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THE COHOMOLOGY RING OF ORBIT SPACES OF FREE $\mathbb{Z}_2$-ACTIONS ON SOME DOLD MANIFOLDS

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Abstract

We determine the possible $\mathbb{Z}_2$-cohomology rings of orbit spaces of free actions of $\mathbb{Z}_2$ (or fixed point free involutions) on the Dold manifold $P(1, n)$, where n is an odd natural number.

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1. Introduction

If G is a topological group and X is a topological space, a natural question concerns the existence of a continuous free action of G on X. A relevant example is the result of John Milnor, which says that the symmetric group $\mathbb{S}_3$ on three elements cannot act freely on the n-sphere $\mathbb{S}^n$. If such an action exists, further natural questions concern properties of the orbit space X/G and its cohomology ring. There are recent results in [4] (X = an arbitrary product of spheres and $G = \mathbb{Z}_2$) and [8] (X = a space of type (a, b) and $G = \mathbb{Z}_2$ or $\mathbb{S}^1$). The cohomology rings of the real, complex and quaternionic projective spaces $\mathbb{R}P^n$, $\mathbb{C}P^n$ and $\mathbb{K}P^n$ are standard examples; as is well known, these spaces are orbit spaces of certain standard free actions of $\mathbb{Z}_2$, $\mathbb{S}^1$ and $\mathbb{S}^3$ on $\mathbb{S}^n$, $\mathbb{S}^{2n+1}$ and $\mathbb{S}^{4n+3}$, respectively.

This paper is devoted to these questions when X is a special Dold manifold and $G = \mathbb{Z}_2$. The Dold manifolds $P(m, n)$ were introduced by Dold [5] for the purpose of finding odd-dimensional generators for the unoriented cobordism ring; they are finite-dimensional approximations to the classifying space $BO(2) = P(\infty, \infty)$ for real 2-plane bundles. Specifically, $P(m, n)$ is the orbit space of the free involution $-1 \times$ (conjugation) acting on $\mathbb{S}^m \times \mathbb{C}P^n$; that is, $P(m, n)$ is a closed smooth $(m + 2n)$-dimensional manifold. In [6], Khare exhibited a fixed point free involution on $P(m, n)$ when n is odd; specifically, the involution

$$\begin{aligned} S : \quad \mathbb{S}^m \times \mathbb{C}P^n &\longrightarrow \mathbb{S}^m \times \mathbb{C}P^n \\ (x, [z_0, z_1, \dots, z_n]) &\longmapsto (x, [-\bar{z}_1, \bar{z}_0, \dots, -\bar{z}_n, \bar{z}_{n-1}]) \end{aligned}$$

induces a free involution on $P(m, n)$.

Thus, the question makes sense for n odd. The main tool in this context is the Leray–Serre spectral sequence associated to the Borel fibration coming from a G-action on X (see [1]). If the action is free, the total space of this fibration has the same homotopy type, and hence the same cohomology, as X/G. Let B_G be the Milnor classifying space for G-principal bundles. If the fundamental group of B_G (which is the base space of the Borel fibration) acts trivially on the cohomology of the fibre (which is X), then the E_2-term of the Leray–Serre spectral sequence assumes a very suitable form, giving a promising scenario to attack this question. Following [5], the ring structure of $H^*(P(m,n);\mathbb{Z}_2)$ is given by

$$H^*(P(m,n);\mathbb{Z}_2) = \frac{\mathbb{Z}_2[c,d]}{\langle c^{m+1}, d^{n+1}\rangle},$$

where $c \in H^1(P(m,n);\mathbb{Z}_2)$ and $d \in H^2(P(m,n);\mathbb{Z}_2)$. Consequently, for any $n \geq 1$, $H^q(P(1,n);\mathbb{Z}_2) = \mathbb{Z}_2$ if $q = 0, 1, \dots, 2n+1$, and $H^q(P(1,n);\mathbb{Z}_2) = \{0\}$ otherwise. On the other hand, the fundamental group of $B_{\mathbb{Z}_2} = RP^\infty$ is $\mathbb{Z}_2$. Therefore, for any $n \geq 1$, the above condition is automatic for $P(1,n)$. In this setting, it is known that if a closed smooth manifold does not bound, then it does not admit a free involution (see [3]). If n is even, $P(m,n)$ may or not bound, depending on the value of m (see [6]). In particular, $P(1,n)$ does not bound if n is even. Further, for n odd and $m \geq 2$, the above condition is not automatic. If n is even, this also happens for all values of m for which $P(m,n)$ bounds (see again [6]), and in these cases we do not even know if free involutions exist. This explains the choice of $P(1,n)$ with n odd; for all other $P(m,n)$, either the question does not make sense or it may be very difficult. Our results can be summarised by the following theorem.

Theorem 1.1. *Let n be an odd natural number and suppose that $G = \mathbb{Z}_2$ acts freely on the Dold manifold $X = P(1,n)$. Then $H^*(X/G;\mathbb{Z}_2)$ is isomorphic to one of the following graded algebras:*

(i) $\mathbb{Z}_2[x,z]/\langle x^2, z^{n+1}\rangle$, *where* $\deg(x) = 1$ *and* $\deg(z) = 2$*;*

(ii) $\mathbb{Z}_2[x,y,z]/\langle x^4, x^2y, y^2 + ax^2 + bxy, z^{(n+1)/2}\rangle$, *where* $a, b \in \mathbb{Z}_2, \deg(x) = \deg(y) = 1$ *and* $\deg(z) = 4$*;*

(iii) $\mathbb{Z}_2[x,y,z,w,v]/\phi(x,y,z,w,v)$, *where*

$$\phi(x,y,z,w,v) = \langle x^5, y^2 + a_1x^2 + b_1xy, z^2 + a_2x^3z + b_2xw, w^2 + a_3x^2v + b_3xyv, \\ v^{(n+1)/4}, x^2y + a_4x^3 + b_4z, yz + a_5x^4 + b_5xz, x^2w, yw + a_6x^3z + b_6xw, zw\rangle$$

and $a_j, b_j \in \mathbb{Z}_2$, $\deg(x) = \deg(y) = 1$, $\deg(z) = 3$, $\deg(w) = 5$, $\deg(v) = 8$ *and* $n \equiv 3 \pmod 4$*;*

(iv) $\mathbb{Z}_2[x,y,z]/\langle x^3, y^2 + ax^2 + bxy, z^{(n+1)/2}\rangle$, *where* $a, b \in \mathbb{Z}_2$, $\deg(x) = \deg(y) = 1$ *and* $\deg(z) = 4$.

Remark 1.2. An open question coming from Theorem 1.1 is to ask which of the four possibilities in the theorem are actually realised; in particular, which one of these four possibilities is the realisation of the Khare involution of [6]. We would like to thank the referee for this remark.

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2. Preliminaries

First we establish some notations. Throughout, H^* will denote the Alexander–Čech cohomology with coefficients in $\mathbb{Z}_2$ in the sense of [9]. The symbol '≅' will denote an appropriate isomorphism between algebraic objects.

Let G be a compact Lie group acting on a paracompact Hausdorff space X. Then one has the Borel fibration

$$\pi : X_G \longrightarrow B_G,$$

with fibre X, where the total space $X_G = (E_G \times X)/G$ is the *Borel construction*. Here, $E_G \longrightarrow B_G$ is the universal G-bundle of Milnor. If G acts freely on X, the natural map $X_G \longrightarrow X/G$ is a homotopy equivalence and thus the cohomology rings $H^*(X_G)$ and $H^*(X/G)$ are isomorphic. Associated to $\pi : X_G \longrightarrow B_G$, one has a first-quadrant spectral sequence, $\{E_r, d_r\}$, converging to $H^*(X_G)$, with

$$E_2^{p,q} = H^p(B_G; \mathcal{H}^q(X)).$$

Here, $\mathcal{H}^q(X)$ denotes $H^q(X)$ twisted by the action of the fundamental group $\pi_1(B_G)$. As mentioned in the introduction, if $\pi_1(B_G)$ acts trivially on $H^*(X)$, the E_2-term has the suitable form

$$E_2^{p,q} = H^p(B_G) \otimes H^q(X).$$

When restricted to the subalgebras $E_2^{*,0}$ and $E_2^{0,*}$, the product structure in the spectral sequence coincides with the cup product on $H^*(B_G)$ and $H^*(X)$, respectively. Also, the homomorphisms

$$H^p(B_G) \cong E_2^{p,0} \twoheadrightarrow E_3^{p,0} \twoheadrightarrow \cdots \twoheadrightarrow E_{p+1}^{p,0} \cong E_\infty^{p,0} \subset H^p(X_G)$$

and

$$H^q(X_G) \twoheadrightarrow E_\infty^{0,q} \cong E_{q+2}^{0,q} \hookrightarrow E_{q+1}^{0,q} \hookrightarrow \cdots \hookrightarrow E_2^{0,q} \cong H^q(X)$$

are, respectively, the homomorphisms

$$\pi^* : H^p(B_G) \longrightarrow H^p(X_G)$$

and

$$i^* : H^q(X_G) \longrightarrow H^q(X),$$

where $i : X \longrightarrow X_G$ is the inclusion map.

From Section 1, in the present case we are supposing that $G = \mathbb{Z}_2$ acts freely on $X = P(1, n)$, where n is an odd natural number. In this case, the following results will be useful.

Proposition 2.1 [2, page 374]. *Suppose that $G = \mathbb{Z}_2$ acts on the finitistic space X. If $H^j(X) = 0$ for $j > N$, then $H^j(X_G) = 0$ for $j > N$.*

Proposition 2.2 [2, page 374]. *Suppose that $G = \mathbb{Z}_2$ acts on the finitistic space X. Suppose that $\sum rk\, H^j(X) < \infty$. Then the following statements are equivalent.*

(a) *G acts trivially on $H^*(X)$ and, with the notation as above, the spectral sequence of the fibration $X \overset{i}{\hookrightarrow} X_G \overset{\pi}{\longrightarrow} B_G$ collapses in the E_2-term.*
(b) *$\sum rk\, H^j(X) = \sum rk\, H^j(X^G)$, where X^G denotes the fixed point set of the action of G on X.*

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3. Proof of the main theorem

The proof is based on hard spectral sequence arguments; for details on spectral sequences, we cite for example [7]. The difficulty lies in the fact that $P(1,n)$ has nonzero $\mathbb{Z}_2$-cohomology in all dimensions $0 \le j \le 2n+1$.

With the hypothesis and notations of Section 2, the first point is that Proposition 2.2 implies that the spectral sequence does not collapse in the E_2-term and, as before mentioned,

$$E_2^{p,q} = H^p(B_G) \otimes_{\mathbb{Z}_2} H^q(X).$$

Let $r \ge 2$ be the smallest natural number such that $d_r \neq 0$. Then $E_2 = \cdots = E_r$. As in Section 1, let $c \in H^1(X)$ and $d \in H^2(X)$ be the generators. By the multiplicative properties of the spectral sequence, we have either $d_r^{0,1}(1 \otimes c) \neq 0$ or $d_r^{0,2}(1 \otimes d) \neq 0$. Thus, d_r can be nontrivial only for $r = 2$ or $r = 3$. In this way, the question is divided into the following cases:

Case 1. $r = 2$, $d_2^{0,1}(1 \otimes c) \neq 0$ and $d_2^{0,2}(1 \otimes d) = 0$;

Case 2. $r = 2$, $d_2^{0,1}(1 \otimes c) = 0$ and $d_2^{0,2}(1 \otimes d) \neq 0$;

Case 3. $r = 3$, $d_3^{0,1}(1 \otimes c) = 0$ and $d_3^{0,2}(1 \otimes d) \neq 0$.

First consider Case 1. Then $d_2^{0,2\ell}(1 \otimes d^\ell) = 0$ for all $\ell \in \{0, 1, \ldots, n\}$ and

$$d_2^{0,2\ell+1}(1 \otimes (c \smile d^\ell)) = d_2^{0,2\ell+1}((1 \otimes c) \cdot (1 \otimes d^\ell)) = t^2 \otimes d^\ell \neq 0.$$

Consequently, the differential

$$d_2^{p,q} : E_2^{p,q} \longrightarrow E_2^{p+2,q-1}$$

is trivial if $q \equiv 0 \pmod 2$ and an isomorphism if $q \equiv 1 \pmod 2$. Thus, $E_3^{p,q} = \{0\}$ for all p if $q \equiv 1 \pmod 2$. Also, $E_3^{p,q} = \{0\}$ for $p \ge 2$ if $q \equiv 0 \pmod 2$. In the remaining cases, $E_3^{p,q} = E_2^{p,q}$. So, we have $E_\infty \cong E_3$ and

$$H^j(X_G) \cong \mathrm{Tot}(E_\infty)^j = \bigoplus_{j=p+q} E_\infty^{p,q} = \begin{cases} \mathbb{Z}_2 & \text{if } 0 \le j \le 2n+1, \\ \{0\} & \text{otherwise.} \end{cases}$$

Next, we compute the ring structure of $H^*(X_G)$. Set $x = \pi^*(t) \in H^1(X_G)$. Then $x \neq 0$, $x \in E_\infty^{1,0}$ and $x^2 = 0$. The element $1 \otimes d \in E_2^{0,2}$ is a permanent cocycle and determines a nonzero element $\mathbf{z} \in E_\infty^{0,2}$. Then $\mathbf{z}^{n+1} = 0$ and, as a graded commutative algebra,

$$\mathrm{Tot}(E_\infty) \cong \frac{\mathbb{Z}_2[x, \mathbf{z}]}{\langle x^2, \mathbf{z}^{n+1} \rangle},$$

where $\deg(x) = 1$ and $\deg(\mathbf{z}) = 2$. Since the composition

$$H^2(X_G) \twoheadrightarrow E_\infty^{0,2} \cong E_4^{0,2} \hookrightarrow E_3^{0,2} \hookrightarrow E_2^{0,2} \cong H^2(X)$$

is the homomorphism $i^* : H^2(X_G) \longrightarrow H^2(X)$, there is a unique nonzero element $z \in H^2(X_G)$ such that $i^*(z) = d$. Clearly, $z^{n+1} = 0$. Therefore,

$$H^*(X_G) \cong \frac{\mathbb{Z}_2[x, z]}{\langle x^2, z^{n+1} \rangle},$$

where $\deg(x) = 1$ and $\deg(z) = 2$. This determines alternative (i) of the theorem.

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Now assume that $r = 2$, $d_2^{0,1}(1 \otimes c) = 0$ and $d_2^{0,2}(1 \otimes d) = t^2 \otimes c$. Then

$$d_2^{0,2\ell}(1 \otimes d^\ell) = \begin{cases} t^2 \otimes (c \smile d^{\ell-1}) & \text{if } \ell \equiv 1 \pmod 2, \\ 0 & \text{if } \ell \equiv 0 \pmod 2, \end{cases}$$

and $d_2^{0,2\ell+1}(1 \otimes (c \smile d^\ell)) = 0$ for all $\ell \in \{0, 1, \dots, n\}$. Hence, the differential

$$d_2^{p,q} : E_2^{p,q} \longrightarrow E_2^{p+2,q-1}$$

is trivial if $q \equiv 1 \pmod 2$ or $q \equiv 0 \pmod 4$, and an isomorphism if $q \equiv 2 \pmod 4$. Thus, $E_3^{p,q} = \{0\}$ for all p if $q \equiv 2 \pmod 4$. Also, $E_3^{p,q} = \{0\}$ for $p \geq 2$ if $q \equiv 1 \pmod 4$. In the remaining cases, $E_3^{p,q} = E_2^{p,q}$.

Consequently, $d_3 = 0$ and hence $E_4 = E_3$; also, we can check that $d_4^{p,2\ell} = 0$ for all ℓ, and $d_4^{p,2\ell+1} = 0$ if $\ell \equiv 0 \pmod 2$. Suppose that $\ell \equiv 1 \pmod 2$ and let $1 \otimes (c \smile d^\ell) \in E_4^{0,2\ell+1}$ be the nonzero element. We have

$$\begin{aligned} d_4^{0,2\ell+1}(1 \otimes (c \smile d^\ell)) &= d_4^{0,2\ell+1}([1 \otimes (c \smile d)] \cdot (1 \otimes d^{\ell-1})) \\ &= d_4^{0,3}(1 \otimes (c \smile d)) \cdot (1 \otimes d^{\ell-1}). \end{aligned}$$

The element $1 \otimes (c \smile d)$ cannot be written as a product of two nonzero elements in E_4. Because of this, the differential d_4 is determined by the possible values of $d_4^{0,3}(1 \otimes (c \smile d))$. Let us consider the following cases:

Subcase 2.1. $d_4^{0,3}(1 \otimes (c \smile d)) = t^4 \otimes 1$;

Subcase 2.2. $d_4^{0,3}(1 \otimes (c \smile d)) = 0$.

First consider Subcase 2.1. Then $d_4^{0,2\ell+1}(1 \otimes (c \smile d^\ell)) = t^4 \otimes d^{\ell-1}$ and we conclude that

$$d_4^{p,q} : E_4^{p,q} \longrightarrow E_4^{p+4,q-3}$$

is trivial if $q \equiv 0 \pmod 2$ or $q \equiv 1 \pmod 4$, and an isomorphism if $q \equiv 3 \pmod 4$. Hence, $E_5^{p,q} = \{0\}$ for all p if $q \equiv 3 \pmod 4$. Also, $E_5^{p,q} = \{0\}$ for $p \geq 4$ if $q \equiv 0 \pmod 4$. In the remaining cases, $E_5^{p,q} = E_4^{p,q}$.

So, we have $E_\infty \cong E_5$ and

$$H^j(X_G) \cong \mathrm{Tot}(E_\infty)^j = \bigoplus_{j=p+q} E_\infty^{p,q} = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } j = 1, 2, 5, 6, \dots, 2n-1, 2n, \\ \mathbb{Z}_2 & \text{for } j = 0, 3, 4, 7, \dots, 2n-2, 2n+1, \\ \{0\} & \text{for } j > 2n+1. \end{cases}$$

Set $x = \pi^*(t) \in H^1(X_G)$. Then $x \neq 0$, $x \in E_\infty^{1,0}$ and $x^4 = 0$. The elements $1 \otimes c \in E_2^{0,1}$ and $1 \otimes d^2 \in E_2^{0,4}$ are permanent cocycles and determine, respectively, the nonzero elements $\mathbf{y} \in E_\infty^{0,1}$ and $\mathbf{z} \in E_\infty^{0,4}$. Therefore, $\mathbf{y}^2 = 0$, $x^2\mathbf{y} = 0$ and $\mathbf{z}^{(n+1)/2} = 0$. Thus, as a graded commutative algebra,

$$\mathrm{Tot}(E_\infty) \cong \frac{\mathbb{Z}_2[x, \mathbf{y}, \mathbf{z}]}{\langle x^4, \mathbf{y}^2, x^2\mathbf{y}, \mathbf{z}^{(n+1)/2} \rangle},$$

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where $\deg(x) = \deg(\mathbf{y}) = 1$ and $\deg(\mathbf{z}) = 4$. Since the composition

$$H^1(X_G) \twoheadrightarrow E_\infty^{0,1} \cong E_3^{0,1} \hookrightarrow E_2^{0,1} \cong H^1(X)$$

is the homomorphism $i^* : H^1(X_G) \longrightarrow H^1(X)$ and $i^* \circ \pi^* = 0$ in positive degrees, we can choose a nonzero element $y \in H^1(X_G)$, $y \neq x$, such that $i^*(y) = c$ and $x^2y = 0$. Then y represents $\mathbf{y}$ and satisfies $xy \neq 0$ and $y^2 = ax^2 + bxy$, with $a, b \in \mathbb{Z}_2$. Similarly, let $z \in H^4(X_G)$ be the unique nonzero element such that $i^*(z) = d^2$. Then z represents $\mathbf{z}$ and satisfies $z^{(n+1)/2} = 0$. Therefore,

$$H^*(X_G) \cong \frac{\mathbb{Z}_2[x, y, z]}{\langle x^4, x^2y, y^2 + ax^2 + bxy, z^{(n+1)/2} \rangle},$$

where $\deg(x) = \deg(y) = 1$, $\deg(z) = 4$ and $a, b \in \mathbb{Z}_2$. This determines alternative (ii) of the theorem.

Now consider Subcase 2.2, that is, $d_4^{0,3}(1 \otimes (c \smile d)) = 0$. So, $d_4 = 0$ and $E_5 = E_4 = E_3$. If $n = 1$, by dimensional reasons, $d_s = 0$ for $s \geq 5$ and thus $E_\infty \cong E_5$. But

$$H^4(X_G) \cong \mathrm{Tot}(E_\infty)^4 = E_\infty^{4,0} \oplus E_\infty^{1,3} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

and this contradicts Proposition 2.1. Thus, Subcase 2.2 does not happen when $n = 1$. Take $n > 1$. The differential d_5 is determined by the possible values of $d_5^{0,4}(1 \otimes d^2)$. If $d_5^{0,4}(1 \otimes d^2) = 0$, then $d_5 = 0$ and $E_6 = E_5 = E_4 = E_3$. It follows that $d_s = 0$ for $s \geq 6$ and so $E_\infty \cong E_3$. But

$$H^{2n+2}(X_G) \cong E_\infty^{2n+2,0} \oplus E_\infty^{2n-1,3} \oplus \cdots \oplus E_\infty^{4,2n-2} \oplus E_\infty^{1,2n+1} \cong \underbrace{\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}_{n+1 \text{ copies}}$$

and this again contradicts Proposition 2.1. Thus, $d_5^{0,4}(1 \otimes d^2) = t^5 \otimes 1$. In this case, we claim that n must be of the form $n \equiv 3 \pmod 4$.

If, on the contrary, $n \equiv 1 \pmod 4$, then

$$d_5^{0,2\ell}(1 \otimes d^\ell) = \begin{cases} t^5 \otimes d^{\ell-2} & \text{for } \ell = 2, 6, \ldots, n-3, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$d_5^{0,2\ell+1}(1 \otimes (c \smile d^\ell)) = \begin{cases} t^5 \otimes (c \smile d^{\ell-2}) & \text{for } \ell = 3, 7, \ldots, n-2, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, $d_5^{p,q}$ is an isomorphism if either $q = 2\ell$ and $\ell \in \{2, 6, \ldots, n-3\}$, or $q = 2\ell + 1$ and $\ell \in \{3, 7, \ldots, n-2\}$; otherwise, it is trivial. Therefore, $E_6^{p,2\ell} = \{0\}$ for all p, if $\ell \equiv 2 \pmod 4$. Also, $E_6^{p,2\ell} = \{0\}$ for $p \geq 5$, if $\ell \equiv 0 \pmod 4$ and $\ell \neq n-1$. In the remaining cases, $E_6^{p,2\ell} = E_5^{p,2\ell}$. Similarly, we have $E_6^{p,2\ell+1} = \{0\}$ for all p, if $\ell \equiv 3 \pmod 4$. Also, $E_6^{p,2\ell+1} = \{0\}$ for $p \geq 5$, if $\ell \equiv 1 \pmod 4$ and $\ell \neq n$. In the remaining cases, $E_6^{p,2\ell+1} = E_5^{p,2\ell+1}$. One can check that $d_s = 0$ for all $s \geq 6$ and so $E_\infty \cong E_6$. But

$$H^{2n+2}(X_G) \cong E_\infty^{1,2n+1} \oplus E_\infty^{4,2n-2} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2,$$

which again contradicts Proposition 2.1.

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In this way, $n \equiv 3 \pmod 4$ and now

$$d_5^{0,2\ell}(1 \otimes d^\ell) = \begin{cases} t^5 \otimes d^{\ell-2} & \text{for } \ell = 2, 6, \ldots, n-1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$d_5^{0,2\ell+1}(1 \otimes (c \smile d^\ell)) = \begin{cases} t^5 \otimes (c \smile d^{\ell-2}) & \text{for } \ell = 3, 7, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that $d_5^{p,q}$ is an isomorphism if either $q = 2\ell$ and $\ell \in \{2, 6, \ldots, n-1\}$, or $q = 2\ell + 1$ and $\ell \in \{3, 7, \ldots, n\}$; otherwise, it is trivial. We have $E_6^{p,2\ell} = \{0\}$ for all p, if $\ell \equiv 2 \pmod 4$. Also, if $\ell \equiv 0 \pmod 4$, $E_6^{p,2\ell} = \{0\}$ for $p \geq 5$. In the remaining cases, $E_6^{p,2\ell} = E_5^{p,2\ell}$. When $q = 2\ell + 1$, we get $E_6^{p,2\ell+1} = \{0\}$ for all p, if $\ell \equiv 3 \pmod 4$. Also, $E_6^{p,2\ell+1} = \{0\}$ for $p \geq 5$, if $\ell \equiv 1 \pmod 4$. In the remaining cases, $E_6^{p,2\ell+1} = E_5^{p,2\ell+1}$.

It follows that the sequence collapses in the E_6-term and

$$H^j(X_G) \cong \bigoplus_{j=p+q} E_\infty^{p,q} = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2, & j \neq 0, 7, 8, 15, \ldots, 2n-6, 2n+1 \text{ and } j < 2n+1, \\ \mathbb{Z}_2, & j = 0, 7, 8, 15, \ldots, 2n-6, 2n+1, \\ \{0\}, & j > 2n+1. \end{cases}$$

Set $x = \pi^*(t) \in H^1(X_G)$. Then $x \neq 0$, $x \in E_\infty^{1,0}$ and $x^5 = 0$. The elements $1 \otimes c \in E_2^{0,1}$, $1 \otimes (c \smile d) \in E_2^{0,3}$, $1 \otimes (c \smile d^2) \in E_2^{0,5}$ and $1 \otimes d^4 \in E_2^{0,8}$ are permanent cocycles and determine nonzero elements $\mathbf{y} \in E_\infty^{0,1}$, $\mathbf{z} \in E_\infty^{0,3}$, $\mathbf{w} \in E_\infty^{0,5}$ and $\mathbf{v} \in E_\infty^{0,8}$, respectively. We conclude that, as a graded commutative algebra,

$$\mathrm{Tot}(E_\infty) \cong \frac{\mathbb{Z}_2[x, \mathbf{y}, \mathbf{z}, \mathbf{w}, \mathbf{v}]}{\langle x^5, \mathbf{y}^2, \mathbf{z}^2, \mathbf{w}^2, \mathbf{v}^{(n+1)/4}, x^2\mathbf{y}, \mathbf{yz}, x^2\mathbf{w}, \mathbf{yw}, \mathbf{zw} \rangle},$$

where $\deg(x) = \deg(\mathbf{y}) = 1$, $\deg(\mathbf{z}) = 3$, $\deg(\mathbf{w}) = 5$ and $\deg(\mathbf{v}) = 8$. Since the composition

$$H^m(X_G) \twoheadrightarrow E_\infty^{0,m} \cong E_{m+2}^{0,m} \hookrightarrow E_{m+1}^{0,m} \hookrightarrow \cdots \hookrightarrow E_2^{0,m} \cong H^m(X)$$

is the homomorphism $i^* : H^m(X_G) \longrightarrow H^m(X)$ and $i^* \circ \pi^* = 0$ in positive degrees, we can choose nonzero elements $y \in H^1(X_G)$, $z \in H^3(X_G)$, $w \in H^5(X_G)$ and $v \in H^8(X_G)$ such that

$$i^*(y) = c, \quad i^*(z) = c \smile d, \quad i^*(w) = c \smile d^2, \quad i^*(v) = d^4, \quad x^2 w = 0 \quad \text{and} \quad v^{(n+1)/4} = 0.$$

The following relations hold in $H^*(X_G)$:

$$\begin{aligned} z^2 &= a_2 x^3 z + b_2 x w, & a_2, b_2 &\in \mathbb{Z}_2, \\ w^2 &= a_3 x^2 v + b_3 x y v, & a_3, b_3 &\in \mathbb{Z}_2, \\ x^2 y &= a_4 x^3 + b_4 z, & a_4, b_4 &\in \mathbb{Z}_2, \\ yz &= a_5 x^4 + b_5 x z, & a_5, b_5 &\in \mathbb{Z}_2, \\ yw &= a_6 x^3 z + b_6 x w, & a_6, b_6 &\in \mathbb{Z}_2. \end{aligned}$$

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Also, in $\mathrm{Tot}(E_\infty)$, we have $zw = 0$ because $i^*(zw) = i^*(z) \smile i^*(w) = c^2 \smile d^3 = 0$ and $i^* : H^8(X_G) \longrightarrow H^8(X)$ is an isomorphism. Therefore,

$$H^*(X_G) \cong \frac{\mathbb{Z}_2[x, y, z, w, v]}{\phi(x, y, z, w, v)},$$

where

$$\begin{aligned}\phi(x, y, z, w, v) = \langle x^5, y^2 + a_1x^2 + b_1xy, z^2 + a_2x^3z + b_2xw, w^2 + a_3x^2v + b_3xyv, v^{(n+1)/4},\\ x^2y + a_4x^3 + b_4z, yz + a_5x^4 + b_5xz, x^2w, yw + a_6x^3z + b_6xw, zw\rangle,\end{aligned}$$

with $\deg(x) = \deg(y) = 1$, $\deg(z) = 3$, $\deg(w) = 5$ and $\deg(v) = 8$. This gives alternative (iii) in the theorem.

Finally, consider Case 3, that is, $r = 3$, $d_3^{0,1}(1 \otimes c) = 0$ and $d_3^{0,2}(1 \otimes d) = t^3 \otimes 1$. Then, for all $\ell \in \{0, 1, \ldots, n\}$,

$$d_3^{0,2\ell}(1 \otimes d^\ell) = \begin{cases} t^3 \otimes d^{\ell-1} & \text{if } \ell \equiv 1 \pmod 2,\\ 0 & \text{if } \ell \equiv 0 \pmod 2,\end{cases}$$

and

$$d_3^{0,2\ell+1}(1 \otimes (c \smile d^\ell)) = d_3^{0,2\ell+1}((1 \otimes c) \cdot (1 \otimes d^\ell)) = \begin{cases} t^3 \otimes (c \smile d^{\ell-1}) & \text{if } \ell \equiv 1 \pmod 2,\\ 0 & \text{if } \ell \equiv 0 \pmod 2.\end{cases}$$

This implies that the differential

$$d_3^{p,q} : E_3^{p,q} \longrightarrow E_3^{p+3,q-2}$$

is trivial if $q \equiv 0 \pmod 4$ or $q \equiv 1 \pmod 4$, and an isomorphism if $q \equiv 2 \pmod 4$ or $q \equiv 3 \pmod 4$. Thus, $E_4^{p,q} = \{0\}$ for all p, if $q \equiv 2 \pmod 4$ or $q \equiv 3 \pmod 4$. Also, $E_4^{p,q} = \{0\}$ for $p \geq 3$, if $q \equiv 0 \pmod 4$ or $q \equiv 1 \pmod 4$. In the remaining cases, $E_4^{p,q} = E_3^{p,q}$. So, we have $E_\infty \cong E_4$ and

$$H^j(X_G) \cong \bigoplus_{j=p+q} E_\infty^{p,q} = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } j = 1, 2, 5, 6, \ldots, 2n-1, 2n,\\ \mathbb{Z}_2 & \text{for } j = 0, 3, 4, 7, \ldots, 2n-2, 2n+1,\\ \{0\} & \text{for } j > 2n+1.\end{cases}$$

As before, set $x = \pi^*(t) \in H^1(X_G)$. Then $x \neq 0$, $x \in E_\infty^{1,0}$ and $x^3 = 0$. The elements $1 \otimes c \in E_2^{0,1}$ and $1 \otimes d^2 \in E_2^{0,4}$ are permanent cocycles and determine nonzero elements $\mathbf{y} \in E_\infty^{0,1}$ and $\mathbf{z} \in E_\infty^{0,4}$, respectively. Clearly, $\mathbf{y}^2 = 0$ and $\mathbf{z}^{(n+1)/2} = 0$. We conclude that, as a graded commutative algebra,

$$\mathrm{Tot}(E_\infty) \cong \frac{\mathbb{Z}_2[x, \mathbf{y}, \mathbf{z}]}{\langle x^3, \mathbf{y}^2, \mathbf{z}^{(n+1)/2}\rangle},$$

where $\deg(x) = \deg(\mathbf{y}) = 1$ and $\deg(\mathbf{z}) = 4$. Choosing nonzero elements $y \in H^1(X_G)$ and $z \in H^4(X_G)$ such that $i^*(y) = c$ and $i^*(z) = d^2$ gives

$$H^*(X_G) \cong \frac{\mathbb{Z}_2[x, y, z]}{\langle x^3, y^2 + ax^2 + bxy, z^{(n+1)/2}\rangle},$$

where $\deg(x) = \deg(y) = 1$, $\deg(z) = 4$ and $a, b \in \mathbb{Z}_2$. This gives alternative (iv) and completes the proof of the theorem.

References

[1] A. Borel, *Seminar on Transformation Groups* (Princeton University Press, Princeton, NJ, 1960).

[2] G. E. Bredon, *Introduction to Compact Transformation Groups* (Academic Press, New York, 1972).

[3] P. E. Conner and E. E. Floyd, *Differentiable Periodic Maps*, Ergebnisse Series, 33 (Springer, Berlin–Heidelberg, 1964).

[4] D. M. Davis, 'Projective product spaces', *J. Topol.* **3** (2010), 265–279.

[5] A. Dold, 'Erzeugende der Thomschen Algebra $\mathfrak{N}$', *Math. Z.* **65** (1956), 25–35.

[6] S. S. Khare, 'On Dold manifolds', *Topology Appl.* **33** (1989), 297–307.

[7] J. McCleary, *A User's Guide to Spectral Sequences*, 2nd edn (Cambridge University Press, New York, 2001).

[8] P. L. Q. Pergher, H. K. Singh and T. B. Singh, 'On Z2 and S1 free actions on spaces of cohomology type (a, b)', *Houston J. Math.* **36**(1) (2010), 137–146.

[9] E. H. Spanier, *Algebraic Topology* (McGraw-Hill, New York, 1966).

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