equipped with an evaluation of x. In our case, this amounts to looking at pointed A-words. In logic on words, these evaluations are usually encoded by extending the alphabet, cf. Straubing (1994, II.2). Instead of A, we consider the alphabet $A \times 2$, which we think of as consisting of two copies of A, that is we identify $A \times 2$ with the set $A \cup \{a' \mid a \in A\}$, and we call the elements of the second copy of A *marked letters*. Assuming $w = a_1 \ldots a_n$ and $1 \leq i \leq n$, we write $w^{(i)}$ for the word $a_1 \ldots a_{i-1}a'_i a_{i+1} \ldots a_n$, that is for the word in $(A \times 2)^*$ having the same shape as w but with the letter in position i marked, and w^0 for the word $a_1 \ldots a_n$ seen as a word in $(A \times 2)^*$. Then we define $L_{\varphi(x)}$ as the set of all words in the alphabet $A \times 2$ *with only one marked letter* such that the underlying word in the alphabet A satisfies φ when the variable x points at the marked position. This approach has the advantage of treating words with evaluations of the free variable as *bona fide* words in an extended alphabet. However, the only words in the alphabet $A \times 2$, which represent structures with an evaluation of the free variable are those with precisely one marked letter.

Now, given $L \subseteq (A \times 2)^*$, denote by $L_\exists$ the language consisting of those words $w = a_1 \ldots a_n$ over A such that there exists $1 \leq i \leq n$ with $a_1 \ldots a_{i-1}a'_i a_{i+1} \ldots a_n \in L$. Observe that $L = L_{\varphi(x)}$ entails $L_\exists = L_{\exists x.\varphi(x)}$, thus recovering the usual existential quantification.

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Example 2.1. Let A be the two-letter alphabet $\{a, b\}$. Consider the language $L \subseteq (A \times 2)^*$ defined by

$$L = \{u \in (A \times 2)^* \mid |u|_a + |u|_{a'} = |u|_b + |u|_{b'}\},$$

where $|u|_a$ is the number of a's appearing in u, and similarly for $|u|_{a'}$, $|u|_b$, and $|u|_{b'}$. Then,

$$\begin{aligned} L_\exists &= \{w \in A^* \mid \exists 1 \le i \le |w| \text{ such that } w^{(i)} \in L\} \\ &= \{w \in A^* \mid |w|_a = |w|_b\} \end{aligned}$$

is the language consisting of all those A-words in which the numbers of a's equals the number of b's.

Amongst the generalisations of the existential quantifier are the *modular quantifiers*. Consider the ring $\mathbb{Z}_q$ of integers modulo q, and pick $p \in \mathbb{Z}_q$. We say that an A-word w satisfies the sentence $\exists_{p \bmod q} x.\varphi(x)$ if there exist p modulo q positions in w for which the formula $\varphi(x)$ holds. Moreover, for an arbitrary language $L \subseteq (A \times 2)^*$, we define $L_{\exists_{p \bmod q}}$ as the set of A-words $w = a_1 \ldots a_n$ such that the cardinality of the set

$$\{1 \le i \le n \mid a_1 \ldots a_{i-1} a_i' a_{i+1} \ldots a_n \in L\} \tag{2}$$

is congruent to p modulo q. Clearly, if the language L is defined by the formula $\varphi(x)$, then $L_{\exists_{p \bmod q}}$ is defined by the formula $\exists_{p \bmod q} x.\varphi(x)$.

Finally, generalising the preceding situations, we can consider an arbitrary *semiring*, that is a tuple $(S, +, \cdot, 0_S, 1_S)$ such that $(S, +, 0_S)$ is a commutative monoid, $(S, \cdot, 1_S)$ is a monoid, the operation $\cdot$ distributes over $+$, and $0_S \cdot s = 0_S = s \cdot 0_S$ for all $s \in S$. If no confusion arises, we will denote a semiring by S only. Given any semiring S and element $k \in S$, we can define a *semiring quantifier* as follows. For $L \subseteq (A \times 2)^*$, an A-word $w = a_1 \ldots a_n$ belongs to the quantified language, denoted by $\mathcal{Q}_k(L)$, provided that

$$\underbrace{1_S + \cdots + 1_S}_{m \text{ times}} = k,$$

where m is the cardinality of the set in (2).

2.2 Stone duality and the Vietoris hyperspace

Stone duality for Boolean algebras (Stone, 1936) establishes a categorical equivalence between the category of Boolean algebras and their homomorphisms, and the opposite of the category BStone of Boolean (Stone) spaces and continuous maps between them.

A *Boolean space* is a compact Hausdorff space in which the clopen (i.e. simultaneously closed and open) subsets form a basis for the topology. There is an obvious forgetful functor $|-|\colon \mathsf{BStone} \to \mathsf{Set}$. When clear from the context, we will omit writing $|-|$. The dual of a Boolean space X is the Boolean algebra $\mathit{Clop}(X)$ of its clopen subsets, equipped with set-theoretic operations. Conversely, given a Boolean algebra B, the dual space X may be taken either as the set of ultrafilters on B (i.e. those proper filters F satisfying $a \in F$ or $\neg a \in F$ for every $a \in B$), or equivalently as the set of all Boolean algebra homomorphisms $h\colon B \to \mathbf{2}$, equipped with the topology generated by the sets

$$\widehat{a} := \{F \mid a \in F\} \cong \{h \mid h(a) = 1\}, \text{ for } a \in B.$$

An example of Boolean space, central to our treatment, is the *Stone–Čech compactification* of an arbitrary set K (regarded as a discrete space). This is the dual space of the Boolean algebra $\mathcal{P}K$, and is denoted by βK. It is well known that the assignment $K \mapsto \beta K$ induces a functor $\beta\colon \mathsf{Set} \to \mathsf{BStone}$, which is left adjoint to the forgetful functor $|-|\colon \mathsf{BStone} \to \mathsf{Set}$.

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Another functor, which played a key rôle in Gehrke et al. (2016) and will serve here as a leading example, is the *Vietoris functor* $\mathcal{V}\colon \mathsf{BStone} \to \mathsf{BStone}$. Given a Boolean space X, consider the collection $\mathcal{V}X$ of all closed subsets of X equipped with the topology generated by the clopen subbasis

$$\{\Diamond V \mid V \in Clop(X)\} \cup \{(\Diamond V)^c \mid V \in Clop(X)\},$$

where $\Diamond V := \{K \in \mathcal{V}X \mid K \cap V \neq \emptyset\}$. The resulting space is called the *Vietoris (hyper)space* of X, and is again a Boolean space. Further, if $f\colon X \to Y$ is a continuous map in BStone, then so is the direct image function $\mathcal{V}X \to \mathcal{V}Y$, $K \mapsto f[K]$. In fact, it is well known that this is the functor part of a monad $\mathcal{V}$ on BStone. The Vietoris hyperspace of an arbitrary topological space was first introduced by Vietoris (1922); for a complete account, including the results stated here without proof, see Michael (1951).

2.3 Boolean spaces with internal monoids

In this section, we give the definition of a *Boolean space with an internal monoid*, or BiM for short (see Definition 2.2 below), a topological recogniser well suited for dealing with non-regular languages. In Gehrke et al. (2016), a *Boolean space with an internal monoid* was defined as a pair (X, M) consisting of a Boolean space X, a dense subspace M equipped with a monoid structure, and a *biaction* (i.e. a pair of compatible left and right actions) of M on X with continuous components extending the obvious biaction of M on itself. Here, we use a small variation and simplification of this notion. Instead of imposing that the monoid is a dense subset of the space, we require a map from the monoid to the space with dense image.

In what follows, for a Boolean space X we will denote by $[X, X]$ the set of continuous endofunctions on X, which comes with the obvious monoid operation $\circ$ given by composition. Given a monoid $(M, \cdot)$, we will denote by $r\colon M \to M^M$ and $l\colon M \to M^M$ the two maps induced from the monoid multiplication via currying, which correspond, respectively, to the obvious right and left actions of M on itself.

Definition 2.2. *A* Boolean space with an internal monoid, *or* BiM, *is a tuple* (X, M, h, ρ, λ), *where X is a Boolean space, M is a monoid, and* $h\colon M \to X$, $\lambda\colon M \to [X, X]$ *and* $\rho\colon M \to [X, X]$ *are functions such that h has a dense image and for all $m \in M$ the following diagrams commute in* Set.

$$\begin{array}{ccc} M & \xrightarrow{h} & X \\ {\scriptstyle l(m)}\downarrow & & \downarrow{\scriptstyle \lambda(m)} \\ M & \xrightarrow{h} & X \end{array} \qquad\qquad \begin{array}{ccc} M & \xrightarrow{h} & X \\ {\scriptstyle r(m)}\downarrow & & \downarrow{\scriptstyle \rho(m)} \\ M & \xrightarrow{h} & X \end{array} \tag{3}$$

If no confusion arises we write (X, M), or even just X, for the BiM (X, M, h, ρ, λ). *A* morphism *between two* BiMs *X and X' is a pair $(\tilde{\psi}, \psi)$, where $\tilde{\psi} : X \to X'$ is a continuous map and $\psi : M \to M'$ is a monoid morphism such that $\tilde{\psi} \circ h = h' \circ \psi$. Note that since the image of h is dense in X, given ψ, $\tilde{\psi}$ is uniquely determined if it exists. Accordingly, we will sometimes just write ψ to designate the pair as well as each of its components. We denote the ensuing category of* BiMs *by* BiM.

Remark 2.3. If (X, M) is a BiM of the form $(\beta(A^*), A^*)$, and (X', M') is any BiM, then every monoid morphism $\psi : A^* \to M'$ yields a (unique) continuous extension $\tilde{\psi} : \beta(A^*) \to X'$ making the pair $(\tilde{\psi}, \psi)$ into a BiM morphism. Thus, BiM morphisms $(\beta(A^*), A^*) \to (X', M')$ are in one-to-one correspondence with monoid morphisms $A^* \to M'$. For this reason, we will often treat these two things as one and the same.

Remark 2.4. It follows from Definition 2.2 that ρ and λ induce commuting right and left M-actions on X, so that h is an M-biaction morphism. Indeed, since h has a dense image in X, it follows that $\rho(m)$ and $\lambda(m)$ are the unique extensions to X of $r(m)$ and $l(m)$, respectively. But the left and right actions of M on itself commute, hence ρ and λ must enjoy the same properties. We also obtain that $(X, \mathsf{Im}(h))$ is a Boolean space with an internal monoid according to the definition in Gehrke et al. (2016).

Remark 2.5. An equivalent way of saying that the diagrams in (3) commute for all $m \in M$ is to say that the following two diagrams commute in Set.

$$\begin{array}{ccc} [X,X] & \xrightarrow{-\circ h} & X^M \\ {\scriptstyle\lambda}\uparrow & & \uparrow{\scriptstyle h\circ -} \\ M & \xrightarrow{l} & M^M \end{array} \qquad \begin{array}{ccc} [X,X] & \xrightarrow{-\circ h} & X^M \\ {\scriptstyle\rho}\uparrow & & \uparrow{\scriptstyle h\circ -} \\ M & \xrightarrow{r} & M^M \end{array}$$

This will come in handy in the proof of Theorem 3.5.

To conclude, we recall the notion of recognition associated with BiMs. Under the bijection between subsets of a given set K and clopens of its Stone–Čech compactification βK, we write $\widehat{L}$ for the clopen corresponding to the subset $L \in \mathcal{P}K$.

Definition 2.6. *Let A be a finite alphabet, $L \in \mathcal{P}(A^*)$ and (X, M, h, ρ, λ) a* BiM. *A morphism of* BiMs $\psi : (\beta(A^*), A^*) \to (X, M)$ *recognises the language L if there is a clopen $C \subseteq X$ such that $\psi^{-1}(C) = \widehat{L}$. Moreover, the* BiM (X, M) *recognises the language L if there exists a* BiM *morphism* $(\beta(A^*), A^*) \to (X, M)$ *recognising L. Finally, if $\mathcal{B} \hookrightarrow \mathcal{P}(A^*)$ is a Boolean subalgebra, the* BiM (X, M) *is said to recognise $\mathcal{B}$ provided it recognises each $L \in \mathcal{B}$.*

Equivalently, a language $L \in \mathcal{P}(A^*)$ is recognised by the morphism of BiMs $\psi : (\beta(A^*), A^*) \to (X, M)$ when there exists a clopen $C \subseteq X$ such that $L = \psi^{-1}(h^{-1}(C))$. This notion of topological recognition is summarised in the following diagram.

$$\begin{array}{rcccl} \widehat{L} = \tilde{\psi}^{-1}(C) \subseteq & \beta(A^*) & \xrightarrow{\tilde{\psi}} & X & \supseteq C \\ & \uparrow & & \uparrow{\scriptstyle h} & \\ \widehat{L} \cap A^* = L = \psi^{-1}(h^{-1}(C)) \subseteq & A^* & \xrightarrow{\psi} & M & \supseteq h^{-1}(C) \end{array}$$

The topology on X specifies which subsets of M can be used for recognition, namely the preimages under h of the clopens in X. However, when M is finite so is X. In fact, in this case X has the same carrier set as M and is equipped with the discrete topology, therefore in the regular setting we recover the usual notion of recognition.

To conclude this section, we provide an example to illustrate how language recognition works concretely in this topological setting.

Example 2.7. Consider the alphabet $A = \{a, b\}$ and the (non-regular) language $L = \{w \in A^* \mid |w|_a = |w|_b\}$ (cf. Example 2.1). The syntactic monoid $A^*/{\sim_L}$ can be identified with $(\mathbb{Z}, +)$, and L coincides with the preimage of $\{0\}$ under the syntactic morphism $\psi : A^* \twoheadrightarrow \mathbb{Z}$. Let $\mathbb{Z}_\infty$ denote the *one-point* (or *Alexandroff*) *compactification* of the discrete space $\mathbb{Z}$. The underlying set of $\mathbb{Z}_\infty$ is $\mathbb{Z} \cup \{\infty\}$, and its open sets are precisely those subsets which are either cofinite (i.e. that have finite complement), or do not contain ∞. It is not difficult to see that $\mathbb{Z}_\infty$ is a Boolean space, and $(\mathbb{Z}_\infty, \mathbb{Z})$ is a BiM. We have a commutative square as follows.

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$$\begin{array}{ccc} \beta(A^*) & \xrightarrow{\tilde{\psi}} & \mathbb{Z}_\infty \\ \uparrow & & \uparrow \\ A^* & \xrightarrow{\psi} & \mathbb{Z} \end{array}$$

The Boolean algebra $\mathcal{B} \subseteq \mathcal{P}(A^*)$ of languages recognised by the morphism of BiMs $\psi\colon (\beta(A^*), A^*) \to (\mathbb{Z}_\infty, \mathbb{Z})$ is given by $\mathcal{B} = \{\psi^{-1}(C \cap \mathbb{Z}) \mid C \text{ is a clopen subset of } \mathbb{Z}_\infty\}$. The subsets of $\mathbb{Z}$ of the form $C \cap \mathbb{Z}$, for $C \subseteq \mathbb{Z}_\infty$ a clopen, are precisely the finite or cofinite subsets (in this sense, the topology of $\mathbb{Z}_\infty$ specifies which subsets of the syntactic monoid can be used for recognition). Note that the Boolean subalgebra of $\mathcal{P}(\mathbb{Z})$ consisting of the finite or cofinite subsets is generated by the singletons $\{n\}$ for $n \in \mathbb{Z}$. In turn, the languages of the form $\psi^{-1}(\{n\})$, for $n \in \mathbb{Z}$, are precisely the quotients of L. Therefore, $\mathcal{B}$ is the Boolean subalgebra of $\mathcal{P}(A^*)$ generated by the quotients of L.

If we adopt the classical notion of recognition for monoids, and allow arbitrary subsets of $\mathbb{Z}$ for recognition purposes, then the syntactic morphism $\psi\colon A^* \twoheadrightarrow \mathbb{Z}$ recognises also the majority language

$$\psi^{-1}(\mathbb{Z}_{\geq 0}) = \{w \in A^* \mid |w|_a \geq |w|_b\}.$$

Thus, $\mathcal{B}$ is strictly contained in the Boolean algebra of languages recognised by the monoid morphism ψ in the classical sense. In fact, the syntactic monoid of the majority language is also $(\mathbb{Z}, +)$, and the distinction between these recognisers can only be made at the topological level (cf. Gehrke et al., 2010, Example 3.2).

2.4 Monads and algebras

We assume the reader is familiar with the basic notions of category theory, and especially with monads as a categorical approach to general algebra. Concerning the latter, we refer the reader to, for example, Mac Lane (1998, Chapter VI) or Borceux (1994, Chapters 3–4).

Consider a monad (T, η, μ) on a category $\mathcal{C}$. Recall that an *Eilenberg–Moore algebra* for T (or a *T-algebra*, for short) is a pair (X, h) where X is an object of $\mathcal{C}$ and $h\colon TX \to X$ is a morphism in $\mathcal{C}$, which behaves well with respect to the unit η and multiplication μ of the monad, that is, $h \circ \eta_X = \mathsf{id}_X$ and $h \circ Th = h \circ \mu_X$. A *morphism* of T-algebras $(X_1, h_1) \to (X_2, h_2)$ is a morphism $f\colon X_1 \to X_2$ in $\mathcal{C}$ satisfying $f \circ h_1 = h_2 \circ Tf$. Let $\mathcal{C}^T$ denote the category of T-algebras. When T is a monad on the category Set of sets and functions, categories of the form Set^T are, up to equivalence, precisely the varieties of (possibly infinite arity) algebras. This correspondence restricts to categories of Eilenberg–Moore algebras for *finitary* monads (i.e. monads preserving filtered colimits) and varieties of algebras in types consisting of finite arity operations. A T-algebra (X, h) is said to be *finitely carried* (or sometimes just *finite*) provided X is finite. We write Set^T_f for the full subcategory of Set^T on the finitely carried objects. The forgetful functor $\mathsf{Set}^T \to \mathsf{Set}$ that sends (X, h) to X restricts to the finitely carried algebras, and gives rise to a functor $\mathsf{Set}^T_f \to \mathsf{Set}_f$.

In Section 5.2, we shall see how several quantifiers in logic on words can be modelled by considering a semiring S and the associated free S-semimodule monad, which we now recall:

Example 2.8. A semiring S induces a functor $\mathcal{S}\colon \mathsf{Set} \to \mathsf{Set}$, which sends a set X to the set of all functions $X \to S$ with *finite support*, that is

$$\mathcal{S}X := \{f\colon X \to S \mid f(x) = 0 \text{ for all but finitely many } x \in X\}.$$

If $\psi\colon X \to Y$ is any function, define $\mathcal{S}\psi\colon \mathcal{S}X \to \mathcal{S}Y$ as $f \mapsto (y \mapsto \sum_{\psi(x)=y} f(x))$. Any element $f \in \mathcal{S}X$ can be represented as a formal sum $\sum_{i=1}^{n} s_i x_i$, where $\{x_1, \ldots, x_n\}$ is the support of f and

$s_i = f(x_i)$ for each i. The functor $\mathcal{S}$ is part of a monad $(\mathcal{S}, \eta, \mu)$ on Set, called the *free S-semimodule monad*, whose unit is

$$\eta_X\colon X \to \mathcal{S}X, \quad \eta_X(x)(x') = \begin{cases} 1 & \text{if } x' = x \\ 0 & \text{otherwise} \end{cases}$$

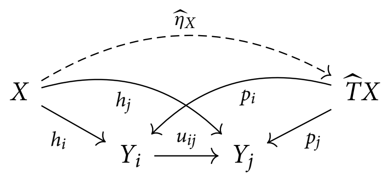
and whose multiplication is

$$\mu_X\colon \mathcal{S}^2X \to \mathcal{S}X, \quad \sum_{i=1}^{n} s_i f_i \mapsto \left(x \mapsto \sum_{i=1}^{n} s_i f_i(x) \right),$$

where the latter is an ordinary sum in the semiring S. The category $\mathsf{Set}^{\mathcal{S}}$ is the category of modules over the semiring S, also known as S-semimodules. For example, if S is the Boolean semiring $\mathbf{2}$ then $\mathcal{S} = \mathcal{P}_f$ (the finite powerset monad), whose Eilenberg–Moore algebras are join semilattices. If S is $(\mathbb{N}, +, \cdot, 0, 1)$ or $(\mathbb{Z}, +, \cdot, 0, 1)$, then the algebras for the monad $\mathcal{S}$ are, respectively, Abelian monoids and Abelian groups.

2.5 Profinite monads

Throughout this subsection we fix a monad T on Set. We begin by recalling the definition of the associated *profinite monad* $\widehat{T}$ on the category of Boolean spaces, following Adámek et al. (2016*b*). First, we provide an intuitive idea of the construction, and then we give the formal definition. Given a Boolean space X, one considers all continuous maps $h_i\colon X \to Y_i$ where the Y_i's are finite sets equipped with Eilenberg–Moore algebra structures $\alpha_i\colon TY_i \to Y_i$, as well as the algebra morphisms $u_{ij}\colon Y_i \to Y_j$ satisfying $u_{ij} \circ h_i = h_j$. Equipping the finite sets Y_i with the discrete topology, one obtains a cofiltered diagram (or inverse limit system) $\mathcal{D}_X$ in BStone, and $\widehat{T}X$ is the limit of this system. It turns out that $\widehat{T}$ is the underlying functor of a monad $(\widehat{T}, \widehat{\eta}, \widehat{\mu})$ on BStone, called the *profinite monad associated with* T. For example, it is not difficult to see how to obtain its unit $\widehat{\eta}_X$ from the universal property of the limit, as in the following diagram, where the morphisms $p_i\colon \widehat{T}X \to Y_i$ are the limit maps.



To give the formal definition of $\widehat{T}$, we introduce the functor $G\colon \mathsf{Set}_f^T \to \mathsf{BStone}$ obtained as the composition of the forgetful functor to Set_f with the embedding of Set_f into BStone:

$$G\colon \mathsf{Set}_f^T \longrightarrow \mathsf{Set}_f \longrightarrow \mathsf{BStone}.$$

The shape of the diagram we constructed above for a Boolean space X is the comma category $X \downarrow G$ whose objects are essentially the maps $h_i\colon X \to G(Y_i, \alpha_i)$, and whose arrows are the maps u_{ij} as above. The diagram $\mathcal{D}_X$ is then given by precomposing the functor G with the codomain functor $cod\colon X \downarrow G \to \mathsf{Set}_f^T$, which maps $h_i\colon X \to G(Y_i, \alpha_i)$ to the algebra (Y_i, α_i).

$$\mathcal{D}_X\colon X \downarrow G \xrightarrow{cod} \mathsf{Set}_f^T \xrightarrow{G} \mathsf{BStone}$$

Formally, for an arbitrary Boolean space X, we have $\widehat{T}X := \lim \mathcal{D}_X$.

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Note that this is the pointwise limit computation of the right Kan extension of G along itself, cf. Mac Lane (1998, X.3). That is, using standard category-theoretic notation, $\widehat{T} = \mathrm{Ran}_G\, G$. It is well known, see, for example, Leinster (2013), that the right Kan extension of a functor G along itself, when it exists, is the functor part of a monad, called the *codensity monad* for G.

Example 2.9. Let $T = \mathcal{P}_f$ be the finite powerset monad on Set, that is the semiring monad associated with the Boolean semiring **2**. The finitely carried T-algebras are the finite join semilattices, cf. Example 2.8. Using the pointwise limit computation described above, and the fact that the Vietoris functor on BStone preserves codirected limits (Engelking, 1989, 3.12.27.(f)), it is not difficult to see that the profinite monad $\widehat{T}$ of T is the Vietoris monad on BStone. Note that, because every Boolean space X is T_1, there is a canonical map $\mathcal{P}_f|X| \to |\mathcal{V}X|$, which views a finite subset of X as a closed subset. In fact, these are the components of a natural transformation. Next, we show that this natural transformation, which 'compares' T and $\widehat{T}$, can be defined for any profinite monad.

The universal property of the right Kan extension, along with the fact that the underlying-set functor $|-|: \mathsf{BStone} \to \mathsf{Set}$ is right adjoint and thus preserves right Kan extensions, allows one to define a natural transformation

$$\tau_X \colon T|X| \to |\widehat{T}X| \tag{4}$$

which was also used in Adámek et al. (2016*b*). Here we give a presentation based on the limit computation of $\widehat{T}X$. Notice that the maps $|h_i|: |X| \to |Y_i|$ are functions into the carrier sets of the Eilenberg–Moore algebras $\alpha_i \colon TY_i \to Y_i$ and thus, by the universal property of the free algebra $T|X|$, we can extend the maps $|h_i|$ to algebra morphisms $h_i^\#$ from $T|X|$ to (Y_i, α_i). The functions $h_i^\#$ form a cone for the diagram $|-| \circ \mathcal{D}_X$ in Set whose limit is $|\widehat{T}X|$, by virtue of the fact that the forgetful functor $|-|: \mathsf{BStone} \to \mathsf{Set}$ preserves limits. By the universal property of the limit, this yields a unique map τ_X as in (4).

The natural transformation τ behaves well with respect to the units and multiplications of the monads T and $\widehat{T}$, in the sense that the next two diagrams commute, see Adámek et al. (2016*a*, Proposition B.7). Thus, the pair $(|-|, \tau)$ is a *monad morphism*, or *monad functor* in the terminology of Street (1972).

$$\begin{array}{ccc} T|X| & \xrightarrow{\tau_X} & |\widehat{T}X| \\ & {\scriptstyle \eta_{|X|}}\nwarrow \quad \nearrow{\scriptstyle |\widehat{\eta}_X|} & \\ & |X| & \end{array} \qquad \begin{array}{ccccc} T^2|X| & & \xrightarrow{\mu_{|X|}} & & T|X| \\ {\scriptstyle T\tau_X}\downarrow & & & & \downarrow{\scriptstyle \tau_X} \\ T|\widehat{T}X| & \xrightarrow{\tau_{\widehat{T}X}} & |\widehat{T}^2X| & \xrightarrow{|\widehat{\mu}_X|} & |\widehat{T}X| \end{array} \tag{5}$$

The fact that $(|-|, \tau)$ is a monad functor entails that the functor $|-|$ lifts to a functor $\widehat{|-|}$ between the categories of Eilenberg–Moore algebras for the monads $\widehat{T}$ and T, as in the next diagram.

$$\begin{array}{ccc} \mathsf{BStone}^{\widehat{T}} & \xrightarrow{\widehat{|-|}} & \mathsf{Set}^T \\ \downarrow & & \downarrow \\ \mathsf{BStone} & \xrightarrow{|-|} & \mathsf{Set} \end{array} \tag{6}$$

As a consequence, we immediately obtain that the set $|\widehat{T}X|$ admits a T-algebra structure, a result also used in Adámek et al. (2016*a*) for finite algebras. This structure is essentially the one obtained by applying the functor $\widehat{|-|}$ to the free $\widehat{T}$-algebra $(\widehat{T}X, \widehat{\mu}_X)$. In more detail,

Lemma 2.10. *Given a Boolean space X, the composite map*

$$T|\widehat{T}X| \xrightarrow{\tau_{\widehat{T}X}} |\widehat{T}^2X| \xrightarrow{|\widehat{\mu}_X|} |\widehat{T}X|$$

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is a T-algebra structure on $|\widehat{T}X|$. Moreover, τ_X is a morphism of T-algebras from the free T-algebra on $|X|$ to $|\widehat{T}X|$ with the above structure.

Proof. This is a straightforward verification using the commutativity of the diagrams in (5). □

Example 2.11. When applied to the finite powerset monad $\mathcal{P}_f$, the previous lemma tells us that the Vietoris space $\mathcal{V}X$ of a Boolean space X is a join semilattice when equipped with the binary operation ∪. Further, the canonical inclusion map $\tau_X \colon \mathcal{P}_f|X| \to |\mathcal{V}X|$ is a semilattice homomorphism. An important property of the map $\tau_X \colon \mathcal{P}_f|X| \to |\mathcal{V}X|$ is that it has dense image, see, for example, Kuratowski (1966, Theorem 4 p. 163). As we shall now see, this feature is common to all profinite monads.

While in some proofs it is essential to keep track of the forgetful functor, we will sometimes omit it in what follows and simply write $\tau_X \colon TX \to \widehat{T}X$. We recall a property of the natural transformation τ, which will be crucial in the following.

Lemma 2.12. *For any Boolean space X, the map $\tau_X \colon TX \to \widehat{T}X$ has dense image. Further, the composite*

$$TM \xrightarrow{Th} TX \xrightarrow{\tau_X} \widehat{T}X$$

has dense image whenever $h \colon M \to X$ is a function with dense image.

Proof. For a proof of the fact that τ_X has dense image, see Reggio (2020, Lemma 2.9). An easy adaptation of the latter proof yields the second part of the statement. □

Remark 2.13. Notice that, for an arbitrary monad T on Set, the components of the natural transformation τ from (4) need not be injective. A counterexample is provided by the powerset monad $\mathcal{P}$ on Set. Indeed, both $\mathcal{P}$ and $\mathcal{P}_f$ generate the same profinite monad, namely the Vietoris monad on BStone. In the case of the monad $\mathcal{P}$, $\tau_X \colon \mathcal{P}X \to \mathcal{V}X$ sends a subset of the Boolean space X to its closure, and this function is injective precisely when X is finite. However, the components of τ are injective if T is finitary and restricts to finite sets, for example, if T is the finite powerset monad on Set. For more details, we refer the reader to Reggio (2020, Section 2.2).

3. Extending Commutative Set-Monads to BiMs

In this section we study liftings of monads from the category of sets to the category of BiMs. Let us fix, throughout the section, a monad T on Set. In Section 2.5, we have seen that the profinite monad $\widehat{T}$ provides a canonical way of extending T to Boolean spaces. On the other hand, in Section 3.1 we consider ways of lifting T to the category of monoids. The combination of these two liftings, the topological and the monoid one, is considered in Section 3.2. In particular, in Theorem 3.5 we give sufficient conditions for T to be extended in a canonical way to the category of BiMs, by combining the aforementioned liftings.

3.1 *Lifting* Set*-monads to the category of monoids*

It is well known that there are two 'canonical' natural transformations of bifunctors

$$\otimes, \otimes' \colon TX \times TY \to T(X \times Y),$$

defined intuitively as follows. If we think of elements of TX as terms $t(x_1, \ldots, x_m)$, then $t(x_1, \ldots, x_m) \otimes s(y_1, \ldots, y_n)$ is defined as

$$t(s((x_1, y_1), \ldots, (x_1, y_n)), \ldots, s((x_m, y_1), \ldots, (x_m, y_n))),$$

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whereas $t(x_1, \ldots, x_m) \otimes' s(y_1, \ldots, y_n)$ is defined as

$$s(t((x_1, y_1), \ldots, (x_m, y_1)), \ldots, t((x_1, y_n), \ldots, (x_m, y_n))).$$

In general $\otimes$ and $\otimes'$ do not coincide, and when they do the monad is called commutative, a notion due to Kock (1970). We give a formal definition in the case of the monad T. Every Set-monad has a unique *strength*, that is a natural transformation $\sigma_{X,Y} \colon X \times TY \to T(X \times Y)$ such that the following diagrams commute.

$$\begin{array}{ccc} X \times Y & \xrightarrow{\mathsf{id}_X \times \eta_Y} & X \times TY \\ & \searrow^{\eta_{X \times Y}} & \downarrow \sigma_{X,Y} \\ & & T(X \times Y) \end{array} \qquad \begin{array}{ccccc} X \times T^2 Y & \xrightarrow{\sigma_{X,TY}} & T(X \times TY) & \xrightarrow{T\sigma_{X,Y}} & T^2(X \times Y) \\ \downarrow \mathsf{id}_X \times \mu_Y & & & & \downarrow \mu_{X \times Y} \\ X \times TY & & \xrightarrow{\sigma_{X,Y}} & & T(X \times Y) \end{array} \tag{7}$$

This natural transformation can be explicitly described as follows. For any $x \in X$, write $f_x \colon Y \to X \times Y$ for the function sending y to (x, y). Then, $\sigma_{X,Y} \colon X \times TY \to T(X \times Y)$ sends a pair (x, s) to the image of s under $Tf_x \colon TY \to T(X \times Y)$. Associated with the strength σ, there is a *costrength* $\sigma'_{X,Y} \colon TX \times Y \to T(X \times Y)$ defined as the composition

$$TX \times Y \xrightarrow{\gamma_{TX,Y}} Y \times TX \xrightarrow{\sigma_{Y,X}} T(Y \times X) \xrightarrow{T\gamma_{Y,X}} T(X \times Y),$$

where $\gamma_{X,Y} \colon X \times Y \to Y \times X$ is the function sending (x, y) to (y, x). The costrength σ' enjoys properties symmetric to those of the strength σ, which are expressed by the following commutative diagrams.

$$\begin{array}{ccc} X \times Y & \xrightarrow{\eta_X \times \mathsf{id}_Y} & TX \times Y \\ & \searrow^{\eta_{X \times Y}} & \downarrow \sigma'_{X,Y} \\ & & T(X \times Y) \end{array} \qquad \begin{array}{ccccc} T^2 X \times Y & \xrightarrow{\sigma'_{TX,Y}} & T(TX \times Y) & \xrightarrow{T\sigma'_{X,Y}} & T^2(X \times Y) \\ \downarrow \mu_X \times \mathsf{id}_Y & & & & \downarrow \mu_{X \times Y} \\ TX \times Y & & \xrightarrow{\sigma'_{X,Y}} & & T(X \times Y) \end{array} \tag{8}$$

The monad T is said to be *commutative* if, for all sets X, Y, the following square commutes.

$$\begin{array}{ccccc} TX \times TY & \xrightarrow{\sigma_{TX,Y}} & T(TX \times Y) & \xrightarrow{T\sigma'_{X,Y}} & T^2(X \times Y) \\ \downarrow \sigma'_{X,TY} & & & & \\ T(X \times TY) & & & & \downarrow \mu_{X \times Y} \\ \downarrow T\sigma_{X,Y} & & & & \\ T^2(X \times Y) & & \xrightarrow{\mu_{X \times Y}} & & T(X \times Y) \end{array} \tag{9}$$

Note that the commutativity of this diagram formalises the aforementioned idea that the natural transformations $\otimes$ and $\otimes'$ coincide. Given a monoid $(M, \cdot, 1)$, one has two possibly different 'canonical' ways of defining a binary operation on TM, obtained as either of the two composites

$$TM \times TM \underset{\otimes'}{\overset{\otimes}{\rightrightarrows}} T(M \times M) \xrightarrow{T(\cdot)} TM. \tag{10}$$

If $e \colon 1 \to M$ denotes the map selecting the unit of the monoid, we can also define a map $1 \to TM$ obtained as the composite $Te \circ \eta_1$. That these data (with either of the two binary operations) give rise to monoid structures on TM is a direct consequence of a more general result by Kock (1970, Theorem 2.1):

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Theorem 3.1. *If T is a commutative* Set*-monad then $\otimes = \otimes'$, and thus for every monoid $(M, \cdot, 1)$ the composition in (10) gives a monoid structure on TM. This yields a lifting of T to a monad on the category of monoids and their homomorphisms.*

3.2 Combining the topological and monoid liftings

In Sections 2.5 and 3.1, respectively, we have seen that a Set-monad T can be lifted to a monad $\widehat{T}$ on the category of Boolean spaces and, if it is commutative, it can also be lifted to a monad on the category of monoids. In this section we show that, if T is commutative and finitary, then the topological and monoid liftings can be combined to obtain a lifting of T to the category of BiMs (see Theorem 3.5 below).

Let T be a commutative Set-monad, (X, M) a BiM and $h\colon M \to X$ the associated function with dense image. We would like to define a structure of BiM on the pair $(\widehat{T}X, TM)$. In particular, we should give a function $TM \to \widehat{T}X$ with dense image. To this aim, we define $\widehat{h}\colon TM \to \widehat{T}X$ as the composition

$$TM \xrightarrow{Th} TX \xrightarrow{\tau_X} \widehat{T}X. \tag{11}$$

By Lemma 2.12, this function has dense image. Further, since both Th and τ_X are T-algebra morphisms, $\widehat{h}$ is also a T-algebra morphism. In order to show that $(\widehat{T}X, TM)$ carries a BiM structure, it remains to define appropriate actions of TM on $\widehat{T}X$. This will occupy us for the rest of the section.

Remark 3.2. (i) If $\alpha\colon TB \to B$ is a T-algebra and A is any set, the set of functions B^A carries a natural 'pointwise' Eilenberg–Moore algebra structure for T, which turns B^A into the A-indexed power of B in Set^T. This T-algebra structure makes all projections $B^A \to B$ algebra morphisms. In fact, a function $C \to B^A$ from a T-algebra C is an algebra morphism precisely when all compositions with the projections $B^A \to B$ are algebra morphisms.

(ii) If $\alpha_i\colon TB_i \to B_i$ for $i \in \{1, 2\}$ are Eilenberg–Moore algebras for T, and $f\colon B_1 \to B_2$ is an algebra morphism, then $\mathsf{Set}(A, f) = f \circ -\colon B_1^A \to B_2^A$ is a T-algebra morphism.

We obtain at once the following fact.

Lemma 3.3. *For any* Set*-monad T, the sets TM^{TM} and $\widehat{T}X^{TM}$ carry T-algebra structures and the function*

$$\widehat{h} \circ -\colon TM^{TM} \to \widehat{T}X^{TM}$$

is a T-algebra morphism.

Proof. With the notation of Remark 3.2.(ii), consider $A = TM$, $\alpha_1 = \mu_M\colon T^2M \to TM$, α_2 the T-algebra structure on $\widehat{T}X$ given as in Lemma 2.10, and $f = \widehat{h}$. □

Thus, by Remark 3.2.(i), also the power algebra $\widehat{T}X^{\widehat{T}X}$ admits a T-algebra structure. Crucially, if the monad T is finitary, the set $[\widehat{T}X, \widehat{T}X]$ of continuous endofunctions on $\widehat{T}X$ is a subalgebra of $\widehat{T}X^{\widehat{T}X}$. This is proved in the following proposition which will allow us to define, in the proof of Theorem 3.5, a biaction of TM on $\widehat{T}X$.

Proposition 3.4. *If T is a finitary* Set*-monad, then $[\widehat{T}X, \widehat{T}X]$ is a subalgebra of the T-algebra $\widehat{T}X^{\widehat{T}X}$. With respect to this structure, the function*

$$- \circ \widehat{h}\colon [\widehat{T}X, \widehat{T}X] \to \widehat{T}X^{TM}$$

is a T-algebra morphism.

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Proof. It suffices to prove the first part of the statement, for then the function $-\circ\widehat{h}$ is a T-algebra morphism because it coincides with the following composition of T-algebra morphisms:

$$[\widehat{T}X, \widehat{T}X] \hookrightarrow \widehat{T}X^{\widehat{T}X} \xrightarrow{\widehat{T}X^{\widehat{h}}} \widehat{T}X^{TM}.$$

Recall from Section 2.5 that $\widehat{T}X$ is the cofiltered limit of finite sets Y_i which carry T-algebra structures $\alpha_i\colon TY_i \to Y_i$. We have the following isomorphisms in the category of sets:

$$\begin{aligned}[\widehat{T}X, \widehat{T}X] &\cong [\widehat{T}X, \lim_i Y_i] \\ &\cong \lim_i [\widehat{T}X, Y_i] \\ &\cong \lim_i [\lim_j Y_j, Y_i] \\ &\cong \lim_i \operatorname{colim}_j\, [Y_j, Y_i].\end{aligned}$$

The second isomorphism follows from the fact that the functor $[\widehat{T}X, -]\colon \mathsf{BStone} \to \mathsf{Set}$ is representable, hence it preserves all limits (Mac Lane, 1998, Theorem V.4.1). For the last isomorphism, we have used the fact that the Y_i are finite spaces, and consequently finitely copresentable (i.e. finitely presentable when regarded as objects of $\mathsf{BStone}^{\mathrm{op}}$). Therefore, the functors $[-, Y_i]\colon \mathsf{BStone} \to \mathsf{Set}$ turn cofiltered limits to filtered colimits.

The sets Y_i carry T-algebra structures and so do the sets $[Y_j, Y_i] \cong Y_i^{Y_j}$ with respect to pointwise operations. Since T is finitary, the forgetful functor $\mathsf{Set}^T \to \mathsf{Set}$ preserves and reflects both filtered colimits and limits (see, e.g. Borceux, 1994, Propositions 3.4.1–3.4.2). Whence, $[\widehat{T}X, \widehat{T}X]$ carries a T-algebra structure. We claim that, with respect to this T-algebra structure, $[\widehat{T}X, \widehat{T}X]$ is a subalgebra of the power algebra $\widehat{T}X^{\widehat{T}X}$.

For each $x \in \widehat{T}X$, write $ev_x\colon [\widehat{T}X, \widehat{T}X] \to \widehat{T}X$ for the function sending f to $f(x)$. By Remark 3.2.(i), the natural inclusion

$$[\widehat{T}X, \widehat{T}X] \hookrightarrow \widehat{T}X^{\widehat{T}X}$$

is a T-algebra morphism if, and only if, each ev_x is a T-algebra morphism. Write $\{\pi_i\colon \widehat{T}X \to Y_i \mid i \in I\}$ for the cone of continuous functions defining $\widehat{T}X$ as the cofiltered limit of finite sets Y_i which carry T-algebra structures. It is not difficult to see that each π_i is a T-algebra morphism; for a proof, see Reggio (2020, Proposition 2.10). It suffices to show that each composition $\pi_i \circ ev_x\colon [\widehat{T}X, \widehat{T}X] \to Y_i$ is a T-algebra morphism, for then ev_x will coincide with the unique T-algebra morphism induced by the universal property of $\widehat{T}X$. For any $j \in I$, denote by $\gamma_j\colon \widehat{T}X^{Y_j} \to Y_i$ the composite

$$\widehat{T}X^{Y_j} \xrightarrow{\pi_i \circ -} Y_i^{Y_j} \xrightarrow{ev_{\pi_j(x)}} Y_i.$$

The map $ev_{\pi_j(x)}$ is a product projection, hence a T-algebra morphism by Remark 3.2.(i). Moreover, $\pi_i \circ -$ is a T-algebra morphism by Remark 3.2.(ii). It follows that each γ_j is a T-algebra morphism. Upon recalling that $[\widehat{T}X, \widehat{T}X] \cong \operatorname{colim}_j\, [Y_j, \widehat{T}X]$ in Set^T, it is not difficult to see that $\pi_i \circ ev_x\colon [\widehat{T}X, \widehat{T}X] \to Y_i$ is the unique T-algebra morphism induced by the cocone $\{\gamma_j\colon [Y_j, \widehat{T}X] \to Y_i \mid j \in I\}$, thus concluding the proof. □

Exploiting the previous observations we can prove the main result of this section:

Theorem 3.5. *Any finitary commutative* Set*-monad* T *can be extended to a monad on* BiM *mapping* (X, M) *to* $(\widehat{T}X, TM)$.

Proof. We first give the definition of the monad on an object (X, M, h, ρ, λ). We will show that this is mapped to a BiM $(\widehat{T}X, TM, \widehat{h}, \widehat{\rho}, \widehat{\lambda})$, where $\widehat{h} = \tau_X \circ Th$ (cf. equation (11)), and $\widehat{\rho}$ and

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$\widehat{\lambda}$ are defined as follows. Recall that $[X, X]$ and $[\widehat{T}X, \widehat{T}X]$ denote the sets of continuous endofunctions on X and $\widehat{T}X$, respectively. To define $\widehat{\rho}$, consider the composite of the following two maps, where $\widehat{T}_{X,X}$ is given by the application of the functor $\widehat{T}$ to a continuous function in $[X, X]$:

$$M \xrightarrow{\ \rho\ } [X, X] \xrightarrow{\ \widehat{T}_{X,X}\ } [\widehat{T}X, \widehat{T}X]. \tag{12}$$

By Proposition 3.4 we know that $[\widehat{T}X, \widehat{T}X]$ is a T-algebra, hence the map in (12) admits a unique extension to an algebra morphism $\widehat{\rho}\colon TM \to [\widehat{T}X, \widehat{T}X]$. The function $\widehat{\lambda}$ is defined similarly, as the unique T-algebra morphism extending $\widehat{T}_{X,X} \circ \lambda$.

In order to prove that $(\widehat{T}X, TM, \widehat{h}, \widehat{\rho}, \widehat{\lambda})$ is a BiM, it remains to prove that the functions $\widehat{h}$, $\widehat{\rho}$ and $\widehat{\lambda}$ make the diagrams in Definition 2.2 commute. Equivalently, by virtue of Remark 2.5, that the next square and the analogous one (with $\widehat{\rho}$ replaced by $\widehat{\lambda}$, and $\widehat{r}$ by $\widehat{l}$) commute,

$$\begin{array}{ccc} [\widehat{T}X, \widehat{T}X] & \xrightarrow{\ -\circ\widehat{h}\ } & \widehat{T}X^{TM} \\ {\scriptstyle \widehat{\rho}}\uparrow & & \uparrow{\scriptstyle \widehat{h}\circ -} \\ TM & \xrightarrow{\ \widehat{r}\ } & TM^{TM} \end{array} \tag{13}$$

where $\widehat{r}$ and $\widehat{l}$ denote the right and left action, respectively, of TM on itself. To this end, notice that the following diagram commutes.

$$\begin{array}{ccccc} [\widehat{T}X, \widehat{T}X] & & \xrightarrow{\ \quad -\circ\widehat{h}\quad\ } & & \widehat{T}X^{TM} \\ {\scriptstyle \widehat{T}_{X,X}}\uparrow & & & \nearrow{\scriptstyle \tau_X\circ T-} & \\ [X, X] & \xrightarrow{\ -\circ h\ } & X^{M} & & \uparrow{\scriptstyle \widehat{h}\circ -} \\ {\scriptstyle \rho}\uparrow & & \uparrow{\scriptstyle h\circ -} & & \\ M & \xrightarrow{\ r\ } & M^{M} & \xrightarrow{\ T_{M,M}\ } & TM^{TM} \end{array} \tag{14}$$

For the upper leftmost trapezoid, recalling that $\widehat{h} = \tau_X \circ Th$, we must prove that for all $f \in [X, X]$ we have

$$\tau_X \circ T(f \circ h) = \widehat{T}f \circ \tau_X \circ Th.$$

In turn, this follows from the fact that $\tau_X \circ Tf = \widehat{T}f \circ \tau_X$ by naturality of τ. The lower rightmost trapezoid commutes by the very definition of $\widehat{h}$, whereas the inner square is a reformulation of the left commuting square in (3), cf. Remark 2.5.

We derive the commutativity of (13) using the universal property of the free T-algebra on M and by observing that (a) in the outer square in (14), the right vertical and the top horizontal arrows are morphisms of T-algebras by Lemma 3.3 and Proposition 3.4, respectively; (b) the map $\widehat{\rho}$ was defined as the unique extension of $\widehat{T}_{X,X} \circ \rho$ to the free algebra TM; (c) the map $\widehat{r}$ is the unique algebra morphism extending $T_{M,M} \circ r$ to TM. To settle item (c), notice that it is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc} TM \times M & \xrightarrow{\ \mathsf{id}_{TM}\times\eta_M\ } & TM \times TM \\ & & \downarrow{\scriptstyle \otimes} \\ {\scriptstyle \mathsf{id}_{TM}\times r}\downarrow & & T(M \times M) \\ & & \downarrow{\scriptstyle T(\cdot)} \\ TM \times M^{M} & \xrightarrow{\ \varepsilon\ } & TM \end{array} \tag{15}$$

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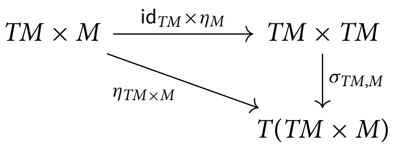
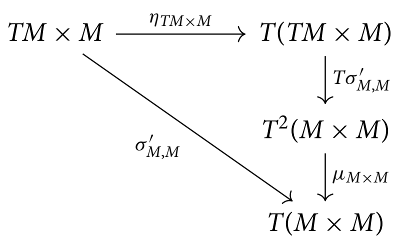
where $\otimes$ denotes either of the two compositions in diagram (9), $\varepsilon(s,f) = T_{M,M}(f)(s)$ for every $(s,f) \in TM \times M^M$, and $\cdot : M \times M \to M$ is the monoid operation of M. Now, observe that the identity

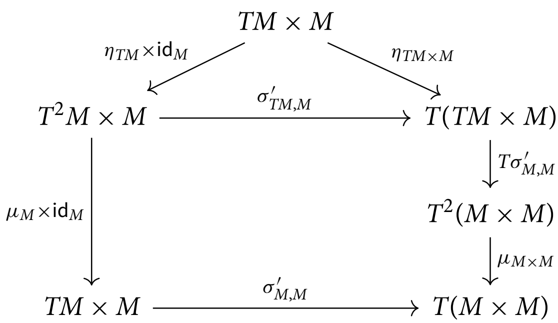
$$\otimes \circ (\mathsf{id}_{TM} \times \eta_M) = \sigma'_{M,M}, \tag{16}$$

where σ' is the costrength of T, holds provided the following two diagrams commute.

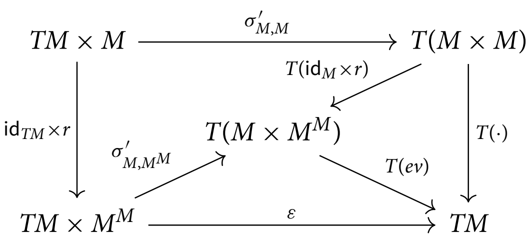




The left-hand triangle commutes by the leftmost diagram in (7). To see that the right-hand triangle commutes, since $(\mu_M \times \mathsf{id}_M) \circ (\eta_{TM} \times \mathsf{id}_M) = \mathsf{id}_{TM\times M}$, it suffices to show that the following diagram commutes.



In turn, the top triangle commutes by the leftmost diagram in (8), while the lower square commutes by the rightmost diagram in (8). Therefore, by equation (16), the commutativity of diagram (15) is equivalent to the commutativity of the outer square below



where $ev\colon M \times M^M \to M$ sends $(m,f) \in M \times M^M$ to $f(m)$. The upper leftmost triangle commutes by naturality of σ', while the rightmost triangle and the lower one are easily seen to be commutative. Hence, item c) above is satisfied and diagram (13) commutes, as was to be proved. Reasoning in a similar manner for the left action, one can see that $(\widehat{T}X, TM, \widehat{h}, \widehat{\rho}, \widehat{\lambda})$ is indeed a BiM.

It is now a matter of straightforward computations to check that the assignment $(X, M) \mapsto (\widehat{T}X, TM)$ yields a monad on the category of BiMs, with unit $(\widehat{\eta}_X, \eta_M)\colon (X, M) \to (\widehat{T}X, TM)$ and multiplication $(\widehat{\mu}_X, \mu_M)\colon (\widehat{T^2}X, T^2M) \to (\widehat{T}X, TM)$. $\square$

Remark 3.6. Assume that the monad T is not commutative and we attempt to use, in the proof of Theorem 3.5, the monoid multiplication on TM given by $\otimes$. All is fine for the right action and

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indeed the right action $\widehat{r}$ of TM on itself is the unique extension of $T_{M,M} \circ r$. However, this is not the case for the left action. Symmetrically, if we chose the multiplication of TM stemming from $\otimes'$, then the left action $\widehat{l}$ would be the extension of the map $T_{M,M} \circ l$, but this property would fail for the right action.

4. Extending the Free Semimodule Monad to BiMs

In Theorem 3.5, we showed how to lift any finitary commutative monad on Set to a monad on BiM. The purpose of the present section is then twofold. On the one hand we provide an example of a family of Set-monads to which this result applies, and on the other hand, we give explicit descriptions of the various objects, maps and actions of the associated monads on BiM. This will be essential for our further work on recognisers in the following sections.

Given a semiring S, recall from Example 2.8 the free S-semimodule monad $\mathcal{S}$ on Set. Notice that $\mathcal{S}$ is a commutative monad if, and only if, S is a commutative semiring, that is the multiplication $\cdot$ is a commutative operation. Indeed, for a monoid M, the two monoid operations one can define on $\mathcal{S}M$ are given as follows. If $f, f' \in \mathcal{S}M$, then one can define $ff'(x)$ either by

$$\sum_{mm'=x} f(m) \cdot f'(m') \quad \text{or} \quad \sum_{m'm=x} f'(m') \cdot f(m),$$

and the two coincide precisely when the semiring is commutative. For this reason, for the rest of the paper we will only consider commutative semirings S. We also consider the associated Set-monad $\mathcal{S}$, along with the profinite monad $\widehat{\mathcal{S}}$ on BStone (cf. Section 2.5).

Throughout this section, we fix an arbitrary finite and commutative semiring S. Let X be a Boolean space, and denote by B its dual algebra. Next, we provide a concrete description of the Boolean space $\widehat{\mathcal{S}}X$ in terms of *measures* on X. For more details and for the proofs of several facts mentioned in this section, the interested reader is referred to Reggio (2020).

Definition 4.1. *Let X be a Boolean space and B the dual algebra. An S*-valued measure *(or just a* measure *when the semiring is clear from the context) on X is a function $\mu\colon B \to S$ which is finitely additive, that is*

(1) $\mu(0) = 0$, and
(2) $\mu(K \vee L) = \mu(K) + \mu(L)$ whenever $K, L \in B$ are disjoint.

We remark that in item 1 the first 0 is the bottom of the Boolean algebra, while the second 0 is in S. Also, one can express item 2 without reference to disjointness:

2'. $\mu(K \vee L) + \mu(K \wedge L) = \mu(K) + \mu(L)$ for all $K, L \in B$.

Note that our notion of measure is not standard, as we only require finite additivity. Also, the measure is only defined on the clopens of the space X. Finally, it takes values in a (finite and commutative) semiring.

Example 4.2. If the semiring S is idempotent (i.e. $s + s = s$ for all $s \in S$), hence a semilattice, the measures $\mu\colon B \to S$ are precisely the homomorphisms of join semilattices which preserve 0. Equivalently, the monoid morphisms $B \to S$ where $(B, \vee, 0)$ is viewed as the monoid part of a Boolean ring. This is the case, for instance, when $S = \mathbf{2}$ is the Boolean semiring. In general, there may be measures that are not monoid morphisms.

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Notation 4.3. Let X be a set and $f\colon X \to S$ a function. If $Y \subseteq X$ is a subset such that the sum $\sum_{x\in Y} f(x)$ exists in S, then we write

$$\int_Y f := \sum_{x\in Y} f(x).$$

If $B \subseteq \mathcal{P}X$, and $\int_Y f$ exists for each $Y \in B$, then $\int f\colon B \to S$ denotes the function taking Y to $\int_Y f$. Note that, whenever $f\colon X \to S$ is finitely supported, the function $\int f\colon \mathcal{P}X \to S$ is well defined.

Suppose X is a Boolean space and B its dual algebra. Using the fact that $\mathcal{S}X$ is dense in $\widehat{\mathcal{S}}X$, it is not difficult to see that the Boolean algebra $\widehat{B}$ dual to $\widehat{\mathcal{S}}X$ is isomorphic to the subalgebra of $\mathcal{P}(\mathcal{S}X)$ generated by the elements of the form

$$[L, k] := \{f \in \mathcal{S}X \mid \int_L f = k\},$$

for $L \in B$ and $k \in S$. For a proof of this fact, see Reggio (2020, Lemma 4.2). Regarding the elements of $\widehat{\mathcal{S}}X$ as Boolean algebra homomorphisms $\varphi\colon \widehat{B} \to \mathbf{2}$, we can define a function

$$\widehat{\mathcal{S}}X \longrightarrow \{\mu\colon B \to S \mid \mu \text{ is a measure on } X\}, \quad \varphi \mapsto \mu_\varphi \tag{17}$$

where μ_φ is the measure sending $L \in B$ to the unique $k \in S$ such that $\varphi[L, k] = 1$. Such a k exists and is unique because the sets $[L, k']$, where L is fixed and k' varies in S, form a finite partition of $\mathcal{S}X$. In turn, the set of all measures on X is equipped with a natural topology, generated by the sets of the form

$$\overline{[L, k]} = \{\mu\colon B \to S \mid \mu \text{ is a measure on } X,\ \mu(L) = k\} \tag{18}$$

for $L \in B$ and $k \in S$ (the notation $\overline{[L, k]}$ is justified by Proposition 4.6 below). With respect to this topology, the space $\widehat{\mathcal{S}}X$ admits the following measure-theoretic characterisation.

Theorem 4.4. (Cf. Reggio, 2020, Theorem 4.3). *Let S be a finite and commutative semiring. For any Boolean space X, the map in (17) yields a homeomorphism between $\widehat{\mathcal{S}}X$ and the space of S-valued measures on X.*

Example 4.5. In view of Examples 2.9 and 4.2, the previous theorem entails that, for any Boolean space X with dual algebra B, the Vietoris space $\mathcal{V}X$ is homeomorphic to the space of all join semilattice homomorphisms $B \to \mathbf{2}$ that preserve 0, with the topology defined in (18). An explicit homeomorphism is given by sending a closed subset $C \subseteq X$ to the join semilattice homomorphism $B \to \mathbf{2}$, which maps $a \in B$ to 1 if $\widehat{a} \cap C \neq \emptyset$, and to 0 otherwise.

The previous result allows for a concrete representation of the map τ_X in (4) which, in turn, yields the following concrete instantiation of Lemma 2.12 (cf. also Remark 2.13).

Proposition 4.6. *If X is a Boolean space, then the function*

$$\tau_X\colon \mathcal{S}X \to \widehat{\mathcal{S}}X,\ f \mapsto \int f$$

embeds $\mathcal{S}X$ in $\widehat{\mathcal{S}}X$ as a dense subspace. Moreover $\overline{[L, k]}$, as defined in (18), is the topological closure of $[L, k]$ whenever L is a clopen of X and $k \in S$.

As follows by the general results in Sections 2.5 and 3, respectively, $\widehat{\mathcal{S}}X$ is a module over the semiring S and it is a Boolean space with an internal monoid if X is. Here, we identify the concrete nature of this structure relative to the incarnation of $\widehat{\mathcal{S}}X$ as the space of measures on X. We state these as lemmas and, indeed, one can prove them directly. However, the results in this section are just special cases of the more general results in Sections 2.5 and 3.

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Lemma 4.7. *Let X be a Boolean space and let $\mu, \nu \in \widehat{\mathcal{S}}X$. Then*

$$\mu + \nu \colon K \mapsto \mu(K) + \nu(K)$$

is again a measure on X and the ensuing binary operation on $\widehat{\mathcal{S}}X$ is continuous. Further, for any $k \in S$,

$$k\mu \colon K \mapsto k \cdot \mu(K)$$

is again a measure on X and the ensuing unary operation on $\widehat{\mathcal{S}}X$ is continuous.

This accounts for the S-semimodule structure on $\widehat{\mathcal{S}}X$. Now, assume that X is not just a Boolean space, but a BiM. To improve readability, we assume $h\colon M \to X$ is injective and identify M with its image. First, we observe that $\mathcal{S}M$ sits as a dense subspace of $\widehat{\mathcal{S}}X$ by composing the map $\mathcal{S}h\colon \mathcal{S}M \to \mathcal{S}X$ with the integration map of Proposition 4.6. This is the concrete incarnation, in the case of the monad $\mathcal{S}$, of Lemma 2.12.

Lemma 4.8. *Let (X, M) be a Boolean space with an internal monoid. Then*

$$\mathcal{S}M \to \widehat{\mathcal{S}}X,\ f \mapsto \int f$$

is the map $\widehat{h}$ from (11) transporting $\mathcal{S}M$ into a dense subspace of $\widehat{\mathcal{S}}X$.

We remark that, since we assumed h is injective, so is the map $\widehat{h}$ in the previous lemma (cf. Remark 2.13). Now, to exhibit the BiM structure of $\widehat{\mathcal{S}}X$, we start by identifying the actions of M on $\widehat{\mathcal{S}}X$.

Lemma 4.9. *Let (X, M) be a Boolean space with an internal monoid. Further, let $\mu \in \widehat{\mathcal{S}}X$ and $m \in M$. Then*

$$m\mu \colon K \mapsto \mu(m^{-1}K),$$

where $m^{-1}K = \{x \in X \mid mx \in K\}$ whenever $K \subseteq X$ is clopen, is again a measure on X. This defines a left action of M on $\widehat{\mathcal{S}}X$ with continuous components. Similarly,

$$\mu m \colon K \mapsto \mu(Km^{-1})$$

defines a right action of M on $\widehat{\mathcal{S}}X$ with continuous components, and these actions are compatible in the sense that $(m\mu)n = m(\mu n)$.

Using the S-semimodule structure of $\widehat{\mathcal{S}}X$ (see Lemma 2.10), along with the biaction of M on $\widehat{\mathcal{S}}X$ provided by the previous lemma, it is easy to obtain the biaction of $\mathcal{S}M$ on $\widehat{\mathcal{S}}X$. The following can be regarded as the specific incarnation of Theorem 3.5.

Proposition 4.10. *Let (X, M) be a Boolean space with an internal monoid. The map*

$$\mathcal{S}M \times \widehat{\mathcal{S}}X \to \widehat{\mathcal{S}}X,\ (f, \mu) \mapsto f\mu := \sum_{m \in M} f(m) \cdot m\mu$$

is a left action of $\mathcal{S}M$ on $\widehat{\mathcal{S}}X$ with continuous components. A right action with continuous components may be defined similarly. These two actions are compatible and provide the BiM *structure on $(\widehat{\mathcal{S}}X, \mathcal{S}M)$.*

Finally, we consider a restriction of the above action of $\mathcal{S}M$ on $\widehat{\mathcal{S}}X$, which we will need for the construction of the space $\Diamond_S X$ in Section 5. This is given by precomposing with the unit of the monad $\widehat{\mathcal{S}}$:

$$\widehat{\eta}_X \colon X \to \widehat{\mathcal{S}}X,\ x \mapsto \mu_x$$

where $\mu_x(K) = 1$ if $x \in K$, and $\mu_x(K) = 0$ otherwise. That is, $\mu_x = \int \chi_x$ where χ_x is the characteristic function of $\{x\}$ into S. It is immediate that this map embeds X as a (closed) subspace of $\widehat{\mathcal{S}}X$. Thus, we obtain an 'action'

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$$\mathcal{S}M \times X \to \widehat{\mathcal{S}}X, \ (f, x) \mapsto f\mu_x.$$

Next, we observe that this 'action' factors through the map $\mathcal{S}X \to \widehat{\mathcal{S}}X$ defined in Proposition 4.6.

Lemma 4.11. *Let (X, M) be a Boolean space with an internal monoid. Consider the map*

$$\mathcal{S}M \times X \to \mathcal{S}X, \ (f, x) \mapsto fx,$$

where $fx(y) := \sum_{mx=y} f(m)$. Then, we have

$$f\mu_x = \int fx.$$

Furthermore, for each $f \in \mathcal{S}M$, the assignment $x \mapsto \int fx$ is continuous.

Example 4.12. We illustrate the actions in Lemma 4.9 and Proposition 4.10 in the case of the Boolean semiring $S = \mathbf{2}$. If (X, M, h, ρ, λ) is any BiM, then $\widehat{\mathcal{S}}X$ is the Vietoris space $\mathcal{V}X$ and $\mathcal{S}M$ is the monoid $\mathcal{P}_f M$ (with monoid operation $\cup$). For every $m \in M$, the component $\lambda(m) \colon X \to X$ of the left action of M on X yields a continuous map

$$\mathcal{V}(\lambda(m)) \colon \mathcal{V}X \to \mathcal{V}X, \ \ C \mapsto \lambda(m)(C),$$

which is the component at m of the left action of M on $\mathcal{V}X$ given in Lemma 4.9. Similarly, the components of the right action of M on $\mathcal{V}X$ are obtained by applying the functor $\mathcal{V}$ to the components $\rho(m) \colon X \to X$ of the right action of M on X. The left action of $\mathcal{P}_f M$ on $\mathcal{V}X$ described in Proposition 4.10 is

$$\mathcal{P}_f M \times \mathcal{V}X \to \mathcal{V}X, \ \ (F, C) \mapsto \bigcup_{m \in F} \lambda(m)(C).$$

(The right action of $\mathcal{P}_f M$ on $\mathcal{V}X$ is defined in a similar way.) Upon restricting the latter action to X, we obtain the map $\mathcal{P}_f M \times X \to \mathcal{V}X$, $(F, x) \mapsto \{\lambda(m)(x) \mid m \in F\}$ described in Lemma 4.11.

5. Recognisers for Operations Given by *S*-Valued Transductions

In this section we will see how we can use the extension of a Set-monad T to BiMs, obtained in Section 3, to generate recognisers for languages obtained by applying an operation modelled by the monad T.

It is by now a standard result in the theory of formal languages that many operations on languages can be modelled using transductions, that is maps of the form $M \to \mathcal{P}N$ for two monoids M and N, see Pin and Sakarovitch (1982). The starting point of this work is the observation that the existential quantifier can also be modelled as a transduction, as we will see in Section 5.2. Furthermore, modular quantifiers $\exists_{p \bmod q}$ of modulus q fit into the same pattern. The only difference is that, instead of using transductions of the form $M \to \mathcal{P}N$, one needs to replace the powerset $\mathcal{P}N$ with the free $\mathbb{Z}_q$-semimodule over N. More generally, we are interested in operations that can be modelled as maps $M \to \mathcal{S}N$ with $\mathcal{S}$ denoting as before the free S-semimodule monad. In category theory these maps are known as Kleisli maps for $\mathcal{S}$, the morphisms in the so-called Kleisli category of $\mathcal{S}$.

We start Section 5.1 by briefly recalling the definition of the Kleisli maps for a monad. Then we present the blueprint of our approach, using an additional assumption on the T-Kleisli map under consideration (namely that it is a monoid morphism), and in Section 5.2 we instantiate T to the free S-semimodule monads for commutative semirings S and adapt the general theory developed previously.

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5.1 Recognising operations modelled by a monad T

Consider a monad (T, η, μ) on a category $\mathcal{C}$. The Kleisli category $\mathsf{Kl}(T)$ for T is equivalent to the category of *free* T-algebras and has played a crucial rôle in program semantics for modelling functions with side effects. Formally, the objects of $\mathsf{Kl}(T)$ are the objects in the underlying category $\mathcal{C}$, and a morphism $X \to Y$ in $\mathsf{Kl}(T)$ (called a *T-Kleisli map*) is a morphism $X \to TY$ in $\mathcal{C}$. One can think of an object X in $\mathsf{Kl}(T)$ as the generator of the free algebra TX. Notice that morphisms $X \to Y$ in $\mathsf{Kl}(T)$ are in one-to-one correspondence with T-algebra morphisms $TX \to TY$ between the corresponding free algebras.

Hereafter, we assume T is an arbitrary commutative and finitary monad on Set, and let A, B be finite sets. We start by observing that a Kleisli map $R\colon A^* \to T(B^*)$ could be used to transform languages in the alphabet B into languages in the alphabet A. Assume that $L = \phi^{-1}(P)$ for some monoid morphism $\phi\colon B^* \to M$ and some $P \subseteq M$. We consider the function

$$A^* \xrightarrow{R} T(B^*) \xrightarrow{T\phi} TM.$$

Since T is a commutative monad, we know that it lifts to the category of monoids and thus we can see $T\phi$ as a monoid morphism. If R is also a monoid morphism, and we will assume this only in this subsection, then so is $T\phi \circ R$, and it could be used for language recognition in the standard way. Assuming that we have a way of turning the recognising sets in M into recognising sets in TM, that is that we have a predicate transformer $\mathcal{P}M \to \mathcal{P}TM$ mapping P to $\widetilde{P}$, we obtain a language $\widetilde{L}$ in A^* as the preimage of $\widetilde{P}$ under the morphism $T\phi \circ R$.

Remark 5.1. In the running example of the next subsection we will need maps R that are not monoid morphisms, and in that setting we will have to use a matrix representation of the transduction instead. Nevertheless, the techniques used in the next subsection can be seen as an adaptation of the theory developed here for the case when R is indeed a monoid morphism.

In this work we go beyond regular languages, so we are interested in languages recognised by a BiM morphism as follows:

$$\begin{array}{ccc} \beta(B^*) & \xrightarrow{\tilde{\phi}} & X \\ \uparrow & & \uparrow{\scriptstyle h} \\ B^* & \xrightarrow{\phi} & M \end{array} \tag{19}$$

We recall that to improve readability, and since $\tilde{\phi}$ is uniquely determined by its restriction to B^*, we sometimes denote such a morphism of BiMs simply by ϕ, instead of $(\tilde{\phi}, \phi)$.

By Theorem 3.5, we know that $(\widehat{T}X, TM)$ is a BiM, and in what follows we use it for recognising A-languages by constructing another BiM morphism $(\beta(A^*), A^*) \to (\widehat{T}X, TM)$ as in Lemma 5.2 below. To this end, we need a way of lifting the Kleisli map $R\colon A^* \to T(B^*)$ to a Kleisli map for the monad $\widehat{T}$. This can be done in a natural way using a natural transformation

$$\tau^{\#}\colon \beta T \to \widehat{T}\beta$$

obtained from the natural transformation $\tau_X\colon T|X| \to |\widehat{T}X|$ defined in (4) using the unit ι and the counit ε of the adjunction $\beta \dashv |-|$. Explicitly, $\tau^{\#}$ is obtained as the composite

$$\beta T \xrightarrow{\beta T \iota} \beta T|-|\beta \xrightarrow{\beta\tau\beta} \beta|-|\widehat{T}\beta \xrightarrow{\varepsilon \widehat{T}\beta} \widehat{T}\beta\,. \tag{20}$$

($\tau^{\#}$ is the *mate* of τ; this is a rather standard construction in category theory, see for example Street, 1972, Theorem 9.) In down-to-earth terms, the component of $\tau^{\#}$ at a set Y is the free extension of the composite

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$$TY \xrightarrow{T\iota_Y} T|\beta Y| \xrightarrow{\tau_{\beta Y}} |\widehat{T}\beta Y|.$$

It follows that the natural transformation $\tau^{\#}\colon \beta T \to \widehat{T}\beta$ also behaves well with respect to the units and multiplications of the monads. That is, in the terminology of Street (1972), the pair $(\beta, \tau^{\#})$ is a *monad opfunctor*. This, in turn, implies that β can be lifted to a functor $\widehat{\beta}$ between the Kleisli categories making the next square commute, where the vertical functors are the free functors from the base to the Kleisli categories.

$$\begin{array}{ccc} \mathsf{Kl}(T) & \xrightarrow{\ \widehat{\beta}\ } & \mathsf{Kl}(\widehat{T}) \\ \uparrow & & \uparrow \\ \mathsf{Set} & \xrightarrow{\ \beta\ } & \mathsf{BStone} \end{array}$$

The functor $\widehat{\beta}$ maps the Kleisli map $R\colon A^* \to T(B^*)$ to the Kleisli map $\widehat{R}\colon \beta(A^*) \to \widehat{T}\beta(B^*)$ given by

$$\widehat{R}\colon \beta(A^*) \xrightarrow{\beta R} \beta T(B^*) \xrightarrow{\tau^{\#}} \widehat{T}\beta(B^*)\,. \tag{21}$$

Lemma 5.2. *Assume $R\colon A^* \to T(B^*)$ is a monoid morphism. If the pair $(\tilde{\phi}, \phi)$ from (19) is a morphism of* BiMs*, then so is the pair $(\widehat{T}\tilde{\phi} \circ \widehat{R}, T\phi \circ R)$ described in the next diagram.*

$$\begin{array}{ccccc} \beta(A^*) & \xrightarrow{\ \widehat{R}\ } & \widehat{T}\beta(B^*) & \xrightarrow{\ \widehat{T}\tilde{\phi}\ } & \widehat{T}X \\ {\scriptstyle \iota}\uparrow & & & & \uparrow{\scriptstyle \tau_X \circ Th} \\ A^* & \xrightarrow{\ R\ } & T(B^*) & \xrightarrow{\ T\phi\ } & TM \end{array}$$

Proof. In the statement of the lemma we have omitted writing the forgetful functor $|-|$ on the top line of the diagram. We will need it nevertheless in the proof. Using the definition of $\widehat{R}$, we need to show that the next diagram commutes:

$$\begin{array}{ccccccc} |\beta(A^*)| & \xrightarrow{|\beta R|} & |\beta T(B^*)| & \xrightarrow{|\tau^{\#}|} & |\widehat{T}\beta(B^*)| & \xrightarrow{|\widehat{T}\tilde{\phi}|} & |\widehat{T}X| \\ & & & & {\scriptstyle \tau\beta}\uparrow & & \uparrow{\scriptstyle \tau} \\ {\scriptstyle \iota}\uparrow & & \uparrow{\scriptstyle \iota T} & & T|\beta(B^*)| & \xrightarrow{T|\tilde{\phi}|} & T|X| \\ & & & \overset{T\iota}{\nearrow} & & \overset{Th}{\nearrow} & \\ A^* & \xrightarrow{\ R\ } & T(B^*) & \xrightarrow{\ T\phi\ } & TM & & \end{array}$$

The two rectangles in the diagram above commute by naturality of ι, respectively τ, and the bottom right rhombus commutes because ϕ is a morphism of BiMs. To prove that the middle trapezoid is commutative, we just have to recall how the transformation $\tau^{\#}$ is defined, see (20). In a 2-categorical terminology, this is a simple exercise involving the mates τ and $\tau^{\#}$:

$$\begin{array}{ccccccc} |-|\beta T & \xrightarrow{|-|\beta T\iota} & |-|\beta T|-|\beta & \xrightarrow{|-|\beta\tau\beta} & |-|\beta|-|\widehat{T}\beta & \xrightarrow{|-|\varepsilon\widehat{T}\beta} & |-|\widehat{T}\beta \\ {\scriptstyle \iota T}\uparrow & & \uparrow{\scriptstyle \iota T|-|\beta} & & \uparrow{\scriptstyle \iota|-|\widehat{T}\beta} & \overset{\mathsf{id}}{\nearrow} & \\ T & \xrightarrow{\ T\iota\ } & T|-|\beta & \xrightarrow{\ \tau\beta\ } & |-|\widehat{T}\beta & & \end{array}$$

The squares commute by naturality of ι, whilst the triangle commutes because $|-|\varepsilon \circ \iota|-| = \mathsf{id}$. □

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5.2 Recognising quantified languages via S-transductions

Here we show how to construct BiMs recognising quantified languages. We point out that the content of this subsection could be easily adapted to arbitrary Kleisli maps for the monads of the form $\widehat{\mathcal{S}}$, for commutative semirings S. We start with a language L in the extended alphabet $(A \times 2)^*$ recognised by a BiM morphism as in the following diagram.

$$\begin{array}{ccc} \beta((A\times 2)^*) & \xrightarrow{\tilde{\phi}} & X \\ \uparrow & & \uparrow{\scriptstyle h} \\ (A\times 2)^* & \xrightarrow{\phi} & M \end{array}$$

In other words, there exists a clopen C in X such that $L = \phi^{-1}(C \cap M)$. The aim of this subsection is to construct recognisers for the quantified languages $L_\exists$ and $L_{\exists_{p \bmod q}}$, as defined in Section 2.1. To this end, using the formal sum notation in the definition of the monad $\mathcal{S}$, we consider the map

$$R\colon A^* \to \mathcal{S}((A \times 2)^*), \quad w \mapsto \sum_{i=1}^{|w|} 1_S \cdot w^{(i)} \tag{22}$$

where $w^{(i)}$ is the word in $(A \times 2)^*$ with the same shape as w but with the letter in position i marked (see Section 2.1). If S is the Boolean semiring $\mathbf{2}$, then R simply associates with each word w the set of all words in $(A \times 2)^*$ with the same shape as w and with exactly one marked letter. The framework developed in the previous subsection does not immediately apply, because R is not a monoid morphism. So, the first step we have to take is to obtain a monoid morphism from R, which will then be used to construct BiM recognisers for quantified languages.

Upon viewing R as an S-transduction (see Sakarovitch, 2009), we observe that it is realised by the rational S-transducer $\mathcal{T}_R$, as shown in Figure 1, in which we have drawn transition maps only for a generic letter $a \in A$.

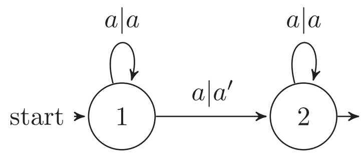

Figure 1. The S-transducer $\mathcal{T}_R$ realising R. All the transitions have weights 1_S, and thus the transducer outputs value 1_S for all pairs of the form $(w, w^{(i)})$, with $w \in A^*$ and $1 \le i \le |w|$.

This transducer provides the following representation of R in terms of a monoid morphism

$$\mathsf{R}\colon A^* \to \mathcal{M}_2(\mathcal{S}((A \times 2)^*)), \tag{23}$$

where $\mathcal{M}_n(\mathcal{S}((A \times 2)^*))$ denotes the set of $n \times n$-matrices over the semimodule $\mathcal{S}((A \times 2)^*)$. For a word $w \in A^*$, the matrix $\mathsf{R}(w)$ has at position (i, j) the formal sum of output words obtained from the transducer $\mathcal{T}_R$ by going from state i to state j while reading input word w. That is, R is given by

$$w \mapsto \begin{pmatrix} 1_S \cdot w^0 & \sum_i 1_S \cdot w^{(i)} \\ 0_S & 1_S \cdot w^0 \end{pmatrix}.$$

The next two examples provide the motivation for considering the particular transduction R in the first place.

Example 5.3. Assume S is the Boolean semiring $\mathbf{2}$. Then $\mathcal{S} = \mathcal{P}_f$ and $R(w) = \{w^{(i)} \mid 1 \le i \le |w|\}$. The language $L_\exists \subseteq A^*$ is recognised by the following composite monoid morphism, that will be denoted by $\phi_\exists$.

$$A^* \xrightarrow{\mathsf{R}} \mathcal{M}_2(\mathcal{P}_f((A \times 2)^*)) \xrightarrow{\mathcal{M}_2(\mathcal{P}_f\phi)} \mathcal{M}_2(\mathcal{P}_f M)$$

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Just observe that, if $L = \phi^{-1}(P)$ for some $P \subseteq M$, then $L_\exists = \phi_\exists^{-1}(\widetilde{P})$, where $\widetilde{P}$ is the set of matrices in $\mathcal{M}_2(\mathcal{P}_f M)$ such that the finite set in position $(1, 2)$ intersects P.

Example 5.4. Assume S is the semiring $\mathbb{Z}_q$. The language $L_{\exists_{p \bmod q}} \subseteq A^*$ is recognised by the following composite monoid morphism, that will be denoted by $\phi_{\exists_{p \bmod q}}$.

$$A^* \xrightarrow{\mathsf{R}} \mathcal{M}_2(\mathcal{S}((A \times 2)^*)) \xrightarrow{\mathcal{M}_2(\mathcal{S}\phi)} \mathcal{M}_2(\mathcal{S}M)$$

Indeed, if $L = \phi^{-1}(P)$ with $P \subseteq M$ then $L_{\exists_{p \bmod q}} = \phi^{-1}_{\exists_{p \bmod q}}(\widetilde{P})$, where $\widetilde{P}$ is the set of matrices in $\mathcal{M}_2(\mathcal{S}M)$ such that the finitely supported function $f\colon \mathbb{Z}_q \to M$ in position $(1, 2)$ has the property that $\int_P f = p$ in $\mathbb{Z}_q$.

In view of Theorem 3.5, we know that whenever (X, M) is a BiM, then so is $(\widehat{\mathcal{S}}X, \mathcal{S}M)$ with the actions of the internal monoid as in Proposition 4.10. Using this fact, we can prove the following lemma.

Lemma 5.5. *If* (X, M) *is a* BiM, *then so is*

$$(\mathcal{M}_n(\widehat{\mathcal{S}}X), \mathcal{M}_n(\mathcal{S}M))$$

for any integer $n \geq 1$.

Proof. The set $\mathcal{M}_n(\widehat{\mathcal{S}}X)$ is a Boolean space with respect to the product topology of $n \times n$ copies of $\widehat{\mathcal{S}}X$. The statement then follows easily upon defining the actions of the monoid $\mathcal{M}_n(\mathcal{S}M)$ on $\mathcal{M}_n(\widehat{\mathcal{S}}X)$ by using the actions of $\mathcal{S}M$ on $\widehat{\mathcal{S}}X$ via matrix multiplication, and the S-semimodule structure of $\widehat{\mathcal{S}}X$. For example, the left action of $(f_{ij})_{i,j} \in \mathcal{M}_n(\mathcal{S}M)$ on $(\mu_{ij})_{i,j} \in \mathcal{M}_n(\widehat{\mathcal{S}}X)$ yields a matrix of measures in $\widehat{\mathcal{S}}X$ having at position (i, j) the measure $\sum_{k=1}^n f_{ik}\mu_{kj}$. □

We next prove a result which entails that the monoid morphisms $\phi_\exists$ and $\phi_{\exists_{p \bmod q}}$ constructed in Examples 5.3 and 5.4 can be extended to BiM morphisms recognising $L_\exists$ and $L_{\exists_{p \bmod q}}$, respectively.

Lemma 5.6. *If the pair* $(\tilde{\phi}, \phi)$ *from* (19) *is a morphism of* BiMs *and* $\mathsf{R}\colon A^* \to \mathcal{M}_n(\mathcal{S}(B^*))$ *is a monoid morphism, then the pair* $(\mathcal{M}_n(\widehat{\mathcal{S}}\tilde{\phi}) \circ \widehat{\mathsf{R}}, \mathcal{M}_n(\mathcal{S}\phi) \circ \mathsf{R})$ *described in the next diagram is a* BiM *morphism,*

$$\begin{array}{ccccc}
\beta(A^*) & \xrightarrow{\widehat{\mathsf{R}}} & \mathcal{M}_n(\widehat{\mathcal{S}}\beta(B^*)) & \xrightarrow{\mathcal{M}_n(\widehat{\mathcal{S}}\tilde{\phi})} & \mathcal{M}_n(\widehat{\mathcal{S}}X) \\
\uparrow & & & & \uparrow{\scriptstyle \mathcal{M}_n(\tau_X \circ \mathcal{S}h)} \\
A^* & \xrightarrow{\mathsf{R}} & \mathcal{M}_n(\mathcal{S}(B^*)) & \xrightarrow{\mathcal{M}_n(\mathcal{S}\phi)} & \mathcal{M}_n(\mathcal{S}M)
\end{array}$$

where $\widehat{\mathsf{R}}$ *is the unique continuous extension of the following composite map:*

$$A^* \xrightarrow{\mathsf{R}} \mathcal{M}_n(\mathcal{S}(B^*)) \xrightarrow{\mathcal{M}_n(\iota)} \mathcal{M}_n(\beta\mathcal{S}(B^*)) \xrightarrow{\mathcal{M}_n(\tau^\#)} \mathcal{M}_n(\widehat{\mathcal{S}}\beta(B^*)).$$

Proof. This follows essentially by Lemma 5.2 by setting $T = \mathcal{S}$, along with the functoriality of $\mathcal{M}_n(-)$. Note that the aforementioned lemma applies to this setting because R is a monoid morphism. □

If we apply the previous lemma to the monoid morphism R in equation (23) we obtain the BiM $(\mathcal{M}_2(\widehat{\mathcal{S}}X), \mathcal{M}_2(\mathcal{S}M))$ which, when instantiated with the appropriate semiring S, recognises the quantified languages $L_\exists$ and $L_{\exists_{p \bmod q}}$.

For instance, suppose the semiring S is $\mathbb{Z}_q$. If L is recognised by a clopen $C \subseteq X$ then, upon recalling from (18) that subbasic clopens of $\widehat{\mathcal{S}}X$ are of the form $\overline{[K, k]}$ for K a clopen of X and

$k \in S$, one can easily prove that the quantified language $L_{\exists_{p \bmod q}}$ is recognised by the clopen subset of $\mathcal{M}_2(\widehat{\mathcal{S}X})$ given by the product $\widehat{\mathcal{S}X} \times \overline{[C, p]} \times \widehat{\mathcal{S}X} \times \widehat{\mathcal{S}X}$, where the elements of the clopen $\overline{[C, p]}$ should appear in position $(1, 2)$ in the matrix view of the space.

However, notice that the image of the morphism $\mathcal{M}_2(\mathcal{S}\tilde{\phi}) \circ \widehat{\mathsf{R}}$ is contained in the subspace of $\mathcal{M}_2(\widehat{\mathcal{S}X})$, which can be represented by the matrix

$$\begin{pmatrix} X & \widehat{\mathcal{S}X} \\ 0 & X \end{pmatrix}.$$

As a consequence, we can use for the same recognition purpose a smaller BiM, through which the morphism $\mathcal{M}_2(\mathcal{S}\tilde{\phi}) \circ \widehat{\mathsf{R}}$ factors. We denote this BiM morphism by

$$\Diamond_S \phi \colon (\beta(A^*), A^*) \to (\Diamond_S X, \Diamond_S M),$$

where

$$\Diamond_S X := \widehat{\mathcal{S}X} \times X \text{ and } \Diamond_S M := \mathcal{S}M \times M,$$

with monoid structure and biactions defined essentially by identifying the products above with upper triangular matrices, and then using the matrix multiplication and the concrete description of several monoid actions from Lemmas 4.9 and 4.11. Using the notations described in these lemmas, the left action of $\Diamond_S M$ on $\Diamond_S X$ can be described by

$$\begin{pmatrix} m & f \\ 0 & m \end{pmatrix} \begin{pmatrix} x & \mu \\ 0 & x \end{pmatrix} = \begin{pmatrix} mx & m\mu + \int fx \\ 0 & mx \end{pmatrix},$$

where $(f, m) \in \Diamond_S M$ and $(\mu, x) \in \Diamond_S X$. Recall from Section 2.1 that the language $\mathcal{Q}_k(L)$ in the alphabet A is obtained by quantifying the language $L \subseteq (A \times 2)^*$ with respect to the quantifier associated with a semiring S and an element $k \in S$. We summarise the preceding observations in the following theorem.

Theorem 5.7. *Let S be a commutative semiring, and $k \in S$. Suppose a language $L \subseteq (A \times 2)^*$ is recognised by the* BiM *morphism $\phi \colon (\beta((A \times 2)^*), (A \times 2)^*) \to (X, M)$. Then the quantified language $\mathcal{Q}_k(L) \subseteq A^*$ is recognised by the* BiM *morphism $\Diamond_S \phi \colon (\beta(A^*), A^*) \to (\Diamond_S X, \Diamond_S M)$.*

As an immediate consequence, taking $S = \mathbf{2}$ the Boolean semiring and $k = 1$, we recover the result in Gehrke et al. (2016, Proposition 13) on existential quantification:

Corollary 5.8. *Consider a formula $\varphi(x)$ with a free first-order variable x. If the language $L_{\varphi(x)} \subseteq (A \times 2)^*$ is recognised by the* BiM *morphism $\phi \colon (\beta((A \times 2)^*), (A \times 2)^*) \to (X, M)$, then the existentially quantified language $L_{\exists x.\varphi(x)} \subseteq A^*$ is recognised by the* BiM *morphism $\Diamond_2 \phi \colon (\beta(A^*), A^*) \to (\mathcal{V}X \times X, \mathcal{P}_f M \times M)$.*

6. Duality-Theoretic Account of the Construction

In Section 5.2, the BiM $\Diamond_S X = \widehat{\mathcal{S}X} \times X$ was defined by means of a matrix representation of a certain S-transduction. This may look like an ad hoc way to obtain a recogniser for the quantified languages. In Sections 6.1 and 6.2, we show that both the space component of $\Diamond_S X$ and the actions of its internal monoid can be derived in a natural way by using duality.

Let S be a finite and commutative semiring, and (X, M) a BiM. As earlier, we denote by B the dual algebra of X. Further, let $\phi \colon (\beta((A \times 2)^*), (A \times 2)^*) \to (X, M)$ be a BiM morphism. We denote by $\mathbb{B}$ the preimage under ϕ of B (equivalently, $\mathbb{B}$ is the image of the Boolean algebra homomorphism $B \to \mathcal{P}((A \times 2)^*)$ dual to ϕ). That is, $\mathbb{B}$ is the Boolean algebra, closed under quotients in $\mathcal{P}((A \times 2)^*)$, of languages recognised by the BiM morphism ϕ.

In Section 5.2, we introduced the map $\Diamond_S\phi$ as a recogniser for the quantified languages obtained from the languages in $\mathbb{B}$. Here we prove, by duality, that $\Diamond_S\phi$ is in fact the minimal possible BiM recogniser for these quantified languages. This will allow us to obtain, in Section 6.3, a Reutenauer-type theorem for $\Diamond_S X$ (see Theorem 6.13). The idea is the following. On the language side, we are interested in the Boolean algebra generated by the languages of the form $\mathcal{Q}_k(L)$, for $k \in S$ and $L \in \mathbb{B}$. This coincides with the Boolean algebra $\mathcal{Q}\mathbb{B}$ obtained as the preimage of $\widehat{B}$, the Boolean algebra of clopens of $\widehat{\mathcal{S}}X$, under the composite

$$\phi_Q \colon A^* \xrightarrow{R} \mathcal{S}((A \times 2)^*) \xrightarrow{\mathcal{S}\phi} \mathcal{S}M \xrightarrow{\int} \widehat{\mathcal{S}}X, \tag{24}$$

where R is as in equation (22), and $\int : \mathcal{S}M \to \widehat{\mathcal{S}}X$ is the integration map (cf. Lemma 4.8). Indeed, suppose $L \in \mathbb{B}$, that is $L = \phi^{-1}(K)$ for some clopen subset $K \subseteq X$. Then, for every $k \in S$,

$$\begin{aligned}\phi_Q{}^{-1}(\overline{[K,k]}) &= \{w \in A^* \mid \int_K \sum_{i=1}^{|w|} 1_S \cdot \phi(w^{(i)}) = k\} \\ &= \{w \in A^* \mid w \in \mathcal{Q}_k(\phi^{-1}(K))\} = \mathcal{Q}_k(L).\end{aligned}$$

The Boolean algebra $\mathcal{Q}\mathbb{B}$ is not closed under quotients. Since we want a BiM recogniser, and not just a 'Boolean space recogniser', we want to recognise the Boolean algebra closed under quotients generated by $\mathcal{Q}\mathbb{B}$. Furthermore, from the viewpoint of logic we are adding one layer of quantifiers. Thus, by inductive hypothesis, it makes sense to include also the languages of the form $L_0 = \{w \in A^* \mid w^0 \in L\}$, for $L \in \mathbb{B}$. (Recall from Section 2.1 that w^0 is the word in $(A \times 2)^*$ having the same shape as w and no marked positions.) These are the languages in the alphabet A, which are recognised by ϕ upon composing with the embedding

$$A^* \xrightarrow{(\ )^0} (A \times 2)^* \hookrightarrow \beta((A \times 2)^*), \quad w \mapsto w^0.$$

Let $\mathbb{B}_0$ be the Boolean algebra that is the preimage of $\mathbb{B}$ under the latter embedding. We thus want a BiM recogniser for $\mathbb{B}'$, the closure under quotients of $<\mathcal{Q}\mathbb{B} \cup \mathbb{B}_0>_{BA}$. We show that:

1. The Boolean algebra $<\mathcal{Q}\mathbb{B} \cup \mathbb{B}_0>_{BA}$ is already closed under quotients, whence $\mathbb{B}' = <\mathcal{Q}\mathbb{B} \cup \mathbb{B}_0>_{BA}$.
2. This allows us to see $\mathbb{B}'$ as a quotient of the coproduct of $\mathcal{Q}\mathbb{B}$ and $\mathbb{B}_0$, hence also of $\widehat{B}$ and B. By describing the quotienting operations on these subalgebras, we can define a compatible quotienting operation on the coproduct, which makes the natural map $\widehat{B} + B \to \mathcal{P}(A^*)$ a homomorphism of Boolean algebras with biactions.
3. Finally, dualising the quotienting operation on $\widehat{B} + B$ we get actions of $\Diamond_S M$ on $\Diamond_S X$, which coincide with those given by matrix multiplication in Section 5.2. Further, we recover $\Diamond_S\phi$ as dual to the homomorphism $\widehat{B} + B \to \mathcal{P}(A^*)$.

To improve readability, throughout this section we omit reference to the semiring S, and write $\Diamond\phi$, $\Diamond X$, $\Diamond M$ instead of $\Diamond_S\phi$, $\Diamond_S X$, $\Diamond_S M$.

6.1 The space $\Diamond X$ by duality

Recall from equation (24) the map

$$\phi_Q \colon A^* \xrightarrow{R} \mathcal{S}((A \times 2)^*) \xrightarrow{\mathcal{S}\phi} \mathcal{S}M \xrightarrow{\int} \widehat{\mathcal{S}}X,$$

which is given, for every $w \in A^*$, by

$$\phi_Q(w) := \int f_w,$$

where

$$f_w := \mathcal{S}\phi(R(w)) = \sum_{1 \le i \le |w|} 1_S \cdot \phi(w^{(i)}).$$

The unique extension of ϕ_Q to a continuous map $\beta(A^*) \to \widehat{\mathcal{S}}X$, which we will denote again by ϕ_Q, coincides with $\widehat{\mathcal{S}}\phi \circ \widehat{R}$, where $\widehat{R}\colon \beta(A^*) \to \widehat{\mathcal{S}}\beta((A \times 2)^*)$ is the Kleisli map from equation (21).

For any $k \in S$ and $L \in \mathbb{B}$, the clopen in $\beta(A^*)$ corresponding to $\mathcal{Q}_k(L)$ is ${\phi_Q}^{-1}(\overline{[K, k]})$, where $K \subseteq X$ is the clopen in X recognising L via ϕ, and $\overline{[K, k]}$ is as in equation (18). By Theorem 4.4, the clopens of $\widehat{\mathcal{S}}X$ are generated by the sets of the form $\overline{[K, k]}$ with $k \in S$ and $K \subseteq X$ clopen, thus we have:

Proposition 6.1. *The Boolean algebra $\mathcal{Q}\mathbb{B}$ of those languages over A which are inverse images of clopens under ϕ_Q is generated by the quantified languages $\mathcal{Q}_k(L)$, for $k \in S$ and $L \in \mathbb{B}$.*

Note that $\mathcal{Q}\mathbb{B}$, as defined in the previous proposition, is *not* closed under quotients. This is the reason we had to make an adjustment between Sections 5.1 and 5.2 above.

We denote by $\mathbb{B}_0$ the Boolean algebra of languages closed under quotients, which is recognised by the BiM morphism

$$\phi_0\colon (\beta(A^*), A^*) \xrightarrow{(\,)^0} (\beta((A \times 2)^*), (A \times 2)^*) \xrightarrow{\phi} (X, M).$$

In other words, $\mathbb{B}_0$ consists of the languages of the form $L_0 := {\phi_0}^{-1}(K)$, obtained as the preimage under $(\,)^0$ of languages $L = \phi^{-1}(K)$ in $\mathbb{B}$. Taking the product map, it now follows that

$$\Diamond\phi = \phi_Q \times \phi_0\colon \beta(A^*) \to \widehat{\mathcal{S}}X \times X, \tag{25}$$

viewed just as a map of Boolean spaces, 'recognises' the Boolean algebra generated by $\mathcal{Q}\mathbb{B} \cup \mathbb{B}_0$, in the sense that the elements of the latter Boolean algebra are exactly those of the form $\Diamond\phi^{-1}(C)$ for some clopen $C \subseteq \widehat{\mathcal{S}}X \times X$. However, since $\mathcal{Q}\mathbb{B}$ is *not* closed under quotients, $<\mathcal{Q}\mathbb{B} \cup \mathbb{B}_0>_{BA}$ need not be closed under quotients, a priori.

The Boolean algebra $\mathbb{B}'$ that we are interested in is the closure under quotients of $<\mathcal{Q}\mathbb{B} \cup \mathbb{B}_0>_{BA}$. The important observation is that $<\mathcal{Q}\mathbb{B} \cup \mathbb{B}_0>_{BA}$ *is already closed under the quotient operations*, thus explaining why $\widehat{\mathcal{S}}X \times X$, along with the above product map, is the right recogniser spacewise.

Proposition 6.2. *The Boolean algebra generated by $\mathcal{Q}\mathbb{B} \cup \mathbb{B}_0$ is closed under quotients. That is,*

$$\mathbb{B}' = <\mathcal{Q}_k(L), L_0 \mid L \in \mathbb{B} \textit{ and } k \in S>_{BA}.$$

Proof. Since $\mathbb{B}_0$ is closed under quotients, it suffices to consider the quotienting of languages of the form $\mathcal{Q}_k(L) = {\phi_Q}^{-1}(\overline{[K, k]})$ where $K \subseteq X$ is the clopen recognising L via ϕ. For $u \in A^*$, we have

$$\begin{aligned} u^{-1}\mathcal{Q}_k(L) &= \{w \in A^* \mid uw \in \mathcal{Q}_k(L)\} \\ &= \{w \in A^* \mid \int f_{uw} \in \overline{[K, k]}\}, \end{aligned}$$

where we used the fact that $\mathcal{Q}_k(L) = {\phi_Q}^{-1}(\overline{[K, k]}) \cap A^*$. Since the free variable in the word uw either occurs in u or in w,

$$f_{uw} = \sum_{1 \le i \le |uw|} 1_S \cdot \phi((uw)^{(i)}) = \sum_{1 \le i \le |w|} 1_S \cdot \phi(u^0 w^{(i)}) + \sum_{1 \le i \le |u|} 1_S \cdot \phi(u^{(i)} w^0) = \phi(u^0) f_w + f_u \phi(w^0),$$

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where $\phi(u^0)f_w$ and $f_u\phi(w^0)$ are obtained from the left and right actions, respectively, of M on $\mathcal{S}M$, cf. Lemma 4.9. Further, because $\int(\phi(u^0)f_w + f_u\phi(w^0)) = \int\phi(u^0)f_w + \int f_u\phi(w^0)$, we have

$$u^{-1}\mathcal{Q}_k(L) = \{w \in A^* \mid \int \phi(u^0)f_w + \int f_u\phi(w^0) \in \overline{[K,k]}\}$$
$$= \bigcup_{k_1+k_2=k} \{w \in A^* \mid \int \phi(u^0)f_w \in \overline{[K,k_1]} \text{ and } \int f_u\phi(w^0) \in \overline{[K,k_2]}\}.$$

Now,

$$\int \phi(u^0)f_w \in \overline{[K,k_1]} \iff \int f_w \in \overline{[\phi(u^0)^{-1}K,k_1]} \tag{26}$$

which in turn is equivalent to $w \in \mathcal{Q}_{k_1}((u^0)^{-1}L)$, which is an element of $\mathcal{Q}\mathbb{B}$. We now proceed with the second condition. Writing $\text{Sup}(f_u) = \{m \in M \mid f_u(m) \neq 0\}$ for the support of f_u, we have

$$\int f_u\phi(w^0) \in \overline{[K,k_2]} \iff \int f_u \in \overline{[K\phi(w^0)^{-1},k_2]} \iff \int_J f_u = k_2$$

where $J = K\phi(w^0)^{-1} \cap \text{Sup}(f_u)$. Whence, $\int f_u\phi(w^0) \in \overline{[K,k_2]}$ if, and only if, there is a set $I \subseteq \text{Sup}(f_u)$ with

- $\int_I f_u = k_2$;
- $m\phi(w^0) \in K$ for each $m \in I$;
- $m\phi(w^0) \in K^c$ for each $m \in I^c = \text{Sup}(f_u) \setminus I$.

Observe that $m\phi(w^0) \in K$ if, and only if, $w \in {\phi_0}^{-1}(m^{-1}K)$. Thus,

$$\{w \in A^* \mid \int f_u\phi(w^0) \in \overline{[K,k_2]}\}$$

is equal to

$$\bigcup_{\substack{I \subseteq \text{Sup}(f_u) \\ \int_I f_u = k_2}} \left([\bigcap_{m\in I} {\phi_0}^{-1}(m^{-1}K)] \cap [\bigcap_{m\in I^c} {\phi_0}^{-1}(m^{-1}K^c)] \right) \tag{27}$$

which is in $\mathbb{B}_0$. Hence, $u^{-1}\mathcal{Q}_k(L)$ belongs to the Boolean algebra generated by $\mathcal{Q}\mathbb{B} \cup \mathbb{B}_0$. □

Corollary 6.3. *The dual space of $\mathbb{B}'$ is a closed subspace of $\widehat{\mathcal{S}X} \times X$. In particular, $\mathbb{B}'$ is recognised as a Boolean algebra by $\widehat{\mathcal{S}X} \times X$.*

Proof. By the previous proposition, $\mathbb{B}' = <\mathcal{Q}\mathbb{B} \cup \mathbb{B}_0>_{BA}$. But $\mathbb{B}_0$ is the preimage under ϕ_0 of the Boolean algebra dual to X, and $\mathcal{Q}\mathbb{B}$ is the preimage under ϕ_Q of the Boolean algebra dual to $\widehat{\mathcal{S}X}$. Thus, $\mathbb{B}'$ is the preimage of the Boolean algebra dual to $\widehat{\mathcal{S}X} \times X$ under the map $\Diamond\phi\colon \beta(A^*) \to \widehat{\mathcal{S}X} \times X$ from (25), and therefore $\mathbb{B}'$ is recognised as a Boolean algebra by $\widehat{\mathcal{S}X} \times X$.

Now, since $\mathbb{B}'$ is the image of the Boolean algebra homomorphism dual to $\Diamond\phi\colon \beta(A^*) \to \widehat{\mathcal{S}X} \times X$, the dual space of $\mathbb{B}'$ is homeomorphic to the image of $\Diamond\phi$, which is a closed subspace of $\widehat{\mathcal{S}X} \times X$. □

6.2 The internal monoid structure of $\Diamond X$ by duality

In Section 5.2, we have described the monoid operation of $\Diamond M = \mathcal{S}M \times M$ and the actions of $\Diamond M$ on $\Diamond X = \widehat{\mathcal{S}X} \times X$ in terms of matrix multiplication. This multiplication was introduced in an ad hoc manner. Here we show that these actions (and, in particular, the monoid operation)

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need not be guessed, as they can be derived by duality. In fact, they are the appropriate actions on $\Diamond X$ for making $\Diamond\phi\colon \beta(A^*) \to \widehat{\mathcal{S}X} \times X$ (see equation (25)) a BiM morphism. For this purpose, we consider the homomorphism dual to $\Diamond\phi$:

$$\varphi\colon \widehat{B} + B \to \mathcal{P}(A^*), \quad \overline{[K,k]} \mapsto {\phi_Q}^{-1}(\overline{[K,k]}), \quad K \mapsto {\phi_0}^{-1}(K).$$

We already know, by Proposition 6.2, that the image of φ is closed under quotients. In fact, Proposition 6.2 tells us that we can define a biaction of $\Diamond M$ on $\widehat{B} + B$ so that φ becomes a homomorphism of Boolean algebras with biactions. Thus, for each $(f, m) \in \Diamond M$, we want to define a 'left quotient' by (f, m), that is the component at (f, m) of a right action on $\widehat{B} + B$, and a 'right quotient' (which is a left action on $\widehat{B} + B$), so that φ becomes a homomorphism of Boolean algebras with biactions.

The monoid morphism from A^* to $\Diamond M$ is given by sending the internal monoid element $u \in A^*$ to the internal monoid element $(f_u, \phi(u^0)) \in \mathcal{S}M \times M$, where f_u is defined as at the beginning of Section 6.1. Now, the component at (f, m) of a 'left quotient' operation on $\widehat{B} + B$ is a homomorphism

$$\Lambda(f, m)\colon \widehat{B} + B \to \widehat{B} + B.$$

Given the nature of coproducts, such a homomorphism is determined by its components $\Lambda_1(f, m)\colon \widehat{B} \to \widehat{B} + B$ and $\Lambda_2(f, m)\colon B \to \widehat{B} + B$. Our goal then, is to show that:

- the computation of quotient operations in the image of φ combined with wanting φ to be a morphism of Boolean algebras with biactions, dictates what $\Lambda_1(f, m)$ and $\Lambda_2(f, m)$ must be;
- the left action of $\Diamond M$ on $\Diamond X$ dual to Λ coincides with the one defined in Section 5.2.

The symmetric facts for the 'right quotient' operation are similar and thus we only consider the 'left quotient'. Also, note that we will not prove directly that the $\Lambda(f, m)$'s that we define are components of a right action on a Boolean algebra, as this will follow from the second bullet point above.

So, we want to define the action such that φ becomes a homomorphism sending the action of $(f_u, \phi(u^0)) \in \mathcal{S}M \times M$ on $\widehat{B} + B$ to the action of the quotient operation $u^{-1}(\ )$ on $\mathcal{P}(A^*)$. The computations in the proof of Proposition 6.2 tell us the components of $u^{-1}{\phi_Q}^{-1}(\overline{[K,k]})$ in $\mathcal{Q}\mathbb{B}$ and in $\mathbb{B}_0$, respectively. Since $\mathcal{Q}\mathbb{B}$ and $\mathbb{B}_0$ are precisely the images under φ of $\widehat{B}$ and B, respectively, the computations tell us how to define $\Lambda_1(f_u, \phi(u^0))$ using components $\Lambda_{11}(f, m)\colon \widehat{B} \to \widehat{B}$ and $\Lambda_{12}(f, m)\colon \widehat{B} \to B$.

By the computation in (26), we have that the component $\Lambda_{11}(f, m)\colon \widehat{B} \to \widehat{B}$ depends only on the second coordinate of the pair $(f_u, \phi(u^0))$ and it sends $\overline{[K,k]}$ to $\overline{[(\phi(u^0))^{-1}K, k]}$. Stating it for an arbitrary element $(f, m) \in \mathcal{S}M \times M$, we have

$$\Lambda_{11}(f, m)\colon \widehat{B} \to \widehat{B}, \quad \overline{[K,k]} \mapsto \overline{[m^{-1}K, k]}.$$

Similarly, the computation in (27), stated for an arbitrary element $(f, m) \in \mathcal{S}M \times M$, yields

$$\Lambda_{12}(f, m)\colon \widehat{B} \to B, \quad \overline{[K,k]} \mapsto \bigcup_{\substack{I \subseteq \mathrm{Sup}(f) \\ \int_I f = k}} \Big([\bigcap_{n \in I} n^{-1}K] \cap [\bigcap_{n \in I^c} n^{-1}K^c] \Big). \tag{28}$$

The above observations imply that:

Proposition 6.4. *The map $\varphi\colon \widehat{B} + B \to \mathcal{P}(A^*)$ is a homomorphism of Boolean algebras with biactions when the left quotient operation $\Lambda(f, m)$ of $\widehat{B} + B$ is defined on $\widehat{B}$ by*

$$\Lambda_1(f, m)\colon \overline{[K,k]} \mapsto \bigvee_{k_1 + k_2 = k} (\Lambda_{11}(\overline{[K,k_1]}) \wedge \Lambda_{12}(\overline{[K,k_2]}))$$

and on B by $\Lambda_2(f, m)\colon K \mapsto m^{-1}K$.

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Next, we show that the maps $\Lambda_{11}(f,m)$ and $\Lambda_{12}(f,m)$ are dual to the summands of the first component of the action of (f,m) on $\Diamond X$, and that $\Lambda_1(f,m)$ and $\Lambda_2(f,m)$ are dual, respectively, to

$$\lambda_1(f,m)\colon \widehat{\mathcal{S}}X \times X \to \widehat{\mathcal{S}}X, \quad (\mu,x) \mapsto m\mu + \int fx$$

and

$$\lambda_2(f,m)\colon \widehat{\mathcal{S}}X \times X \to X, \quad (\mu,x) \mapsto mx.$$

Lemma 6.5. *The homomorphism $\Lambda_{11}(f,m)\colon \widehat{B} \to \widehat{B}$ given by $\overline{[K,k]} \mapsto \overline{[m^{-1}K,k]}$ is dual to the continuous function $\lambda_{11}(f,m)\colon \widehat{\mathcal{S}}X \to \widehat{\mathcal{S}}X$ given by $\mu \mapsto m\mu$, where*

$$m\mu\colon B \to S,\ K \mapsto \mu(m^{-1}K).$$

Proof. The function $\lambda_{11}(f,m)$ is dual to $\Lambda_{11}(f,m)$ if, and only if, for all $\mu \in \widehat{\mathcal{S}}X$ and all $\overline{[K,k]} \in \widehat{B}$ we have

$$\lambda_{11}(f,m)\mu \in \overline{[K,k]} \quad\Longleftrightarrow\quad \mu \in \Lambda_{11}(f,m)\overline{[K,k]}.$$

But $\lambda_{11}(f,m)\mu = m\mu$, so

$$\begin{aligned}
\lambda_{11}(f,m)\mu \in \overline{[K,k]} \quad&\Longleftrightarrow\quad m\mu \in \overline{[K,k]}\\
&\Longleftrightarrow\quad m\mu(K) = k\\
&\Longleftrightarrow\quad \mu(m^{-1}K) = k\\
&\Longleftrightarrow\quad \mu \in \overline{[m^{-1}K,k]} = \Lambda_{11}(f,m),
\end{aligned}$$

as was to be proved. □

Lemma 6.6. *The homomorphism $\Lambda_{12}(f,m)\colon \widehat{B} \to B$ given as in (28) is dual to the continuous function $\lambda_{12}(f,m)\colon X \to \widehat{\mathcal{S}}X$ given by $x \mapsto \int fx$, where*

$$\int fx\colon B \to S,\ K \mapsto \int_{Kx^{-1}} f.$$

Proof. Let $x \in X$ and $[K,k] \in \widehat{B}$. Then,

$$\begin{aligned}
\int fx \in [K,k] \quad&\Longleftrightarrow\quad \int_K fx = k\\
&\Longleftrightarrow\quad \int_{Kx} f = k\\
&\Longleftrightarrow\quad \sum_{x\in n^{-1}K} f(n) = k,
\end{aligned}$$

and the latter is true if, and only if, there exists $I \subseteq \mathrm{Sup}(f)$ with $\int_I f = k$, $x \in n^{-1}K$ for each $n \in I$, and $x \notin n^{-1}K$ for each $n \in \mathrm{Sup}(f) \setminus I$. That is,

$$\int fx \in [K,k] \quad\Longleftrightarrow\quad x \in \Lambda_{12}(f,m)[K,k].$$

Therefore, the homomorphism $\Lambda_{12}(f,m)$ is dual to the continuous map $\lambda_{12}(f,m)$. □

Lemma 6.7. *The homomorphism $\Lambda_1(f,m)\colon \widehat{B} \to \widehat{B} + B$ given as in Proposition 6.4 is dual to the continuous function $\lambda_1(f,m)\colon \widehat{\mathcal{S}}X \times X \to \widehat{\mathcal{S}}X$ given by $(\mu,x) \mapsto m\mu + \int fx$.*

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Proof. Let $(\mu, x) \in \widehat{\mathcal{S}X} \times X$ and $[K, k] \in \widehat{B}$. Then,

$$\begin{aligned}
\lambda_1(f,m)(\mu,x) \in [K,k] &\iff \lambda_{11}(f,m)\mu \ + \ \lambda_{12}(f,m)x \in [K,k] \\
&\iff \exists k_1, k_2 \left(k_1 + k_2 = k,\ \lambda_{11}(f,m)\mu \in [K,k_1],\ \text{and}\ \lambda_{12}(f,m)x \in [K,k_2]\right) \\
&\iff \exists k_1, k_2 \left(k_1 + k_2 = k, \mu \in \Lambda_{11}(f,m)[K,k_1], \text{and}\ x \in \Lambda_{12}(f,m)[K,k_2]\right) \\
&\iff (\mu,x) \in \Lambda_1[K,k],
\end{aligned}$$

as was to be shown. □

It is straightforward to see that $\Lambda_2(f,m)\colon K \mapsto m^{-1}K$ is dual to $\lambda_2(f,m)\colon x \mapsto mx$. Since the continuous map $\lambda_1(f,m) \times \lambda_2(f,m)\colon \Diamond X \to \Diamond X$ coincides with the component at (f,m) of the left action of $\Diamond M$ on $\Diamond X$, defined in Section 5.2 through matrix multiplication, we conclude that:

Corollary 6.8. *The dual of the left quotienting operation Λ on $\widehat{B} + B$ defined in Proposition 6.4 is the left action of $\Diamond M$ on $\Diamond X$ defined in Section 5.2.*

A similar result holds of course for the right action, and the monoid operation of $\Diamond M$ can be recovered by restricting the actions on $\Diamond X$ to $\Diamond M$. As a consequence, we have

Theorem 6.9. *Let $\phi\colon (\beta((A \times 2)^*), (A \times 2)^*) \to (X, M)$ be a* BiM *morphism. The homomorphism of Boolean algebras with biactions*

$$\varphi\colon \widehat{B} + B \to \mathcal{P}(A^*),\ \ \overline{[K,k]} \mapsto {\phi_Q}^{-1}(\overline{[K,k]}),\ \ K \mapsto {\phi_0}^{-1}(K)$$

obtained by equipping $\widehat{B} + B$ with the biaction of $\Diamond M$ as indicated in Proposition 6.4, is dual to the BiM *morphism*

$$\Diamond\phi\colon (\beta(A^*), A^*) \to (\Diamond X, \Diamond M)$$

defined in Section 5.2.

6.3 A Reutenauer theorem for $\Diamond X$

In Reutenauer (1979), Reutenauer characterised the Boolean algebra of languages recognised by the Schützenberger product of two monoids. In this last subsection, we prove a Reutenauer-type result (Theorem 6.13 below) characterising the Boolean algebra closed under quotients generated by all languages recognised by the BiM $\Diamond X$ through *length preserving* morphisms.

Theorem 5.7 above tells us that the BiM $\Diamond X$ recognises all the quantified languages. This amounts to the 'soundness' of the construction. However, note that the BiM $(\beta(A^*), A^*)$ also recognises all the quantified languages; in fact, it recognises any A-language whatsoever. Crucially, the Reutenauer-type result ensures that the BiM $\Diamond X$ is optimal with respect to recognising the quantified languages, thus establishing the 'completeness' of the construction.

Definition 6.10. *We call a* BiM *morphism $\psi\colon (\beta(A^*), A^*) \to (\Diamond X, \Diamond M)$* length preserving *provided, for each $a \in A$, we have that*

$$\pi_1 \circ \psi(a)\colon M \to S$$

is the characteristic function χ_{m_a} for some single $m_a \in M$. That is, $\pi_1 \circ \psi(a)(m_a) = 1$ and $\pi_1 \circ \psi(a)(m) = 0$ for all $m \in M$ with $m \neq m_a$.

Remark 6.11. A monoid morphism $h\colon A^* \to C^*$ is called *length preserving* provided $h(a) \in C$ for all $a \in A$. In other words, h sends irreducible elements of the monoid A^* to irreducible elements of C^*. The irreducible elements of the monoid $\mathcal{S}M$ are precisely the characteristic functions of single elements of M, whence the definition above can be seen as an extension of the usual notion of length preserving monoid morphism.

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Recall that, given any BiM morphism $\phi\colon (\beta((A\times 2)^*), (A\times 2)^*) \to (X, M)$, we obtain a BiM morphism

$$\Diamond\phi\colon (\beta(A^*), A^*) \to (\Diamond X, \Diamond M), \quad w \mapsto \left(\int f_w, \phi(w^0)\right).$$

Upon defining $f_a := \pi_1 \circ \Diamond\phi(a)$, we have $f_a = \chi_{m_a}$ where $m_a = \phi(a, 1)$. Hence, $\Diamond\phi$ is length preserving. It is now a matter of a straightforward computation to prove the following proposition.

Proposition 6.12. *Let* (X, M) *be a* BiM. *Every length preserving* BiM *morphism* $(\beta(A^*), A^*) \to (\Diamond X, \Diamond M)$ *is of the form* $\Diamond\phi$ *for some* BiM *morphism* $\phi\colon (\beta((A\times 2)^*), (A\times 2)^*) \to (X, M)$.

Proof. Consider an arbitrary length preserving BiM morphism $\psi\colon (\beta(A^*), A^*) \to (\Diamond X, \Diamond M)$. We define $\phi\colon (\beta((A\times 2)^*), (A\times 2)^*) \to (X, M)$ by

$$\begin{aligned} \phi\colon (A\times 2)^* &\to M, \\ (a, 0) &\mapsto \pi_2 \circ \psi(a) \\ (a, 1) &\mapsto m_a \end{aligned}$$

where $m_a \in M$ is such that $\pi_1 \circ \psi(a) = \chi_{m_a}$. The universal property of the Stone–Čech compactification guarantees that ϕ is a BiM morphism with the topological component $\tilde{\phi} = \beta\phi$. It now suffices to show that $\psi(a) = \Diamond\phi(a)$ for each $a \in A$:

$$\Diamond\phi(a) = (f_a, \phi_0(a)) = (\chi_{\phi(a,1)}, \phi(a, 0)) = (\chi_{m_a}, \pi_2 \circ \psi(a)) = (\pi_1 \circ \psi(a), \pi_2 \circ \psi(a)) = \psi(a). \square$$

We thus obtain the following Reutenauer-type result for the BiM $\Diamond X$:

Theorem 6.13. *Let* X *be a* BiM, *and* A *a finite alphabet. The Boolean subalgebra closed under quotients of* $\mathcal{P}(A^*)$ *generated by all languages over* A, *which are recognised by a length preserving* BiM *morphism into* $\Diamond X$ *is generated as a Boolean algebra by the languages over* A *recognised by* X, *and the languages* $\mathcal{Q}_k(L)$ *for* L *a language over* $A \times 2$ *recognised by* X.

Proof. Let us denote by $\mathbb{B}''$ the Boolean algebra generated by the languages over A recognised by X, and the languages $\mathcal{Q}_k(L)$ for L a language over $A \times 2$ recognised by X. We must prove that $\mathbb{B}''$ coincides with the Boolean subalgebra closed under quotients of $\mathcal{P}(A^*)$ generated by all languages over A, which are recognised by a length preserving BiM morphism into $\Diamond X$.

If $L' \in \mathcal{P}(A^*)$ is recognised by a length preserving BiM morphism $\psi\colon (\beta(A^*), A^*) \to (\Diamond X, \Diamond M)$, then by Proposition 6.12 there is a BiM morphism $\phi\colon (\beta((A\times 2)^*), (A\times 2)^*) \to (X, M)$ such that $\Diamond\phi = \psi$. That is, L' lies in the Boolean algebra called $\mathbb{B}'$ in the beginning of this section. Since $\mathbb{B}' \subseteq \mathbb{B}''$ by Proposition 6.2, we have $L' \in \mathbb{B}''$.

For the other direction, if L is a language over $A \times 2$ recognised by X, then $\mathcal{Q}_k(L)$ is recognised by $\Diamond X$ through a length preserving morphism in view of Theorem 5.7. Finally, suppose L is a language over A recognised by $\eta\colon \beta(A^*) \to X$ through the clopen K. Consider any function $\phi\colon \beta((A\times 2)^*) \to M$ satisfying $\phi(a, 0) = \eta(a)$ for each $a \in A$. Then $L = \Diamond\phi^{-1}(\widehat{\mathcal{S}}X \times K)$, showing that L is recognised by $\Diamond X$ through a length preserving morphism. $\square$

7. Conclusion

In this paper, we have provided a general construction for recognisers, which captures the action of quantifier-like operations on arbitrary languages of words, drawing heavily on a combination of categorical and duality-theoretic tools.

This paper is a stepping stone in a long-term research programme aimed at finding meaningful ultrafilter equations that characterise logically defined classes of non-regular languages. The next step is to understand the effect on equations of the constructions introduced here.

The generic development of Section 5 allows this work to be extended to encompass a wider range of operations on languages modelled by rational transducers which, by the Kleene–Schützenberger theorem (see, e.g. Sakarovitch, 2009), admit a matrix representation. Also, the duality-theoretic account in Section 6 leads to a Reutenauer-type characterisation theorem, akin to the one in Gehrke et al. (2016). It would be interesting to identify a common framework for our contributions and the recent work (Chen and Urbat, 2016).

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Conflicts of interest

The authors declare none.

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