Proceedings of the Royal Society of Edinburgh, **142A**, 329–357, *2012*

Trace inequalities for weighted Hardy–Sobolev spaces in the non-diagonal case

Carme Cascante and **Joaquin M. Ortega**
Departament de Matemàtica Aplicada i Anàlisi,
Facultat de Matemàtiques, Universitat de Barcelona,
Gran Via 585, 08071 Barcelona, Spain
(cascante@ub.edu; ortega@ub.edu)

(MS received 22 October 2010; accepted 26 April 2011)

We give characterizations of the positive Borel measures μ on $\mathbb{S}^n$ so that the weighted Hardy–Sobolev spaces $H^p_s(w)$ are imbedded in $L^q(\mathrm{d}\mu)$, for a range of $s > 0$, $0 < p, q < +\infty$, $q \neq p$, where w is a doubling weight in the unit sphere of $\mathbb{C}^n$.

1. Introduction

This paper is devoted to the study of trace inequalities for some weighted Hardy–Sobolev spaces $H^p_s(w)$ on the unit ball $\mathbb{B}^n$. Before we state our main results, we introduce some notation. We shall write $\zeta\bar{\eta}$ to indicate the complex inner product in $\mathbb{C}^n$ given by $\zeta\bar{\eta} = \sum_{i=1}^n \zeta_i\bar{\eta}_i$ if $\zeta = (\zeta_1, \dots, \zeta_n)$, $\eta = (\eta_1, \dots, \eta_n)$. If $\zeta \in \mathbb{S}^n$, the unit sphere in $\mathbb{C}^n$, and $r > 0$, then the non-isotropic ball is given by $B(\zeta, r) = \{\eta \in \mathbb{S}^n,\ |1 - \zeta\bar{\eta}| < r\}$. For any $\alpha > 1$, and $\zeta \in \mathbb{S}^n$, the admissible region $D_\alpha(\zeta)$ is given by

$$D_\alpha(\zeta) = \{z \in \mathbb{B}^n;\ |1 - z\bar{\zeta}| < \tfrac{1}{2}\alpha(1 - |z|^2)\}.$$

If f is a function defined on $\mathbb{B}^n$, the admissible maximal function $M_\alpha f$ is the function on $\mathbb{S}^n$ defined by $M_\alpha f(\zeta) = \sup_{z \in D_\alpha(\zeta)} |f(z)|$.

If w is a weight in $\mathbb{S}^n$, the weighted Hardy–Sobolev space $H^p_s(w)$, $0 \leqslant s$, $0 < p < +\infty$, consists of those functions f holomorphic in $\mathbb{B}^n$ such that if $f(z) = \sum_k f_k(z)$ is its homogeneous polynomial expansion, and

$$\mathcal{R}^s f(z) := (I + R)^s f(z) = \sum_k (1 + k)^s f_k(z),$$

we have that

$$\|f\|_{H^p_s(w)} := \|M_\alpha[\mathcal{R}^s f]\|_{L^p(w)} < +\infty.$$

Observe that, for $s = 1$, $\mathcal{R} = I + R$, where R is the usual radial derivative operator.

We shall say that a finite positive Borel measure on $\mathbb{S}^n$ is a q-trace measure for $H^p_s(w)$ if and only if there exists $C > 0$ such that, for any $f \in H^p_s(w)$,

$$\|M_\alpha[f]\|_{L^q(\mathrm{d}\mu)} \leqslant C\|f\|_{H^p_s(w)}. \tag{1.1}$$

The main goal of this work is to obtain characterizations for the q-trace measures for $H^p_s(w)$. This will be done for a certain range of indices and properties of the weight w.

https://doi.org/10.1017/S0308210510001654 Published online by Cambridge University Press

If $0 < s < n$ and $p > 1$, for any function g in $H^p_s(w)$ there exists $f \in L^p(w)$ such that

$$g(z) = C_s[f](z) := \int_{\mathbb{S}^n} \frac{f(\zeta)}{(1 - z\bar{\zeta})^{n-s}}\,\mathrm{d}\sigma(\zeta), \tag{1.2}$$

and $\|g\|_{H^p_s(w)} \simeq \|f\|_{L^p(w)}$ (see, for example, [5]). Consequently, μ is a q-trace measure for $H^p_s(w)$ if there exists $C > 0$ such that

$$\|M_\alpha[C_s[f]]\|_{L^q(\mathrm{d}\mu)} \leqslant C\|f\|_{L^p(w)}. \tag{1.3}$$

We next recall the relationship between the above problem and its real non-isotropic counterpart. We denote by K_s the non-isotropic potential operator defined, if ν is a finite positive Borel measure on $\mathbb{S}^n$, by

$$K_s[\nu](\zeta) = \int_{\mathbb{S}^n} \frac{1}{|1 - \zeta\bar{\eta}|^{n-s}}\,\mathrm{d}\nu(\eta), \quad \zeta \in \bar{\mathbb{B}}^n.$$

When $\mathrm{d}\nu = f\,\mathrm{d}\sigma$, we shall just write $K_s[f] := K_s[f\,\mathrm{d}\sigma]$.

Any measure μ on $\mathbb{S}^n$ for which there exists $C > 0$ such that

$$\|K_s[f]\|_{L^q(\mathrm{d}\mu)} \leqslant C\|f\|_{L^p(w)} \tag{1.4}$$

will be called a q-trace measure for the non-isotropic weighted potential space $K_s[L^p(w)]$. Since $M_\alpha[C_s[f]] \leqslant CK_s[|f|]$, it is clear that any q-trace measure for the space $K_s[L^p(w)]$ is also a q-trace measure for the weighted Hardy–Sobolev space $H^p_s(w)$. On the other hand, the two problems are not, in general, equivalent, as was observed in [5].

The q-trace measures for the Riesz potentials on $\mathbb{R}^n$ for any relative position of q and p have been thoroughly studied. We recall that, when $0 < s < n$, $I_s(x) = c(n, s)|x|^{s-n}$ is the Riesz kernel on $\mathbb{R}^n$, and $I_s * L^p(\mathbb{R}^n)$ is the potential space of functions defined by

$$I_s[f](x) = \int_{\mathbb{R}^n} I_s(|x - y|) f(y)\,\mathrm{d}y.$$

A positive Borel measure on $\mathbb{R}^n$ is a q-trace measure for $I_s * L^p(\mathbb{R}^n)$ if and only if

$$\biggl(\int_{\mathbb{R}^n} |I_s[f](x)|^q\,\mathrm{d}\mu(x)\biggr)^{1/q} \leqslant C\biggl(\int_{\mathbb{R}^n} |f(x)|^p\,\mathrm{d}x\biggr)^{1/p}. \tag{1.5}$$

When $1 < p < q$, it was shown by Adams [2, theorem 7.2.2] that the characterization for a measure μ to satisfy (1.5) is that

$$\sup_{x\in\mathbb{R}^n, r>0} \frac{\mu(B(x, r))}{r^{(n-sp)q/p}} < +\infty, \tag{1.6}$$

where here $B(x, r)$ is the Euclidean ball of radius $r > 0$ centred at $x \in \mathbb{R}^n$.

In the case $p = q$, a capacitary characterization in this case can be found in [2, theorem 7.2.1]: (1.5) holds for $p = q$ if and only if

$$\sup_G \frac{\mu(G)}{C_{s,p}(G)} < +\infty,$$

where the supremum is taken over all open sets $G \subset \mathbb{R}^n$ of positive Riesz capacity. Other non-capacitary characterizations have been given in [11].

We observe that, for the case $q = 1$, duality gives that (1.5) holds if and only if the energy,

$$\mathcal{E}_{sp}[\mu] := \|I_s[\mu]\|_{L^{p'}(\mathrm{d}x)} < +\infty.$$

In the upper triangle case $1 < q < p$, a capacitary characterization was obtained in [14]. We also recall a non-capacitary condition [6, p. 393] in terms of the real Wolff potential defined by

$$\mathcal{W}_{sp}[\mu](x) = \int_0^{+\infty} \left(\frac{\mu(B(x,r))}{r^{n-sp}} \right)^{p'-1} \frac{\mathrm{d}r}{r},$$

where $p' = p/(p-1)$. Namely, (1.5) holds if and only if

$$\mathcal{W}_{sp}[\mu] \in L^{q(p-1)/(p-q)}(\mathrm{d}\mu).$$

In recent years there has been some work concerning weighted versions of the real q-trace problems. In [1] an extension was obtained of the characterization of the q-trace measures for the weighted space $I_s[L^p(w)]$ for the case when $1 < p < +\infty$, $p \leqslant q$, where w is a weight in the Muckenhoupt class $\mathcal{A}_p(\mathbb{R}^n)$. In [7] a characterization was obtained, for $q < p$ and $p > 1$, that extends the corresponding unweighted case and includes the case $q < 1$.

For the more difficult problem of the characterization of q-trace measures for the Hardy–Sobolev spaces H^p_s on the unit ball, the results have so far remained incomplete and only include some particular ranges of s, q and p. For the case when $p \leqslant 1$, the p-trace measures for H^p_s were considered in [3]. If $1 < p = q$ and $n - sp < 1$, the p-trace measures for H^p_s coincide with the p-trace measures for $K_s[L^p]$ [10]. If $1 < q < p$ and $n - sp < 1$, a characterization in terms of a non-isotropic Wolff potential was obtained in [6] (see the proof of theorem 1.4, below). Interesting results for a related problem on 2-Carleson measures for H^2_s when $n - 2s \geqslant 1$ have been obtained in [20] (see also [17] for the case $n - 2s = 1$).

The paper has two aims. On the one hand we extend to the weighted Hardy–Sobolev spaces some of the characterizations obtained in the unweighted case. On the other hand, we give some results for weighted spaces $H^p_s(w)$ in other ranges of s, p and q, which are new even when $w \equiv 1$ and which correspond to some situations when $q < 1$. For our first aim, the methods use some of the ideas on the unweighted case, although with more technicalities. Towards the second aim, when $q < 1$, where the results are new even in the unweighted case, some new techniques are needed.

Before we state our main results, we recall that a weight w defined on $\mathbb{S}^n$ satisfies a doubling condition of order τ for some $\tau > 0$ if there exists $C > 0$ such that, for any $\zeta \in \mathbb{S}^n$ and $0 < r < R$,

$$w(B(\zeta,R)) := \int_{B(\zeta,R)} w(\eta)\,\mathrm{d}\sigma(\eta) \leqslant C \left(\frac{R}{r}\right)^{\tau} w(B(\zeta,r)).$$

The class of such doubling weights will be denoted by D_τ.

A weight w is in the A_p class in $\mathbb{S}^n$, $1<p<+\infty$, if there exists $C>0$ such that, for any non-isotropic ball $B\subset\mathbb{S}^n$,

$$\left(\frac{1}{|B|}\int_B w\,\mathrm{d}\sigma\right)\left(\frac{1}{|B|}\int_B w^{-1/(p-1)}\,\mathrm{d}\sigma\right)^{p-1}\leqslant C,$$

where $|B|$ denotes the Lebesgue measure of the ball B.

We also point out that if $p>1$, any weight w in A_p satisfies a doubling condition of order τ, for some $\tau<pn$. This fact can be found, for instance, in [16], where it is proved that if $w\in A_p$, there exists $p_1<p$ such that $w\in A_{p_1}$ (see the corollary on p. 202 therein) and that any weight in $w\in A_{p_1}$ satisfies a doubling condition of order np_1 [16, (1.5), p. 196].

Observe that, as a consequence of Hölder's inequality and (1.2), for any doubling weight in D_τ with $\tau-sp<0$ and $p>1$, the space $H_s^p(w)$ consists of bounded functions. On the other hand, if $p\leqslant 1$, it can be also shown, using different arguments (see § 2) that if $\tau-sp\leqslant 0$, $H_s^p(w)$ consists also of bounded functions. Consequently, the q-trace measures for $H_s^p(w)$ in this range of indices are just the finite ones. Therefore, the following theorems are of interest when either $\tau-sp>0$ if $p\leqslant 1$ or $\tau-sp\geqslant 0$ if $p>1$. We also observe that any q-trace measure for $H_s^p(w)$ must be finite, which can be deduced by applying the definition to a constant function. Consequently, we shall suppose from now on that any of the measures μ considered here are finite.

The next two theorems concern the case $p\leqslant q$ and will be the main aim of § 2. Let us recall that if $1<p<+\infty$, $0<s<n$ [1], the non-isotropic weighted Riesz capacity of a set $E\subset\mathbb{S}^n$ is defined by

$$C_{sp}^w(E)=\inf\{\|f\|_{L^p(w)}^p;\ f\geqslant 0,\ K_s[f]\geqslant 1 \text{ on } E\}.$$

THEOREM 1.1. *Let $p\leqslant q<+\infty$, $p\leqslant 1$, $s>0$, let w be a doubling weight and let μ be a finite positive Borel measure on $\mathbb{S}^n$. The following statements are equivalent:*

(i) *μ is a q-trace measure for $H_s^p(w)$;*

(ii) *there exist $C>0$ and $\delta>0$ such that, for any $\zeta\in\mathbb{S}^n$, $0<r<\delta$,*

$$\mu(B(\zeta,r))\leqslant C\left(\frac{w(B(\zeta,r))}{r^{sp}}\right)^{q/p}. \tag{1.7}$$

THEOREM 1.2. *Let $1<p<q<+\infty$, $0<s<n$, let w be an $\mathcal{A}_p$-weight and let μ be a finite positive Borel measure on $\mathbb{S}^n$.*

(i) *If there exists $C>0$ such that, for any $\zeta\in\mathbb{S}^n$, $r>0$,*

$$\mu(B(\zeta,r))\leqslant CC_{sp}^w(B(\zeta,r))^{q/p}, \tag{1.8}$$

then μ is a q-trace measure for $H_s^p(w)$.

(ii) *If, in addition, $w\in D_\tau$, $0\leqslant\tau-sp<1$, then μ is a q-trace measure for $H_s^p(w)$ if and only if condition (1.8) holds.*

For $q < p$, we have obtained some partial characterizations, which in some cases when $q < 1$ are new even for the case $w \equiv 1$, that we summarize in the following theorems, which are the main subject of § 3.

THEOREM 1.3. *Let $0 < q < p \leqslant 1$, $0 < s$, w be a weight satisfying a doubling condition of order τ and μ be a finite positive Borel measure on $\mathbb{S}^n$. Assume that, in addition, $0 \leqslant \tau - sp < p$. We then have that the following assertions are equivalent:*

(i) *μ is a q-trace measure for $H_p^s(w)$;*

(ii) *for any fixed $\delta > 0$, if r_B denotes the radius of the non-isotropic ball B, then*

$$\sup_{r_B<\delta} \frac{\mu(B)}{w(B)r_B^{-sp}} \chi_B(\zeta) \in L^{q/(p-q)}(\mathrm{d}\mu). \tag{1.9}$$

In [5] a weighted Wolff-type potential of a positive measure ν on $\mathbb{S}^n$, given by

$$\mathcal{W}_{sp}^w[\nu](\zeta) = \int_0^K \left(\frac{\nu(B(\zeta,t))}{t^{n-sp}}\right)^{(p'-1)} \left(\frac{1}{t^n}\int_{B(\zeta,t)} w^{-(p'-1)}\right)\frac{\mathrm{d}t}{t}, \tag{1.10}$$

is defined. Here $K > 2$ is a fixed constant.

THEOREM 1.4. *Let $0 < q < p$, $1 < p$, $\tau, s > 0$, such that $0 \leqslant \tau - sp < 1$, let μ be a finite positive Borel measure on $\mathbb{S}^n$ and let w be a weight in $\mathcal{A}_p$, $w \in D_\tau$. If $q \leqslant 1$, we assume in addition that one of the following conditions is satisfied:*

(a) *if $1 < p \leqslant 2$, $\tau < (p-1)n + s$;*

(b) *if $p > 2$, $\tau < n + s(p-1)$.*

Then the following conditions are equivalent:

(i) *there exists $C > 0$ such that*

$$\|M_\alpha[f]\|_{L^q(\mathrm{d}\mu)} \leqslant C\|f\|_{H_s^p(w)};$$

(ii) *there exists $C > 0$ such that*

$$\|K_s[f]\|_{L^q(\mathrm{d}\mu)} \leqslant C\|f\|_{L^p(w)};$$

(iii) $\mathcal{W}_{sp}^w[\mu] \in L^{q(p-1)/(p-q)}(\mu)$.

Finally, a remark on notation: we shall adopt the convention of using the same letter for various absolute constants whose values may change in each occurrence, and we shall write $A \preceq B$ if there exists an absolute constant M such that $A \leqslant MB$. We shall say that two quantities A and B are equivalent if both $A \preceq B$ and $B \preceq A$, and in that case we shall write $A \simeq B$.

2. q-trace measures for $H^p_s(w)$, $p \leqslant q$

2.1. A necessary condition

The following proposition gives that, for any $p, q > 0$, (1.7) is a necessary condition for a measure to be a q-trace for $H^p_s(w)$.

Proposition 2.1. *Let $0 < p, q < +\infty$, $0 < s < n$. Let μ be a positive finite Borel measure on $\mathbb{S}^n$ and let w be a doubling weight on $\mathbb{S}^n$. Assume that there exists $C > 0$ such that*

$$\|M_\alpha[f]\|_{L^q(\mathrm{d}\mu)} \leqslant C\|f\|_{H^p_s(w)}$$

for any $f \in H^p_s(w)$. Thus, there exist $C > 0$, $\delta > 0$, such that, for any $\zeta \in \mathbb{S}^n$, $0 < r < \delta$,

$$\mu(B(\zeta, r)) \leqslant C\left(\frac{w(B(\zeta, r))}{r^{sp}}\right)^{q/p}. \tag{2.1}$$

Proof. Observe that, since μ is finite, the constant δ in the proposition is not relevant. It is sufficient to prove (2.1) for $\delta < 1$, since if $\delta > 1$, and $1 \leqslant r < \delta$,

$$\begin{aligned}\mu(B(\zeta, r)) \leqslant \mu(\mathbb{S}^n) &\preceq \left(\frac{w(B(\zeta, 1))}{\delta^{sp}}\right)^{q/p} \\ &\preceq \left(\frac{w(B(\zeta, r))}{r^{sp}}\right)^{q/p}.\end{aligned}$$

Let $\zeta \in \mathbb{S}^n$ and $0 < r < 1$ be fixed. If $z \in \bar{\mathbb{B}}^n$, let

$$F(z) = \frac{1}{(1-(1-r)z\bar{\zeta})^N},$$

with $N > 0$ to be chosen later. If $\eta \in B(\zeta, r)$ and $z \in D_\alpha(\eta)$, then $|1-(1-r)z\bar{\zeta}| \preceq r$ and, consequently,

$$\frac{\mu(B(\zeta, r))}{r^{Nq}} \leqslant C\int_{B(\zeta, r)} |M_\alpha F(\eta)|^q \,\mathrm{d}\mu(\eta).$$

On the other hand, since

$$\left|\mathcal{R}^s \frac{1}{(1-(1-r)z\bar{\zeta})^N}\right| \preceq \frac{1}{|1-(1-r)z\bar{\zeta}|^{N+s}},$$

we have that

$$\begin{aligned}\|F\|^p_{H^p_s(w)} &\leqslant C\int_{\mathbb{S}^n} \frac{1}{|1-(1-r)\eta\bar{\zeta}|^{(N+s)p}} w(\eta)\,\mathrm{d}\sigma(\eta) \\ &= \int_{B(\zeta, r)} \frac{1}{|1-(1-r)\eta\bar{\zeta}|^{(N+s)p}} w(\eta)\,\mathrm{d}\sigma(\eta) \\ &\quad + \sum_{k\geqslant 1}\int_{B(\zeta, 2^{k+1}r)\setminus B(\zeta, 2^k r)} \frac{1}{|1-(1-r)\eta\bar{\zeta}|^{(N+s)p}} w(\eta)\,\mathrm{d}\sigma(\eta).\end{aligned}$$

If $k \geqslant 1$ and $\eta \in B(\zeta, 2^{k+1}r) \setminus B(\zeta, 2^k r)$, then $|1-(1-r)\eta\bar{\zeta}| \simeq 2^k r$. This estimate, together with the fact that w is doubling, shows that the above is bounded by

$$\sum_{k\geqslant 0} \frac{w(B(\zeta, 2^{k+1}r))}{(2^k r)^{(N+s)p}} \preceq \frac{w(B(\zeta, r))}{r^{(N+s)p}} \sum_{k\geqslant 0} \left(\frac{C}{2^{(N+s)p}}\right)^k,$$

which gives the desired estimate, provided N is chosen to be sufficiently large. □

2.2. Proof of theorem 1.1

Proof. Assume $p \leqslant q < +\infty$, $p \leqslant 1$, $s > 0$. Let w be a doubling weight and let μ be a finite positive Borel measure on $\mathbb{S}^n$. By proposition 2.1, condition (i) implies (ii).

The proof that condition (ii) implies (i) follows the ideas in [3] closely, and we include it for the sake of completeness. The definition of the space $H^p_s(w)$ yields that if $f \in H^p_s(w)$, the function $F = \mathcal{R}^s f$ satisfies that $M_\alpha F \in L^p(w)$. Using the integral formula

$$f(z) = \frac{1}{\Gamma(s)} \int_0^1 \left(\log\frac{1}{t}\right)^{s-1} F(tz)\,\mathrm{d}t,$$

it is then enough to show that

$$\left(\int_{\mathbb{S}^n} |M_\alpha[G](\zeta)|^q \,\mathrm{d}\mu(\zeta)\right)^{1/q} \lesssim \|F\|_{H^p(w)},$$

where the function G is defined by

$$G(z) = \int_0^1 (1-t)^{s-1} |F(tz)|\,\mathrm{d}t.$$

If $E \subset \mathbb{S}^n$, we recall that $T(E)$ is the tent over E defined by

$$T(E) = \mathbb{B}^n \setminus \left(\bigcup_{\zeta\notin E} D_\alpha(\zeta)\right).$$

A tent atom with respect to a doubling weight w is a non-negative function a defined on $\mathbb{B}^n$ satisfying

(a) $\sup a \subset T(B(\zeta, R))$,

(b) $a(z) \leqslant w(B(\zeta, R))^{-1}$.

In [4, lemma 2.1], a dyadic decomposition on homogeneous spaces is constructed. In our setting of the unit sphere, if w is a doubling weight in $\mathbb{S}^n$, and g is a function defined on $\mathbb{B}^n$ satisfying that $M_\alpha[g] \in L^1(w)$, this decomposition says that there exists a sequence of tent atoms $(a_j)_j$, supported in $T(B(\zeta_j, \delta_j))$, $\delta_j < 2$, and a sequence of non-negative numbers $(\lambda_j)_j$ such that

$$|g| \leqslant \sum_j \lambda_j a_j \quad \text{and} \quad \sum_j \lambda_j \leqslant \|M_\alpha[g]\|_{L^1(w)}.$$

We apply the above decomposition to the function $|F|^p$, in order to obtain that there exists a sequence of tent atoms $(a_j)_j$ supported in $T(B(\zeta_j,\delta_j))$, and a sequence of non-negative numbers $(\lambda_j)_j$ such that

$$|F|^p \leqslant \sum_j \lambda_j a_j \quad \text{and} \quad \sum_j \lambda_j \leqslant \int_{\mathbb{S}^n} M_\alpha[F]^p w \, \mathrm{d}\sigma = \|F\|^p_{H^p(w)}.$$

Next, since $p \leqslant 1$, Minkowski's inequality yields

$$\biggl(\int_0^1 (1-t)^{s-1}\biggl(\sum_j \lambda_j a_j(t\zeta)\biggr)^{1/p} \mathrm{d}t\biggr)^p \leqslant \sum_j \lambda_j \biggl(\int_0^1 (1-t)^{s-1} a_j(t\zeta)^{1/p} \, \mathrm{d}t\biggr)^p.$$

Consequently,

$$\begin{aligned}
\biggl(\int_{\mathbb{S}^n} & M_\alpha[G](\zeta)^q \, \mathrm{d}\mu(\zeta)\biggr)^{p/q} \\
&\leqslant \biggl(\int_{\mathbb{S}^n}\biggl(\int_0^1 (1-t)^{s-1}\biggl(\sum_j \lambda_j a_j(t\zeta)\biggr)^{1/p} \mathrm{d}t\biggr)^q \mathrm{d}\mu(\zeta)\biggr)^{p/q} \\
&\leqslant \biggl(\int_{\mathbb{S}^n}\biggl(\sum_j \lambda_j \biggl(\int_0^1 (1-t)^{s-1} a_j(t\zeta)^{1/p} \, \mathrm{d}t\biggr)^p\biggr)^{q/p} \mathrm{d}\mu(\zeta)\biggr)^{p/q}. \qquad (2.2)
\end{aligned}$$

Since $q/p \leqslant 1$, applying Minkowski's inequality again yields that the above is bounded by

$$\sum_j \lambda_j \biggl(\int_{\mathbb{S}^n}\biggl(\int_0^1 (1-t)^{s-1} a_j(t\zeta)^{1/p} \, \mathrm{d}t\biggr)^q \mathrm{d}\mu(\zeta)\biggr)^{p/q}. \qquad (2.3)$$

The fact that each a_j is a tent atom supported in $T(B(\zeta_j,\delta_j))$ gives that any of the integrals

$$\int_0^1 (1-t)^{s-1} a_j(t\zeta)^{1/p} \, \mathrm{d}t$$

in the previous sum are in fact bounded by

$$\frac{\chi_{B(\zeta_j,\delta_j)}(\zeta)}{w(B(\zeta_j,\delta_j))^{1/p}} \int_{1-C\delta_j}^1 (1-t)^{s-1} \, \mathrm{d}t.$$

In consequence, the above estimate and the hypothesis on μ gives that (2.3) is bounded up to a constant by

$$\begin{aligned}
\sum_j \lambda_j & \biggl(\biggl(\delta_j^{sp} \frac{1}{w(B(\zeta_j,\delta_j))}\biggr)^{q/p} \mu(B(\zeta_j,\delta_j))\biggr)^{p/q} \\
&\preceq \sum_j \lambda_j \biggl(\biggl(\delta_j^{sp} \frac{1}{w(B(\zeta_j,\delta_j))}\biggr)^{q/p}\biggl(\frac{w(B(\zeta_j,\delta_j))}{\delta_j^{sp}}\biggr)^{q/p}\biggr)^{p/q} \\
&= C\sum_j \lambda_j \preceq \|F\|_{H^p(w)}.
\end{aligned}$$

□

https://doi.org/10.1017/S0308210510001654 Published online by Cambridge University Press

2.3. The case $1 < p \leqslant q$

The methods in [1] can easily be adapted to obtain an estimate of the non-isotropic weighted Riesz capacities of balls in $\mathbb{S}^n$, namely, if $p > 1$, $w \in A_p(\mathbb{S}^n)$, $\zeta \in \mathbb{S}^n$, then for every r, $0 < r \leqslant 2$,

$$C_{sp}^w(B(\zeta, r)) \simeq \left(\int_r^K t^{(sp-n)/(p-1)} \frac{1}{t^n} \int_{B(\zeta,t)} w^{-(p'-1)} \frac{\mathrm{d}t}{t} \right)^{1-p}.$$

We observe that, since $s < n$, in the above integral we could have integrated on the interval $(r, +\infty)$, obtaining an equivalent expression. If $w \equiv 1$, and $n - sp > 0$, we recover the well-known estimate $C_{sp}(B(\zeta, r)) \simeq r^{n-sp}$, $\zeta \in \mathbb{S}^n$, $r \leqslant 2$.

For a general weight, $C_{sp}^w(B(\zeta, r)) \preceq w(B(\zeta, r))/r^{sp}$, and both quantities are not, in general, equivalent, as the following example shows: we define, for $\eta \in \mathbb{S}^n$ fixed, the weight $w(\zeta) = |1 - \zeta\bar{\eta}|^{sp-n}$. It may immediately be seen that $w \in A_p$, provided $0 < s < n$. On the other hand, for any $0 < r < \frac{1}{2}$, $C_{s,p}^w(B(\eta, r)) \simeq (\log 1/r)^{1-p}$ and $w(B(\eta, r))/r^{sp} \simeq 1$.

The proof of theorem 1.2(i) is a consequence of a non-isotropic version of a result in [1], which gives a characterization of the q-trace measures for weighted Riesz potentials in $\mathbb{R}^n$. This non-isotropic version, which can be proved analogously to its real counterpart, is as follows.

Theorem 2.2 (Adams [1]). *Let $1 < p \leqslant q < \infty$, $0 < s < n$, let w be a weight in $\mathcal{A}_p$ and let μ be a non-negative Borel measure on $\mathbb{S}^n$. The following assertions are equivalent:*

(a) *there exists $C > 0$ such that, for any $f \geqslant 0$,*

$$\|K_s[f]\|_{L^q(\mathrm{d}\mu)} \leqslant C\|f\|_{L^p(w)};$$

(b) *there exists $C > 0$ such that, for any open set $E \subset \mathbb{S}^n$, $\mu(E) \leqslant CC_{sp}^w(E)^{q/p}$.*

If in addition $p < q$, then (a) holds if and only if condition (1.8) is satisfied.

Observe that (1.8) for $p < q$ is just condition (b), but only for balls.

2.4. Proof of theorem 1.2

The proof of assertion (i) is an immediate consequence of the inequality

$$|M_\alpha[C_s[f]]| \preceq K_s[|f|]$$

and theorem 2.2.

To prove part (ii) of the theorem, we shall need the following theorem, proved in [5].

Theorem 2.3 (Cascante *et al.* [5, theorem 3.4]). *Let ν be a positive measure on $\mathbb{S}^n$ and w be a weight in $\mathbb{B}^n$. Let $0 < s < n$ and $1 < p < +\infty$ and assume that $w \in D\tau$, where $0 \leqslant \tau - sp < 1$. Then there exists a holomorphic function $\mathcal{U}_{sp}^w[\nu]$ on*

$\mathbb{B}^n$ *such that*

(i) *for any* $\eta \in \mathbb{S}^n$,

$$\lim_{\rho\to 1} \operatorname{Re} \mathcal{U}^w_{sp}[\nu](\rho\eta) \succeq \mathcal{W}^w_{sp}[\nu](\eta),$$

(ii) $\|\mathcal{U}^w_{sp}[\nu]\|^p_{H^p_s(w)} \preceq \mathcal{E}^w_{sp}[\nu]$.

Next we shall show the necessity of (1.8) in condition (ii) of theorem 1.2.

It is shown in [5, theorem 3.2] that, provided w is an $\mathcal{A}_p$-weight, and $\mathcal{W}^w_{sp}[\nu]$ is the Wolff potential defined in (1.10), the following weighted Wolff-type estimate holds:

$$\mathcal{E}^w_{sp}(\nu) \simeq \int_{\mathbb{S}^n} \mathcal{W}^w_{sp}(\nu)(\zeta)\, \mathrm{d}\nu(\zeta). \tag{2.4}$$

In [1] Adams also proved a weighted extremal theorem for the weighted Riesz capacities, which can be easily adapted to the non-isotropic case.

THEOREM 2.4. *If* $G \subset \mathbb{S}^n$ *is open, there exists a positive capacitary measure* ν_G *on* $\mathbb{S}^n$ *such that*

(α) $\operatorname{supp} \nu_G \subset \bar{G}$,

(β) $\nu_G(\bar{G}) = C^w_{sp}(G) = \mathcal{E}^w_{sp}(\nu_G)$,

(γ) $\mathcal{W}^w_{sp}(\nu_G)(\zeta) \geqslant C$, *for* C^w_{sp}*-a.e.* $\zeta \in G$,

(δ) $\mathcal{W}^w_{sp}(\nu_G)(\zeta) \leqslant C$, *for any* $\zeta \in \operatorname{supp} \nu_G$.

Observe that $\mathcal{W}^w_{sp}(\nu_G) \succeq 1$ except on a set of C^w_{sp}-capacity zero, and then also a.e. with respect to the Lebesgue measure on $\mathbb{S}^n$ [5, theorem 3.5]. Also observe that from (α), (β) and (γ) we have

$$\int_{\mathbb{S}^n} \mathcal{W}^w_{sp}(\nu)\, \mathrm{d}\nu_G \leqslant C C^w_{sp}(G).$$

Next, if G is an open set in $\mathbb{S}^n$ and ν_G is the capacitary measure of theorem 2.4, let $F(z) = \mathcal{U}^{w\lambda}_{sp}(\nu_G)(z)$ be the holomorphic function obtained in theorem 2.3. The fact that ν_G is extremal gives that

$$\lim_{r\to 1} \operatorname{Re} F(r\zeta) \geqslant C\mathcal{W}^w_{sp}(\nu_G)(\zeta) \geqslant C,$$

for a.e. $x \in G$ with respect to the Lebesgue measure on G. Hence, if P is the Poisson–Szegő kernel, then

$$|F(z)| = \left| P\Big[\lim_{r\to 1} F(r\cdot)\Big](z)\right| \geqslant \left| P\Big[\operatorname{Re} \lim_{r\to 1} F(r\cdot)\Big](z)\right| \geqslant C$$

for any $z \in T(G)$ and, consequently, $|M_\alpha F(\zeta)| \geqslant C$ on G. Since μ is a q-trace measure for $H^p_s(w)$, we then have

$$\mu(G) \leqslant \int_G |M_\alpha F(\zeta)|^q \, \mathrm{d}\mu(\zeta) \leqslant C\|F\|^q_{H^p_s(w)} \leqslant C\mathcal{E}^w_{sp}(\nu_G)^{q/p} \leqslant C C^w_{sp}(G)^{q/p}.$$

In particular, for any $\zeta \in \mathbb{S}^n$, and $r > 0$, we have that

$$\mu(B(\zeta, r)) \leqslant C C^w_{sp}(B(\zeta, r))^{q/p},$$

as we wanted to prove.

REMARK 2.5. The arguments we have applied are valid for $p = q > 1$. In consequence, under the hypotheses of theorem 1.2, that is, $w \in \mathcal{A}_p \cap D_\tau$, $0 \leqslant \tau - sp < 1$, we have that μ is a p-trace measure for $H^p_s(w)$ if and only if $\mu(G) \preceq C^w_{sp}(G)$, for any open set $G \subset \mathbb{S}^n$.

3. q-trace measures for $H^p_s(w)$, $q < p$

3.1. The case $q < p < 1$

Our first observation is that, for $p < 1$, unlike what happens when $p \leqslant q$, the characterization in terms of balls, (1.7), is not sufficient in general if $q < p$, even in the unweighted case.

PROPOSITION 3.1. *Let $0 < q < p < 1$, and $s > 0$ such that $sp \leqslant n$. Then there exists a positive Borel measure on $\mathbb{S}^n$ such that $\mu(B(\zeta, r)) \leqslant C r^{(n-sp)q/p}$ for any $\zeta \in \mathbb{S}^n$ and $r < 1$, and that μ is not a q-trace measure for $H^p_s(\mathbb{B}^n)$.*

Proof. Let $E \subset \mathbb{S}^n$ be a compact set such that the non-isotropic Hausdorff measure $\mathcal{H}^{(n-sp)q/p}(E) \in (0, +\infty)$. Since $q/p < 1$, we then have that $\mathcal{H}^{n-sp}(E) = 0$. Theorem 2 in [9] gives that there exists $f \in H^p_s(\mathbb{B}^n)$ such that E coincides with the exceptional set for the function f, that is, for any $\zeta \in E$ the admissible limit $M_\alpha[f](\zeta) = +\infty$. On the other hand, Frostman's theorem gives that there exists a non-trivial non-negative Borel measure on $\mathbb{S}^n$ supported on the set E, satisfying $\mu(B(\zeta, r)) \leqslant C r^{(n-sp)q/p}$ (for a proof of Frostman's theorem for compact sets see, for example, [12, Theorem 8.17] and the references therein). Since

$$\int_{\mathbb{S}^n} |M_\alpha f|^q \, \mathrm{d}\mu \equiv +\infty,$$

μ cannot be a q-trace measure for $H^p_s(\mathbb{B}^n)$, that is, the estimate

$$\int_{\mathbb{S}^n} |M_\alpha[f]|^q \, \mathrm{d}\mu \leqslant C \|f\|^q_{H^p_s(\mathbb{B}^n)}$$

cannot hold for any $f \in H^p_s(\mathbb{B}^n)$. But by construction it satisfies the growth condition on balls. □

In order to deal with this case $q < p$, we shall use a non-isotropic dyadic decomposition of the unit sphere, which plays a similar role to the dyadic decomposition in $\mathbb{R}^n$. We recall [15,19], that if $\lambda > 1$ is big enough, for any (large negative) integer m, there exists a sequence of points ζ^k_j and a family of sets $\mathcal{D}_m = \{E^k_j\}$, where $k = m, m+1, \dots$ and $j = 1, 2, \dots$ such that

(i) $B(\zeta^k_j, \lambda^k) \subset E^k_j \subset B(\zeta^k_j, \lambda^{k+1})$,

(ii) for each $k \geqslant m$, the sets $\{E^k_j\}_j$ are pairwise disjoint in j and $\mathbb{S}^n = \bigcup_j E^k_j$,

(iii) if $k < l$, then either $E^k_j \cap E^l_i = \emptyset$ or $E^k_j \subset E^l_i$.

https://doi.org/10.1017/S0308210510001654 Published online by Cambridge University Press

The family $\mathcal{D} = \bigcup_m \mathcal{D}_m$ is called a dyadic decomposition of $\mathbb{S}^n$, and we denote the sets in $\mathcal{D}$ by cubes Q; if $B(\zeta_j^k, \lambda^k) \neq B(\zeta_j^k, \lambda^{k+1})$, i.e. if $\lambda^k \leqslant 2$, we define $r_Q = \lambda^k$. Otherwise, $r_Q = 2$.

Lemma 3.2. *Let μ be a finite positive Borel measure on $\mathbb{S}^n$ and $\delta > 0$. If $\mathcal{D} = \bigcup_m \mathcal{D}_m$ is a dyadic decomposition of $\mathbb{S}^n$, then the following conditions are equivalent:*

(i)
$$\sup_{r_B<\delta} \frac{\mu(B)}{w(B) r_B^{-sp}} \chi_B \in L^{q/(p-q)}(\mathrm{d}\mu);$$

(ii)
$$\sup_{Q\in\mathcal{D}_m} \frac{\mu(Q)}{w(Q) r_Q^{-sp}} \chi_Q \in L^{q/(p-q)}(\mathrm{d}\mu),$$

with an $L^{q/(p-q)}(\mathrm{d}\mu)$-norm bounded independently of m.

Proof. First of all, observe that in (i) the fixed constant δ is not relevant. Indeed, since μ is finite, we have that, for $\delta \leqslant r_B < \delta_1$, $\mu(B)/w(B) r_B^{-sp}$ is bounded. We also remark that (ii) can be replaced by an analogous condition, taking only $r_Q < \delta$.

We begin by proving that (i) implies (ii). The properties of the dyadic family $\mathcal{D}$, and the fact that w satisfies a doubling condition, easily show that if we set

$$f_m = \sup_{Q\in\mathcal{D}_m} \frac{\mu(Q)}{w(Q) r_Q^{-sp}} \chi_Q,$$

then the following estimates hold for each non-positive integer m:

$$f_m \preceq \sup_{r_B\leqslant 3} \frac{\mu(B)}{w(B) r_B^{-sp}} \chi_B \preceq \sup_k f_k, \tag{3.1}$$

and the left-hand side of the estimate gives (ii).

Next, assume that (ii) holds. Using again the properties of the dyadic decomposition and the doubling condition satisfied by the weight, it can be shown that there exists $C > 0$, such that, for any $k < m$, $f_m \leqslant C f_k$. We deduce that if

$$\sup_m \int_{\mathbb{S}^n} \left(\sup_{Q\in\mathcal{D}_m} \frac{\mu(Q)}{w(Q) r_Q^{-sp}} \chi_Q \right)^{q/(p-q)} \mathrm{d}\mu < +\infty,$$

then the non-decreasing sequence of functions defined by $h_k = \max(f_k, \dots, f_0)$, $k \leqslant 0$, satisfies $h_k \leqslant C f_k$, and consequently, by the monotone convergence theorem,

$$\begin{aligned} \int_{\mathbb{S}^n} \left(\sup_{r_B\leqslant 3} \frac{\mu(B)}{w(B) r_B^{-sp}} \chi_B \right)^{q/(p-q)} \mathrm{d}\mu &\preceq \int_{\mathbb{S}^n} (\sup_k f_k)^{q/(q-p)} \,\mathrm{d}\mu \\ &= \lim_k \int_{\mathbb{S}^n} h_k^{q/(q-p)} \,\mathrm{d}\mu \\ &< +\infty. \end{aligned}$$

□

We next state a non-isotropic version of [18, theorem 3(c), (d)]. The proof relies on the fact that the non-isotropic dyadic maximal function defined by

$$M[f](\zeta) = \sup_{Q\in\mathcal{D}_m} \frac{1}{|Q|}\int_Q f$$

is of strong type (p, p), with constants independent of m. The proof of this fact can be shown by adapting the classical case to the non-isotropic situation.

THEOREM 3.3. *Let μ be a non-negative Borel measure on $\mathbb{S}^n$, $(t_Q)_{Q\in\mathcal{D}}$ a sequence of non-negative real numbers, $0 < q < p < +\infty$. There exists a constant C such that the inequality*

$$\Big\|\sum_{Q\in\mathcal{D}_m} \gamma_Q t_Q \chi_Q\Big\|_{L^q(\mathrm{d}\mu)} \leqslant C\Big(\sum_Q \gamma_Q^p\Big)^{1/p} \tag{3.2}$$

holds for any large negative m, and any sequence $(\gamma_Q)_{Q\in\mathcal{D}_m}$ of non-negative real numbers, if and only if one of the following conditions holds:

(α) *if $\max(q, 1) < p$, there exists $C > 0$ such that, for any m,*

$$\int_{\mathbb{S}^n}\Big(\sum_{Q\in\mathcal{D}_m} t_Q^{p'}\mu(Q)^{p'-1}\chi_Q\Big)^{q(p-1)/(p-q)}\mathrm{d}\mu \leqslant C,$$

where $p' = p/(p-1)$ is the conjugate of p;

(β) *if $0 < q < p \leqslant 1$, there exists $C > 0$ such that, for any m,*

$$\int_{\mathbb{S}^n}\Big(\sup_{Q\in\mathcal{D}_m}\mu(Q)t_Q^p\chi_Q\,\mathrm{d}\mu\Big)^{q/(p-q)}\mathrm{d}\mu \leqslant C.$$

3.2. Proof of theorem 1.3

Assume first that (ii) of theorem 1.3 holds, that is,

$$\sup_{r_B<\delta}\frac{\mu(B)}{w(B)r_B^{-sp}}\chi_B \in L^{q/(p-q)}(\mathrm{d}\mu).$$

Then Hölder's inequality with exponent $p/q > 1$ gives that

$$\begin{aligned}\int_{\mathbb{S}^n}|M_\alpha[f](\zeta)|^q\,\mathrm{d}\mu(\zeta) &\leqslant \bigg(\int_{\mathbb{S}^n}|M_\alpha[f](\zeta)|^p\frac{\mathrm{d}\mu(\zeta)}{\sup_{r_B<\delta}(\mu(B)/w(B)r_B^{-sp})\chi_B(\zeta)}\bigg)^{q/p}\\ &\quad\times\bigg(\int_{\mathbb{S}^n}\Big(\sup_{r_B<\delta}\frac{\mu(B)}{w(B)r_B^{-sp}}\chi_B\Big)^{q/(p-q)}\mathrm{d}\mu(\zeta)\bigg)^{(p-q)/p}\\ &\preceq \bigg(\int_{\mathbb{S}^n}|M_\alpha[f](\zeta)|^p\frac{\mathrm{d}\mu(\zeta)}{\sup_{r_B<\delta}(\mu(B)/w(B)r_B^{-sp})\chi_B(\zeta)}\bigg)^{q/p}.\end{aligned}$$

If we define

$$\mathrm{d}\mu_1(\zeta) = \frac{\mathrm{d}\mu(\zeta)}{\sup_B(\mu(B)/w(B)r_B^{-sp})\chi_B(\zeta)},$$

then the measure μ_1 satisfies, for any $r_B < \delta$,

$$\mu_1(B) = \int_B \mathrm{d}\mu_1(\zeta) \leqslant \int_B \frac{w(B) r_B^{-sp}}{\mu(B)} \, \mathrm{d}\mu(\zeta) = \frac{w(B)}{r_B^{sp}}.$$

Hence, since $p \leqslant 1$, theorem 1.1 gives that there exists $C > 0$ such that, for any $f \in H_s^p(w)$,

$$\biggl(\int_{\mathbb{S}^n} |M_\alpha[f](\zeta)|^p \, \mathrm{d}\mu_1(\zeta) \biggr)^{1/p} \leqslant C \|f\|_{H_s^p(w)}.$$

This completes the proof of the sufficiency of condition (1.9).

Next assume that (i) of theorem 1.3 holds. By Lemma 3.2, in order to complete the proof of the theorem, we must show that

$$\sup_{Q \in \mathcal{D}_m} \mu(Q) \frac{\mu(Q)}{w(Q) r_Q^{-sp}} \chi_Q \in L^{q/(p-q)}(\mathrm{d}\mu),$$

with the $L^{q/(p-q)}(\mathrm{d}\mu)$-norm bounded independently of m. By the observation at the beginning of the proof of lemma 3.2, we may assume that the radius r_Q is strictly less than 1.

Let m be a large negative integer, and let $Q \in \mathcal{D}_m$, $\zeta_Q \in Q$, $r_Q < 1$ and $\beta < 1$ be as chosen later. Let $(\lambda_Q)_{Q \in \mathcal{D}_m}$ be a sequence of non-negative real numbers in l^p. We consider the holomorphic function on $\mathbb{B}^n$ defined by

$$F(z) = \sum_{Q \in \mathcal{D}_m, r_Q < 1} \lambda_Q \frac{1}{(1 - z(1 - r_Q)\bar{\zeta}_Q)^\beta}.$$

Since $\beta < 1$, the real part of the above functions is non-negative, and consequently we have that, for any $\zeta \in \mathbb{S}^n$,

$$\begin{aligned} \operatorname{Re} M_\alpha[F](\zeta) &\geqslant M_\alpha \biggl[\sum_{Q \in \mathcal{D}_m, r_Q < 1} \lambda_Q \operatorname{Re} \frac{1}{(1 - z(1 - r_Q)\bar{\zeta}_Q)^\beta} \biggr](\zeta) \\ &\simeq \sum_{Q \in \mathcal{D}_m, r_Q < 1} \lambda_Q M_\alpha \biggl[\frac{1}{|1 - z(1 - r_Q)\bar{\zeta}_Q|^\beta} \biggr](\zeta) \\ &\succeq \sum_{Q \in \mathcal{D}_m, r_Q < 1} \lambda_Q \frac{1}{r_Q^\beta} \chi_Q(\zeta). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\biggl\| \sum_{Q \in \mathcal{D}_m, r_Q < 1} \lambda_Q \frac{1}{(1 - z(1 - r_Q)\bar{\zeta}_Q)^\beta} \biggr\|_{H_s^p(w)} \\ &\quad = \biggl\| M_\alpha \biggl[\mathcal{R}^s \biggl(\sum_{Q \in \mathcal{D}_m, r_Q < 1} \lambda_Q \frac{1}{(1 - z(1 - r_Q)\bar{\zeta}_Q)^\beta} \biggr) \biggr] \biggr\|_{L^p(w)} \\ &\quad \preceq \biggl(\biggl(\int_{\mathbb{S}^n} \sum_{Q \in \mathcal{D}_m, r_Q < 1} \lambda_Q M_\alpha \frac{1}{|1 - z(1 - r_Q)\bar{\zeta}_Q|^{\beta + s}}(\eta) \biggr)^p w(\eta) \, \mathrm{d}\sigma(\eta) \biggr)^{1/p}. \end{aligned}$$

But, since

$$\sup_{z\in D_\alpha(\eta)} \frac{1}{|1-(1-r_Q)z\bar{\zeta}_Q|^{\beta+s}} \preceq \frac{1}{|1-\eta(1-r_Q)\bar{\zeta}_Q|^{\beta+s}},$$

and $p \leqslant 1$, we have that

$$\begin{aligned}\Bigg\| \sum_{Q\in\mathcal{D}_m, r_Q<1} \lambda_Q \frac{1}{(1-z(1-r_Q)\bar{\zeta}_Q)^\beta} \Bigg\|_{H_s^p(w)} &\preceq \bigg(\sum_{Q\in\mathcal{D}_m, r_Q<1} \lambda_Q^p \int_Q \frac{w(\eta)\,\mathrm{d}\sigma(\eta)}{|1-(1-r_Q)\eta\bar{\zeta}_Q|^{(\beta+s)p}} \bigg)^{1/p} \\ &\preceq \bigg(\sum_{Q\in\mathcal{D}_m, r_Q<1} \lambda_Q^p \frac{w(Q)}{r_Q^{(\beta+s)p}} \bigg)^{1/p},\end{aligned}$$

where the last estimate holds provided we choose $\beta < 1$ such that $(\beta+s)p - \tau > 0$ (which is possible since by hypothesis $\tau - sp < p$). Altogether, applying (1.1), we have that

$$\Bigg\| \sum_{Q\in\mathcal{D}_m, r_Q<1} \lambda_Q \frac{\chi_Q}{r_Q^\beta} \Bigg\|_{L^q(\mathrm{d}\mu)} \preceq \bigg(\sum_{Q\in\mathcal{D}_m, r_Q<1} \lambda_Q^p \frac{w(Q)}{r_Q^{(\beta+s)p}} \bigg)^{1/p}.$$

Defining

$$\gamma_Q := \lambda_Q \frac{w(Q)^{1/p}}{r_Q^{\beta+s}},$$

we can rewrite the latter estimate as

$$\Bigg\| \sum_{Q\in\mathcal{D}_m, r_Q<1} \gamma_Q \frac{r_Q^s}{w(Q)^{1/p}} \chi_Q \Bigg\|_{L^q(\mathrm{d}\mu)} \preceq \bigg(\sum_{Q\in\mathcal{D}_m, r_Q<1} \gamma_Q^p \bigg)^{1/p},$$

which, by theorem 3.3, is equivalent to the condition

$$\sup_{Q\in\mathcal{D}_m, r_Q<1} \frac{\mu(Q)}{w(Q) r_Q^{-sp}} \chi_Q \in L^{q/(p-q)}(\mu),$$

with the $L^{q/(p-q)}(\mu)$-norm bounded independently of m.

3.3. Proof of theorem 1.4 for $q > 1$

The methods for the proof of theorem 1.4 when $q > 1$ will use duality, and consequently cannot be used for $q \leqslant 1$. Although the proof that we shall give for $q \leqslant 1$ can also be applied to the case $q > 1$, we include both proofs, since some additional restrictions on τ, s and p have to be considered when $q \leqslant 1$.

If ν is a positive Borel measure on $\mathbb{S}^n$, $1 < p < +\infty$, we recall that the (s,p)-energy of ν with weight w is defined by

$$\mathcal{E}_{sp}^w(\nu) = \int_{\mathbb{S}^n} (K_s[\nu](\zeta))^{p'} w(\zeta)^{-(p'-1)}\,\mathrm{d}\sigma(\zeta). \tag{3.3}$$

https://doi.org/10.1017/S0308210510001654 Published online by Cambridge University Press

If we write $(K_s[\nu])^{p'} = (K_s[\nu])^{p'-1}K_s[\nu]$, Fubini's theorem gives that

$$\mathcal{E}_{sp}^w(\nu) = \int_{\mathbb{S}^n} K_s[w^{-1}K_s[\nu]]^{p'-1}(\zeta)\,\mathrm{d}\nu(\zeta).$$

The first observation is a reformulation of assertion (ii) in theorem 1.4, which will be useful in the remainder of the section.

Lemma 3.4. *Let $0 < s < n$, $0 < p, q < +\infty$, let w be a weight on $\mathbb{S}^n$ and μ be a positive Borel measure on $\mathbb{S}^n$. Then the following conditions are equivalent:*

(a) *there exists $C > 0$ such that, for any f,*

$$\left(\int_{\mathbb{S}^n} K_s[f](\zeta)^q\,\mathrm{d}\mu(\zeta)\right)^{1/q} \leqslant C\|f\|_{L^p(w)};$$

(b) *there exists $C > 0$ such that, for any f,*

$$\left(\int_{\mathbb{S}^n} K_s[fw^{-(p'-1)}](\zeta)^q\,\mathrm{d}\mu(\zeta)\right)^{1/q} \leqslant C\|f\|_{L^p(w^{-(p'-1)})}.$$

Proof. This is an immediate consequence of the fact that if we set $|g| = |f|w^{-(p'-1)}$, then $|g|^p w = |f|^p w^{-(p'-1)}$. □

The following theorem, which is the non-weighted version of theorem 1.4 for $q > 1$, was proved in [6].

Theorem 3.5 (Cascante *et al*. [6, theorems 3.3 and 4.1]). *Let $1 < q < p$, $s > 0$, with $0 \leqslant n - sp < 1$, and let μ be a positive Borel measure on $\mathbb{S}^n$. Then the following conditions are equivalent:*

(α) *there exists $C > 0$ such that, for any $f \in L^p$,*

$$\left(\int_{\mathbb{S}^n} |M_\alpha[C_s[f]](\zeta)|^q\,\mathrm{d}\mu(\zeta)\right)^{1/q} \leqslant C\|f\|_{L^p};$$

(β) *there exists $C > 0$ such that, for any $f \in L^p$,*

$$\left(\int_{\mathbb{S}^n} |K_s[f](\zeta)|^q\,\mathrm{d}\mu(\zeta)\right)^{1/q} \leqslant C\|f\|_{L^p};$$

(γ) *$\mathcal{W}_{sp}[\mu] \in L^{q(p-1)/(p-q)}(\mu)$.*

To prove theorem 1.4 for $1 < q$, we shall follow the ideas of the previous theorem, which deal with the technicalities arising from the fact that we are dealing with a weight. We shall show that (i) ⇔ (ii) and (ii) ⇔ (iii). The proof that (ii) ⇒ (i) is a consequence of the pointwise estimate $|M_\alpha[C_s[f]]| \preceq K_s[|f|]$.

Next we shall show that (i) ⇒ (ii). Assume first that $\mathcal{E}_{sp}^w(\mathrm{d}\mu) < +\infty$ and let f be a bounded non-negative function on $\mathbb{S}^n$. We shall consider the holomorphic potential $\mathcal{U}_{sp}^w[f\,\mathrm{d}\mu]$ given in theorem 2.3, applied to the measure $f\,\mathrm{d}\mu$. Using the weighted version of Wolff's estimate, Hölder's inequality and condition (i) of theorem 2.3, we have

$$\begin{aligned}\mathcal{E}^w_{sp}(f\,\mathrm{d}\mu) &\simeq \int_{\mathbb{S}^n} \mathcal{W}^w_{sp}(f\,\mathrm{d}\mu)(\zeta)f(\zeta)\,\mathrm{d}\mu(\zeta)\\ &\preceq \int_{\mathbb{S}^n} M_\alpha[\mathcal{U}^w_{sp}[f\,\mathrm{d}\mu]](\zeta)f(\zeta)\,\mathrm{d}\mu(\zeta)\\ &\preceq \|f\|_{L^{q'}(\mathrm{d}\mu)}\|M_\alpha[\mathcal{U}^w_{sp}[f\,\mathrm{d}\mu]]\|_{L^q(\mathrm{d}\mu)}.\end{aligned}$$

The fact that we are assuming that (i) holds, together with property (ii) of theorem 2.3, gives that the above is bounded by

$$\|f\|_{L^{q'}(\mathrm{d}\mu)}\mathcal{E}^w_{sp}(f\,\mathrm{d}\mu)^{1/p};$$

thus,

$$\mathcal{E}^w_{sp}(f\,\mathrm{d}\mu)^{1/p'} \preceq \|f\|_{L^{q'}(\mathrm{d}\mu)}.$$

Since, by definition, $\mathcal{E}^w_{sp}(f\,\mathrm{d}\mu) = K_s[w^{-1}K_s[f\,\mathrm{d}\mu]]^{p'-1}(\zeta)$, the latter estimate can be rewritten using duality as

$$\int_{\mathbb{S}^n} K_s[g](\zeta)^q\,\mathrm{d}\mu(\zeta) \preceq \|g\|^q_{L^p(w)},$$

which is (ii). The fact that any measure μ that satisfies (i) also satisfies $\mathcal{E}^w_{sp}(\mathrm{d}\mu) < +\infty$ can be argued as in [6].

Next we prove that (iii) $\Rightarrow$ (ii). We shall follow closely the proofs of [6, theorems 3.3 and 4.1]. By duality, since $q > 1$, in order to prove (ii), we have to check that, for any $g \geqslant 0$,

$$\|K_s[g\,\mathrm{d}\mu]\|_{L^{p'}(w^{-(p'-1)})} \preceq \|g\|_{L^{q'}(\mathrm{d}\mu)}.$$

But the left-hand side of the estimate that we have to check is just the weighted energy $\mathcal{E}^w_{sp}(g\,\mathrm{d}\mu)^{1/p'}$. By applying the weighted version of Wolff's estimate, the above estimate can be rewritten as

$$\int_{\mathbb{S}^n} \mathcal{W}^w_{sp}[g\,\mathrm{d}\mu]g\,\mathrm{d}\mu \preceq \|g\|^{p'}_{L^{q'}(\mathrm{d}\mu)}. \tag{3.4}$$

If $M^{\mathrm{HL}}_\mu[g]$ denotes the centred non-isotropic Hardy–Littlewood maximal function defined by

$$M^{\mathrm{HL}}_\mu[g](\zeta) = \sup_{r>0}\frac{1}{\mu(B(\zeta,r))}\int_{B(\zeta,r)} g(\eta)\,\mathrm{d}\mu(\eta),$$

we then have that the following pointwise estimate holds:

$$\mathcal{W}^w_{sp}[g\,\mathrm{d}\mu](\zeta) \preceq (M^{\mathrm{HL}}_\mu[g](\zeta))^{p'-1}\mathcal{W}^w_{sp}[\mu](\zeta).$$

Consequently, applying Hölder's inequality with exponent $r = q'/(p'-1) > 1$ to (3.4), we obtain

$$\int_{\mathbb{S}^n} \mathcal{W}^w_{sp}[g\,\mathrm{d}\mu] g\,\mathrm{d}\mu \preceq \biggl(\int_{\mathbb{S}^n} (M^{\mathrm{HL}}_\mu[g](\zeta))^{q'}\,\mathrm{d}\mu(\zeta)\biggr)^{1/r} \biggl(\int_{\mathbb{S}^n} (g(\zeta)\mathcal{W}^w_{sp}[\mu](\zeta))^{r'}\,\mathrm{d}\mu(\zeta)\biggr)^{1/r'}.$$

The fact that the centred non-isotropic Hardy–Littlewood maximal operator M^{HL}_μ is bounded in $L^{q'}(\mathrm{d}\mu)$, together with Hölder's inequality with exponent $q'/r' > 1$ gives finally that the above estimate is bounded, up to a constant, by

$$\|g\|^{p'}_{L^{q'}(\mathrm{d}\mu)} \biggl(\int_{\mathbb{S}^n} (\mathcal{W}^w_{sp}[\mu](\zeta))^{q(p-1)/(p-q)}\,\mathrm{d}\mu(\zeta)\biggr)^{(p-q)/q(p-1)} \preceq \|g\|^{p'}_{L^{q'}(\mathrm{d}\mu)},$$

that is, we have proved (3.4).

We next prove that (ii) $\Rightarrow$ (iii). Assume that (ii) holds. By lemma 3.4, we have that

$$\biggl(\int_{\mathbb{S}^n} |K_s[fw^{-(p'-1)}]|^q\,\mathrm{d}\mu\biggr)^{1/q} \leqslant C\|f\|_{L^p(w^{-(p'-1)})},$$

for any $f \geqslant 0$. If $(\lambda_Q)_Q$ is a sequence of non-negative real numbers, let

$$f = \sup_{Q\in\mathcal{D}_m} \lambda_Q \chi_Q.$$

Then

$$\|f\|_{L^p(w^{-(p'-1)})} \leqslant \biggl(\sum_Q \lambda_Q^p w^{-(p'-1)}(Q)\biggr)^{1/p}.$$

We also have that

$$\sum_Q \lambda_Q r_Q^{s-n} w^{-(p'-1)}(Q)\chi_Q \leqslant K_s[fw^{-(p'-1)}].$$

Combining these expressions, we obtain

$$\biggl\|\sum_Q \lambda_Q r_Q^{s-n} w^{-(p'-1)}(Q)\chi_Q\biggr\|_{L^q(\mathrm{d}\mu)} \leqslant C\biggl(\sum_Q \lambda_Q^p w^{-(p'-1)}(Q)\biggr)^{1/p}.$$

Equivalently,

$$\biggl\|\sum_Q \gamma_Q r_Q^{s-n} (w^{-(p'-1)}(Q))^{1-1/p}\chi_Q\biggr\|_{L^q(\mathrm{d}\mu)} \leqslant C\biggl(\sum_Q \gamma_Q^p\biggr)^{1/p}.$$

By (α) in theorem 3.3 we obtain that

$$\sum_{Q\in\mathcal{D}_m} \biggl(\frac{\mu(Q)}{r_Q^{n-sp}}\biggr)^{p'-1} \frac{w^{-(p'-1)}(Q)}{r_Q^n}\chi_Q \in L^{q(p-1)/(p-q)}(\mathrm{d}\mu).$$

https://doi.org/10.1017/S0308210510001654 Published online by Cambridge University Press

We observe that the left-hand side of the above estimate is a discrete version of the Wolff potential $\mathcal{W}^w_{sp}[\mu]$. The continuous version of the above conclusion can be obtained with similar methods to those used in the proof of [6, theorem 3.3]. The arguments are based on an estimate of the continuous version of the Wolff potential in terms of an average of discrete Wolff potential associated to a collection of shifted dyadic lattices $\mathcal{D}_t$, an estimate that holds because, since $q > 1$, we also have that $q(p-1)/(p-q) > 1$. Hence, using Hölder's inequality allows us to finish the theorem. We refer the reader to [6] for the details.

3.4. Proof of theorem 1.4 for $q \leqslant 1$

In the proof of theorem 1.4 when $q \leqslant 1$, we shall use an alternative argument. We shall see that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii).

As in the previous case, the fact that (ii) $\Rightarrow$ (i) in theorem 1.4 is a consequence of the estimate $M_\alpha[C_s[f]] \preceq K_s[|f|]$.

Next we shall show that (iii) $\Rightarrow$ (ii). We need some technical lemmas.

Lemma 3.6. *Let $1 < p < +\infty$, $s, \tau > 0$, and let $w \in \mathcal{A}_p$ satisfy a doubling condition of order τ. We then obtain that there exists a $C > 0$ such that, for any $\zeta \in \mathbb{S}^n$, $r > 0$, $k \geqslant 1$,*

$$w^{-(p'-1)}\left(B\left(\zeta, \frac{r}{2^k}\right)\right) \leqslant C 2^{k(\tau-np)/(p-1)} w^{-(p'-1)}(B(\zeta, r)).$$

Since w is in D_τ,

$$\frac{1}{r^n} w(B(\zeta, r)) \leqslant C 2^{k(\tau-n)} \frac{1}{(r/2^k)^n} w\left(B\left(\zeta, \frac{r}{2^k}\right)\right).$$

The fact that $w \in \mathcal{A}_p$ allows the above estimate to be rewritten as

$$\left(\frac{1}{r^n} w^{-(p'-1)}(B(\zeta, r))\right)^{-(p-1)} \leqslant C 2^{k(\tau-n)} \left(\frac{1}{(r/2^k)^n} w^{-(p'-1)}\left(B\left(\zeta, \frac{r}{2^k}\right)\right)\right)^{-(p-1)}.$$

Equivalently,

$$2^{kn} w^{-(p'-1)}\left(B\left(\zeta, \frac{r}{2^k}\right)\right) \leqslant C 2^{k(\tau-n)/(p-1)} w^{-(p'-1)}(B(\zeta, r)),$$

which gives

$$w^{-(p'-1)}\left(B\left(\zeta, \frac{r}{2^k}\right)\right) \leqslant C 2^{k(\tau-np)/(p-1)} w^{-(p'-1)}(B(\zeta, r)).$$

As a consequence of the above lemma we have the following.

Lemma 3.7. *Let $1 < p < +\infty$, $s, \tau > 0$, and let $w \in \mathcal{A}_p$ satisfy a doubling condition of order τ. Assume that $\tau - n < s(p-1)$. We then have that*

$$\frac{1}{r^{n-s}} \preceq \frac{1}{w^{-(p'-1)}(B(\eta, r))} \int_0^r \frac{1}{t^{n-s}} w^{-(p'-1)}(B(\eta, t)) \frac{\mathrm{d}t}{t} \preceq \frac{1}{r^{n-s}},$$

with constants independent of η and $r > 0$.

Observe that this last lemma implies that the pair $(K_s, w^{-(p'-1)})$ satisfies the so-called logarithmic bounded oscillation property (LBO) [8, (3.1)], i.e.

$$\sup_{\eta\in B(\zeta,r)} \frac{1}{w^{-(p'-1)}(B(\eta,r))}\int_0^r \frac{1}{t^{n-s}} w^{-(p'-1)}(B(\eta,t))\frac{\mathrm{d}t}{t} \preceq \inf_{\eta\in B(\zeta,r)} \frac{1}{w^{-(p'-1)}(B(\eta,r))}\int_0^r \frac{1}{t^{n-s}} w^{-(p'-1)}(B(\eta,t))\frac{\mathrm{d}t}{t}.$$

We also observe that the hypothesis in theorem 1.4 for the case where $q \leqslant 1$ implies that for any p we always have that $\tau - n < s(p-1)$, and consequently the pair $(K_s, w^{-(p'-1)})$ satisfies the LBO condition.

Now we can follow with the proof of (iii) $\Rightarrow$ (ii). We shall sketch some of the ideas of the proof of [8, theorem 3.3], a version of this assertion is considered in $\mathbb{R}^n$. In order to show the implication, we first observe that it is enough to prove that if $\mathrm{d}\mu_1 = \mathrm{d}\mu/(\mathcal{W}^w_{sp}[\mu])^{p-1}$, then the following estimate holds:

$$\int_{\mathbb{S}^n} K_s[fw^{-(p'-1)}]^p(\zeta)\,\mathrm{d}\mu_1(\zeta) \preceq \|f\|^p_{L^p(w^{-(p'-1)})}. \tag{3.5}$$

Indeed, if we assume for a moment that (3.5) holds, then Hölder's inequality with exponent p/q and (iii) yield

$$\begin{aligned}\biggl(\int_{\mathbb{S}^n} K_s[fw^{-(p'-1)}]^q(\zeta)\,\mathrm{d}\mu(\zeta)\biggr)^{1/q} &\leqslant \biggl(\int_{\mathbb{S}^n} K_s[fw^{-(p'-1)}]^p(\zeta)\,\mathrm{d}\mu_1(\zeta)\biggr)^{1/p} (\mathcal{W}^w_{sp}[\mu]^{q(p-1)/(p-q)})^{(p-q)/qp}\\ &\preceq \biggl(\int_{\mathbb{S}^n} K_s[fw^{-(p'-1)}]^p(\zeta)\,\mathrm{d}\mu_1(\zeta)\biggr)^{1/p}\\ &\preceq \|f\|_{L^p(w^{-(p'-1)})}.\end{aligned}$$

So we are left to prove (3.5). Applying duality, (3.5) holds if and only if

$$\mathcal{E}^w_{sp}[g\,\mathrm{d}\mu_1] \preceq \|g\|^{p'}_{L^{p'}(\mathrm{d}\mu_1)}. \tag{3.6}$$

The arguments in [8, theorem 3.3] can be reproduced in our situation to show that the above continuous energy can be rewritten as a supremum over certain 'translations' of non-isotropic dyadic energies introduced in [6].

Consequently, in order to prove (3.6), it is enough to show that the discrete version (3.7) holds, with constants independent of m, namely,

$$\mathcal{E}^{w\mathcal{D}_m}_{sp}[g\,\mathrm{d}\mu_1] = \int_{\mathbb{S}^n}\biggl(\sum_{Q\in\mathcal{D}_m} r_Q^{s-n}\int_Q g\,\mathrm{d}\mu_1\chi_Q\biggr)^{p'}\mathrm{d}\sigma \preceq \|g\|^{p'}_{L^{p'}(\mathrm{d}\mu_1)}. \tag{3.7}$$

Again applying duality, in order to prove (3.7), it is enough to prove the following lemma.

LEMMA 3.8. *If we define*

$$K_s^{\mathcal{D}_m}[fw^{-(p'-1)}](\zeta) = \sum_{Q\in\mathcal{D}_m} r_Q^{s-n}\int_Q fw^{-(p'-1)}\chi_Q(\zeta),$$

then we have that

$$\int_{\mathbb{S}^n} K_s^{\mathcal{D}_m}[fw^{-(p'-1)}]^p(\zeta)\,\mathrm{d}\mu_1(\zeta) \preceq \|f\|^p_{L^p(w^{-(p'-1)})}. \tag{3.8}$$

Proof. Since

$$\mathrm{d}\mu_{\mathcal{D}_m} = \mathrm{d}\mu_1 \preceq \frac{\mathrm{d}\mu}{\mathcal{W}_{sp}^{w,\mathcal{D}_m}[\mu]^{p-1}},$$

in order to prove the lemma we shall show that the estimate

$$\int_{\mathbb{S}^n} K_s^{\mathcal{D}_m}[fw^{-(p'-1)}]^p(\zeta)\,\mathrm{d}\mu_{\mathcal{D}_m}(\zeta) \preceq \|f\|^p_{L^p(w^{-(p'-1)})}$$

holds.

Next, the pair $(K_s, w^{-(p'-1)})$ satisfies an LBO property, and in consequence, the non-isotropic version of [8, lemma 2.2] gives that the above discrete version holds if and only if, for any $P\in\mathcal{D}_m$,

$$\sum_{Q\subset P} r_Q^{(s-n)p'}\mu_{\mathcal{D}_m}(Q)^{p'}w^{-(p'-1)}(Q) \preceq \mu_{\mathcal{D}_m}(P).$$

Let us check this last estimate holds. If we define $\mathcal{W}_{sp}^{w,\mathcal{D}_m}[\mu]$ by

$$\mathcal{W}_{sp}^{w,\mathcal{D}_m}[\mu](\zeta) = \sum_{Q\in\mathcal{D}_m} r_Q^{(s-n)p'}w^{-(p'-1)}(Q)(\mu(Q))^{p'-1}\chi_Q(\zeta),$$

then, by Hölder's inequality,

$$\mu_{\mathcal{D}_m}^{p'}(Q) \leqslant \mu_{\mathcal{D}_m}^{p'-1}(Q)\int_Q \frac{\mathrm{d}\mu}{\mathcal{W}_{sp}^{w,\mathcal{D}_m}[\mu]^p}.$$

Consequently, for any $P\in\mathcal{D}_m$,

$$\begin{aligned}
&\sum_{Q\subset P} r_Q^{(s-n)p'}\mu_{\mathcal{D}_m}(Q)^{p'}w^{-(p'-1)}(Q)\\
&\qquad\leqslant \sum_{Q\subset P} r_Q^{(s-n)p'}\mu_{\mathcal{D}_m}(Q)^{p'-1}w^{-(p'-1)}(Q)\\
&\qquad\qquad\times\int_Q \frac{\mathrm{d}\mu(\eta)}{\sum_{Q'\subset P} r_{Q'}^{(s-n)p'}\mu(Q')w^{-(p'-1)(Q')}\chi_{Q'}(\eta)} \leqslant \mu_{\mathcal{D}_m}(P).
\end{aligned}$$

For the proof of (i) $\Rightarrow$ (iii) of theorem 1.4 when $q<1$, we shall construct adequate holomorphic test functions and use a pointwise equivalence between the weighted Wolff potential $\mathcal{W}_{sp}^w[\nu]$ and the non-isotropic Riesz potential $V_{sp}^w[\nu]$ defined below. This pointwise estimate adapts to the weighted non-isotropic case the pointwise inequality relating the nonlinear Riesz potential and the Wolff potential obtained in $\mathbb{R}^n$ and for the unweighted case in [13, theorem 6.2]. We recall that if ν is a non-negative Borel measure on $\mathbb{S}^n$, $V^w_{sp}[\nu]$ is the non-isotropic weighted Riesz potential defined by

$$V^w_{sp}[\nu](\zeta) := K_s[w^{-1}K_s\nu]^{p'-1}. \tag{3.9}$$

We have that $\mathcal{W}^w_{sp}[\nu] \leqslant CV^w_{sp}[\nu]$, and as we have already said, the weighted version of the fundamental Wolff's theorem establishes that in average the converse is also true. The following result shows that in some particular cases both nonlinear potentials are pointwise equivalent.

Proposition 3.9. *Let $1 < p < +\infty$, $\tau > 0$, $s > 0$ and let w be a weight in $\mathcal{A}_p$, satisfying a doubling condition of order τ. Let ν be a positive Borel measure on $\mathbb{S}^n$. Assume that, in addition, one of the following conditions is satisfied:*

(i) *if $1 < p \leqslant 2$, $\tau < (p-1)n + s$;*

(ii) *if $p > 2$, $\tau < n + s(p-1)$.*

Then $V^w_{sp}[\nu] \simeq \mathcal{W}^w_{sp}[\nu]$.

We first observe that if $w \equiv 1$, then $\tau = n$ and, in the case $p \leqslant 2$, condition (i) can be rewritten as $p > 2 - s/n$. If $p > 2$, then (ii) is always satisfied. Since the estimate $\mathcal{W}^w_{sp}[\nu] \leqslant CV^w_{sp}[\nu]$ holds for any positive measure, we have only to prove that, under the hypothesis of the proposition, the other pointwise estimate also holds.

Proof. We have that

$$V^w_{sp}[\nu](\zeta) \simeq V_1(\zeta) + V_2(\zeta),$$

where

$$V_1(\zeta) = \int_0^\infty \frac{1}{r^{n-s}} \int_{|1-\zeta\bar{\eta}|<r} \left(w^{-1}(\eta) \int_{|1-\zeta\bar{z}|\geqslant 2Cr} \frac{\mathrm{d}\nu(z)}{|1-\eta\bar{z}|^{n-s}} \right)^{1/(p-1)} \mathrm{d}\sigma(\eta) \frac{\mathrm{d}r}{r}$$

and

$$V_2(\zeta) = \int_0^\infty \frac{1}{r^{n-s}} \int_{|1-\zeta\bar{\eta}|<r} \left(w^{-1}(\eta) \int_{|1-\zeta\bar{z}|< 2Cr} \frac{\mathrm{d}\nu(z)}{|1-\eta\bar{z}|^{n-s}} \right)^{1/(p-1)} \mathrm{d}\sigma(\eta) \frac{\mathrm{d}r}{r}.$$

We begin by estimating the function V_1. Since $|1-\zeta\bar{\eta}| \leqslant r$ and $|1-\zeta\bar{z}| \geqslant 2Cr$, we have that $|1-\eta\bar{z}| \simeq |1-\zeta\bar{z}|$. We consider separately the two conditions given in the hypothesis. In case (i), Hölder's inequality with exponent $1/(p-1) \geqslant 1$ gives

$$\begin{aligned}
&\left(\int_{|1-\zeta\bar{z}|\geqslant 2Cr} \frac{\mathrm{d}\nu(z)}{|1-\zeta\bar{z}|^{n-s}} \right)^{1/(p-1)} \\
&\qquad \simeq \left(\int_{2Cr}^\infty \frac{1}{t^{n-s}} \int_{|1-\zeta\bar{z}|<t} \mathrm{d}\nu(z) \frac{\mathrm{d}t}{t} \right)^{1/(p-1)} \\
&\qquad \preceq \int_{2Cr}^\infty \frac{\nu(B(\zeta,t))^{1/(p-1)}}{t^{(n-s-\varepsilon)/(p-1)}} \left(\int_{2Cr}^\infty \frac{\mathrm{d}t}{t^{\varepsilon(1/(p-1))'+1}} \right)^{(2-p)/(p-1)} \frac{\mathrm{d}t}{t} \\
&\qquad \simeq \frac{1}{r^{\varepsilon/(p-1)}} \int_{2Cr}^\infty \frac{\nu(B(\zeta,t))^{1/(p-1)}}{t^{(n-s-\varepsilon)/(p-1)}} \frac{\mathrm{d}t}{t},
\end{aligned}$$

where $\varepsilon > 0$ is small enough to satisfy

$$s - \frac{\varepsilon}{p-1} > \frac{\tau - n}{p-1},$$

which is possible since we are assuming that $\tau < (p-1)n + s$ and since $p \leqslant 2$, $(p-1)n + s \leqslant n + s(p-1)$. Hence, Fubini's theorem and lemma 3.6 give

$$\begin{aligned} V_1(\zeta) &\preceq \int_0^\infty \int_0^{t/2C} r^{s-(\varepsilon/(p-1))} \frac{1}{r^n} \int_{B(\zeta,r)} w^{-(p'-1)} \frac{\mathrm{d}r}{r} \frac{\nu(B(\zeta,t))^{1/(p-1)}}{t^{(n-s-\varepsilon)/(p-1)}} \frac{\mathrm{d}t}{t} \\ &\preceq \int_0^\infty \frac{\nu(B(\zeta,t))^{1/(p-1)}}{t^{-s+(n-s)/(p-1)}} \fint_{B(\zeta,t)} w^{-(p'-1)} \frac{\mathrm{d}t}{t} \\ &\preceq \mathcal{W}^w_{sp}[\nu](\zeta). \end{aligned}$$

In case (ii), $1/(p-1) < 1$. Then

$$\begin{aligned} \biggl(\int_{|1-\zeta\bar{z}|\geqslant 2Cr} \frac{\mathrm{d}\nu(z)}{|1-\eta\bar{z}|^{n-s}}\biggr)^{1/(p-1)} &\simeq \biggl(\int_{2Cr}^\infty \frac{1}{t^{n-s}} \int_{|1-\zeta\bar{z}|<t} \mathrm{d}\nu(z) \frac{\mathrm{d}t}{t}\biggr)^{1/(p-1)} \\ &\preceq \int_{2Cr}^\infty \frac{1}{t^{(n-s)/(p-1)}} \nu(B(\zeta,t))^{1/(p-1)} \frac{\mathrm{d}t}{t}, \end{aligned}$$

and consequently, Fubini's theorem gives

$$\begin{aligned} V_1(\zeta) &\preceq \int_0^\infty \frac{1}{r^{n-s}} \int_{2Cr}^\infty \frac{1}{t^{(n-s)/(p-1)}} \nu(B(\zeta,t))^{1/(p-1)} \frac{\mathrm{d}t}{t} \int_{B(\zeta,r)} w^{-(p'-1)} \frac{\mathrm{d}r}{r} \\ &= \int_0^\infty \int_0^{t/2C} r^s \frac{1}{r^n} \int_{B(\zeta,r)} w^{-(p'-1)} \frac{\mathrm{d}r}{r} \frac{\nu(B(\zeta,t))^{1/(p-1)}}{t^{(n-s)/(p-1)}} \frac{\mathrm{d}t}{t}. \end{aligned}$$

Since by hypothesis we are assuming that $s > (\tau - n)/(p-1)$, lemma 3.6 gives that the above is bounded, up to a constant, by

$$\begin{aligned} &\int_0^\infty t^s \frac{1}{t^n} \int_{B(\zeta,t)} w^{-(p'-1)} \frac{\nu(B(\zeta,t))^{1/(p-1)}}{t^{(n-s)/(p-1)}} \frac{\mathrm{d}t}{t} \\ &\qquad = C \int_0^\infty \frac{\nu(B(\zeta,t))^{1/(p-1)}}{t^{(n-sp)/(p-1)}} \frac{1}{t^n} \int_{B(\zeta,t)} w^{-(p'-1)} \frac{\mathrm{d}t}{t} \\ &\qquad = C\mathcal{W}^w_{sp}[\nu](\zeta). \end{aligned}$$

We now estimate the function V_2. Again we consider separately the two conditions given in the hypothesis.

In case (i), let I_1 be the function defined by

$$I_1(\zeta, r) = \int_{|1-\zeta\bar{\eta}|<r} \biggl(w^{-1}(\eta) \int_{|1-\eta\bar{z}|\leqslant 2Cr} \frac{\mathrm{d}\nu(z)}{|1-\zeta\bar{z}|^{n-s}}\biggr)^{1/(p-1)} \mathrm{d}\sigma(\eta).$$

Since $1/(p-1) > 1$, Minkowski's inequality yields

$$\begin{aligned} I_1^{p-1} &= \left\| \int_{|1-\zeta\bar z|<2Cr} \frac{\mathrm{d}\nu(z)}{|1-\eta\bar z|^{n-s}} \right\|_{L^{1/(p-1)}(\chi_{|1-\zeta\bar\eta|<r} w^{-(p'-1)}(\eta)\,\mathrm{d}\sigma(\eta))} \\ &\preceq \int_{|1-\zeta\bar z|<2Cr} \mathrm{d}\nu(z) \biggl(\int_{|1-\zeta\bar\eta|<Cr} \frac{w^{-(p'-1)}(\eta)\,\mathrm{d}\sigma(\eta)}{|1-\eta\bar z|^{(n-s)/(p-1)}} \biggr)^{p-1}. \end{aligned}$$

Since $|1-\zeta\bar z| < 2Cr$ and $|1-\zeta\bar\eta| < Cr$, we have that $|1-z\bar\eta| < Cr$. Hence,

$$\begin{aligned} \int_{|1-\zeta\bar\eta|<Cr} \frac{w^{-(p'-1)}(\eta)\,\mathrm{d}\sigma(\eta)}{|1-\eta\bar z|^{(n-s)/(p-1)}} &\leqslant \int_{|1-\eta\bar z|<Cr} \frac{w^{-(p'-1)}(\eta)\,\mathrm{d}\sigma(\eta)}{|1-\eta\bar z|^{(n-s)/(p-1)}} \\ &\simeq \int_0^{Cr} t^{n-(n-s)/(p-1)} \frac{1}{t^n} \int_{B(z,t)} w^{-(p'-1)} \frac{\mathrm{d}t}{t} \\ &\preceq r^{n-(n-s)/(p-1)} \frac{1}{r^n} \int_{B(z,r)} w^{-(p'-1)} \frac{\mathrm{d}r}{r}, \end{aligned}$$

where in the last estimate we have used lemma 3.6, since, by hypothesis,

$$n - \frac{n-s}{p-1} > \frac{\tau - n}{p-1}.$$

Thus,

$$I_1^{p-1} \preceq r^{-(n-s)} \int_{|1-\zeta\bar z|<2Cr} \biggl(\int_{B(\zeta,r)} w^{-(p'-1)} \biggr)^{p-1} \mathrm{d}\nu(z),$$

and consequently,

$$\begin{aligned} V_2(\zeta) &= \int_0^{+\infty} \frac{1}{r^{n-s}} I_1(\zeta,r) \frac{\mathrm{d}r}{r} \\ &\preceq \int_0^{\infty} r^{(sp-n)/(p-1)} \nu(B(\zeta,r))^{1/(p-1)} \frac{1}{r^n} \int_{B(z,r)} w^{-(p'-1)} \frac{\mathrm{d}r}{r} \\ &\preceq \mathcal{W}_{sp}^{w}(\zeta). \end{aligned}$$

In case (ii), Hölder's inequality with exponent $p-1$ gives

$$\begin{aligned} &\int_{|1-\zeta\bar\eta|<r} w^{-(p'-1)}(\eta) \biggl(\int_{|1-\zeta\bar z|\leqslant 2Cr} \frac{\mathrm{d}\nu(z)}{|1-\eta\bar z|^{n-s}} \biggr)^{1/(p-1)} \mathrm{d}\sigma(\eta) \\ &\quad\leqslant \biggl(\int_{B(\zeta,r)} w^{-(p'-1)} \biggr)^{(p-2)/(p-1)} \\ &\qquad \times \biggl(\int_{|1-\zeta\bar\eta|<r} w^{-(p'-1)}(\eta) \int_{|1-\zeta\bar z|<2Cr} \frac{\mathrm{d}\nu(z)}{|1-\eta\bar z|^{n-s}}\, \mathrm{d}\sigma(\eta) \biggr)^{1/(p-1)}. \qquad (3.10) \end{aligned}$$

Next, $B(\zeta,r)\subset B(z,Cr)$, and consequently the fact that we are assuming that $s>(\tau-n)/(p-1)$ gives, by lemma 3.6, that

$$\begin{aligned}\int_{|1-\zeta\bar\eta|<r}\frac{w^{-(p'-1)}(\eta)}{|1-\eta\bar z|^{n-s}}\,\mathrm{d}\sigma(\eta) &\leqslant \int_{|1-z\bar\eta|<Cr}\frac{w^{-(p'-1)}(\eta)}{|1-\eta\bar z|^{n-s}}\,\mathrm{d}\sigma(\eta)\\ &\preceq \int_0^r \fint_{B(z,t)}\frac{w^{-(p'-1)}}{t^{-s}}\frac{\mathrm{d}t}{t}\\ &\preceq r^s\fint_{B(z,t)} w^{-(p'-1)}\\ &\preceq \frac{1}{r^{n-s}}\int_{B(\zeta,r)} w^{-(p'-1)}.\end{aligned}$$

Finally, we obtain that (3.10) is bounded up to a constant by

$$\frac{1}{r^{(n-s)/(p-1)}}\mu(B(\zeta,r))^{1/(p-1)}\int_{B(\zeta,r)} w^{-(p'-1)},$$

and, consequently,

$$\begin{aligned}V_2(\zeta) &\preceq \int_0^\infty \frac{r^s}{r^{(n-s)/(p-1)}}\frac{1}{r^n}\int_{B(\zeta,r)} w^{-(p'-1)}\nu(B(\zeta,r))^{1/(p-1)}\frac{\mathrm{d}r}{r}\\ &= C\int_0^\infty\left(\frac{\nu(B(\zeta,r))}{r^{n-sp}}\right)^{1/(p-1)}\frac{1}{t^n}\int_{B(\zeta,t)} w^{-(p'-1)}\frac{\mathrm{d}r}{r}\\ &\preceq \mathcal{W}^w_{sp}[\nu](\zeta).\end{aligned}$$

□

We can now prove that (i) ⇒ (iii). Assume that (i) holds, and let ν be a non-negative measure on $\mathbb{S}^n$. If $\mathcal{U}^w_{sp}[\nu]$ are the holomorphic functions obtained in theorem 2.3, we have that

$$\left(\int_{\mathbb{S}^n}\mathcal{W}^w_{sp}[\nu]^q\,\mathrm{d}\mu\right)^{1/q}\preceq \|M_\alpha\mathcal{U}^w_{sp}[\nu]\|_{L^q(\mathrm{d}\mu)}\preceq\|\mathcal{U}^w_{sp}[\nu]\|_{H^p_s(w)}\preceq\mathcal{E}^w_{sp}(\nu)^{1/p}.$$

Theorem 3.9 gives that $\mathcal{W}^w_{sp}[\nu]\simeq V^w_{sp}[\nu]$, and from the above estimate we deduce that, for every non-negative measure ν,

$$\left(\int_{\mathbb{S}^n}(K_s[[w^{-1}K_s\nu]^{p'-1}])^q\,\mathrm{d}\mu\right)^{1/q}\preceq\left(\int_{\mathbb{S}^n}K_s[\nu]^{p'}w^{-(p'-1)}\,\mathrm{d}\sigma\right)^{1/p}. \tag{3.11}$$

For a fixed large negative m, let $\mathcal{D}=\bigcup_m\mathcal{D}_m$ be the dyadic decomposition introduced in § 3.1.

Let γ be the measure defined by $\mathrm{d}\gamma=w^{-(p'-1)}\,\mathrm{d}\sigma$, and let $(\lambda_Q)_Q$ be a sequence of non-negative real numbers satisfying

$$\sum_{Q\in\mathcal{D}_m}\lambda_Q^p\gamma(Q)<+\infty.$$

We consider the set

$$E = \left\{ f \in L^p(\mathrm{d}\gamma);\ \sum_{Q \in \mathcal{D}_m} \lambda_Q r_Q^s \frac{w^{-(p'-1)}(Q)}{r_Q^n} \chi_{T(Q)}(z) \leqslant K_s[f\,\mathrm{d}\gamma](z),\ z \in \mathbb{B}^n \right\}.$$

We have that E is a convex set in $L^p(\mathrm{d}\gamma)$, which is a uniformly convex Banach space. Consequently (see, for example, [2, corollary 1.3.4]), there exists a unique function $f_E \in \bar{E}$ with least $L^p(\mathrm{d}\gamma)$-norm. Next we shall show that there exists a constant $C > 0$ depending only on the dimension n such that the function defined by $f = C \sup_{Q \in \mathcal{D}_m} \lambda_Q \chi_Q$ is in E. We check that

$$\sum_{Q \in \mathcal{D}_m} \lambda_Q r_Q^s \frac{w^{-(p'-1)}(Q)}{r_Q^n} \chi_{T(Q)}(z) \leqslant K_s[f\,\mathrm{d}\gamma](z)$$

for any $z \in \mathbb{B}^n$, that is, $f \in E$. Indeed, for m fixed, and any j, there exists a unique $Q_j \in \mathcal{D}_m$ such that the point $z/|z| \in Q_j$. The fact that the weight $w^{-(p'-1)}$ satisfies a doubling condition gives that

$$\int_{Q_j} \frac{w^{-(p'-1)}}{r_{Q_j}^{n-s}} \preceq \int_{Q_j \setminus Q_{j-1}} \frac{w^{-(p'-1)}}{r_{Q_j}^{n-s}},$$

and, consequently,

$$\lambda_{Q_j} \int_{Q_j} \frac{w^{-(p'-1)}}{r_{Q_j}^{n-s}} \preceq \int_{Q_j \setminus Q_{j-1}} \Big(\sup_{Q \in \mathcal{D}_m} \lambda_Q \chi_Q \Big) \frac{w^{-(p'-1)}}{r_{Q_j}^{n-s}}.$$

This estimate implies that $\sup_{Q \in \mathcal{D}_m} \lambda_Q \chi_Q \in E$.

From the minimality of the norm of the extremal function f_E, we deduce that

$$\int_{\mathbb{S}^n} |f_E|^p \,\mathrm{d}\gamma \preceq \int_{\mathbb{S}^n} \Big| \sup_{Q \in \mathcal{D}_m} \lambda_Q \chi_Q \Big|^p \,\mathrm{d}\gamma \leqslant \sum_{Q \in \mathcal{D}_m} \lambda_Q^p \gamma(Q).$$

We shall check that this extremal function can be written as $f_E = h^{p'-1}$, where $h = K_s[\nu]$ for some non-negative measure. Postponing the proof of this fact, we finish the proof of the theorem as follows. The fact that for $z \in \mathbb{B}^n$ the operator $f \in L^p(\mathrm{d}\gamma) \to K_s[f\,\mathrm{d}\gamma](z)$ is continuous (just a consequence of Hölder's inequality) gives that the function $f_E \in E$. And since we are assuming that $f_E \geqslant 0$, the monotone convergence theorem gives that, for any $\zeta \in \mathbb{S}^n$,

$$\sum_{Q \in \mathcal{D}_m} \lambda_Q r_Q^s \frac{w^{-(p'-1)}(Q)}{r_Q^n} \chi_Q(\zeta) \leqslant K_s[f_E\,\mathrm{d}\gamma](\zeta).$$

We apply (3.11) to this measure ν to obtain

$$\left(\int_{\mathbb{S}^n} (K_s[f_E\,\mathrm{d}\gamma])^q \,\mathrm{d}\mu \right)^{1/q} \preceq \left(\int_{\mathbb{S}^n} |f_E|^p \,\mathrm{d}\gamma \right)^{1/p}.$$

Hence, we have that

$$\Big\| \sum_{Q \in \mathcal{D}_m} \lambda_Q r_Q^{s-n} \gamma(Q) \chi_Q \Big\|_{L^q(\mathrm{d}\mu)} \preceq \Big(\sum_{Q \in \mathcal{D}_m} \lambda_Q^p \gamma(Q) \Big)^{1/p}.$$

Applying [18, theorem 3.c], we obtain that

$$\sum_{Q\in\mathcal{D}_m}\left(\frac{\mu(Q)}{r_Q^{n-sp}}\right)^{p'-1}\frac{w^{-(p'-1)}(Q)}{r_Q^n}\chi_Q\in L^{q(p-1)/(p-q)}(\mathrm{d}\mu).$$

We observe that the left-hand side of the above estimate is a discrete version of the Wolff potential $\mathcal{W}^w_{sp}[\mu]$. We have thus proved a discrete version of in theorem 1.4(iii).

The arguments used to deduce the continuous version that we have used for the case $q>1$ cannot be applied in our new setting. The continuous version for $q\leqslant 1$ is a consequence of a more subtle argument that can be obtained using similar methods to those in [8]. The idea of the proof is again based on an estimate of the continuous version of the Wolff potential in terms of an average of discrete Wolff potentials associated to a collection of shifted dyadic lattices $\mathcal{D}_t$. Since, in general, $q(p-1)/(p-q)$ is not bigger that 1, we cannot apply Hölder's inequality as in the case $q>1$. The proof of the estimate requires the use of inequalities in spaces of mixed norm together with a dyadic vector-valued Fefferman–Stein maximal theorem. In order to apply the results, we again use that either hypothesis (a) or hypothesis (b) when $q\leqslant 1$ implies that the pair $(K_s,\mathrm{d}\gamma)$ satisfies the LBO condition. We refer the reader to [8] to check the precise calculations.

It remains to prove that $f_E=h^{p'-1}$, where $h=K_s[\nu]$ for some non-negative measure. We shall follow some of the ideas in [2, theorem 2.2.7].

Let $\psi\in L^p(\mathrm{d}\gamma)$ such that $K_s[\psi\,\mathrm{d}\gamma]\geqslant 0$. We then have that, for any $t\geqslant 0$, $f_E+t\psi\in\bar{E}$ and, consequently,

$$\int_{\mathbb{S}^n}|f_E+t\psi|^p\,\mathrm{d}\gamma\geqslant\int_{\mathbb{S}^n}|f_E|^p\,\mathrm{d}\gamma.$$

Next, differentiating

$$\int_{\mathbb{S}^n}|f_E+t\psi|^p\,\mathrm{d}\gamma$$

at $t=0$, we get that, for any $\psi\in L^p(\mathrm{d}\gamma)$ such that $K_s[\psi\mathrm{d}\gamma]\geqslant 0$,

$$\int_{\mathbb{S}^n}|f_E|^{p-2}f_E\psi\,\mathrm{d}\gamma\geqslant 0. \tag{3.12}$$

We define $h=|f_E|^{p-2}f_E$, a function that is in $L^{p'}(\mathrm{d}\gamma)$, since $|h|^{p'}=|f_E|^p$. We now check that there exists a positive measure ν_E such that $h=K_s[\nu_E]$. Indeed, we define a distribution on $\mathbb{S}^n$, ν_E, in the following way. Let ω be a test function on $\mathbb{S}^n$, and set $\omega=K_s[\theta\,\mathrm{d}\gamma]$ for some $\theta\in L^p(\mathrm{d}\gamma)$. If we define

$$\langle\nu_E,\omega\rangle=\int_{\mathbb{S}^n}h\theta\,\mathrm{d}\gamma,$$

we obtain

$$|\langle\nu_E,\omega\rangle|\leqslant\|h\|_{L^{p'}(\mathrm{d}\gamma)}\|\theta\|_{L^p(\mathrm{d}\gamma)}.$$

Observe that ν_E is well defined, since if $K_s[\theta\,\mathrm{d}\gamma]=K_s[\theta_1\,\mathrm{d}\gamma]$, then $K_s[(\theta-\theta_1)\,\mathrm{d}\gamma]=0$, and by (3.12) we have that

$$\int_{\mathbb{S}^n}h(\theta-\theta_1)\,\mathrm{d}\gamma=0.$$

Since $w^{-(p'-1)}$ is a weight in $A^{p'}$ we deduce, using [5, lemma 2.1], that there exists $p_1 > p$ such that $\|\theta\|_{L^p(\mathrm{d}\gamma)} \preceq \|\theta\|_{L^{p_1}(\mathrm{d}\sigma)}$. Consequently,

$$|\langle \nu_E, \omega\rangle| \preceq \|h\|_{L^{p'}(\mathrm{d}\gamma)}\|\theta\|_{L^{p_1}(\mathrm{d}\sigma)},$$

and $|\langle \nu_E, \omega\rangle|$ is bounded by the norm of the function ω in the non-isotropic Sobolev space $K_s[L^{p_1}[\mathrm{d}\sigma]]$. Thus, ν_E is a distribution.

Next, if $\omega_1 = K_s[\theta_1\,\mathrm{d}\gamma]$ is a non-negative test function, again using (3.12), we obtain

$$\langle \nu_E, \omega_1\rangle = \int_{\mathbb{S}^n} h\theta_1\,\mathrm{d}\gamma \geqslant 0.$$

Thus, we conclude that the distribution ν_E is a non-negative measure, as we wanted to prove. □

Acknowledgements

Both authors were partly supported by DGICYT (Grant nos MTM2011-27932-C02-01, MTM2007-30904-E) and DURSI (Grant no. 2009SGR1303). The authors thank the referee for valuable suggestions and comments.

References

1 D. R. Adams. Weighted nonlinear potential theory. *Trans. Am. Math. Soc.* **297** (1986), 73–94.

2 D. R. Adams and L. I. Hedberg. *Function spaces and potential theory* (Springer, 1996).

3 P. Ahern. Exceptional sets for holomorphic Sobolev functions. *Michigan Math. J.* **35** (1988), 29–41.

4 P. Ahern and A. Nagel. Strong L^p estimates for maximal functions with respect to singular measures; with applications to exceptional sets. *Duke Math. J.* **53** (1986), 359–393.

5 C. Cascante and J. M. Ortega. Carleson measures for weighted Hardy–Sobolev spaces. *Nagoya Math. J.* **186** (2007), 29–68.

6 C. Cascante, J. M. Ortega and I. E. Verbitsky. Trace inequalities of Sobolev type in the upper triangle case. *Proc. Lond. Math. Soc.* **80** (2000), 391–414.

7 C. Cascante, J. M. Ortega and I. E. Verbitsky. Nonlinear potentials and two weight trace inequalities for general dyadic and radial kernels. *Indiana Univ. Math. J.* **53** (2004), 845–882.

8 C. Cascante, J. M. Ortega and I. E. Verbitsky. On L^p–L^q trace inequalities. *J. Lond. Math. Soc.* **74** (2006), 497–511.

9 W. S. Cohn. Nonisotropic Hausdorff measure and exceptional sets for holomorphic Sobolev functions. *Illinois J. Math.* **33** (1989), 673–690.

10 W. S. Cohn and I. E. Verbitsky. Trace inequalities for Hardy–Sobolev functions in the unit ball of $\mathbb{C}^n$. *Indiana Univ. Math. J.* **43** (1994), 1079–1097.

11 R. Kerman and E. T. Sawyer. The trace inequality and eigenvalue estimates for Schrödinger operators. *Annales Inst. Fourier* **36** (1986), 207–228.

12 P. Mattila. *Geometry of sets and measures in Euclidean spaces: fractals and rectifiability*, Cambridge Studies in Advanced Mathematics, vol. 44 (Cambridge University Press, 1995).

13 V. G. Maz′ya and V. P. Khavin. A nonlinear potential theory. *Usp. Mat. Nauk* **27** (1972), 67–138.

14 V. G. Maz′ya and Y. Netrusov. Some counterexamples for the theory of Sobolev spaces on bad domains. *Potent. Analysis* **4** (1995), 47–65.

15 E. T. Sawyer and R. L. Wheeden. Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces. *Am. J. Math.* **114** (1992), 813–874.

16 E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43 (Princeton University Press, 1993).

17 E. Tchoundja. Carleson measures for the generalized Bergman spaces via a $T(1)$-type theorem. *Ark. Mat.* **46** (2008), 377–406.

18 I. E. Verbitsky. Imbedding and multiplier theorems for discrete Littlewood–Paley spaces. *Pac. J. Math.* **176** (1996), 529–556.

19 I. E. Verbitsky and R. L. Wheeden. Weighted norm inequalities for integral operators. *Trans. Am. Math. Soc.* **350** (1998), 3371–3391.

20 A. Volberg and B. Wick. Bergman-type singular operators and the characterization of Carleson measures for Besov–Sobolev spaces on the complex ball. Preprint, arXiv:0910.1142 (2009).

(*Issued 6 April 2012*)