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A NOTE ON ENTROPY OF AUTO-EQUIVALENCES: LOWER BOUND AND THE CASE OF ORBIFOLD PROJECTIVE LINES

KOHEI KIKUTA, YUUKI SHIRAISHI
AND ATSUSHI TAKAHASHI

Abstract. Entropy of categorical dynamics is defined by Dimitrov–Haiden–Katzarkov–Kontsevich. Motivated by the fundamental theorem of the topological entropy due to Gromov–Yomdin, it is natural to ask an equality between the entropy and the spectral radius of induced morphisms on the numerical Grothendieck group. In this paper, we add two results on this equality: the lower bound in a general setting and the equality for orbifold projective lines.

§1. Introduction

It is interesting to bring some dynamical view points into the category theory. Motivated by the classical theory of dynamical systems, the notion of entropy of categorical dynamical systems (entropy of endofunctors for short) is defined by Dimitrov *et al.* [DHKK]. The entropy of endofunctors is actually similar to the topological entropy in the sense of sharing many properties (Lemmas 2.7–2.9). Moreover, the entropy of the derived pullback of a surjective endomorphism of a smooth projective variety over $\mathbb{C}$ is equal to its topological entropy [KT]. In other words, the entropy of endofunctors can be thought of as a categorical generalization of the topological entropy.

In this paper, we add two results on the entropy of endofunctors. The first one is that, for the perfect derived categories $\mathrm{per}(B)$ of a smooth proper differential graded algebra B, the lower bound of the entropy $h(F)$ of an endofunctor F is given by the natural logarithm of the spectral radius $\rho(\mathcal{N}(F))$ on the numerical Grothendieck group, called the (numerical) Gromov–Yomdin type inequality (See also [KT, Conjecture 5.3]). It is motivated by the fundamental theorem of the topological entropy for complex dynamics on algebraic varieties due to Gromov–Yomdin [Gro1, Gro2, Yom]:

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THEOREM (Theorem 2.13). *For each endofunctor F of* $\mathrm{per}(B)$ *admitting left or right adjoint functors, such that $F^n B \not\cong 0$ for $n \geqslant 0$, we have*

$$h(F) \geqslant \log \rho(\mathcal{N}(F)). \tag{1}$$

For the proof, we use some norm inspired by the theory of dynamical degree and algebraic cycles due to Truong [Tru]. Ikeda shows this inequality by the mass growth for Bridgeland's stability conditions [Ike].

The equality in the Gromov–Yomdin type inequality is now known to hold for elliptic curves [Kik], varieties with the ample (anti-)canonical sheaf [KT] and abelian surfaces [Yos], which gives some applications to the topological entropy of dynamics on moduli spaces of stable objects in the sense of Bridgeland [Ouc1, Yos]. But, in general, it does not hold for some Calabi–Yau varieties [Fan, Ouc2]. As a corollary of the first main theorem, it is easy to show the equality for derived categories of hereditary finite-dimensional algebras (Proposition 2.14, Corollary 2.15).

The second result of this paper claims the equality for the derived category $\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})$ of an orbifold projective line $\mathbb{P}^1_{A,\Lambda}$ introduced by Geigle–Lenzing [GL]. Orbifold projective lines are important and interesting objects since they are not only in the next class to hereditary finite-dimensional algebras but few examples whose homological and classical mirror symmetry are well understood (cf. [IST, IST2, IT, Kea, Ros, ST, Tak1, Tak2, Ued]):

THEOREM (Theorem 3.10). *For each auto-equivalence F of $\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})$, we have*

$$h(F) = \log \rho(\mathcal{N}(F)). \tag{2}$$

Moreover, the spectral radius $\rho(\mathcal{N}(F))$ is an algebraic integer, and $h(F) = 0$ if $\chi_A \neq 0$.

It is an important and interesting problem to find a characterization of endofunctors attaining the lower bound of the inequality (1).

§2. Preliminaries

2.1 Notations and terminologies

Throughout this paper, we work over the base field $\mathbb{C}$ and all triangulated categories are $\mathbb{C}$-linear and not equivalent to the zero category. The translation functor on a triangulated category is denoted by [1]. All (triangulated) functors are $\mathbb{C}$-linear.

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A triangulated category $\mathcal{T}$ is called *split-closed* if every idempotent in $\mathcal{T}$ splits, namely, if it contains all direct summands of its objects, and it is called *thick* if it is split-closed and closed under isomorphisms. For an object $M \in \mathcal{T}$, we denote $\langle M \rangle$ by the smallest thick triangulated subcategory containing M. An object $G \in \mathcal{T}$ is called a *split-generator* if $\langle G \rangle = \mathcal{T}$. A triangulated category $\mathcal{T}$ is said to be *of finite type* if for all $M, N \in \mathcal{T}$ we have $\sum_{n\in\mathbb{Z}} \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{T}}(M, N[n]) < \infty$.

2.2 Complexity

From now on, $\mathcal{T}, \mathcal{T}'$ denote triangulated categories of finite type.

DEFINITION 2.1. [DHKK, Definition 2.1] For each $M, N \in \mathcal{T}$, define the function $\delta_{\mathcal{T},t}(M, N) : \mathbb{R} \longrightarrow \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ in t by (i) $\delta_{\mathcal{T},t}(M, N) := 0$ if $N \simeq 0$, (ii)

$$\delta_{\mathcal{T},t}(M, N) := \inf \left\{ \sum_{i=1}^{p} \exp(n_i t) \;\middle|\; \begin{array}{ccccc} 0 & \longrightarrow & A_1 & \cdots & A_{p-1} \longrightarrow N \oplus N' \\ & \nwarrow \;\; \swarrow & & & \nwarrow \;\; \swarrow \\ & M[n_1] & & \cdots & M[n_p] \end{array} \right\}$$

if $N \in \langle M \rangle$ and (iii) $\delta_{\mathcal{T},t}(M, N) := \infty$ if $N \notin \langle M \rangle$. The function $\delta_{\mathcal{T},t}(M, N)$ is called the *complexity* of N with respect to M.

REMARK 2.2. If $\mathcal{T}$ has a split-generator G and $M \in \mathcal{T}$ is not isomorphic to a zero object, then an inequality $1 \leqslant \delta_{\mathcal{T},0}(G, M) < \infty$ holds.

We recall some basic properties of the complexity.

LEMMA 2.3. *Let $M_1, M_2, M_3, M_4 \in \mathcal{T}$.*

(i) *If $M_2 \in \langle M_1 \rangle$ and $M_2 \ncong 0$, then $0 < \delta_{\mathcal{T},t}(M_1, M_2)$.*
(ii) *If $M_1 \cong M_3$, then $\delta_{\mathcal{T},t}(M_1, M_2) = \delta_{\mathcal{T},t}(M_3, M_2)$.*
(iii) *If $M_2 \cong M_3$, then $\delta_{\mathcal{T},t}(M_1, M_2) = \delta_{\mathcal{T},t}(M_1, M_3)$.*
(iv) *If $M_2 \ncong 0$, then $\delta_{\mathcal{T},t}(M_1, M_3) \leqslant \delta_{\mathcal{T},t}(M_1, M_2)\delta_{\mathcal{T},t}(M_2, M_3)$.*
(v) *We have $\delta_{\mathcal{T},t}(M_4, M_2) \leqslant \delta_{\mathcal{T},t}(M_4, M_1) + \delta_{\mathcal{T},t}(M_4, M_3)$ for an exact triangle $M_1 \to M_2 \to M_3$.*
(vi) *We have $\delta_{\mathcal{T}',t}(F(M_1), F(M_2)) \leqslant \delta_{\mathcal{T},t}(M_1, M_2)$ for any triangulated functor $F : \mathcal{T} \longrightarrow \mathcal{T}'$.*

LEMMA 2.4. *Let $\mathcal{D}^b(\mathbb{C})$ be the bounded derived category of finite-dimensional $\mathbb{C}$-vector spaces. For $M \in \mathcal{D}^b(\mathbb{C})$, we have the following inequality*

$$\delta_{\mathcal{D}^b(\mathbb{C}),t}(\mathbb{C}, M) = \sum_{l\in\mathbb{Z}} (\dim_{\mathbb{C}} H^l(M)) \cdot e^{-lt}. \tag{3}$$

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2.3 Entropy of endofunctors

Endofunctor F means triangulated functor $F:\mathcal{T}\to\mathcal{T}$. We assume that all endofunctors of $\mathcal{T}$ satisfy that $F^nG\ncong 0$ for $n\geqslant 0$ (if $\mathcal{T}$ has a split-generator G).

DEFINITION 2.5. [DHKK, Definition 2.4] Let G be a split-generator of $\mathcal{T}$ and F an endofunctor of $\mathcal{T}$. The *entropy* of F is the function $h_t(F):\mathbb{R}\longrightarrow\{-\infty\}\cup\mathbb{R}$ given by

$$h_t(F):=\lim_{n\to\infty}\frac{1}{n}\log\delta_{\mathcal{T},t}(G,F^nG). \tag{4}$$

It follows from [DHKK, Lemma 2.5] that the entropy is well defined and does not depend on the choice of split-generators.

LEMMA 2.6. *Let G,G' be split-generators of $\mathcal{T}$ and F an endofunctor of $\mathcal{T}$. The entropy $h_t(F)$ of F is given by*

$$h_t(F)=\lim_{n\to\infty}\frac{1}{n}\log\delta_{\mathcal{T},t}(G,F^nG'). \tag{5}$$

The three lemmas below show that the entropy of endofunctors is similar to the topological entropy.

LEMMA 2.7. *Let G be a split-generator of $\mathcal{T}$ and F_1,F_2 endofunctors of $\mathcal{T}$.*

(i) *If $F_1\cong F_2$, then $h_t(F_1)=h_t(F_2)$.*
(ii) *We have $h_t(F_1^m)=mh_t(F_1)$ for $m\geqslant 1$.*
(iii) *We have $h_t(F_1F_2)=h_t(F_2F_1)$.*
(iv) *If $F_1F_2\cong F_2F_1$, then $h_t(F_1F_2)\leqslant h_t(F_1)+h_t(F_2)$.*
(v) *If $F_1=F_2[m]$ $(m\in\mathbb{Z})$, then $h_t(F_1)=h_t(F_2)+mt$.*

LEMMA 2.8. *Let F_i be an endofunctor of $\mathcal{T}_i$ with a split-generator G_i $(i=1,2)$. If there exists a fully faithful functor $F':\mathcal{T}_2\to\mathcal{T}_1$, which has left and right adjoint functors, such that $F'F_2\simeq F_1F'$, then $h_t(F_2)\leqslant h_t(F_1)$.*

LEMMA 2.9. *Let F_i be an endofunctor of $\mathcal{T}_i$ with a split-generator G_i $(i=1,2)$. If there exists an essentially surjective functor $F':\mathcal{T}_1\to\mathcal{T}_2$ such that $F'F_1\simeq F_2F'$, then $h_t(F_2)\leqslant h_t(F_1)$.*

As a corollary of Lemma 2.9, we have the following

COROLLARY 2.10. *Let F' an auto-equivalence of $\mathcal{T}$. The entropy is a class function, namely, $h_t(F'FF'^{-1})=h_t(F)$.*

Let B be a smooth proper differential graded (dg) $\mathbb{C}$-algebra B and $\mathrm{per}(B)$ the perfect derived category of dg B-modules, the full triangulated subcategory of the derived category $\mathcal{D}(B)$ of dg B-modules containing B closed under isomorphisms and taking direct summands. By definition, B is a split-generator of $\mathrm{per}(B)$.

The following proposition enables us to compute entropy.

PROPOSITION 2.11. [DHKK, Theorem 2.7] *Let G, G' be split-generators of $\mathrm{per}(B)$ and F an endofunctor of $\mathrm{per}(B)$. The entropy $h_t(F)$ is given by*

$$h_t(F) = \lim_{n\to\infty} \frac{1}{n} \log \delta'_{\mathrm{per}(B),t}(G, F^n G'), \tag{6}$$

where

$$\delta'_{\mathrm{per}(B),t}(M, N) := \sum_{m\in\mathbb{Z}} \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{per}(B)}(M, N[m])e^{-mt}, \quad M, N \in \mathrm{per}(B). \tag{7}$$

Proof. The following is proven in the proof of [DHKK, Theorem 2.7].

LEMMA 2.12. *For each $M \in \mathrm{per}(B)$, there exist $C_1(t), C_2(t)$ for $t \in \mathbb{R}$ such that*

$$C_1(t)\delta_{\mathrm{per}(B),t}(G, M) \leqslant \delta'_{\mathrm{per}(B),t}(G, M) \leqslant C_2(t)\delta_{\mathrm{per}(B),t}(G, M).$$

In particular, for each $M \in \mathrm{per}(B)$ we have

$$\lim_{n\to\infty} \frac{1}{n} \log \delta_{\mathrm{per}(B),t}(G, M) = \lim_{n\to\infty} \frac{1}{n} \log \delta'_{\mathrm{per}(B),t}(G, M). \tag{8}$$

Together with Lemma 2.6, we have

$$h_t(F) = \lim_{n\to\infty} \frac{1}{n} \log \delta_{\mathrm{per}(B),t}(G, F^n G') = \lim_{n\to\infty} \frac{1}{n} \log \delta'_{\mathrm{per}(B),t}(G, F^n G').$$

We finished the proof of the proposition. ∎

In order to state the first main theorem, we prepare some terminologies. For $M, N \in \mathrm{per}(B)$, set

$$\chi(M, N) := \sum_{n\in\mathbb{Z}} (-1)^n \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{per}(B)}(M, N[n]). \tag{9}$$

It naturally induces a bilinear form on the Grothendieck group $K_0(\mathrm{per}(B))$ of $\mathrm{per}(B)$, called the *Euler form*, which is denoted by the same letter χ. Then the *numerical Grothendieck group* $\mathcal{N}(\mathrm{per}(B))$ is defined as the quotient of $K_0(\mathrm{per}(B))$ by the radical of χ (which is well defined by the Serre duality). It is important to note that $\mathcal{N}(\mathrm{per}(B))$ is a free abelian group of finite rank by Hirzebruch–Riemann–Roch theorem [Shk, Lun]. If an endofunctor F of $\mathrm{per}(B)$ admits left or right (hence both by the Serre duality) adjoint functors, it respects the radical of χ. Therefore, it induces an endomorphism $\mathcal{N}(F)$ on $\mathcal{N}(\mathrm{per}(B))$. Note that an endofunctor lifting to a dg endofunctor of the dg category $\mathrm{per}_{dg}(B)$ admits adjoint functors. The *spectral radius* $\rho(\mathcal{N}(F))$ of $\mathcal{N}(F)$ is the maximum of absolute values of eigenvalues of $\mathbb{C}$-linear endomorphism $\mathcal{N}(F)\otimes_{\mathbb{Z}}\mathbb{C}$. Set $\delta_{\mathcal{T}}:=\delta_{\mathcal{T},0}$, $\delta'_{\mathcal{T}}:=\delta'_{\mathcal{T},0}$, $h:=h_0$.

Inspired by the theory of dynamical degree and algebraic cycles due to Truong (cf. [Tru, eq. (3.2)]), we show the following:

THEOREM 2.13. *For each endofunctor F of* $\mathrm{per}(B)$ *admitting left or right adjoint functors, we have*

$$h(F)\geqslant \log\rho(\mathcal{N}(F)). \tag{10}$$

Proof. Let $v_1,\ldots,v_p$ $(v_i=[M_i], M_i\in\mathrm{per}(B))$ be a fixed basis of $\mathcal{N}(\mathrm{per}(B))$. Set $M_0:=\oplus_i M_i, \mathcal{N}(\mathrm{per}(B))_{\mathbb{R}}:=\mathcal{N}(\mathrm{per}(B))\otimes_{\mathbb{Z}}\mathbb{R}, \mathcal{N}(F)_{\mathbb{R}}:=\mathcal{N}(F)\otimes_{\mathbb{Z}}\mathbb{R}, \chi_{\mathbb{R}}:=\chi\otimes_{\mathbb{Z}}\mathbb{R}$. Define a norm $\|\cdot\|$ on $\mathcal{N}(\mathrm{per}(B))_{\mathbb{R}}$ by

$$\|v\|:=\sum_{i=1}^{p}|\chi_{\mathbb{R}}(v_i,v)|,\quad v\in\mathcal{N}(\mathrm{per}(B))_{\mathbb{R}}, \tag{11}$$

which induces an operator norm of $\mathcal{N}(F)_{\mathbb{R}}$, that is,

$$\|\mathcal{N}(F)_{\mathbb{R}}\|:=\sup_{\|v\|=1}\|\mathcal{N}(F)_{\mathbb{R}}v\|.$$

By the compactness of the subset $\{\|v\|=1\}\subset\mathcal{N}(\mathrm{per}(B))_{\mathbb{R}}$, there exists a positive number $C>0$ such that

$$\sum_{i,j=1}^{p}|\chi(v_i,\mathcal{N}(F^n)v_j)|\geqslant C\cdot\|\mathcal{N}(F^n)_{\mathbb{R}}\|.$$

Note that $B\oplus M_0$ is a split-generator of $\mathrm{per}(B)$. By Proposition 2.11, the statement follows from

$$\begin{aligned} h(F)&=\lim_{n\to\infty}\frac{1}{n}\log\delta'_{\mathrm{per}(B)}(B\oplus M_0,F^n(B\oplus M_0))\\ &\geqslant\lim_{n\to\infty}\frac{1}{n}\log\delta'_{\mathrm{per}(B)}(M_0,F^nM_0)\end{aligned}$$

$$\geqslant \lim_{n\to\infty}\frac{1}{n}\log\sum_{i,j}|\chi(v_i,\mathcal{N}(F^n)v_j)|$$
$$\geqslant \lim_{n\to\infty}\frac{1}{n}\log\|\mathcal{N}(F)^n_{\mathbb{R}}\| = \log\rho(\mathcal{N}(F)). \qquad \square$$

Let $\mathrm{Auteq}(\mathcal{T})$ be the group of (natural isomorphism classes of) auto-equivalences of a triangulated category $\mathcal{T}$.

PROPOSITION 2.14. *Let B be a hereditary finite-dimensional $\mathbb{C}$-algebra. For each auto-equivalence $F \in \mathrm{Auteq}(\mathrm{per}(B))$, we have*

$$h(F) = \log\rho(\mathcal{N}(F)). \tag{12}$$

Proof. Due to Theorem 2.13, we only need to show the upper bound. Let $P_1,\ldots,P_{\dim_{\mathbb{C}} B}$ be indecomposable modules. Each auto-equivalence F sends an indecomposable object to an indecomposable one. Since B is hereditary, there exists $m\in\mathbb{Z}$ such that the indecomposable object $F^n(P_i)[m]$ is isomorphic to an object concentrated in degree zero, namely, a B-module. By Proposition 2.11, we have

$$h(F) = \lim_{n\to\infty}\frac{1}{n}\log\delta'_{\mathrm{per}(B)}\left(B, F^n\left(\bigoplus_{i=1}^{\dim_{\mathbb{C}} B} P_i\right)\right)$$
$$= \lim_{n\to\infty}\frac{1}{n}\log\sum_{i=1}^{\dim_{\mathbb{C}} B}|\chi(B,F^n(P_i))| \leqslant \log\rho(\mathcal{N}(F)). \qquad \square$$

COROLLARY 2.15. *Suppose that $B=\mathbb{C}\vec{\Delta}$ for some Dynkin quiver $\vec{\Delta}$. Then, we have*

$$h(F) = \log\rho(\mathcal{N}(F)) = 0. \tag{13}$$

Proof. It is known by [MY, Theorem 3.8], that

$$\mathrm{Auteq}(\mathrm{per}(B)) \cong \langle \mathcal{S}_B, \mathcal{S}_B[-1]\rangle \times \mathrm{Aut}(\vec{\Delta}), \tag{14}$$

where $\mathcal{S}_B$ is the Serre functor of $\mathrm{per}(B)$ and $\mathrm{Aut}(\vec{\Delta})$ is the finite subgroup of $\mathrm{Auteq}(\mathrm{per}(B))$ consisting of automorphisms of $\vec{\Delta}$. Again, by [MY, Theorem 3.8], $\mathcal{S}_B$ is of finite order up to translation. The statement follows from Lemma 2.7(ii), (iv) and (v). $\square$

§3. Orbifold projective lines

In this section, we shall show the Gromov–Yomdin type theorem for the entropy of an auto-equivalenceon the derived category $\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})$ of coherent

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sheaves on an orbifold projective line $\mathbb{P}^1_{A,\Lambda}$. We first recall the definition of orbifold projective line in [GL].

Let $r \geqslant 3$ be a positive integer. Let $A = (a_1, \ldots, a_r)$ be a multiplet of positive integers and $\Lambda = (\lambda_1, \ldots, \lambda_r)$ a multiplet of pairwise distinct elements of $\mathbb{P}^1(\mathbb{C})$ normalized such that $\lambda_1 = \infty$, $\lambda_2 = 0$ and $\lambda_3 = 1$.

In order to introduce an orbifold projective line, we prepare some notations.

DEFINITION 3.1. Let r, A and Λ be as above.

(i) Define a ring $R_{A,\Lambda}$ by

$$R_{A,\Lambda} := \mathbb{C}[X_1, \ldots, X_r]/I_\Lambda, \tag{15a}$$

where I_Λ is an ideal generated by $r-2$ homogeneous polynomials

$$X_i^{a_i} - X_2^{a_2} + \lambda_i X_1^{a_1}, \quad i = 3, \ldots, r. \tag{15b}$$

(ii) Denote by L_A an abelian group generated by r-letters $\vec{X}_i$, $i = 1, \ldots, r$ defined as the quotient

$$L_A := \bigoplus_{i=1}^{r} \mathbb{Z}\vec{X}_i / M_A, \tag{16a}$$

where M_A is the subgroup generated by the elements

$$a_i\vec{X}_i - a_j\vec{X}_j, \quad 1 \leqslant i < j \leqslant r. \tag{16b}$$

(iii) Set

$$a := \text{l.c.m}(a_1, \ldots, a_r), \qquad \mu_A := 2 + \sum_{i=1}^{r}(a_i - 1),$$
$$\chi_A := 2 + \sum_{i=1}^{r}\left(\frac{1}{a_i} - 1\right). \tag{17}$$

We then consider the following quotient stack:

DEFINITION 3.2. Let r, A and Λ be as above. Define a stack $\mathbb{P}^1_{A,\Lambda}$ by

$$\mathbb{P}^1_{A,\Lambda} := [(\mathrm{Spec}(R_{A,\Lambda}) \backslash \{0\}) / \mathrm{Spec}(\mathbb{C}L_A)], \tag{18}$$

which is called the *orbifold projective line* of type (A, Λ).

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The orbifold projective line is a Deligne–Mumford stack whose coarse moduli space is a smooth projective line $\mathbb{P}^1$.

Denote by $\mathrm{gr}^{L_A}(R_{A,\Lambda})$ the abelian category of finitely generated L_A-graded $R_{A,\Lambda}$-modules and denote by $\mathrm{tor}^{L_A}(R_{A,\Lambda})$ the full subcategory of $\mathrm{gr}^{L_A}(R_{A,\Lambda})$ whose objects are finite-dimensional L_A-graded $R_{A,\Lambda}$-modules. It is known (cf. [GL, Section 1.8]) that the abelian category $\mathrm{Coh}(\mathbb{P}^1_{A,\Lambda})$ of coherent sheaves is given by

$$\mathrm{Coh}(\mathbb{P}^1_{A,\Lambda}) = \mathrm{gr}^{L_A}(R_{A,\Lambda})/\mathrm{tor}^{L_A}(R_{A,\Lambda}). \tag{19}$$

Denote by $\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})$ the bounded derived category $\mathcal{D}^b(\mathrm{Coh}(\mathbb{P}^1_{A,\Lambda}))$.

For each $\vec{x} \in L_A$, set

$$\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{x}) := [R_{A,\Lambda}(\vec{x})] \in \mathrm{Coh}(\mathbb{P}^1_{A,\Lambda}), \tag{20}$$

where $(R_{A,\Lambda}(\vec{x}))_{\vec{y}} := (R_{A,\Lambda})_{\vec{x}+\vec{y}}$.

Set $\vec{x}_i := [\vec{X}_i]$ $(i = 1, \ldots r)$ and $\vec{c} := a_1\vec{x}_1 = \cdots = a_r\vec{x}_r$. An element $\vec{x} \in L_A$ has the unique expression of the form

$$\vec{x} = l\vec{c} + \sum_{i=1}^{r} p_i\vec{x}_i, \qquad 0 \leqslant p_i \leqslant a_i - 1. \tag{21}$$

We say that $\vec{x}$ is *positive* if $\vec{x} \neq 0$, $l \geqslant 0$ and $p_i \geqslant 0$ for $i = 1, \ldots, r$.

For a $\mathbb{C}$-module M, set $M^* := \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$.

PROPOSITION 3.3. [GL, Sections 1.8.1 and 2.2] *We have the following:*

(i) *For $\vec{x}, \vec{y} \in L_A$ with $\vec{x} - \vec{y}$ positive,*

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{x}), \mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{y})) = 0. \tag{22}$$

(ii) *Set $\vec{\omega} := (r-2)\vec{c} - \sum_{i=1}^{r} \vec{x}_i \in L_A$. For $M_1, M_2 \in \mathrm{Coh}(\mathbb{P}^1_{A,\Lambda})$, we have the Serre duality isomorphism:*

$$\mathrm{Ext}^1(M_2, M_1) \cong \mathrm{Hom}(M_1, M_2 \otimes \mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{\omega}))^*. \tag{23}$$

(iii) *The category $\mathrm{Coh}(\mathbb{P}^1_{A,\Lambda})$ is hereditary, namely, $\mathrm{Ext}^i(M_1, M_2) = 0$ for $M_1, M_2 \in \mathrm{Coh}(\mathbb{P}^1_{A,\Lambda})$ if $i \neq 0, 1$.*

REMARK 3.4. It follows from Proposition 3.3(iii) that each indecomposable object of $\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})$ is of the form $M[n]$ for some $M \in \mathrm{Coh}(\mathbb{P}^1_{A,\Lambda})$ and $n \in \mathbb{Z}$.

PROPOSITION 3.5. [GL, Sections 1.8.1 and 4.1] *The following sequences are full strongly exceptional collections:*

$$\begin{aligned}(E_1,\ldots,E_{\mu_A}) := (&\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}, \mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{x}_1),\ldots,\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}((a_1-1)\vec{x}_1),\ldots,\\ &\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{x}_r),\ldots,\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}((a_r-1)\vec{x}_r),\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{c}))\\ (E^*_{\mu_A},\ldots E^*_1) := (&\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(-\vec{c}),\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(-(a_r-1)\vec{x}_r),\ldots,\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(-\vec{x}_r),\ldots,\\ &\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(-(a_1-1)\vec{x}_1),\ldots,\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(-\vec{x}_1),\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}).\end{aligned}$$

In particular, $G := \bigoplus_{i=1}^{\mu_A} E_i$ *and* $G^* := \bigoplus_{i=1}^{\mu_A} E_i^*$ *are split-generators of* $\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})$.

It follows from Proposition 3.5 that $\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda}) \cong \mathrm{per}(\mathrm{End}(G))$ since the triangulated category $\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})$ is algebraic. Denote by $\mathcal{N}(\mathbb{P}^1_{A,\Lambda})$ its numerical Grothendieck group.

DEFINITION 3.6. [GL, Section 2.5] Take $[1:\lambda] \in \mathbb{P}^1 \setminus \{\lambda_1,\ldots,\lambda_r\}$. Define S and $S_{i,j}$ for $i=1,\ldots,r$ and $j=0,\ldots,a_i-1$ by the following exact sequences:

$$0 \to \mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{0}) \xrightarrow{X_1^{a_1}-\lambda X_2^{a_2}} \mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{c}) \to S \to 0, \tag{24}$$

$$0 \to \mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(j\vec{x}_i) \xrightarrow{X_i} \mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}((j+1)\vec{x}_i) \to S_{i,j} \to 0. \tag{25}$$

DEFINITION 3.7. [GL, Sections 1.8.2 and 2.8] The *rank* and *degree* are homomorphisms defined as follows:

$$\begin{aligned}&\mathrm{rk}:\mathcal{N}(\mathbb{P}^1_{A,\Lambda}) \to \mathbb{Z}, \qquad \mathrm{rk}([\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}]) := 1, \qquad \mathrm{rk}([S]) := 0\\ &\qquad\qquad \text{and} \quad \mathrm{rk}([S_{i,j}]) := 0,\end{aligned} \tag{26}$$

$$\begin{aligned}&\deg:\mathcal{N}(\mathbb{P}^1_{A,\Lambda}) \to \mathbb{Z}, \qquad \deg([\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{x}_i)]) := \frac{a}{a_i}\\ &\qquad\qquad \text{and} \quad \deg([\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}]) := 0.\end{aligned} \tag{27}$$

DEFINITION 3.8. Denote by $\mathrm{Pic}(\mathbb{P}^1_{A,\Lambda})$ the group consisting of (isomorphism classes of) indecomposable objects in $\mathrm{Coh}(\mathbb{P}^1_{A,\Lambda})$ of rank one with multiplication induced by the tensor product.

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LEMMA 3.9. [GL, Section 2.1] *There is an isomorphism of abelian groups*

$$L_A \cong \mathrm{Pic}(\mathbb{P}^1_{A,\Lambda}), \qquad \vec{x}_i \mapsto \mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{x}_i). \tag{28}$$

One of our results is the following Gromov–Yomdin type theorem for an orbifold projective line:

THEOREM 3.10. *For each auto-equivalence F of $\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})$, we have*

$$h(F) = \log \rho(\mathcal{N}(F)). \tag{29}$$

Moreover, $\rho(\mathcal{N}(F))$ is an algebraic integer and $h(F) = 0$ if $\chi_A \neq 0$.

3.1 Proof of Theorem 3.10 for the case $\chi_A \neq 0$

It is important to note that Lenzing–Meltzer [LM, Proposition 4.2] shows that, if $\chi_A \neq 0$,

$$\mathrm{Auteq}(\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})) \simeq (\mathrm{Aut}(\mathbb{P}^1_{A,\Lambda}) \ltimes \mathrm{Pic}(\mathbb{P}^1_{A,\Lambda})) \times \mathbb{Z}[1]. \tag{30}$$

3.1.1 Case $\chi_A > 0$

Geigle–Lenzing (cf. [GL, Section 5.4.1]) gives an equivalence of triangulated categories

$$\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda}) \cong \mathcal{D}^b(\mathbb{C}\vec{\Delta}_A), \tag{31}$$

where $\vec{\Delta}_A$ is the extended Dynkin quiver below.

A	$(1, a_2, a_3)$	$(2, 2, a_3)$	(2,3,3)	(2,3,4)	(2,3,5)
$\vec{\Delta}_A$	$\widetilde{A}_{a_1,a_2}$	$\widetilde{D}_{a_3}$	$\widetilde{E}_6$	$\widetilde{E}_7$	$\widetilde{E}_8$

This equivalence with Proposition 2.14 yields $h(F) = \log \rho(\mathcal{N}(F))$. Then, [MY, Theorem 4.2, Theorem 4.5] show that $\rho(\mathcal{N}(F)) = 1$. We have finished the proof. □

3.1.2 Case $\chi_A < 0$

We shall prove that $h(F) = 0$ for each $F \in (\mathrm{Aut}(\mathbb{P}^1_{A,\Lambda}) \ltimes \mathrm{Pic}(\mathbb{P}^1_{A,\Lambda})) \times \mathbb{Z}[1]$ if $\chi_A \leqslant 0$.

Choose $\{[\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}], [S_{1,1}], \ldots, [S_{i,j}], \ldots, [S_{r,a_r-1}], [S]\}$ as a basis of $\mathcal{N}(\mathbb{P}^1_{A,\Lambda})$.

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LEMMA 3.11. *For $f \in \mathrm{Aut}(\mathbb{P}^1_{A,\Lambda})$, the automorphism $\mathcal{N}(f^*)$ is a composition of permutations exchanging $[S_{i,j}]$ and $[S_{i',j}]$ for $j = 1, \ldots a_{i-1}$ if $a_i = a_{i'}$ and fixing $[\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}]$ and $[S]$. In particular, we have $\rho(\mathcal{N}(f^*)) = 1$.*

Proof. This is a direct consequence of [LM, Proposition 3.1]. Note also that $r \geqslant 3$ since $\chi_A \leqslant 0$ and hence $\mathrm{Aut}(\mathbb{P}^1_{A,\Lambda})$ is a finite group. □

LEMMA 3.12. *For $\mathcal{L} \in \mathrm{Pic}(\mathbb{P}^1_{A,\Lambda})$, we have $\rho(\mathcal{N}(- \otimes \mathcal{L})) = 1$.*

Proof. Set $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{x})$ for some $\vec{x} \in L_A$. By [GL, (2.5.3) and (2.5.4)], for $i = 1, \ldots, r$ and $j = 1, \ldots, a_i - 1$,

(32)
$$S \otimes \mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{x}) \cong S, \qquad S_{i,j} \otimes \mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{x}) \cong S_{i,j+p_i} \quad \text{for } \vec{x} = l\vec{c} + \sum_{i=1}^{r} p_i \vec{x}_i.$$

It follows from the above isomorphisms that the representation matrix of $\mathcal{N}(- \otimes \mathcal{L})$ in the basis becomes an upper triangular matrix whose diagonal entries are all 1. Hence, its spectral radius is equal to 1. □

Each auto-equivalence F is represented as $F = f^*(- \otimes \mathcal{L})[m]$ (cf. (30)). Since Lemma 2.7(v) gives $h(F) = h(f^*(- \otimes \mathcal{L}))$, we may assume $F = f^*(- \otimes \mathcal{L})$.

PROPOSITION 3.13. *We have*

$$h(f^*(- \otimes \mathcal{L})) = \log \rho(\mathcal{N}(f^*(- \otimes \mathcal{L}))) = 0. \tag{33}$$

Proof. Take G and G^* as in Proposition 3.5. By Proposition 2.11,

$$h(F) = \lim_{n \to \infty} \frac{1}{n} \log \delta'_{\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})}(G, F^n G^*).$$

By straightforward calculation,

$$F^n G^* = \mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \cdots \otimes \mathcal{L}_n \otimes (f^*)^n G^*, \qquad \mathcal{L}_k := (f^*)^k \mathcal{L}.$$

Note that $f^*(G^*) = G^*$ and $\deg(f^*\mathcal{L}) = \deg(\mathcal{L})$ by Lemma 3.11.

Suppose that $\deg(\mathcal{L}) > 0$. For $n \gg 0$ and $\vec{z} \in L_A$, we have

$$\deg(\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \cdots \otimes \mathcal{L}_n \otimes \mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{z})) = n \deg(\mathcal{L}) + \deg(\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{z})) \gg 0.$$

Therefore, Proposition 3.3(i) and (ii) yield

$$\begin{aligned}&\operatorname{Ext}^1(G, \mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \cdots \otimes \mathcal{L}_n \otimes G^*)\\ &\quad\cong \operatorname{Hom}(\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \cdots \otimes \mathcal{L}_n \otimes G^*, G \otimes \mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{\omega}))^* = 0.\end{aligned}$$

Suppose that $\deg(\mathcal{L}) \leqslant 0$. We choose $\vec{z} \in L_A$ so that $\deg(\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{z})) \gg 0$. The objects $G' := G \otimes \mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{z})$ and $G'' := G^* \otimes \mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(-\vec{z})$ are also split-generators. Therefore, Proposition 3.3(i) yields

$$\operatorname{Hom}(G', \mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \cdots \otimes \mathcal{L}_n \otimes G'') = 0.$$

Hence it follows from Proposition 3.3(iii), Lemmas 3.11 and 3.12 that

$$\begin{aligned}h(F) &= \lim_{n\to\infty} \frac{1}{n} \log |\chi(G', F^n G'')|\\ &\leqslant \log \rho(\mathcal{N}(F)) = \log \rho(\mathcal{N}(f^*(- \otimes \mathcal{L}))) = 0.\end{aligned}$$ ∎

To summarize, we have finished the proof of Theorem 3.10 for the case $\chi_A < 0$.

3.2 Proof of Theorem 3.10 for the case $\chi_A = 0$

Define a homomorphism $\nu : \mathcal{N}(\mathbb{P}^1_{A,\Lambda}) \to \mathbb{Z}^2$ by

$$\nu([M]) := (\operatorname{rk}([M]), \deg([M]))$$

and a skew symmetric bilinear form on $\mathbb{Z}^2$ by

$$\chi' : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}, \qquad ((r_1, d_1), (r_2, d_2)) \mapsto r_1 d_2 - r_2 d_1. \tag{34}$$

LEMMA 3.14. *For $M_1, M_2 \in \mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})$, we have*

$$\sum_{j=1}^{a} \chi(M_1(j\vec{\omega}), M_2) = \chi'(\nu(M_1), \nu(M_2)). \tag{35}$$

Proof. It follows from [GL, Section 2.9] with $\chi_A = 0$. ∎

Lemma 3.14 gives the following natural group homomorphism:

$$\varphi : \operatorname{Auteq}(\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})) \to \operatorname{Aut}_{\mathbb{Z}}(\mathbb{Z}^2, \chi') \cong \mathrm{SL}(2, \mathbb{Z}). \tag{36}$$

Denote by $\operatorname{Pic}^0(\mathbb{P}^1_{A,\Lambda}) \subset \operatorname{Pic}(\mathbb{P}^1_{A,\Lambda})$ the subgroup consisting of elements with degree zero.

PROPOSITION 3.15. *There exists the following exact sequence:*

$$\begin{aligned}\{1\} \to (\mathrm{Aut}(\mathbb{P}^1_{A,\Lambda}) \ltimes \mathrm{Pic}^0(\mathbb{P}^1_{A,\Lambda})) \times \mathbb{Z}[2] &\to \mathrm{Auteq}(\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})) \\ &\xrightarrow{\varphi} \mathrm{SL}(2,\mathbb{Z}) \to \{1\}.\end{aligned} \tag{37}$$

Proof. This is a direct consequence of [LM, Theorem 6.3]. □

LEMMA 3.16. *The map* $h : \mathrm{Auteq}(\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})) \longrightarrow \mathbb{R}_{\geqslant 0}$, $F \mapsto h(F)$ *factors through* $\mathrm{SL}(2,\mathbb{Z})$.

Proof. Choose $\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{z})$ so that $\deg(\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{z})) \gg 0$ and set $G' := G \otimes \mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{z})$, $G'' := G^* \otimes \mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(-\vec{z})$. By Lemma 2.7(v), we can assume that an element $F \in \mathrm{Auteq}(\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda}))$ is of the form $F = F'F_1$ with $F' \in \mathrm{Auteq}(\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda}))$ and $F_1 \in \mathrm{Aut}(\mathbb{P}^1_{A,\Lambda}) \ltimes \mathrm{Pic}^0(\mathbb{P}^1_{A,\Lambda})$. Then there exist $F_2, \ldots, F_n \in \mathrm{Aut}(\mathbb{P}^1_{A,\Lambda}) \ltimes \mathrm{Pic}^0(\mathbb{P}^1_{A,\Lambda})$ such that $F^n = (F'F_1)^n = F'^n F_n \cdots F_1$. We have

$$\delta_{\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})}(G', F^n G'') \leqslant \delta_{\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})}(G', F'^n G')\delta_{\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})}(G', F_n \cdots F_1 G''),$$

and hence,

$$h(F) \leqslant h(F') + \lim_{n\to\infty} \frac{1}{n} \log \delta_{\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})}(G', F_n \cdots F_1 G'').$$

The functor F_i is of the form $f_i^*(- \otimes \mathcal{L}_i)$ for some $f_i \in \mathrm{Aut}(\mathbb{P}^1_{A,\Lambda})$ and $\mathcal{L}_i \in \mathrm{Pic}^0(\mathbb{P}^1_{A,\Lambda})$. For arbitrary $\vec{y} \in L_A$, we have $\deg(F_n \cdots F_1(\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{y}))) = \deg(\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{y}))$. Therefore, it follows from Proposition 3.3(i), (iii) that

$$\delta'_{\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})}(G', F_n \cdots F_1 G'') = |\chi(G', F_n \cdots F_1 G'')|.$$

Lemmas 2.12, 3.11 and 3.12 yield

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \log \delta_{\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})}(G', F_n \cdots F_1 G'') &= \lim_{n\to\infty} \frac{1}{n} \log \delta'_{\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})}(G', F_n \cdots F_1 G'') \\ &= \lim_{n\to\infty} \frac{1}{n} \log |\chi(G', F_n \cdots F_1 G'')| = 0,\end{aligned}$$

and hence $h(F) \leqslant h(F')$. We also have $h(F') \leqslant h(F)$ since $F' = FF_1^{-1}$ and F_1^{-1} belongs to $\mathrm{Aut}(\mathbb{P}^1_{A,\Lambda}) \ltimes \mathrm{Pic}^0(\mathbb{P}^1_{A,\Lambda})$. □

PROPOSITION 3.17. *We have* $h(F) = \log \rho(\mathcal{N}(F))$.

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Proof. Since $h(F[1]) = h(F)$, we may assume that $\mathrm{tr}(\varphi(F)) \geqslant 0$. It is easy to calculate $h(F) = 0$ if $\mathrm{tr}(\varphi(F)) = 0, 1$ since $\varphi(F)$ is of finite order and hence F is of finite order up to $\mathrm{Aut}(\mathbb{P}^1_{A,\Lambda}) \ltimes \mathrm{Pic}^0(\mathbb{P}^1_{A,\Lambda})$. If $\mathrm{tr}(\varphi(F)) = 2$, then $F = (- \otimes \mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{x}))F'$ with $F' \in \mathrm{Aut}(\mathbb{P}^1_{A,\Lambda}) \ltimes \mathrm{Pic}^0(\mathbb{P}^1_{A,\Lambda})$ for some $\vec{x} \in L_A$. It follows from Proposition 3.13 that

$$h(F) = h(- \otimes \mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{x})) = 0 = \log \rho(\mathcal{N}(F)).$$

Suppose now that $\mathrm{tr}(\varphi(F)) > 2$.

LEMMA 3.18. *For indecomposable objects $M_1, M_2 \in \mathrm{Coh}(\mathbb{P}^1_{A,\Lambda})$, we have*

$$\mathrm{Ext}^1(M_1, M_2) = 0 \quad \text{if } \chi'(\nu(M_1), \nu(M_2)) > 0. \tag{38}$$

Proof. The statement follows from the slope stability for orbifold projective lines [GL, Proposition 5.2] and Proposition 3.3(ii). ∎

LEMMA 3.19. [Kik, Proposition 4.6] *Assume that* $\mathrm{tr}(\varphi(F)) > 2$. *There exists a sequence of positive integers $\mathbf{m} = (m_{2l}, \ldots, m_1)$ with $l \geqslant 1$ such that $\varphi(F)$ is conjugate in* $\mathrm{SL}(2, \mathbb{Z})$ *to*

$$\begin{pmatrix} 1 & m_{2l-1} \\ m_{2l} & 1 + m_{2l-1}m_{2l} \end{pmatrix} \cdots \begin{pmatrix} 1 & m_1 \\ m_2 & 1 + m_1 m_2 \end{pmatrix}. \tag{39}$$

For each sequence of positive integers $\mathbf{m} = (m_{2l}, \ldots, m_1)$ with $l \geqslant 1$, take $F_{\mathbf{m}} \in \mathrm{Auteq}(\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda}))$ so that

$$\varphi(F_{\mathbf{m}}) = \begin{pmatrix} 1 & m_{2l-1} \\ m_{2l} & 1 + m_{2l-1}m_{2l} \end{pmatrix} \cdots \begin{pmatrix} 1 & m_1 \\ m_2 & 1 + m_1 m_2 \end{pmatrix}.$$

For positive $\vec{x}, \vec{y} \in L_A$, an elementary calculation gives

$$\chi'(\nu(\mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(-\vec{x})), \nu(F^n_{\mathbf{m}} \mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{y}))) > 0.$$

Take G, G^* as in Proposition 3.5 and a positive $\vec{x} \in L_A$. By Proposition 3.3(iii), Lemma 3.18, we obtain

$$\begin{aligned} h(F_{\mathbf{m}}) &= \lim_{n \to \infty} \frac{1}{n} \log \delta'_{\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})}(G^* \otimes \mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(-\vec{x}), F^n_{\mathbf{m}}(G \otimes \mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{x}))) \\ &= \lim_{n \to \infty} \frac{1}{n} \log |\chi(G^* \otimes \mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(-\vec{x}), F^n_{\mathbf{m}}(G \otimes \mathcal{O}_{\mathbb{P}^1_{A,\Lambda}}(\vec{x})))| \\ &\leqslant \log \rho(\mathcal{N}(F_{\mathbf{m}})). \end{aligned}$$

LEMMA 3.20. *We have*

$$\rho(\mathcal{N}(F)) = \rho(\varphi(F)). \tag{40}$$

In particular, $\rho(\mathcal{N}(F))$ is an algebraic integer.

Proof. The inequality $\rho(\mathcal{N}(F)) \geqslant \rho(\varphi(F))$ follows from the commutativity: $\varphi(F) \circ \nu = \nu \circ \mathcal{N}(F)$. The fact that φ factors thorough the surjection $\mathrm{Aut}_{\mathbb{Z}}(\mathcal{N}(\mathbb{P}^1_{A,\Lambda}), \chi) \to \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^2, \chi')$ [LM, Theorem 7.3] yields the reversed inequality. ∎

Since $\varphi(F_{\mathbf{m}})$ is conjugate to $\varphi(F)$, it follows from Lemmas 3.16 and 3.20 that

$$\begin{aligned} h(F) = h(F_{\mathbf{m}}) \leqslant \log \rho(\mathcal{N}(F_{\mathbf{m}})) &= \log \rho(\varphi(F_{\mathbf{m}})) \\ &= \log \rho(\varphi(F)) = \log \rho(\mathcal{N}(F)). \end{aligned}$$

By Theorem 2.13, we finished the proof of Proposition 3.17, hence of Theorem 3.10. ∎

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Kohei Kikuta
Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka
Osaka
560-0043
Japan
k-kikuta@cr.math.sci.osaka-u.ac.jp

Yuuki Shiraishi
International college
Osaka University
Toyonaka
Osaka
560-0043
Japan
shiraishi@cbcmp.icou.osaka-u.ac.jp

Atsushi Takahashi
Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka
Osaka
560-0043
Japan
takahashi@math.sci.osaka-u.ac.jp