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ON SUFFICIENT CONDITIONS FOR THE COMPARISON IN THE EXCESS WEALTH ORDER AND SPACINGS

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Abstract

The purpose of this paper is twofold. On the one hand, we provide sufficient conditions for the excess wealth order. These conditions are based on properties of the quantile functions which are useful when the dispersive order does not hold. On the other hand, we study sufficient conditions for the comparison in the increasing convex order of spacings of generalized order statistics. These results will be combined to show how we can provide comparisons of quantities of interest in reliability and insurance.

Keywords: Excess wealth order; generalized order statistics; spacings

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1. Introduction

In the context of stochastic orders the comparison of the dispersion of random variables is usually carried out in terms of the so-called dispersive order; see Shaked and Shanthikumar (2007). Given two random variables X and Y with distribution functions F and G, respectively, X is said to be less than Y in the *dispersive order*, denoted by $X \leq_{\text{disp}} Y$, if

$$G^{-1}(p) - F^{-1}(p) \text{ is increasing in } p \in (0, 1),$$

where F^{-1} and G^{-1} are the quantile functions of X and Y, respectively, defined as $F^{-1}(p) \equiv \inf\{x \in \mathbb{R} \mid F(x) \geq p\}$ for any value $p \in (0, 1)$, and analogously for G^{-1}.

The dispersive order is the strongest partial order to compare two random variables in terms of their variability. When this order does not hold, it is possible to continue comparing the random variables in these terms by means of some other criteria, such as the excess wealth order. The excess wealth order is defined through the excess wealth transform. Given a random variable X with distribution function F, the excess wealth transform, W_X, associated to X is defined as

$$W_X(p) = \mathbb{E}[(X - F^{-1}(p))_+] = \int_p^1 (F^{-1}(q) - F^{-1}(p))\, \mathrm{d}q \quad \text{for } p \in (0, 1), \tag{1.1}$$

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where $(x)_+ = 0$ if $x < 0$ and $(x)_+ = x$ if $x \geq 0$. When X has a finite mean, W_X is well defined and measures the thickness of the upper tail from a fixed quantile $F^{-1}(p)$.

Now given two random variables X and Y, with finite means, X is said to be less than Y in the excess wealth order, denoted by $X \leq_{\text{ew}} Y$, if

$$W_X(p) \leq W_Y(p) \quad \text{for all } p \in (0, 1).$$

The excess wealth order was independently introduced by Fernandez-Ponce *et al.* (1998) and Shaked and Shanthikumar (1998) and has received great attention in the literature. Properties and applications in several contexts such as reliability, risks, and auction theory can be found in Belzunce (1999), Denuit and Vermandele (1999), Kochar *et al.* (2002), Li (2005), Sordo (2008), (2009), Hu *et al.* (2012), Kochar and Xu (2013), Balakrishnan and Zao (2013), Singpurwalla and Gordon (2014), and Sordo *et al.* (2015).

Among the previous criteria, we have the following relationship:

$$X \leq_{\text{disp}} Y \quad \Longrightarrow \quad X \leq_{\text{ew}} Y. \tag{1.2}$$

One of the main problems related to the applicability of the excess wealth is the evaluation of incomplete integrals of quantile functions, which is not possible in most cases. As we see from (1.2), in these cases, we can check the dispersive order as a sufficient condition for the excess wealth order. However, there are examples, as we will see later, where the excess wealth order holds but the dispersive order does not. Therefore, it is of interest to provide sufficient conditions for the comparison of the excess wealth transforms, which do not involve the computation of incomplete integrals of the survival or quantile functions, when the dispersive order does not hold, which is one of the purposes of this paper.

In order to provide applications, these sufficient conditions are used to compare, in the excess wealth order, the first generalized order statistics (GOSs) drawn from two independent and identically distributed (i.i.d.) random samples. The order of the first GOS in the excess wealth order turns out to be a sufficient condition for the comparison of spacings (or differences between successive GOSs) in the increasing convex order.

The organization of this paper is as follows. In Section 2 we describe several sets of sufficient conditions for the excess wealth order. These conditions are written in terms of the difference of the quantile functions, and describe situations where the difference is nonmonotone. Applications to some parametric models are provided. In Section 3 we compare the minimum of two random vectors of GOSs in terms of the excess wealth order and spacings of GOSs in terms of the increasing convex order. In Section 4 we provide applications of these results to compare interarrival times of repairs for items under a minimal repair policy and for the comparison of reinsurance premiums of two portfolios of risks under the ECOMOR (excédent du coût moyen relatif) treaty.

2. Some sufficient conditions for the excess wealth order

In this section we provide several results where we describe sufficient conditions for the excess wealth order of two random variables. These results are of interest when there are no closed expressions for the excess wealth transforms or such expressions are not easy to deal with. It is worthwhile to mention that these conditions neither imply nor are implied by the dispersive order. Examples where these results are applied are also given. First we recall the definition of the stochastic order, which is required in the proof of the next theorem.

Let X and Y be two random variables, we say that X is less than Y in the *stochastic order*, denoted by $X \leq_{\text{st}} Y$, if

$$F^{-1}(p) \leq G^{-1}(p) \quad \text{for all } p \in (0, 1).$$

It is trivially verified that $X \leq_{\text{st}} Y$ implies that $\mathbb{E}[X] \leq \mathbb{E}[Y]$.

Theorem 2.1. *Let X and Y be two random variables, with distribution functions F and G, and finite means. If there exists a value $p_0 \in (0, 1)$ such that $G^{-1}(p) - F^{-1}(p) \leq \mathbb{E}[Y] - \mathbb{E}[X]$ for all $p \in (0, p_0)$ and $G^{-1}(p) - F^{-1}(p)$ is increasing on $[p_0, 1)$, then*

$$X \leq_{\text{ew}} Y.$$

Proof. First, we consider a value $p \in [p_0, 1)$. It is easy to see (see Belzunce *et al.* (2003)) that the quantile function of $(X - F^{-1}(p))_+$ is given by

$$F_+^{-1}(q; p) = (F^{-1}(q) - F^{-1}(p))_+ \quad \text{for } q \in (0, 1),$$

and similarly for the quantile function of $(Y - G^{-1}(p))_+$. Now, given that $G^{-1}(p) - F^{-1}(p)$ is increasing in $p \in [p_0, 1)$, we have

$$(X - F^{-1}(p))_+ \leq_{\text{st}} (Y - G^{-1}(p))_+$$

and, therefore,

$$\mathbb{E}[(X - F^{-1}(p))_+] \leq \mathbb{E}[(Y - G^{-1}(p))_+].$$

We consider now a value $p \in (0, p_0)$ and observe that the quantile function of

$$\min\{X, F^{-1}(p)\} - \mathbb{E}[X]$$

is given by

$$F_p^{-1}(q) = \begin{cases} F^{-1}(q) - \mathbb{E}[X] & \text{if } 0 < q < p, \\ F^{-1}(p) - \mathbb{E}[X] & \text{if } p \leq q < 1, \end{cases}$$

and similarly for $\min\{Y, G^{-1}(p)\} - \mathbb{E}[Y]$. Given that $G^{-1}(p) - F^{-1}(p) \leq \mathbb{E}[Y] - \mathbb{E}[X]$ for all $p \in (0, p_0)$, we have

$$\min\{X, F^{-1}(p)\} - \mathbb{E}[X] \geq_{\text{st}} \min\{Y, G^{-1}(p)\} - \mathbb{E}[Y]$$

and, therefore,

$$\mathbb{E}[\min\{X, F^{-1}(p)\}] - \mathbb{E}[X] \geq \mathbb{E}[\min\{Y, G^{-1}(p)\}] - \mathbb{E}[Y].$$

From the equality $(x - t)_+ = x - \min\{x, t\}$, we have

$$\mathbb{E}[(X - F^{-1}(p))_+] \leq \mathbb{E}[(Y - G^{-1}(p))_+]. \qquad \square$$

Remark 2.1. If there exists a point p_0 such that $G^{-1}(p) \leq F^{-1}(p)$ for all $p \in (0, p_0)$ and it holds that $\mathbb{E}[X] \leq \mathbb{E}[Y]$, then the assumption $G^{-1}(p) - F^{-1}(p) \leq \mathbb{E}[Y] - \mathbb{E}[X]$ for all $p \in (0, p_0)$ is trivially satisfied.

We now give an example where we can apply the previous result. We consider the case of two Davies distributed random variables, a generalization of the well known Pareto distribution and, therefore, a more flexible model which can be used to provide a better fit; see Hankin and Lee (2006).

Example 2.1. (*Davies distributions.*) We consider two Davies distributed random variables X and Y with quantile functions given by

$$F^{-1}(p) = C_1 \frac{p^{\lambda_1}}{(1-p)^{\theta_1}} \quad \text{for } p \in (0,1)$$

and

$$G^{-1}(p) = C_2 \frac{p^{\lambda_2}}{(1-p)^{\theta_2}} \quad \text{for } p \in (0,1),$$

respectively, and denoted by $X \sim D(\lambda_1, \theta_1, C_1)$ and $Y \sim D(\lambda_2, \theta_2, C_2)$, where $\lambda_1, \theta_1, C_1, \lambda_2, \theta_2, C_2 > 0$. Provided $\theta_1, \theta_2 < 1$, the means are finite and are given by $\mathbb{E}[X] = C_1 B(1+\lambda_1, 1-\theta_1)$ and $\mathbb{E}[Y] = C_2 B(1+\lambda_2, 1-\theta_2)$, where $B(a,b)$ is the beta function.

Belzunce *et al.* (2014) proved that if $\lambda_1 \leq \lambda_2$ and $\theta_1 \leq \theta_2$, there exists a p_0 such that $G^{-1}(p) \leq F^{-1}(p)$ for all $p \in (0, p_0)$, and $G^{-1}(p) - F^{-1}(p)$ is increasing on $[p_0, 1)$. Therefore, if $\mathbb{E}[X] = C_1 B(1+\lambda_1, 1-\theta_1) \leq \mathbb{E}[Y] = C_2 B(1+\lambda_2, 1-\theta_2)$, we have $X \leq_{\text{ew}} Y$. It is also possible to see that, under the previous conditions on the parameters, $G^{-1}(p) - F^{-1}(p)$ is decreasing on $(0, p_0)$ and, therefore, $X \not\leq_{\text{disp}} Y$ or $X \not\geq_{\text{disp}} Y$.

In the previous example, we observe that the unimodality of $G^{-1}(p) - F^{-1}(p)$ and the condition $\lim_{p\to 0^+}(G^{-1}(p) - F^{-1}(p)) \leq \mathbb{E}[Y] - \mathbb{E}[X]$ imply the conditions assumed in Theorem 2.1, therefore we can establish the following result.

Corollary 2.1. *Let X and Y be two random variables with distribution functions F and G, respectively and finite means such that* $\lim_{p\to 0^+}(G^{-1}(p) - F^{-1}(p)) \leq \mathbb{E}[Y] - \mathbb{E}[X]$. *We assume that there exists a value $p_0 \in (0,1)$ such that $G^{-1}(p) - F^{-1}(p)$ is decreasing on $(0, p_0)$, $G^{-1}(p) - F^{-1}(p)$ is increasing on $[p_0, 1)$, then*

$$X \leq_{\text{ew}} Y.$$

This is the case of two Weibull distributed random variables, as we see next.

Example 2.2. (*Weibull distributions.*) Let X and Y be two Weibull distributed random variables with strictly positive parameters α_1, β_1 and α_2, β_2, respectively, denoted by $X \sim W(\alpha_1, \beta_1)$ and $Y \sim W(\alpha_2, \beta_2)$, with quantile functions

$$F^{-1}(p) = \alpha_1(-\log(1-p))^{1/\beta_1} \quad \text{for } p \in (0,1)$$

and

$$G^{-1}(p) = \alpha_2(-\log(1-p))^{1/\beta_2} \quad \text{for } p \in (0,1),$$

respectively.

Belzunce *et al.* (2014) proved that if $\beta_2 < \beta_1$, then $G^{-1}(p) - F^{-1}(p)$ is initially decreasing and later increasing with $\lim_{p\to 0^+}(G^{-1}(p) - F^{-1}(p)) = 0$ and $\lim_{p\to 1^-}(G^{-1}(p) - F^{-1}(p)) = +\infty$. Therefore, if we assume that $\mathbb{E}[X] = \alpha_1\Gamma((\beta_1+1)/\beta_1) \leq \alpha_2\Gamma((\beta_2+1)/\beta_2) = \mathbb{E}[Y]$ then $X \leq_{\text{ew}} Y$ but $X \not\leq_{\text{disp}} Y$ or $X \not\geq_{\text{disp}} Y$.

In the absolutely continuous case, for random variables with interval supports, taking derivatives on the difference of the quantile functions, it is easy to see that the previous corollary can be written as follows.

Corollary 2.2. *Let X and Y be two random variables, with interval supports, with distribution functions F and G, density functions f and g, respectively and finite means such that $\lim_{p\to 0^+}(G^{-1}(p) - F^{-1}(p)) \le \mathbb{E}[Y] - \mathbb{E}[X]$. We assume that there exits a value $p_0 \in (0, 1)$ such that $g(G^{-1}(p)) \ge f(F^{-1}(p))$ on $(0, p_0)$ and $g(G^{-1}(p)) \le f(F^{-1}(p))$ on $[p_0, 1)$, then $X \le_{\text{ew}} Y$.*

Next we consider a model specified by the quantile function to show how the previous result can be applied.

Example 2.3. (*Govindarajulu distributions.*) We consider two Govindarajulu distributed random variables X and Y with quantile functions given by (see Govindarajulu (1977))

$$F^{-1}(p) = \theta_1 + \sigma_1((\beta_1 + 1)p^{\beta_1} - \beta_1 p^{\beta_1+1}) \quad \text{for } p \in (0, 1)$$

and

$$G^{-1}(p) = \theta_2 + \sigma_2((\beta_2 + 1)p^{\beta_2} - \beta_2 p^{\beta_2+1}) \quad \text{for } p \in (0, 1),$$

respectively, and denoted by $X \sim G(\beta_1, \sigma_1, \theta_1)$ and $Y \sim G(\beta_2, \sigma_2, \theta_2)$, where all parameters are nonnegative. We study the monotonicity of $G^{-1}(p) - F^{-1}(p)$ in terms of the crossing points of $f(F^{-1}(p))$ and $g(G^{-1}(p))$ or, equivalently, of the hazard rate functions $f(F^{-1}(p))/(1 - p)$ and $g(G^{-1}(p))/(1 - p)$ evaluated at the quantiles, where f and g are the density functions associated to F^{-1} and G^{-1}, respectively.

It is not difficult to see that there is a crossing point at

$$p_0 = \left(\frac{(\beta_2 + 1)\beta_2\sigma_2}{(\beta_1 + 1)\beta_1\sigma_1}\right)^{1/(\beta_1-\beta_2)} \in (0, 1),$$

in the sense stated in Corollary 2.2, if the following conditions hold:

$$\beta_1 < \beta_2, \qquad (\beta_2 + 1)\beta_2\sigma_2 > (\beta_1 + 1)\beta_1\sigma_1. \tag{2.1}$$

In order to apply the previous theorem, we need $\lim_{p\to 0^+}(G^{-1}(p) - F^{-1}(p)) \le \mathbb{E}[Y] - \mathbb{E}[X]$, which is equivalent to

$$\sigma_1(\beta_2 + 2) \le \sigma_2(\beta_1 + 2). \tag{2.2}$$

Therefore, under (2.1) and (2.2), we have $X \le_{\text{ew}} Y$ but $X \not\le_{\text{disp}} Y$ or $X \not\ge_{\text{disp}} Y$.

We observe that if $\beta_1 \le (>)\beta_2$ and $(\beta_1 + 1)\beta_1\sigma_1 \ge (<)(\beta_2 + 1)\beta_2\sigma_2$, then $X \ge_{\text{disp}} Y$ ($X \le_{\text{disp}} Y$).

To finish we provide a generalization of the main theorem. This result is interesting from a mathematical point of view rather than from a practical point of view, but we include it for the sake of completeness.

Theorem 2.2. *Let X and Y be two random variables with quantile functions F^{-1} and G^{-1}, respectively, and finite means. We assume that there exists a value $p_0 \in (0, 1)$ such that $G^{-1}(p) - F^{-1}(p) \le \mathbb{E}[Y] - \mathbb{E}[X]$ on $(0, p_0)$ and $G^{-1}(p) - F^{-1}(p)$ has $n \ge 1$ relative extremes on the interval $(p_0, 1)$ at points $p_0 < p_1 < p_2 < \cdots < p_n < 1$. Then we have $X \le_{\text{ew}} Y$ if and only if one of the following sets of conditions is satisfied:*

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(i) *the number of relative extremes is even, $n = 2m$, such that p_1 is a maximum, and for $j = 1, \ldots, m$ we have*

$$\mathbb{E}[(X - F^{-1}(p_{2j-1}))_+] \leq \mathbb{E}[(Y - G^{-1}(p_{2j-1}))_+];$$

(ii) *the number of relative extremes is odd, $n = 2m + 1$, such that p_1 is a minimum, and for $j = 1, \ldots, m$, we have*

$$\mathbb{E}[(X - F^{-1}(p_{2j}))_+] \leq \mathbb{E}[(Y - G^{-1}(p_{2j}))_+].$$

Proof. Consider the functions

$$\delta(q, p) = G^{-1}(q) - F^{-1}(p) - (G^{-1}(p) - F^{-1}(p)), \qquad \Delta(p) = \int_p^1 \delta(u, p)\,\mathrm{d}u.$$

It is clear that if $G^{-1}(p) - F^{-1}(p)$ is decreasing (increasing) over an interval (u, u'), then $\delta(q, p) \leq (\geq) 0$ for every $p, q \in (u, u')$ such that $p < q$.

We show that under the initial set of conditions and the conditions stated in Theorem 2.2(i), we have $X \leq_{\mathrm{ew}} Y$ or, equivalently, by (1.1), that $\Delta(p) \geq 0$ for every $p \in (0, 1)$.

From the proof of Theorem 2.1, we have $\Delta(p) \geq 0$ for any p in the intervals $(0, p_0)$ and $(p_n, 1)$. Now, we show that $\Delta(p)$ is increasing for $p \in (p_{n-1}, p_n)$. Taking $p < q \in (p_{n-1}, p_n)$, we have

$$\Delta(p) \leq \int_q^1 \delta(u, p)\,\mathrm{d}u \leq \Delta(q),$$

where the first inequality follows from that fact that $\delta(u, p) \leq 0$ for any $u \in (p, q)$ and the second one from the fact that $\delta(u, p) - \delta(u, q) = \delta(q, p) \leq 0$. Therefore, $\Delta(p)$ is increasing in (p_{n-1}, p_n). Now given that we assume that $\Delta(p_{n-1}) \geq 0$ then $\Delta(p) \geq 0$ in (p_{n-1}, p_n). In a similar way it can be seen that $\Delta(p)$ is decreasing in (p_{n-2}, p_{n-1}), and, therefore, $\Delta(p) \geq 0$ in (p_{n-2}, p_{n-1}). Repeating this argument, for the remaining intervals (p_i, p_{i+1}) for $i = 0, \ldots, n-3$, we obtain $\Delta(p) \geq 0$ for every $p \in (0, 1)$.

The proof for the case Theorem 2.2(ii) follows under similar arguments. □

Remark 2.2. Note that the proof (and therefore the theorem) can be written by taking $p_0 = 0$. We have just to add the condition $\lim_{p \to 0^+}(G^{-1}(p) - F^{-1}(p)) \leq \mathbb{E}[Y] - \mathbb{E}[X]$ in Theorem 2.2(ii) in order to have $\lim_{p \to 0^+} \Delta(p) \geq 0$.

3. Stochastic comparisons of spacings

Order statistics, record values, and spacings (the differences between successive order statistics or record values) have found important applications in many areas of science. An extensive review of theoretical results and applications can be found in the volumes of Balakrishnan and Rao (1998a), (1998b). Due to closed similarity between some distributional, structural, and dependence properties of order statistics and record values, Kamps (1995a), (1995b) introduced the model of GOSs which includes, as special cases, random vectors of order statistics and record values and, moreover, some other models of interest such as sequential order statistics and progressively type-II censored order statistics. In this section we consider the comparison of spacings of GOSs. In the next section, as a particular case, we focus on the comparison of the spacing of ordinary order statistics, which provides an interesting application in insurance.

Now we present the definition of GOSs, due to Kamps (1995a), (1995b).

Definition 3.1. Let $n \in \mathbb{N}$, $k \geq 1$, $m_1, \ldots, m_{n-1} \in \mathbb{R}$, $M_r = \sum_{j=r}^{n-1} m_j$, $1 \leq r \leq n-1$, be parameters such that $\gamma_r = k + n - r + M_r \geq 1$ for all $r \in \{1, \ldots, n-1\}$, and let $\tilde{m} = (m_1, \ldots, m_{n-1})$ if $n \geq 2$ ($\tilde{m} \in \mathbb{R}$ arbitrary, if $n = 1$). If the random variables $U_{(r,n,\tilde{m},k)}$, $r = 1, \ldots, n$, possess a joint density of the form

$$h(u_1, \ldots, u_n) = k\left(\prod_{j=1}^{n-1} \gamma_j\right)\left(\prod_{j=1}^{n-1} (1-u_j)^{m_j}\right)(1-u_n)^{k-1},$$

defined on the cone $0 \leq u_1 \leq \cdots \leq u_n \leq 1$, then they are called uniform GOSs. Now, for a given distribution function F, the random variables

$$X_{(r,n,\tilde{m},k)} = F^{-1}(U_{(r,n,\tilde{m},k)}), \qquad r = 1, \ldots, n,$$

are called the GOSs based on F and the random variables

$$D_{r,n}^X = X_{(r,n,\tilde{m},k)} - X_{(r-1,n,\tilde{m},k)}, \qquad r = 2, \ldots, n,$$

are called the simple spacings of the GOSs $\{X_{(r,n,\tilde{m},k)}, r = 1, \ldots, n\}$.

Stochastic comparisons of GOSs and their spacings have been discussed rather extensively during the past decade; see, for example, Franco *et al.* (2002), Belzunce *et al.* (2005), Hu and Zhuang (2005), Belzunce *et al.* (2008), Zhuang and Hu (2009), Xie and Hu (2009), (2010), Xie and Zhuang (2011), and Balakrishnan *et al.* (2012). In this section we are interested in comparing the size of simple spacings of GOSs based on two independent distributions F and G. In this respect, Belzunce *et al.* (2005) proved that

$$X \leq_{\text{disp}} Y \quad \Longrightarrow \quad D_{r,n}^X \leq_{\text{st}} D_{r,n}^Y, \qquad r = 2, \ldots, n. \tag{3.1}$$

Our purpose is to compare the size of simple spacings of GOSs based on two independent distributions F and G when the underlying random variables X and Y fail to be ordered in the dispersive order. One possibility is to use the following result of Qiu and Wang (2007):

$$X \leq_{\text{ew}} Y \quad \Longrightarrow \quad \mathbb{E}[D_{n-1,n}^X] \leq \mathbb{E}[D_{n-1,n}^Y].$$

Unfortunately, this result only compares the sizes of the last spacings. In addition, a more informative comparison can be made in terms of the increasing convex order (also called stop-loss order) which is defined as follows.

Definition 3.2. Given two random variables X and Y with distribution functions F and G, respectively, we say that X is less than Y in the *increasing convex order*, denoted by $X \leq_{\text{icx}} Y$, if

$$\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$$

for every increasing convex function ϕ for which the previous expectations exist.

Kochar *et al.* (2007) investigated the increasing convex order of the last spacings of two vectors of usual order statistics. Here we extend this study in two directions: firstly, we consider GOSs rather than usual order statistics and secondly, we investigate the increasing convex order among any simple spacing (not necessarily the last ones). We first prove some results about the excess wealth order that will be used later on.

Lemma 3.1. *Let X and Y be two continuous random variables with strictly increasing distribution functions F and G, respectively. Let $h = F^{-1}G$. Then, $X \leq_{\text{ew}} Y$ implies that*

$$\mathbb{E}[\phi(h(Y) - h(x)) \mid Y > x] \leq \mathbb{E}[\phi(Y - x) \mid Y > x] \quad \textit{for all } x \in \mathbb{R} \tag{3.2}$$

and for any increasing convex function ϕ.

Proof. Suppose that $X \leq_{\text{ew}} Y$ and let ϕ be an increasing convex function. From Belzunce (1999), it follows that

$$\mathbb{E}[\phi(X - F^{-1}(p))_+] \leq \mathbb{E}[\phi(Y - G^{-1}(p))_+] \quad \text{for } p \in (0, 1). \tag{3.3}$$

Using the fact that

$$\mathbb{E}[\phi(X - F^{-1}(p))_+] = \mathbb{E}[\phi(X - F^{-1}(p)) \mid X > F^{-1}(p)](1 - p) + \phi(0)p,$$

we see that (3.3) is the same as

$$\begin{aligned} &\mathbb{E}[\phi(X - F^{-1}(p)) \mid X > F^{-1}(p)] \\ &\qquad \leq \mathbb{E}[\phi(Y - G^{-1}(p)) \mid Y > G^{-1}(p)] \quad \text{for } p \in (0, 1). \end{aligned} \tag{3.4}$$

Now, by setting $p = G(x)$ and $h = F^{-1} \circ G$ it is obvious that (3.4) is equivalent to

$$\mathbb{E}[\phi(X - h(x)) \mid X > h(x)] \leq \mathbb{E}[\phi(Y - x) \mid Y > x] \quad \text{for all } x \in \mathbb{R}. \tag{3.5}$$

Using the fact that $X \equiv_{\text{st}} h(Y)$, with h strictly increasing, it follows that (3.5) can be written as (3.2), which completes the proof of the lemma. □

The following result, concerning the minimum from two vectors of GOSs, parallels Balakrishnan *et al.* (2012, Lemma 3.10) for the excess wealth order.

Lemma 3.2. *Let X and Y be two continuous random variables with respective distribution functions F and G. Let $X_{(r,n,\tilde{m},k)}$ and $Y_{(r,n,\tilde{m},k)}$, $r = 1, \dots, n$, be GOSs based on F and G, respectively, with parameter $\gamma_1 = k + n - 1 + M_1$. Similarly, let $X_{(r,n',\tilde{m}',k)}$ and $Y_{(r,n',\tilde{m}',k)}$, $r = 1, \dots, n'$, be GOSs based on F and G, respectively, with parameter $\gamma_1' = k + n' - 1 + M_1'$. Let $\gamma_1' \leq \gamma_1$. If $X_{(1,n,\tilde{m},k)} \leq_{\text{ew}} Y_{(1,n,\tilde{m},k)}$ then $X_{(1,n',\tilde{m}',k)} \leq_{\text{ew}} Y_{(1,n',\tilde{m}',k)}$.*

Proof. Let $\bar{F}_{(1,n,\tilde{m},k)}$ and $\bar{G}_{(1,n,\tilde{m},k)}$ be the survival functions of $X_{(1,n,\tilde{m},k)}$ and $Y_{(1,n,\tilde{m},k)}$, respectively. From Shaked and Shanthikumar (2007, Equation (3.C.1)), the condition that $X_{(1,n,\tilde{m},k)} \leq_{\text{ew}} Y_{(1,n,\tilde{m},k)}$ is equivalent to

$$\int_{\bar{F}^{-1}_{(1,n,\tilde{m},k)}(p)}^{\infty} \bar{F}_{(1,n,\tilde{m},k)}(t)\,\mathrm{d}t \leq \int_{\bar{G}^{-1}_{(1,n,\tilde{m},k)}(p)}^{\infty} \bar{G}_{(1,n,\tilde{m},k)}(t)\,\mathrm{d}t \quad \text{for all } p \in (0, 1), \tag{3.6}$$

where $\bar{F}_{(1,n,\tilde{m},k)}$ and $\bar{G}_{(1,n,\tilde{m},k)}$ are, respectively, the survival functions of

$$X_{(1,n,\tilde{m},k)} \quad \text{and} \quad Y_{(1,n,\tilde{m},k)},$$

given by

$$\bar{F}_{(1,n,\tilde{m},k)}(t) = (\bar{F}(t))^{\gamma_1}, \qquad \bar{G}_{(1,n,\tilde{m},k)}(t) = (\bar{G}(t))^{\gamma_1},$$

and $F^{-1}_{(1,n,\tilde{m},k)}(p)$ and $G^{-1}_{(1,n,\tilde{m},k)}(p)$ are, respectively, the quantile functions of $X_{(1,n,\tilde{m},k)}$ and $Y_{(1,n,\tilde{m},k)}$, given by

$$\bar{F}^{-1}_{(1,n,\tilde{m},k)}(p) = \bar{F}^{-1}((1-p)^{1/\gamma_1}), \qquad \bar{G}^{-1}_{(1,n,\tilde{m},k)}(p) = \bar{G}^{-1}((1-p)^{1/\gamma_1}) \quad \text{for } \ p \in (0,1).$$

Therefore, a condition equivalent to (3.6) is

$$\int_{\bar{F}^{-1}((1-p)^{1/\gamma_1})}^{\infty} (\bar{F}(t))^{\gamma_1}\,\mathrm{d}t \le \int_{\bar{G}^{-1}((1-p)^{1/\gamma_1})}^{\infty} (\bar{G}(t))^{\gamma_1}\,\mathrm{d}t \quad \text{for all } p \in (0,1)$$

which holds if and only if

$$\int_0^{(1-p)^{1/\gamma_1}} u^{\gamma_1}\,\mathrm{d}[\bar{F}^{-1}(u) - \bar{G}^{-1}(u)] \ge 0 \quad \text{for all } p \in (0,1),$$

which is equivalent to

$$\int_0^{q} u^{\gamma_1}\,\mathrm{d}[\bar{F}^{-1}(u) - \bar{G}^{-1}(u)] \ge 0 \quad \text{for all } q \in (0,1).$$

From Barlow and Proschan (1975, Lemma 7.1(b), p. 120), it follows that

$$\int_0^{q} u^{\gamma_1'}\,\mathrm{d}[\bar{F}^{-1}(u) - \bar{G}^{-1}(u)] \ge 0 \quad \text{for all } q \in (0,1),\ 1 \le \gamma_1' \le \gamma_1,$$

which implies that

$$\int_0^{(1-p)^{1/\gamma_1'}} u^{\gamma_1'}\,\mathrm{d}[\bar{F}^{-1}(u) - \bar{G}^{-1}(u)] \ge 0 \quad \text{for all } p \in (0,1),\ 1 \le \gamma_1' \le \gamma_1,$$

or, equivalently, for all $1 \le \gamma_1' \le \gamma_1$,

$$\int_{\bar{F}^{-1}((1-p)^{1/\gamma_1'})}^{\infty} (\bar{F}(t))^{\gamma_1'}\,\mathrm{d}t \le \int_{\bar{G}^{-1}((1-p)^{1/\gamma_1'})}^{\infty} (\bar{G}(t))^{\gamma_1'}\,\mathrm{d}t \quad \text{for all } p \in (0,1),$$

and this means that $X_{(1,n',\tilde{m}',k)} \le_{\mathrm{ew}} Y_{(1,n',\tilde{m}',k)}$. □

Now we establish the main results of this section.

Theorem 3.1. *Let X and Y be two continuous random variables with strictly increasing distribution functions F and G, respectively. Let $X_{(r,n,\tilde{m},k)}$ and $Y_{(r,n,\tilde{m},k)}$, $r = 1, \ldots, n$, be GOSs based on F and G, respectively, with $m_i \ge -1$ for all i and let $D^X_{r,n}$ and $D^Y_{r,n}$, $r = 2, \ldots, n$, be the corresponding spacings. If $X_{(1,n,\tilde{m},k)} \le_{\mathrm{ew}} Y_{(1,n,\tilde{m},k)}$ then*

$$D^X_{r,n} \le_{\mathrm{icx}} D^Y_{r,n} \quad \textit{for } r = 2, \ldots, n.$$

Proof. We consider an increasing convex function ϕ, and prove that

$$\mathbb{E}[\phi(X_{(r,n,\tilde{m},k)} - X_{(r-1,n,\tilde{m},k)})] \le \mathbb{E}[\phi(Y_{(r,n,\tilde{m},k)} - Y_{(r-1,n,\tilde{m},k)})] \quad \text{for } 2 \le r \le n.$$

To establish this, consider the strictly increasing function $h = F^{-1}_{(r,n,\tilde{m},k)} G_{(r,n,\tilde{m},k)} = F^{-1}G$ for $r = 1, \ldots, n$. Since $X_{(r,n,\tilde{m},k)} \equiv_{\mathrm{st}} h(Y_{(r,n,\tilde{m},k)})$ for $r = 1, \ldots, n$, and the vectors $(X_{(r-1,n,\tilde{m},k)}, X_{(r,n,\tilde{m},k)})$ and $(h(Y_{(r-1,n,\tilde{m},k)}), h(Y_{(r,n,\tilde{m},k)}))$ for $2 \le r \le n$ have the same copula (this follows from the fact that two random vectors of GOSs with the same set of

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parameters and possibly based on different distributions have the same copula; see Belzunce *et al.* (2008) and from the strict increase of h), we obtain

$$\phi(X_{(r,n,\tilde{m},k)} - X_{(r-1,n,\tilde{m},k)}) \equiv_{\text{st}} \phi(h(Y_{(r,n,\tilde{m},k)}) - h(Y_{(r-1,n,\tilde{m},k)})) \qquad 2 \le r \le n.$$

Therefore, for $r = 2, \ldots, n$, we have

$$\begin{aligned}\mathbb{E}[\phi(X_{(r,n,\tilde{m},k)} - X_{(r-1,n,\tilde{m},k)})] &= \mathbb{E}[\phi(h(Y_{(r,n,\tilde{m},k)}) - h(Y_{(r-1,n,\tilde{m},k)}))] \\ &= \int \mathbb{E}[\phi(h(Y_{(r,n,\tilde{m},k)}) - h(t)) \mid Y_{(r-1,n,\tilde{m},k)} = t]g_{(r-1,n,\tilde{m},k)}(t)\,\mathrm{d}t, \end{aligned} \tag{3.7}$$

where $g_{(r-1,n,\tilde{m},k)}(t)$ is the density function of $Y_{(r-1,n,\tilde{m},k)}$. From Balakrishnan *et al.* (2012, Equation (3.8)), it follows that

$$[Y_{(r,n,\tilde{m},k)} \mid Y_{(r-1,n,\tilde{m},k)} = t] \simeq_{\text{st}} [Y_{(1,n-r+1,\tilde{m}',k)} \mid Y_{(1,n-r+1,\tilde{m}',k)} > t], \tag{3.8}$$

where $\tilde{m}' = (m'_1, \ldots, m'_{n-r})$ (recall that $\tilde{m} = (m_1, \ldots, m_{n-1})$) is such that $m'_j = m_{n-j}$ for $j = 1, \ldots, n-r$. Using the fact that $m_i \ge -1$ for all i, it follows that $m_1 + \cdots + m_{r-1} \ge 1 - r$, i.e. $M_1 - M_r \ge 1 - r$ which implies that $\gamma'_1 = k + n - r + M_r \le k + n - 1 + M_1 = \gamma_1$.

Then we can use Lemma 3.2 to find that the assumption $X_{(1,n,\tilde{m},k)} \le_{\text{ew}} Y_{(1,n,\tilde{m},k)}$ implies that

$$X_{(1,n-r+1,\tilde{m}',k)} \le_{\text{ew}} Y_{(1,n-r+1,\tilde{m}',k)}$$

which in turn implies, by Lemma 3.1, that

$$\begin{aligned}&\mathbb{E}[\phi(h(Y_{(1,n-r+1,\tilde{m}',k)}) - h(t)) \mid Y_{(1,n-r+1,\tilde{m}',k)} > t] \\ &\quad \le \mathbb{E}[\phi(Y_{(1,n-r+1,\tilde{m}',k)} - t) \mid Y_{(1,n-r+1,\tilde{m}',k)} > t],\end{aligned} \tag{3.9}$$

where $h = F^{-1}_{(1,n-r+1,\tilde{m}',k)} G_{(1,n-r+1,\tilde{m}',k)} = F^{-1}G$. Now, taking into account that ϕh is increasing, it follows from (3.8) and (3.9) that (3.7) is equivalent to

$$\begin{aligned}&\int \mathbb{E}[\phi(h(Y_{(1,n-r+1,\tilde{m}',k)}) - h(t)) \mid Y_{(1,n-r+1,\tilde{m}',k)} > t]g_{(r-1,n,\tilde{m},k)}(t)\,\mathrm{d}t \\ &\quad \le \int \mathbb{E}[\phi(Y_{(1,n-r+1,\tilde{m}',k)} - t) \mid Y_{(1,n-r+1,\tilde{m}',k)} > t]g_{(r-1,n,\tilde{m},k)}(t)\,\mathrm{d}t.\end{aligned} \tag{3.10}$$

By repeating the argument, we see that the right-hand side of (3.10) can be expressed as

$$\begin{aligned}&\int \mathbb{E}[\phi(Y_{(r,n,\tilde{m},k)} - Y_{(r-1,n,\tilde{m},k)}) \mid Y_{(r-1,n,\tilde{m},k)} = t]g_{(r-1,n,\tilde{m},k)}(t)\,\mathrm{d}t \\ &\quad = \mathbb{E}[\phi(Y_{(r,n,\tilde{m},k)} - Y_{(r-1,n,\tilde{m},k)})],\end{aligned}$$

which completes the proof of the theorem. □

It is easy to see that $X \le_{\text{disp}} Y$ implies that $X_{(1,n,\tilde{m},k)} \le_{\text{ew}} Y_{(1,n,\tilde{m},k)}$. However, we know from (3.1) that under the dispersive order, the spacings of GOSs are ordered in the stochastic order (which is stronger than the increasing convex order). Therefore, our interest is to study sufficient conditions for the comparisons of spacings in the increasing convex order when the dispersive order does not hold. The following result addresses this issue.

Corollary 3.1. *Let X and Y be two continuous random variables with strictly increasing distribution functions F and G, respectively. Let $X_{(r,n,\tilde{m},k)}$ and $Y_{(r,n,\tilde{m},k)}$, $r = 1, \dots, n$, be GOSs based on F and G, respectively, with $m_i \geq -1$ for all i and let $D^X_{r,n}$ and $D^Y_{r,n}$, $r = 2, \dots, n$, be the corresponding spacings. If*

$$\mathbb{E}[X_{(1,n,\tilde{m},k)}] \leq \mathbb{E}[Y_{(1,n,\tilde{m},k)}]$$

and there exists a value $p_0 \in (0, 1)$ such that $G^{-1}(p) \leq F^{-1}(p)$ for all $p \in (0, p_0)$ and $G^{-1}(p) - F^{-1}(p)$ is increasing on $[p_0, 1)$, then

$$D^X_{r,n} \leq_{\text{icx}} D^Y_{r,n} \quad r = 2, \dots, n.$$

Proof. Taking into account that $G^{-1}(F(x)) = G^{-1}_{(1,n,\tilde{m},k)}(F_{(1,n,\tilde{m},k)}(x))$, the proof follows from Theorem 2.1. □

4. Applications

In this section we provide two applications of previous results.

4.1. Record values and interarrival times of a minimal repair process

Given a sequence of i.i.d. random variables with common distribution F, the *record times* are defined by

$$L(1) = 1, \qquad L(n) = \min\{j > L(n-1) \mid X_j > X_{L(n-1)}\}, \quad n = 2, 3, \dots.$$

The sequence of *record values* is then defined as $X(n) \equiv X_{L(n)}$, $n = 1, 2, \dots$. A generalization of record values is the case in which $k \in \mathbb{N}$, resulting in the so-called k-records. In insurance, the record values of a time series of claims data describe the successive largest insurance claims. In reliability, record values are equally distributed as the times of repair of an item which is being continuously minimally repaired.

Kochar (1990) showed that the dispersive order is a sufficient condition to compare spacings of record values in the stochastic order. The following result shows that it is also possible to compare spacings of record values in the increasing convex order when the underlying random variables are ordered in the excess wealth order. The proof follows easily from Theorem 3.1 by setting $k = 1$ and $m_i = -1$ for all $i = 1, \dots, n-1$ (which gives the first n record values) and taking into account that the first record values are equally distributed as the distribution from which the record values are arising from.

Corollary 4.1. *Let X and Y be two random variables with continuous and strictly increasing distribution functions F and G, respectively. Let $X_{L(1)}, X_{L(2)}, \dots$ and $Y_{L(1)}, Y_{L(2)}, \dots$ be the sequences of record values arising from F and G, respectively. If $X \leq_{\text{ew}} Y$ then*

$$X_{L(r)} - X_{L(r-1)} \leq_{\text{icx}} Y_{L(r)} - Y_{L(r-1)} \quad \textit{for all } r = 1, 2, \dots.$$

4.2. Application to ECOMOR reinsurance treaty

The usual order statistics from a sample of i.i.d. random variables with common distribution F are a particular case of GOSs based on F when $k = 1$ and $m_i = 0$ for all $i = 1, \dots, n-1$. In this case, Theorem 3.1 states that given two random vectors of order statistics, if the minima from the random vectors are ordered in the excess wealth order, then the corresponding consecutive spacings are ordered in the increasing convex order. Specifically, we have the following result.

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Corollary 4.2. *Let X and Y be two continuous random variables with strictly increasing distribution functions F and G, respectively. Let $(X_{1:n}, \dots, X_{n:n})$ and $(Y_{1:n}, \dots, Y_{n:n})$ be two random vectors of order statistics based on two i.i.d. samples drawn from F and G, respectively. If $X_{1:n} \le_{\text{ew}} Y_{1:n}$ then*

$$X_{r:n} - X_{r-1:n} \le_{\text{icx}} Y_{r:n} - Y_{r-1:n} \quad \textit{for } 2 \le r \le n.$$

Observe, in particular, that $X_{1:n} \le_{\text{ew}} Y_{1:n}$ implies that

$$\mathbb{E}[X_{r:n} - X_{r-1:n}] \le \mathbb{E}[Y_{r:n} - Y_{r-1:n}] \quad \text{for } 2 \le r \le n$$

and, consequently, it also implies that

$$\mathbb{E}[X_{r:n} - X_{k:n}] \le \mathbb{E}[Y_{r:n} - Y_{k:n}] \quad \text{for all } 1 \le k \le r \le n. \tag{4.1}$$

We apply this result to the ECOMOR reinsurance treaty proposed by Thépaut (1950) and studied, among others, by Embrechts *et al.* (1997) and Jiang and Tang (2008). Following Asimit and Jones (2008), we consider a portfolio of n insurance contracts with associated i.i.d. loss random variables X_i, $i = 1, \dots, n$. Let $X_{k:n} \le \cdots \le X_{n:n}$ the corresponding $n - k + 1$ order statistics for some $1 < k < n$. Then the reinsurance amount covered by the ECOMOR reinsurance treaty is given by

$$R^X_{k,n}(t) = \sum_{r=k+1}^{n} (X_{r:n} - X_{k:n})$$

and the corresponding net premium is

$$\mathbb{E}[R^X_{k,n}(t)] = \sum_{r=k+1}^{n} \mathbb{E}(X_{r:n} - X_{k:n}).$$

From (4.1), it follows that

$$X_{1:n} \le_{\text{ew}} Y_{1:n} \quad \Longrightarrow \quad \mathbb{E}[R^X_{k,n}(t)] \le \mathbb{E}[R^Y_{k,n}(t)] \quad \text{for all } 1 < k < n. \tag{4.2}$$

From Shaked and Shanthikumar (2007, Theorem 3.B.26 and Equation (3.C.9)), it follows that $X \le_{\text{disp}} Y$ implies that $X_{1:n} \le_{\text{ew}} Y_{1:n}$ for all $n \ge 1$. By combining this fact with (4.2), we see that

$$X \le_{\text{disp}} Y \quad \Longrightarrow \quad \mathbb{E}[R^X_{k,n}(t)] \le \mathbb{E}[R^Y_{k,n}(t)] \quad \text{for all } 1 < k < n.$$

The following result follows from (4.2) and the Corollary 3.1. It is useful for comparing ECOMOR net premiums of portfolios based on independent distributions which fail to be ordered in the dispersive order.

Corollary 4.3. *Let X and Y be two continuous random variables with strictly increasing distribution functions F and G, respectively. Let $X_{(r,n)}$ and $Y_{(r,n)}$, $r = 1, \dots, n$, be the ordinary order statistics based on F and G, respectively. If $\mathbb{E}[X_{(1,n)}] \le \mathbb{E}[Y_{(1,n)}]$ and there exists a value $p_0 \in (0, 1)$ such that $G^{-1}(p) \le F^{-1}(p)$ for all $p \in (0, p_0)$ and $G^{-1}(p) - F^{-1}(p)$ is increasing on $[p_0, 1)$ then*

$$\mathbb{E}[R^X_{k,n}(t)] \le \mathbb{E}[R^Y_{k,n}(t)] \quad \textit{for all } 1 < k < n.$$

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Example 4.1. Suppose that X and Y are two Weibull distributed random variables with strictly positive parameters α_1, β_1 and α_2, β_2, respectively, denoted, as in Example 2.2, by $X \sim W(\alpha_1, \beta_1)$ and $Y \sim W(\alpha_2, \beta_2)$. Now let $X_{(1,n)}$ and $Y_{(1,n)}$ be the first-order statistics of two independent samples based on F and G, respectively. It is easy to see that $X_{(1,n)}$ and $Y_{(1,n)}$ are also Weibull random variables, with $X_{(1,n)} \sim W(\alpha_1 n^{\beta_1}, \beta_1)$ and $Y_{(1,n)} \sim W(\alpha_2 n^{\beta_2}, \beta_2)$.

Therefore, using the Example 2.2, from the Corollary 4.3, it follows that

$$\left.\begin{array}{r} \beta_1 > \beta_2 \\ \alpha_1 n^{\beta_1} \Gamma\left(\dfrac{\beta_1 + 1}{\beta_1}\right) \leq \alpha_2 n^{\beta_2} \Gamma\left(\dfrac{\beta_2 + 1}{\beta_2}\right) \end{array}\right\} \implies \mathbb{E}[R^X_{k,n}(t)] \leq \mathbb{E}[R^Y_{k,n}(t)]$$

for all $1 < k < n$.

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