Proceedings of the Edinburgh Mathematical Society (2018) **61**, 1–11
doi:10.1017/S0013091516000584

ON THE RANGE INCLUSION FOR NORMAL DERIVATIONS ON C^*-ALGEBRAS

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(Received 28 August 2015)

Abstract For a von Neumann subalgebra $A \subseteq \mathcal{B}(\mathcal{H})$ and any two elements $a, b \in A$ with a normal, such that the corresponding derivations d_a and d_b satisfy the condition $\|d_b(x)\| \leq \|d_a(x)\|$ for all $x \in A$, there exist completely bounded $(a)'$-bimodule map $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that $d_b|A = \varphi d_a|A = d_a\varphi|A$. (In particular $d_b(A) \subseteq d_a(A)$.) Moreover, if A is a factor, then φ can be taken to be normal and these equalities hold on $\mathcal{B}(\mathcal{H})$ instead of just on A. This result is not true for general (even primitive) C^*-algebras $\mathcal{A}$.

Keywords: C^*-algebra, derivation, completely bounded map

2010 *Mathematics subject classification:* Primary 46L57; 47B47; Secondary 47B15

1. Introduction

Derivations have continuously attracted attention (see for example, [**2**, Chapter 4]). Even determining the norm of an inner derivation on a C^*-algebra A turned out to be a much deeper and more interesting problem (connected with the structure of A) than it might have seemed at first sight (see [**3**] and the references therein).

For a C^*-algebra $A \subseteq \mathcal{B}(\mathcal{H})$ and elements $a, b \in A$ we consider the corresponding inner derivations d_a and d_b on $\mathcal{B}(\mathcal{H})$, where $d_a(x) := ax - xa$. In particular, we study the implications of the condition

$$\|d_b(x)\| \leq \kappa \|d_a(x)\| \ (\forall x \in A), \tag{1.1}$$

where κ is a constant, in the case when a is normal. (Note that κ can be assumed to be 1 if we replace a by κa.) The first systematic study of the case when $A = \mathcal{B}(\mathcal{H})$ was by Johnson and Williams in [**12**], who showed that, if a is normal, (1.1) is equivalent to the range inclusion

$$d_b(\mathcal{B}(\mathcal{H})) \subseteq d_a(\mathcal{B}(\mathcal{H})). \tag{1.2}$$

If a is not normal, (1.2) does not always imply (1.1), for by [**11**] it does not even imply that b is in the bicommutant of a. Conversely, if $A = \mathcal{B}(\mathcal{H})$, it turned out that (1.1) implies (1.2) under milder requirements on a than normality [16]. The work of Johnson and Williams was continued by several researchers (see example [5], [6], [8], [13]). In [13] Kissin and Shulman proved that if A is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ such that $d_a(A) \subseteq A$, $d_b(A) \subseteq A$ and a is normal, then the range inclusion $d_b(A) \subseteq d_a(A)$ implies the inequality (1.1) with a suitable constant κ. Here we consider the reverse question, whether (1.1) implies the inclusion $d_b(A) \subseteq d_a(A)$? We have found, somewhat surprisingly, that contrary to the case of $\mathcal{B}(\mathcal{H})$, in a general C^*-algebra A (even a homogeneous one) (1.1) does not necessarily imply the range inclusion $d_b(A) \subseteq d_a(A)$, but it does imply if A is a von Neumann algebra. The latter fact is a part of the main result of §2 (Theorem 2.1). Condition (1.1) obviously implies the existence of a unique bounded linear operator $\varphi : d_a(A) \to d_b(A)$ such that $\varphi d_a = d_b$. This map φ must be a bimodule homomorphism over the commutant $(a)' \cap A$ of a in A since both d_a and d_b are such homomorphisms. When $A = \mathcal{B}(\mathcal{H})$ and a is normal, $(a)'$ is locally cyclic. Therefore, φ is automatically completely bounded by a result of Smith [21] and consequently can be extended to $\mathcal{B}(\mathcal{H})$ by [18, Theorem 8.2]. In Theorem 2.1 we will show that a similar conclusion holds in a general (not necessarily injective) von Neumann algebra. Then in §3, we formulate an analogous, but weaker, result for prime C^*-algebras.

$\overline{S}$ denotes the weak* closure of a subset S in $\mathcal{B}(\mathcal{H})$ and $[a, x] := ax - xa$.

2. The von Neumann algebra case

Theorem 2.1. *Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and let $a, b \in A$ be such that a is normal and the corresponding derivations d_a and d_b satisfy the condition*

$$\|d_b(x)\| \leq \|d_a(x)\| \quad \text{for all } x \in A. \tag{2.1}$$

Then there exists a completely bounded $(a)'$-bimodule map $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that $\varphi(A) \subseteq A$ and $d_b|A = \varphi d_a|A = d_a\varphi|A$. (Hence in particular $d_b(A) \subseteq d_a(A)$.) Moreover, if A is a factor, then a weak continuous φ can be found such that $d_b = \varphi d_a = d_a \varphi$ on $\mathcal{B}(\mathcal{H})$.*

For a proof we need some preparation. The following lemma is proved in [16, 4.7].

Lemma 2.2. *Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a C^*-algebra, J a closed ideal in A, and let $a, b \in A$ satisfy $\|[b, x]\| \leq \|[a, x]\|$ for all $x \in A$. Then the same inequality holds for all $x \in \overline{A}$ and also for all cosets $\dot{x} \in A/J$.*

A function $f : K \to \mathbb{C}$ defined on a subset $K \subseteq \mathbb{C}$ is called a *Schur function* if there exists a constant κ such that for every sequence $(\lambda_i) \subseteq K$ the (infinite) matrix with the entries

$$\mu_{i,k} := \frac{f(\lambda_i) - f(\lambda_k)}{\lambda_i - \lambda_k}, \quad \text{where } \mu_{i,k} \text{ is interpreted as 0 if } \lambda_i = \lambda_k,$$

is a Schur multiplier on $B(\ell^2)$ with the multiplier norm at most κ. The smallest such κ is called the Schur constant of f.

https://doi.org/10.1017/S0013091516000584 Published online by Cambridge University Press

Now we can state a consequence of the main result of Johnson and Williams [**12**] and Lemma 2.2.

Theorem 2.3. *If A is a prime C^*-algebra and $a, b \in A$ are such that $\|[b,x]\| \leq \|[a,x]\|$ for all $x \in A$ and a is normal, then $b = f(a)$ for a Schur function f on the spectrum $\sigma(a)$ of a. Moreover, the Schur constant of f is ≤ 2.*

Proof. By [**7**, Proposition 3.1] there exists a separable prime C^*-subalgebra B of A containing a and b (an elementary proof of this is in [**15**, Lemma 3.2]). Then B is primitive by [**19**, p. 102], hence we may assume that B is an irreducible C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$. By Lemma 2.2 $\|[b,x]\| \leq \|[a,x]\|$ for all $x \in \overline{B} = \mathcal{B}(\mathcal{H})$, and hence by [**12**, Theorem 3.6] there exists a Schur function f (with $\kappa \leq 2$) on $\sigma(a)$ such that $b = f(a)$. □

Proof of Theorem 2.1. Let Δ be the maximal ideal space of the center Z of A. For each $t \in \Delta$ we denote by $A(t)$ the quotient $A/(tA)$ and by $x(t)$ the coset in $A(t)$ of an element $x \in A$. By [**9**, Lemma 10] the function $t \mapsto \|x(t)\|$ is continuous. Also by [**9**] each $A(t)$ is a prime C^*-algebra (in fact, by [**10**] each $A(t)$ is primitive, but the proof of that is much harder). By Lemma 2.2 the condition (2.1) is also satisfied in $A(t)$, and hence by Theorem 2.3 there exists a Schur function f_t on $\sigma(a(t))$ with Schur constant less than or equal to 2 such that $b(t) = f_t(a(t))$.

Given $\varepsilon > 0$ and $t \in \Delta$, there is an element of the form $a_{t,\varepsilon} = \sum \lambda_i e_i \in A$, where $\lambda_i \in \sigma(a(t))$ and the e_i are mutually orthogonal spectral projections of a in A with the sum $\sum_i e_i = 1$, such that $\|a(s) - a_{t,\varepsilon}(s)\| < \varepsilon$ for all s in a clopen neighborhood U_t of t. To show this, first choose $\mu_i \in \sigma(a)$ and (mutually orthogonal) projections $e_i \in A$ with the sum 1 such that $\|a - \sum_i \mu_i e_i\| < \varepsilon/2$. Then $\|a(t)e_i(t) - \mu_i e_i(t)\| < \varepsilon/2$ for each i, which implies that the distance of μ_i to the spectrum of $a(t)e_i(t)$ in the algebra $e_i(t)A(t)e_i(t)$ is less than $\varepsilon/2$ whenever $e_i(t) \neq 0$. Since this spectrum is contained in $\sigma(a(t))$, we can choose for each such i an element $\lambda_i \in \sigma(a(t))$ such that $|\lambda_i - \mu_i| < \varepsilon/2$. Then $\|\sum(\lambda_i - \mu_i)e_i(t)\| = \max_i |\lambda_i - \mu_i| < \varepsilon/2$, hence also $\|a(t) - \sum \lambda_i e_i(t)\| < \varepsilon$. By continuity this inequality persists in a neighborhood U_t of t. Moreover, since $b(t) = f_t(a(t))$ and f_t is continuous (in fact differentiable at each non-isolated point of $\sigma(a)$ by [**12**]), we can achieve (by choosing a possibly smaller neighborhood U_t) that $\|b(s) - b_{t,\varepsilon}(s)\| < \varepsilon$ for all $s \in U_t$, where $b_{t,\varepsilon} := \sum f_t(\lambda_i)e_i$. Covering Δ with finitely many such neighborhoods U_t and considering the corresponding partition of Δ, we see that there exist finitely many (say n) central projections $p_j \in A$ with the sum 1 and with the following property: for each j there exist finitely many scalars $\lambda_{i,j} \in \sigma(a(t_j))$, mutually orthogonal spectral projections $e_{i,j}$ of a with $\sum_i e_{i,j} = p_j$ and a Schur function $f_j := f_{t_j}$ (with the corresponding Schur constant at most 2) such that

$$\left\| \left(a - \sum_i \lambda_{i,j} e_{i,j} \right) p_j \right\| < \varepsilon \text{ and } \left\| \left(b - \sum_i f_j(\lambda_{i,j}) e_{i,j} \right) p_j \right\| < \varepsilon. \tag{2.2}$$

Let $a_{j,\varepsilon} = \sum_i \lambda_{i,j} e_{i,j}$, $b_{j,\varepsilon} = \sum_i f_j(\lambda_{i,j}) e_{i,j}$ and define $\varphi_{j,\varepsilon} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by

$$\varphi_{j,\varepsilon}(x) = \sum_{i,k} \frac{f_j(\lambda_{i,j}) - f_j(\lambda_{k,j})}{\lambda_{i,j} - \lambda_{k,j}} p_j e_{i,j} x e_{k,j} p_j, \quad (e_{k,j} p_j = e_{k,j}),$$

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where the quotient is interpreted as 0 if $\lambda_{i,j} = \lambda_{k,j}$. Observe that

$$d_{a_{j,\varepsilon}} \varphi_{j,\varepsilon} = d_{b_{j,\varepsilon}} = \varphi_{j,\varepsilon} d_{a_{j,\varepsilon}}. \tag{2.3}$$

To prove that set $\{\varphi_{j,\varepsilon} : \varepsilon \in (0,1]\}$ is bounded, fix j and denote $\lambda_{i,j}$ simply by λ_i, $e_{i,j}$ by e_i and f_j by f. Set

$$\mu_{i,k} = \frac{f(\lambda_i) - f(\lambda_k)}{\lambda_i - \lambda_k}.$$

Since f is a Schur function with the constant at most 2, by [**18**, Corollary 8.8] there exist vectors $\zeta_k = (\zeta_{m,k})$ and $\tau_k = (\tau_{m,k})$ in ℓ^2 such that $\mu_{i,k} = \langle \zeta_k, \tau_i \rangle$ and $\|\zeta_k\|, \|\tau_k\| \leq 2$. Now, for all $x \in \mathcal{B}(\mathcal{H})$ with $\|x\| = 1$ and all vectors $\xi, \eta \in \mathcal{H}$ we can estimate (using the Cauchy–Schwarz inequality several times)

$$\begin{aligned}
|\langle \varphi_{j,\varepsilon}(x)\eta, \xi\rangle| &= \left|\sum_{i,k} \mu_{i,k} \langle x e_k \eta, e_i \xi\rangle\right| = \left|\sum_{i,k} \langle \zeta_k, \tau_i\rangle \langle x e_k \eta, e_i \xi\rangle\right| \\
&= \left|\sum_{i,k}\sum_m \zeta_{m,k}\overline{\tau}_{m,i} \langle x e_k \eta, e_i \xi\rangle\right| = \left|\sum_m \left\langle x \sum_k \zeta_{m,k} e_k \eta, \sum_i \tau_{m,i} e_i \xi \right\rangle\right| \\
&\leqslant \|x\| \sum_m \left\|\sum_k \zeta_{m,k} e_k \eta\right\| \left\|\sum_i \tau_{m,i} e_i \xi\right\| \\
&= \sum_m \left(\sum_k |\zeta_{m,k}|^2 \|e_k \eta\|^2\right)^{1/2} \left(\sum_i |\tau_{m,i}|^2 \|e_i \xi\|^2\right)^{1/2} \\
&\leqslant \left(\sum_m \sum_k |\zeta_{m,k}|^2 \|e_k \eta\|^2\right)^{1/2} \left(\sum_m \sum_i |\tau_{m,i}|^2 \|e_i \xi\|^2\right)^{1/2} \\
&= \left(\sum_k \|\zeta_k\|^2 \|e_k \eta\|^2\right)^{1/2} \left(\sum_i \|\tau_i\|^2 \|e_i \xi\|^2\right)^{1/2} \\
&\leqslant 4 \left(\sum_k \|e_k \eta\|^2 \|\right)^{1/2} \left(\sum_i \|\eta_i \xi\|^2\right)^{1/2} = 4\|\eta\|\|\xi\|.
\end{aligned}$$

This implies that $\frac{1}{4}\varphi_{j,\varepsilon}$ is a contraction and a similar computation (replacing $x \in \mathcal{H}$ with $x \in \mathcal{H}^n$, $n \in \mathbb{N}$) shows that it is a complete contraction. Since the sum $\varphi_\varepsilon := \sum_{j=1}^n \varphi_{j,\varepsilon}$ is orthogonal (because the p_j are mutually orthogonal), it follows that the map $\frac{1}{4}\varphi_\varepsilon$ is completely contractive and clearly it is an $(a)'$-bimodule map. Now let φ be a weak* limit point of the net $(\varphi_\varepsilon)_{\varepsilon \to 0}$. Using (2.2) and (2.3) it is not hard to verify that $\|(\varphi_\varepsilon d_a - d_b)|A\| \to 0$ as $\varepsilon \to 0$, hence $\varphi d_a|A = d_b|A = d_a\varphi|A$. Also, $\varphi(A) \subset A$ follows from the definition of φ.

Although the maps d_a, d_b and φ are defined on all of $\mathcal{B}(\mathcal{H})$, the identity $d_b = d_a\varphi$ can be verified only on A since the projections p_j do not commute with all $\mathcal{B}(\mathcal{H})$. However, if A is a factor, then there is only one non-zero p_j, which is equal to the identity, and in this

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case $d_a\varphi = d_b = \varphi d_a$ on $\mathcal{B}(\mathcal{H})$. Moreover, if $\varphi = \varphi_{\rm n} + \varphi_{\rm s}$ is the decomposition of φ into the normal and singular parts, then writing the identity $d_b = d_a\varphi$ as $d_b - d_a\varphi_{\rm n} = d_a\varphi_{\rm s}$, the left-hand side of this identity is normal while the right is singular, so both must be 0. Since $\varphi_{\rm n}$ is normal and an $(a)'$-bimodule map, it has the form $\varphi_n(x) = \sum_{j\in\mathbb{J}} a_i x b_j$ for some index set $\mathbb{J}$ and elements $a_j, b_j \in (a)'' \subseteq A$ such that the sums $\sum_{j\in\mathbb{J}} a_j a_j^*$ and $\sum_{j\in\mathbb{J}} b_j^* b_j$ are weak* convergent [**21**], and hence $\varphi(A) \subseteq A$. Thus φ can be replaced by $\varphi_{\rm n}$. □

Remark 2.4. In Theorem 2.1, if A is not necessarily a factor, we may decompose $\psi := \varphi|A$ into the normal part $\psi_{\rm n}$ and the singular part $\psi_{\rm s}$ and we still have $\psi_{\rm n}(A) \subseteq A$, but $\psi_{\rm n}$ is not necessarily an $(a)'$-bimodule map (only an A^c-bimodule map, where $A^c := (a)' \cap A$ (since this hold for ψ).

A derivation d on a C^*-algebra A is called *normal* if $d^*d = dd^*$, where $d^*(x) := d(x^*)^*$. If d is an inner derivation (that is, if $d(x) = [a, x]$ for a fixed $a \in A$), then by a short direct calculation one can show that the condition $d^*d = dd^*$ means that $a^*a - aa^*$ is in the center Z of A. In the case $Z = \mathbb{C}$ it has been observed already in [**13**, p. 194] that this implies that a is normal. The same conclusion holds in a general C*-algebra, for $a^*a - aa^*$ commutes in particular with a, hence by the Kleinecke–Shirokov theorem [**4**, p. 91], $a^*a - aa^*$ is quasi-nilpotent, and hence 0 (since it is skew-adjoint).

If d_1 and d_2 are derivations on a C^*-algebra $A \subseteq \mathcal{B}(\mathcal{H})$ such that $d_2(A) \subseteq d_1(A)$, then by [**13**, Theorem 6.5] there exists a constant κ such that $\|d_2(x)\| \leqslant \|d_1(x)\|$ for all $x \in A$. The two derivations extend weak* continuously to derivations on $\overline{A}$ (see [**20**, Theorem 2.2.2]), which are necessarily inner [**20**, Theorem 2.5.1] and thus of the form d_a, d_b for elements $a, b \in \overline{A}$. Since d_1 is normal so is its extension d_a, and hence by the previous paragraph a must be normal. Thus from Theorem 2.1 and Lemma 2.2 we deduce the following consequence.

Corollary 2.5. *Let d_1 and d_2 be derivations on a C^*-algebra $A \subseteq \mathcal{B}(\mathcal{H})$ and assume that d_1 is normal. If $d_2(A) \subseteq d_1(A)$, then there exists a completely bounded $(a)'$-bimodule map φ on $\mathcal{B}(\mathcal{H})$ such that $\varphi(\overline{A}) \subseteq \overline{A}$ and $d_b|\overline{A} = d_a\varphi|\overline{A} = \varphi d_a|\overline{A}$, where d_a and d_b are the weak* continuous extensions to $\overline{A}$ of d_1 and d_2.*

Now we show by a simple example that Theorem 2.1 does not hold for general (even homogeneous) C^*-algebras.

Example 2.6. Let $A = C([-1, 1], \mathrm{M}_2(\mathbb{C}))$ be the C*-algebra of all continuous functions from the interval $[-1, 1]$ into 2×2 complex matrices. Let $a, b \in A$ be defined by $a(t) = |t|u$ and $b(t) = tu$ $(t \in [-1, 1])$, where $u \in A$ is the unitary

$$u(t) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

For each $x \in A$ we have

$$\|d_b(x)\| = \sup_{-1\leqslant t\leqslant 1} |t|\|ux(t) - x(t)u\| = \|ax - xa\| = \|d_a(x)\|.$$

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But nevertheless, $d_b(A)$ is not contained in $d_a(A)$. To show this, let

$$p = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{and hence} \quad d_b(p) = t\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Each $x \in A$ is of the form

$$x = \frac{1}{2}\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

where α, β, γ and δ are complex-valued continuous functions on $[-1, 1]$, and

$$d_a(x) = |t|\begin{bmatrix} 0 & \beta \\ -\gamma & 0 \end{bmatrix}.$$

If $d_b(p) = d_a(x)$, then (considering elements in position $(1, 2)$) $|t|\beta = t$ for all $t \in [-1, 1]$, which is impossible for a continuous function β.

3. The case of prime C^*-algebras

We will show by an example below that Theorem 2.1 cannot be generalized to primitive C^*-algebras, but the following weaker result still holds.

Proposition 3.1. *Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a prime C^*-algebra and let $a, b \in A$ with a normal. Then there exists a constant κ such that*

$$\|d_b(x)\| \le \kappa \|d_a(x)\| \text{ for all } x \in A \tag{3.1}$$

if and only if there exists a bounded net (φ_j) of completely bounded $(a)'$-bimodule maps $\varphi_j : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that $\|\varphi_j(d_a(x)) - d_b(x)\| \xrightarrow{j} 0$ for each $x \in \mathcal{B}(\mathcal{H})$ and $\varphi_j(A) \subseteq A$. In fact, if A is primitive, a sequence (φ_j) with the required properties can be found such that $\|\varphi_j d_a - d_b\|_{\mathrm{cb}} \xrightarrow{\varepsilon \to 0} 0$.

Proof. If a net (φ_j) with the required properties exists, then clearly (3.1) is satisfied. Conversely, suppose that (3.1) holds. It suffices to find a net (φ_j) with the required properties (and $\|\varphi_j\|_{\mathrm{cb}} \le 9$) for each separable prime C^*-subalgebra of A containing a and b, since each separable subalgebra of A is contained in a prime separable C^*-subalgebra by [7]. Thus we may assume that A is primitive. By Lemma 2.2, $b = f(a)$ for a Schur function f and we may assume that the Schur constant of f is 1.

Now, to construct the appropriate maps $\varphi_j : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, we first cover the plane $\mathbb{C}$ with a grid of small closed rectangles with sides parallel to the coordinate axes, so that the intersection of any two rectangles is either empty or a common edge. Then each rectangle intersects at most eight other rectangles. By taking slightly larger open rectangles, we can cover $\sigma(a)$ by finitely many such open rectangles $\{U_i\}_{i=1}^n$, each still intersecting only eight other rectangles. Further, given $\varepsilon > 0$, we may assume that the rectangles are so small that $|\lambda - \mu| < \varepsilon$ and $|f(\lambda) - f(\mu)| < \varepsilon$ whenever λ and μ are both contained in a union of two intersecting rectangles U_i. Let $\{g_i\}_{i=1}^n$ be a partition of unity subordinate

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to the covering $\{U_i\}_{i=1}^n$. Choose, for each i, a point $\lambda_i \in U_i$; then a and $b = f(a)$ are approximated (in norm) by elements

$$a_\varepsilon = \sum_{i=1}^n \lambda_i g_i(a) \text{ and } b_\varepsilon = \sum_{i=1}^n f(\lambda_i) g_i(a).$$

Set

$$\mu_{i,k} = \frac{f(\lambda_i) - f(\lambda_k)}{\lambda_i - \lambda_k}$$

(regarded as 0 if $\lambda_i = \lambda_k$) and define

$$\varphi_\varepsilon : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}), \quad \varphi_\varepsilon(x) = \sum_{i,k=1}^n \mu_{i,k} g_i(a) x g_k(a).$$

To show that the net (φ_ε) is bounded, as in the proof of Theorem 2.1 we use the fact that there exist vectors $\zeta_k = (\zeta_{m,k}), \tau_k = (\tau_{m,k}) \in \ell^2$ with $\|\zeta_k\|, \|\tau_k\| \leq 1$ such that $\mu_{i,k} = \langle \zeta_k, \tau_i \rangle$ (since f has the Schur constant 1). Then for all x in the unit ball of $\mathcal{B}(\mathcal{H})$ and $\xi, \eta \in \mathcal{H}$ we compute:

$$\begin{aligned} |\langle \varphi_\varepsilon(x)\eta, \xi \rangle| &= \left| \sum_{i,k} \mu_{i,k} \langle x g_k(a)\eta, g_i(a)\xi \rangle \right| = \left| \sum_m \langle x \sum_k \zeta_{m,k} g_k(a)\eta, \sum_i \tau_{m,i} g_i(a)\xi \rangle \right| \\ &\leqslant \sum_m \left\| \sum_k \zeta_{m,k} g_k(a)\eta \right\| \left\| \sum_i \tau_{m,i} g_i(a)\xi \right\|. \end{aligned}$$

For each k let $L(k) = \{i : g_k g_i \neq 0\}$; thus, by the definition of the functions g_k the set $L(k)$ contains at most nine elements. We can now estimate

$$\begin{aligned} \left\| \sum_k \zeta_{m,k} g_k(a)\eta \right\|^2 &= \sum_k \sum_{i \in L(k)} \zeta_{m,k} \overline{\zeta}_{m,i} \langle g_i(a) g_k(a)\eta, \eta \rangle \\ &\leqslant \sum_k \sum_{i \in L(k)} \|\zeta_{m,k} g_k(a)\eta\| \|\zeta_{m,i} g_i(a)\eta\| \\ &\leqslant \frac{1}{2} \sum_k \sum_{i \in L(k)} (\|\zeta_{m,k} g_k(a)\eta\|^2 + \|\zeta_{m,i} g_i(a)\eta\|^2) \\ &= \frac{9}{2} \sum_k \|\zeta_{m,k} g_k(a)\eta\|^2 + \frac{1}{2} \sum_k \sum_{i \in L(k)} \|\zeta_{m,i} g_i(a)\eta\|^2. \end{aligned}$$

Since each i is contained in at most nine sets $L(k)$, the last double sum is dominated by $9 \sum_k \|\zeta_{m,k} g_k(a)\eta\|^2$, and hence it follows that

$$\left\| \sum_k \zeta_{m,k} g_k(a)\eta \right\|^2 \leqslant 9 \sum_k \|\zeta_{m,k} g_k(a)\eta\|^2.$$

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A similar inequality also holds for $\|\sum_i \tau_{k,i} g_i(a)\xi\|^2$, and we can therefore continue the above estimate of $|\langle\varphi_\varepsilon(x)\eta,\xi\rangle|$ as

$$\begin{aligned}|\langle\varphi_\varepsilon(x)\eta,\xi\rangle| &\leqslant 9\sum_m \left(\sum_k |\zeta_{m,k}|^2\|g_k(a)\eta\|^2\right)^{1/2}\left(\sum_i |\tau_{m,i}|^2\|g_i(a)\xi\|^2\right)^{1/2}\\ &\leqslant 9\left(\sum_{m,k} |\zeta_{m,k}|\|g_k(a)\eta\|^2\right)^{1/2}\left(\sum_{m,i} |\tau_{m,i}|^2\|g_i(a)\xi\|^2\right)^{1/2}\\ &\leqslant 9\left(\sum_k \|g_k(a)\eta\|^2\right)^{1/2}\left(\sum_i \|g_i(a)\xi\|^2\right)^{1/2}.\end{aligned}$$

Since the functions $g_k g_i$ are non-negative,

$$\sum_k \|g_k(a)\eta\|^2 \leqslant \sum_{k,i}\langle g_i(a)\eta, g_k(a)\eta\rangle = \left\|\sum_k g_k(a)\eta\right\|^2 = \|\eta\|^2,$$

and hence (with a similar estimate for $\sum_i \|g_i(a)\xi\|^2$) we finally deduce that

$$|\langle\varphi_\varepsilon(x)\eta,\xi\rangle| \leq 9\|\eta\|\|\xi\|.$$

This implies that $\|\varphi_\varepsilon\| \leqslant 9$ and the same arguments also apply to $\|\varphi_\varepsilon\|_{\text{cb}}$.

We will show that $\|\varphi_\varepsilon d_{a_\varepsilon} - d_{b_\varepsilon}\| \to 0$ as $\varepsilon \to 0$. Using the definition of a_ε and that $\sum_i g_i(a) = 1$, we have $d_{a_\varepsilon}(x) = \sum_{i,k}(\lambda_i - \lambda_k)g_i(a)xg_k(a)$ for each $x \in A$ and a similar formula also holds for $d_{b_\varepsilon}(x)$. Using that $\sum_m g_m(a) = 1$ and that $g_i(a)g_m(a) = 0$ if $m \notin L(i)$, we now compute

$$\varphi_\varepsilon(d_{a_\varepsilon}(x)) = \sum_{i,k}\frac{f(\lambda_i)-f(\lambda_k)}{\lambda_i-\lambda_k}g_i(a)\left(\sum_{m\in L(i),l\in L(k)}(\lambda_m-\lambda_l)g_m(a)xg_l(a)\right)g_k(a)$$

and

$$\begin{aligned}d_{b_\varepsilon}(x) &= \sum_{i,k}(f(\lambda_i)-f(\lambda_k))g_i(a)xg_k(a)\\ &= \sum_{i,k}\frac{f(\lambda_i)-f(\lambda_k)}{\lambda_i-\lambda_k}g_i(a)\left(\sum_{m\in L(i),l\in L(k)}(\lambda_i-\lambda_k)g_m(a)xg_l(a)\right)g_k(a),\end{aligned}$$

and hence

$$\begin{aligned}&(\varphi_\varepsilon d_{a_\varepsilon} - d_{b_\varepsilon})(x)\\ &\quad= \sum_{i,k}\mu_{i,k}g_i(a)t\left(\sum_{m\in L(i),l\in L(k)}((\lambda_m-\lambda_i)-(\lambda_l-\lambda_k))g_m(a)xg_l(a)\right)g_k(a).\end{aligned}$$

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Since $|\lambda_m - \lambda_i| < \varepsilon$ and $|\lambda_l - \lambda_k| < \varepsilon$ if $m \in L(i)$ and $l \in L(k)$, we conclude (by essentially the same computation as above) that

$$\|\varphi_\varepsilon d_{a_\varepsilon} - d_{b_\varepsilon}\| \leqslant 2 \cdot 9 \cdot 9\varepsilon\|\varphi_\varepsilon\|\|x\| \leqslant 1458\varepsilon.$$

The required sequence of maps is therefore $\varphi_{1/j}$ $(j = 1, 2, \ldots)$. □

In the context of Proposition 3.1, in general $d_b(A) \nsubseteq d_a(A)$ (hence there does not exist $\varphi : A \to A$ satisfying $d_b = d_a\varphi$), as shown by the following example.

Example 3.2. Let $A = C^*(s)$ be the C^*-algebra generated by the unilateral shift s on $\mathcal{H} = \ell^2$ and let a be the diagonal operator with the entries $\alpha_n = \frac{1}{(2n+1)\pi/2}$ on the diagonal. A is irreducible [**17**, Theorem 3.5.5] and contains all compact operators, and in particular $a \in A$. It follows from [**14**, Theorem 3.8 and Corollary 3.6(ii)] that the function $f(t) = t^2 \sin(1/t)$ (with $f(0) := 0$) is a Schur function on any compact interval in $\mathbb{R}$, and thus the operator $b := f(a)$ satisfies (3.1). Clearly, b is a diagonal operator with the entries

$$\beta_n = f(\alpha_n) = \frac{(-1)^n}{(2n+1)^2\pi^2/4}$$

along the diagonal. For every matrix $x = [x_{i,j}] \in \mathcal{B}(\ell^2)$, the matrix of $d_a(x)$ is $[(\alpha_i - \alpha_j)x_{i,j}]$, and similarly $d_b(x) = [(\beta_i - \beta_j)x_{i,j}]$. We claim that $d_b(s) \notin d_a(A)$. Suppose the contrary: that $d_b(s) = d_a(x)$ for some $x \in A$; that is, that $[(\alpha_i - \alpha_j)x_{i,j}] = [(\beta_i - \beta_j)\delta_{i,j+1}]$, where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$. (We have used the fact that the matrix of s is $[\delta_{i,j+1}]$.) Then x is the matrix that has the entries

$$x_{i+1,i} = \gamma_i := \frac{\beta_{i+1} - \beta_i}{\alpha_{i+1} - \alpha_i} = \frac{(-1)^i}{\pi}\frac{(2i+1)^2 + (2i+3)^2}{(2i+1)(2i+3)}$$

just below the main diagonal and zeros elsewhere. However, since the quotient algebra $A/\mathcal{K}(\mathcal{H})$ is commutative [**17**, Theorem 3.5.11], the operator $[x, s]$ must be compact. In particular, the entries on the positions $(i+2, i)$ of the matrix of $[x, s]$ must tend to 0 as $i \to \infty$. But this entries are $\gamma_{i+1} - \gamma_i$ and have two accumulation points $\pm(4/\pi)$, which is a contradiction.

Problem. Suppose that a, b are elements of a simple unital W^*-algebra A such that (1.1) holds and a is normal. Is it then necessarily the case that $d_b(A) \subseteq d_a(A)$?

Remark 3.3. The topic of range inclusion of derivations is connected with the perturbation theory of normal operators (see [**1**, **14**]), which works well for a quite general class of functions (in a certain Besov space). In our context, if a function f on $\sigma(a)$ is sufficiently nice, then there exists a completely bounded map φ on $\mathcal{B}(\mathcal{H})$ such that $d_{f(a)} = \varphi d_a = d_a\varphi$ and $\varphi(A) \subseteq A$ for every unital C^*-subalgebra A of $\mathcal{B}(\mathcal{H})$ containing a. For example, if $a^* = a$, then it suffices that f is a restriction of a compactly supported function on $\mathbb{R}$, denoted again by f, such that f''' is continuous, hence $t^3\hat{f}(t)$ is bounded, where $\hat{f}$ is the Fourier transform of f. For convenience we sketch the proof, although a

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similar argument has already been used by others in different contexts. We use that

$$f(a) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(t)\mathrm{e}^{\mathrm{i}ta}$$

to express

$$d_{f(a)}(x) = [f(a), x] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(t)[\mathrm{e}^{\mathrm{i}ta}, x]\,\mathrm{d}t.$$

Then from

$$[\mathrm{e}^{\mathrm{i}ta}, x] = (\mathrm{e}^{\mathrm{i}ta}x\mathrm{e}^{-\mathrm{i}ta} - x)\mathrm{e}^{\mathrm{i}ta} = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{e}^{\mathrm{i}sa}x\mathrm{e}^{-\mathrm{i}sa})\mathrm{d}s\mathrm{e}^{\mathrm{i}ta} = \mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}sa}[a, x]\mathrm{e}^{\mathrm{i}(t-s)a}\,\mathrm{d}s$$

it follows that $[f(a), x] = \varphi([a, x]) = [a, \varphi(x)]$, where

$$\varphi(x) = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(t) \int_0^t \mathrm{e}^{\mathrm{i}sa}x\mathrm{e}^{\mathrm{i}(t-s)a}\,\mathrm{d}s\ \mathrm{d}t \quad (x \in \mathcal{M}_n(\mathcal{B}(\mathcal{H})),\ n \in \mathbb{N}). \tag{3.2}$$

Thus

$$\|\varphi\| \leqslant \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} |\hat{f}(t)t|\,\mathrm{d}t \right) < \infty,$$

where the last inequality is a consequence of boundedness of $t^3\hat{f}(t)$. Approximating the first integral in (3.2) by an integral over a finite interval and then the double integral by Riemann sums, we see that $\varphi(A) \subseteq A$. The condition that f''' exists and is continuous is probably much too strong and we may instead ask the following.

Problem. What minimal smoothness properties must a Schur function f have in order that $d_{f(a)}(A) \subseteq d_a(A)$ for all unital (simple) C^*-algebras A? Is it sufficient that f can be extended to a continuously differentiable function on $\mathbb{C}$?

Acknowledgements. The author acknowledges the financial support from the Slovenian Research Agency (research core funding no. P1-0288).

References

1. A. B. Aleksandrov, V. V. Peller, D. Potapov and F. Sukochev, Functions of normal operators under perturbations, *Adv. Math.* **226** (2011), 5216–5251.
2. P. Ara and M. Mathieu, *Local multipliers of C^*-algebras*, Springer Monographs in Mathematics (Springer, 2003).
3. R. J. Archbold, E. Kaniuth and D. W. B. Somerset, Norms of inner derivations for multiplier algebras of C^*-algebras and group C^*-algebras II, *Adv. Math.* **280** (2015), 225–255.
4. F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete Volume 80 (Springer, 1973).
5. M. Brešar, The range and kernel inclusion of algebraic derivations and commuting maps, *Q. J. Math.* **56** (2005), 31–41.
6. M. Brešar, B. Magajna and S. Špenko, Identifying derivations through the spectra of their values, *Integ. Eqns Operat. Theory* **73**, (2012), 395–411.
7. G. A. Elliott and L. Zsido, Almost uniformly continuous automorphism groups of operator algebras, *J. Operat. Theory* **8** (1982), 227–277.

8. C. K. Fong, Range inclusion for normal derivations, *Glasgow Math. J.* **25** (1984), 255–262.
9. J. Glimm, A Stone-Weierstrass theorem for C^*-algebras, *Ann. Math.* **72** (1960), 216–244.
10. H. Halpern, Irreducible module homomorphisms of a von Neumann algebra into its center, *Trans. Am. Math. Soc.* **140** (1969), 195–221.
11. Y. Ho, Commutants and derivation ranges, *Tohoku Math. J.* **27** (1975), 509–514.
12. B. E. Johnson and J.P. Williams, The range of a normal derivation, *Pac. J. Math.* **58** (1975), 105–122.
13. E. Kissin and V. S. Shulman, On the range inclusion of normal derivations: variations on a theme by Johnson, Williams and Fong, *Proc. Lond. Math. Soc.* **83** (2001), 176–198.
14. E. Kissin and V. S. Shulman, Classes of operator smooth functions, I, operator Lipschitz functions, *Proc. Edinb. Math. Soc.* **48** (2005), 151–173.
15. B. Magajna, The Haagerup norm on the tensor product of operator modules, *J. Funct. Analysis* **129** (1995), 325–348.
16. B. Magajna, Variance of operators and derivations, *J. Math. Analysis Applic.* **433** (2016), 1–30.
17. G. J. Murphy, *C^*-algebras and operator theory* (Academic Press, 1990).
18. V. I. Paulsen, *Completely bounded maps and operator algebras,* Cambridge Studies in Advanced Mathematics, Volume 78 (Cambridge University Press, 2002).
19. G. K. Pedersen, *C^*-algebras and their automorphism groups* (Academic Press, 1979).
20. A. M. Sinclair and Roger R. Smith, *Hochschild cohomology of von Neumann algebras*, London Mathematical Society Lecture Notes Series, Volume 203 (Cambridge Universit Press, 1995).
21. R. R. Smith, Completely bounded module maps and the Haagerup tensor product, *J. Funct. Analysis* **102** (1991), 156–175.