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Dimension of ergodic measures and currents on $\mathbb{CP}(2)$

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Abstract. Let f be a holomorphic endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. We estimate the local directional dimensions of closed positive currents S with respect to ergodic dilating measures ν. We infer several applications. The first one is an upper bound for the lower pointwise dimension of the equilibrium measure, towards a Binder–DeMarco's formula for this dimension. The second one shows that every current S containing a measure of entropy $h_\nu > \log d$ has a directional dimension >2, which answers a question of de Thélin–Vigny in a directional way. The last one estimates the dimensions of the Green current of Dujardin's semi-extremal endomorphisms.

Key words: dimension theory, Lyapunov exponents, normal forms
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1. *Introduction*

We study the dimension of ergodic measures for holomorphic endomorphisms of $\mathbb{P}^2$ and the dimension of currents containing such measures.

Let us recall dynamical properties of these mappings (see [**18**] for a detailed exposition). The topological entropy of an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$ is equal to $2 \log d$. Its Green current is defined by $T := \lim_n (1/d^n) f^{n*}\omega$, where ω is the Fubini–Study (1, 1)-form of $\mathbb{P}^2$. The support of T is the subset of points $x \in \mathbb{P}^2$ such that $\{f^n, n \geq 1\}$ is not equicontinuous in any neighbourhood of x. The equilibrium measure of f is $\mu := T \wedge T$. This is a mixing invariant measure, it equidistributes the repulsive cycles of f and its Lyapunov exponents are $\geq \frac{1}{2} \log d$. This is also the unique measure of maximal entropy $h_\mu = 2 \log d$.

The ergodic measures ν satisfying $\log d < h_\nu \leq 2 \log d$ are called *measures of large entropy*. Their support is contained in the support of μ [**10**, **16**] and their exponents are larger than or equal to $\frac{1}{2}(h_\nu - \log d)$ [**11**, **21**]. Examples are constructed in [**21**] by using coding techniques. The ergodic measures ν with positive exponents are called *dilating*.

1.1. *Dimension of dilating measures.* An open problem is to find a formula for the dimension of dilating measures (see [**23**, Question 2.17] for μ). Difficulties are due to the facts that f is not invertible and not conformal. The dimension of a probability measure ν is defined by (see [**26**, **29**])

$$\dim_H(\nu) := \inf\{\dim_H(A),\ A \text{ Borel set of } \mathbb{P}^2,\ \nu(A) = 1\}.$$

The lower and upper pointwise dimensions of ν at x are

$$\underline{d_\nu}(x) := \liminf_{r\to 0} \frac{\log(\nu(B_x(r)))}{\log r}, \quad \overline{d_\nu}(x) := \limsup_{r\to 0} \frac{\log(\nu(B_x(r)))}{\log r}.$$

These functions are ν-almost everywhere constant when ν is ergodic. If $a \le \underline{d_\nu} \le \overline{d_\nu} \le b$, then $a \le \dim_H(\nu) \le b$. If ν is dilating and $\lambda_1 \ge \lambda_2$ denote its exponents, then

$$\frac{h_\nu}{\lambda_1} \le \underline{d_\nu} \le \overline{d_\nu} \le \frac{h_\nu}{\lambda_2}. \tag{1.1}$$

Binder–DeMarco [**5**] conjectured the formula for the measure μ,

$$\dim_H(\mu) = \frac{\log d}{\lambda_1} + \frac{\log d}{\lambda_2}, \tag{1.2}$$

which generalizes the one-dimensional Mañé's formula [**25**]. Let us note that, by (1.1), this formula is true when $\lambda_1 = \lambda_2$. In view of that conjecture, the following upper bound was proved in [**5**] for polynomial mappings.

$$\dim_H(\mu) \le 4 - \frac{2(\lambda_1 + \lambda_2) - \log d^2}{\lambda_1}. \tag{1.3}$$

This was extended in [**17**] to meromorphic mappings. Moreover, for every dilating measure ν, in [**20**], it was established that

$$\underline{d_\nu} \ge \frac{\log d}{\lambda_1} + \frac{h_\nu - \log d}{\lambda_2}, \tag{1.4}$$

which yields half of the conjecture

$$\dim_H(\mu) \ge \underline{d_\mu} \ge \frac{\log d}{\lambda_1} + \frac{\log d}{\lambda_2}. \tag{1.5}$$

At this stage, by combining (1.3) and (1.4), the conjecture is true for every endomorphism of $\mathbb{P}^2$ satisfying $\frac{1}{2}\log d = \lambda_2 \le \lambda_1$, for which

$$\dim_H(\mu) = \frac{\log d}{\lambda_1} + 2. \tag{1.6}$$

In this article, we prove lower and upper estimates for the directional dimensions of currents containing dilating measures. Our techniques allow to show the following upper bound for the lower dimension of such measures, towards a Binder–DeMarco's formula for $\underline{d_\mu}$. The exponents *do not resonate* if $\lambda_1 \ne k\lambda_2$ for every $k \ge 2$.

THEOREM 1.1. *Let f be an endomorphism of $\mathbb{P}^2$ of degree $d \ge 2$. Let ν be a dilating measure of exponents $\lambda_1 \ge \lambda_2$ whose support is contained in the support of μ. Then*

$$\underline{d_\nu} \le \frac{\log d}{\lambda_1} + \frac{\log d}{\lambda_2} + 2\left(1 - \frac{\lambda_2}{\lambda_1}\right).$$

Moreover, if the exponents do not resonate, then

$$\underline{d_\nu} \le \frac{\log d}{\lambda_1} + \frac{\log d}{\lambda_2} + 2\min\left(1 - \frac{\lambda_2}{\lambda_1}; \frac{\lambda_1}{\lambda_2} - 1\right).$$

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The next sections deal with the dimension of currents. In §1.2, we recall properties of the trace measure. The directional dimensions are more precise than the dimension of the trace measure. We present our estimates in §§1.3, 1.4 and 1.5.

1.2. *Currents and the dimension of their trace measure.* A positive closed current S of bidegree (1, 1) on a complex manifold is locally equal (in the sense of distributions) to $i\partial\overline{\partial}u$, where u is a plurisubharmonic function (see [**14**, Ch. 3]). In particular, S is locally equal to

$$S_{1,1}\frac{i}{2}\,dz\wedge d\overline{z}+S_{1,2}\frac{i}{2}\,dz\wedge d\overline{w}+S_{2,1}\frac{i}{2}\,dw\wedge d\overline{z}+S_{2,2}\frac{i}{2}\,dw\wedge d\overline{w},$$

where $S_{1,1}$, $S_{2,2}$ are positive measures and $S_{1,2}=\overline{S}_{2,1}$ is a complex measure dominated by $S_{1,1}+S_{2,2}$. The *trace measure* of S on $\mathbb{P}^2$ is the positive measure $S\wedge\omega$, where ω is the Fubini–Study form on $\mathbb{P}^2$. This is a probability measure (up to a multiplicative positive constant) equivalent to $S_{1,1}+S_{2,2}$ in every local coordinate as before. It is known that $S\wedge\omega(B_x(r))\leq c(x)r^2$ for every $x\in\mathbb{P}^2$, and hence the pointwise dimension of $S\wedge\omega$ satisfies

$$\text{for all } x\in\mathbb{P}^2,\quad \underline{d_S}(x):=\liminf_{r\to 0}\frac{\log S\wedge\omega(B_x(r))}{\log r}\geq 2.$$

This bound can be improved for the Green current of endomorphisms f of $\mathbb{P}^2$. Indeed, if we set

$$d_\infty:=\lim_n\|Df^n\|_\infty^{1/n}\quad\text{and}\quad \gamma_0:=\min\{1,\log d/\log d_\infty\},$$

then T has local γ-Hölder potentials u for every $\gamma<\gamma_0$ (see [**18**, Proposition 1.18]). This implies that $T\wedge\omega(B_x(r))\leq c_\gamma(x)r^{2+\gamma}$ for every $x\in\mathbb{P}^2$ and every $\gamma<\gamma_0$ (see [**27**, Théorème 1.7.3]). Hence

$$\text{for all } x\in\mathbb{P}^2,\quad \underline{d_T}(x)\geq 2+\gamma_0. \tag{1.7}$$

The geometric structure of the Green current is not well understood. A way to get information is to study its dimension with respect to ergodic measures. For positive closed currents, there are dimensional constraints to contain measures of large entropy. For instance, any current of integration on an algebraic subset of $\mathbb{P}^2$ cannot contain any such measure: this comes from Gromov's iterated graph argument and the relative variational principle (see [**8**] and [**18**, §1.7]). Further, de Thélin–Vigny [**13**] proved that if a current S contains a measure of large entropy (or even dilating) with exponents $\lambda_1\geq\lambda_2$, then, for every $\epsilon>0$, there exists x in the support of ν such that

$$\overline{d_S}(x)\geq 2\frac{\lambda_2}{\lambda_1}+\frac{h_\nu-\log d}{\lambda_2}-\epsilon. \tag{1.8}$$

This inequality quantifies in an ergodic way (with exponents and entropy) the thickness of currents containing large entropy measures. Our results will, in particular, improve (1.7) and (1.8) in a directional way, when the exponents do not resonate.

1.3. *Directional dimensions of currents.* Let (Z,W) be holomorphic coordinates near x in $\mathbb{P}^2$. The lower pointwise directional dimension of S with respect to Z is defined by

$$\underline{d_{S,Z}}(x):=\liminf_{r\to 0}\frac{\log[S\wedge((i/2)\,dZ\wedge d\overline{Z})(B_x(r))]}{\log r}.$$

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We use a similar definition for the upper pointwise dimension by taking a lim sup. Geometrically, the positive measure $S \wedge ((i/2)\, dZ \wedge d\overline{Z})$ is the average of the slices of the current S transversally to the coordinate Z (see §A.2). Moreover,

$$\underline{d_S}(x) = \min\{\underline{d_{S,Z}}(x), \underline{d_{S,W}}(x)\}. \tag{1.9}$$

In this article, we shall work with coordinates (Z, W) coming from a normal form theorem for the inverse branches $f_{\hat{x}}^{-n}$ along generic orbits $\hat{x}$. We shall call them Oseledec–Poincaré coordinates. When the exponents $\lambda_1 \geq \lambda_2$ do not resonate, this theorem provides coordinates $(Z_{\hat{x}}^{\epsilon}, W_{\hat{x}}^{\epsilon})$ near x such that

$$Z^{\epsilon}_{\hat{f}^{-n}(\hat{x})} \circ f_{\hat{x}}^{-n} \simeq e^{-n\lambda_1 \pm n\epsilon} \times Z^{\epsilon}_{\hat{x}}, \quad W^{\epsilon}_{\hat{f}^{-n}(\hat{x})} \circ f_{\hat{x}}^{-n} \simeq e^{-n\lambda_2 \pm n\epsilon} \times W^{\epsilon}_{\hat{x}}.$$

In the next statements, the functions $O(\epsilon)$ are of the form $\epsilon M(\epsilon)$, where M is a positive function which depends on the exponents and the entropy of ν, and on the degree of f. Concerning Theorem 1.2 and Corollaries 1.4 and 1.5, the functions $\underline{d_{T,Z}}, \underline{d_{T,W}}, \overline{d_{T,Z}}, \overline{d_{T,W}}$ are ν-almost everywhere constant.

1.4. *Estimates for directional dimensions of currents.* We begin with lower estimates for the directional dimensions of the Green current T with respect to dilating measures ν contained in μ (examples are measures of large entropy).

THEOREM 1.2. *Let f be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Let ν be a dilating measure whose exponents $\lambda_1 > \lambda_2$ do not resonate and whose support is contained in the support of μ. Then, for every $\epsilon > 0$ and for ν-almost every x, there exist holomorphic coordinates (Z, W) near x such that*

$$\underline{d_{T,Z}}(x) \leq 2\frac{\lambda_1}{\lambda_2} + \frac{\log d}{\lambda_2} + O(\epsilon),$$
$$\underline{d_{T,W}}(x) \leq 2 + \frac{\log d}{\lambda_2} + O(\epsilon).$$

The proof relies on pluripotential theory (μ is the Monge–Ampère mass $T \wedge T$) and on a study of the Jacobians of $T \wedge (i/2)\, dZ_{\hat{x}}^{\epsilon} \wedge d\overline{Z_{\hat{x}}^{\epsilon}}$ and $T \wedge (i/2)\, dW_{\hat{x}}^{\epsilon} \wedge d\overline{W_{\hat{x}}^{\epsilon}}$ with respect to f. This study is given by Proposition 3.3, which will be also crucial for proving Theorem 1.1.

The next result concerns upper estimates for currents S containing dilating measures.

THEOREM 1.3. *Let f be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Let S be a $(1, 1)$-closed positive current on $\mathbb{P}^2$. We assume that the support of S contains a measure of large entropy ν whose exponents satisfy $\lambda_1 > \lambda_2$ and do not resonate. Then, for every $\epsilon > 0$, there exist $x \in \operatorname{Supp} \nu$ and holomorphic coordinates (Z, W) near x such that*

$$\overline{d_{S,Z}}(x) \geq 2 + \frac{h_\nu - \log d}{\lambda_2} - O(\epsilon),$$
$$\overline{d_{S,W}}(x) \geq 2\frac{\lambda_2}{\lambda_1} + \frac{h_\nu - \log d}{\lambda_2} - O(\epsilon).$$

In particular, S has a local directional dimension >2 at some $x \in \operatorname{Supp} \nu$.

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Theorem 1.3 specifies in a directional way de Thélin–Vigny [**13**, Theorem 2]. Indeed, the estimate concerning the Z-coordinate improves (1.8) by replacing λ_2/λ_1 by one, which answers a question of [**13**] in a directional way. Our proof follows the strategy of [**13**] by taking into account the normal form theorem. We obtain more precise lower bounds depending on $\underline{d_\nu}$ and we deduce Theorem 1.3 from the lower estimate (1.4) on $\underline{d_\nu}$. We shall begin with a proof for the Green current T (Theorem 4.6). In this case, the exposition is simpler because the directional dimensions of T are constant ν-almost everywhere. The case of general currents S needs a localization at some point in the support of S (Theorem 4.7).

Theorems 1.2 and 1.3 immediately imply the following result. Let us set

$$\gamma_1 := \frac{h_\mu - \log d}{\lambda_2} = \frac{\log d}{\lambda_2}.$$

COROLLARY 1.4. *Assume that the exponents of* μ *satisfy* $\lambda_1 > \lambda_2$ *and do not resonate. Then, for every* $\epsilon > 0$ *and for* μ*-almost every* x*, there exist holomorphic coordinates* (Z, W) *near* x *such that*

$$2 + \gamma_1 - O(\epsilon) \le \overline{d_{T,Z}}(x), \quad \underline{d_{T,W}}(x) \le 2 + \gamma_1 + O(\epsilon).$$

This shows that, modulo $O(\epsilon)$, either the directional dimensions $\overline{d_{T,Z}}(x)$ and $\underline{d_{T,W}}(x)$ are distinct and separated by $2 + \gamma_1$ or they are equal to $2 + \gamma_1$. Let us observe that the first estimate improves (1.7), since $\lambda_1 \le \log d_\infty$ and $\lambda_1 > \lambda_2$ imply that $\gamma_1 > \gamma_0$.

We note that a bound >2 for the dimension of the trace measure of the Green currents $T^\pm$ is proved in [**12**] for invertible and meromorphic mappings of Kähler surfaces. The proof relies on the laminar properties of $T^\pm$ and coding techniques.

1.5. *Semi-extremal endomorphisms.* An endomorphism f is *extremal* if the exponents of μ satisfy $\lambda_1 = \lambda_2 = \frac{1}{2}\log d$. Articles [**1**, **4**, **17**] characterize these endomorphisms by several equivalent properties: $\dim_H(\mu) = 4$; $\mu \ll \mathrm{Leb}_{\mathbb{P}^2}$; T is a positive smooth form on some open subset of $\mathbb{P}^2$; and f is a Lattès map. Articles [**3**, **15**] show other characterizations.

An endomorphism f is *semi-extremal* if the exponents of μ satisfy $\lambda_1 > \lambda_2 = \frac{1}{2}\log d$. Formula (1.6) of §1.1 implies that these endomorphisms satisfy Conjecture (1.2), i.e.,

$$\dim_H(\mu) = \frac{\log d}{\lambda_1} + 2.$$

Dujardin [**19**] proved that if $\mu \ll T \wedge \omega$, then f is semi-extremal. Examples satisfying this condition are suspensions of one-dimensional Lattès maps, and, more generally, the endomorphisms with an invariant pencil of lines on which f induces a one-dimensional Lattès map [**22**]. One may ask if there exist other examples. In view of a possible characterization, the next result provides necessary conditions about the dimension of those endomorphisms. It follows from Theorem 1.3 and (1.6).

COROLLARY 1.5. *Assume that* $\mu \ll T \wedge \omega$*, that* $\overline{d_\mu} = \underline{d_\mu}$ *and that the exponents* $\lambda_1 > \lambda_2$ *of* μ *do not resonate. For every* $\epsilon > 0$ *and for* μ*-almost every* $x \in \mathbb{P}^2$*, there exist holomorphic coordinates* (Z, W) *near* x *such that*

$$4 - O(\epsilon) \le \overline{d_{T,Z}}(x) \quad \text{and} \quad 2 + \frac{\log d}{\lambda_1} - O(\epsilon) \le \overline{d_{T,W}}(x) \le 2 + \frac{\log d}{\lambda_1}.$$

The first estimate provides a maximal dimension for T, which could be explained by the presence of a one-dimensional Lattès map inside the dynamics of f. However, it seems difficult to produce such a Lattès map as well as an invariant pencil of lines. Note that Corollary 1.5 provides a situation in which the dimensions $\overline{d_{T,Z}}(x)$ and $\overline{d_{T,W}}(x)$ are not equal. To outline the proof of Corollary 1.5, we shall see that (1.6) implies that $\min\{\overline{d_{T,Z}}(x), \overline{d_{T,W}}(x)\} \leq 2 + (\log d)/\lambda_1$ and then we use Theorem 1.3 to verify that the minimum concerns the W-coordinate.

1.6. *Organization of the article.* Section 2 is devoted to normal forms and to the geometry of inverse branches. Theorem 1.2 is proved in §3. We show Theorem 1.3 and Corollary 1.5 in §4, and we show Theorem 1.1 in §5. Technical results are put together in the Appendix.

2. *Normal forms and Oseledec–Poincaré coordinates*

2.1. *Natural extension and normal forms.* Let f be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Let $\mathcal{C}_f$ be its critical set, which is an algebraic subset of $\mathbb{P}^2$. If ν is a dilating measure, then $x \mapsto \log|\mathrm{Jac}\, f(x)| \in L^1(\nu)$, which implies that $\nu(\mathcal{C}_f) = 0$. Let X be the f-invariant Borel set $\mathrm{Supp}(\nu) \setminus \bigcup_{n\in\mathbb{Z}} f^n(\mathcal{C}_f)$ and let

$$\hat{X} := \{\hat{x} = (x_n)_{n\in\mathbb{Z}} \in X^{\mathbb{Z}},\ x_{n+1} = f(x_n)\}.$$

Let $\hat{f}$ be the left shift on $\hat{X}$ and $\pi_0(\hat{x}) := x_0$. There exists a unique $\hat{f}$-invariant measure $\hat{\nu}$ on $\hat{X}$ such that $(\pi_0)_*\hat{\nu} = \nu$. We set $\hat{x}_n := \hat{f}^n(\hat{x})$ for every $n \in \mathbb{Z}$. A function $\alpha : \hat{X} \to]0, +\infty]$ is *ϵ-tempered* if $\alpha(\hat{f}^{\pm 1}(\hat{x})) \geq e^{-\epsilon}\alpha(\hat{x})$. For every $\hat{x} \in X$, we denote by $f_{\hat{x}}^{-n}$ the inverse branch of f^n defined in a neighbourhood of x_0 with values in a neighbourhood of x_{-n}. Articles [**2**, **24**] provide normal forms for these mappings. Let $d(\cdot, \cdot)$ be the Fubini–Study distance on $\mathbb{P}^2$.

THEOREM 2.1. [**2**, Proposition 4.3] *Let ν be a dilating measure with exponents $\lambda_1 > \lambda_2$ and let $\epsilon > 0$. There exists an $\hat{f}$-invariant Borel set $\hat{F} \subset \hat{X}$ such that $\hat{\nu}(\hat{F}) = 1$ and that satisfies the following properties. There exist ϵ-tempered functions $\eta_\epsilon, \rho_\epsilon : \hat{F} \to]0, 1]$ and $\beta_\epsilon, L_\epsilon, M_\epsilon : \hat{F} \to [1, +\infty[$ and, for every $\hat{x} \in \hat{F}$, there exists a holomorphic mapping*

$$\xi_{\hat{x}}^\epsilon : B_{x_0}(\eta_\epsilon(\hat{x})) \to \mathbb{D}^2(\rho_\epsilon(\hat{x}))$$

such that the following diagram commutes for every $n \geq n_\epsilon(\hat{x})$

$$\begin{array}{ccc}
B_{x_{-n}}(\eta_\epsilon(\hat{x}_{-n})) & \xleftarrow{\ f_{\hat{x}}^{-n}\ } & B_{x_0}(\eta_\epsilon(\hat{x})) \\
\downarrow {\scriptstyle \xi_{\hat{x}_{-n}}^\epsilon} & & {\scriptstyle \xi_{\hat{x}}^\epsilon} \downarrow \\
\mathbb{D}^2(\rho_\epsilon(\hat{x}_{-n})) & \xleftarrow[\ R_{n,\hat{x}}\ ]{} & \mathbb{D}^2(\rho_\epsilon(\hat{x}))
\end{array}$$

and such that:

(1) *for all $(p, q) \in B_{x_0}(\eta_\epsilon(\hat{x}))$, $\frac{1}{2}d(p, q) \leq |\xi_{\hat{x}}^\epsilon(p) - \xi_{\hat{x}}^\epsilon(q)| \leq \beta_\epsilon(\hat{x})d(p, q)$;*

(2) $\mathrm{Lip}(f_{\hat{x}}^{-n}) \leq L_\epsilon(\hat{x})e^{-n\lambda_2+n\epsilon}$ *on $B_{x_0}(\eta_\epsilon(\hat{x}))$;*

(3) *if $\lambda_1 \notin \{k\lambda_2, k \geq 2\}$, $R_{n,\hat{x}}(z, w) = (\alpha_{n,\hat{x}}z, \beta_{n,\hat{x}}w)$,*
if $\lambda_1 = k\lambda_2$, where $k \geq 2$, $R_{n,\hat{x}}(z, w) = (\alpha_{n,\hat{x}}z, \beta_{n,\hat{x}}w) + (\gamma_{n,\hat{x}}w^k, 0)$, with:

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(a) $e^{-n\lambda_1 - n\epsilon} \le |\alpha_{n,\hat{x}}| \le e^{-n\lambda_1 + n\epsilon}$ *and* $|\gamma_{n,\hat{x}}| \le M_\epsilon(\hat{x}) e^{-n\lambda_1 + n\epsilon}$*; and*
(b) $e^{-n\lambda_2 - n\epsilon} \le |\beta_{n,\hat{x}}| \le e^{-n\lambda_2 + n\epsilon}$.

Remark 2.2. The diagram commutes for every $n \in [\![1, \ldots, n_\epsilon(\hat{x})]\!]$ for the germs of the mappings (see [**2**]). The integer $n_\epsilon(\hat{x})$ is actually the smallest integer such that $L_\epsilon(\hat{x}) e^{-n\lambda_2 + n\epsilon} \le e^{-n\epsilon}$, so that $L_\epsilon(\hat{x}) e^{-n\lambda_2 + n\epsilon} \eta_\epsilon(\hat{x}) \le e^{-n\epsilon} \eta_\epsilon(\hat{x}) \le \eta_\epsilon(\hat{x}_{-n})$. Item 2 thus ensures that $f_{\hat{x}}^{-n}(B_{x_0}(\eta_\epsilon(\hat{x}))) \subset B_{x_{-n}}(\eta_\epsilon(\hat{x}_{-n}))$, as indicated in the diagram.

We shall need the following lemma. Let $n_1(L)$ be the smallest integer n satisfying $L/4 \le e^{n\epsilon}$. The first item uses the upper bound for $\mathrm{Lip}(f_{\hat{x}}^{-n})$ provided by Theorem 2.1. The second item comes from [**17**, Proposition 3.1].

LEMMA 2.3. *Let* $\hat{x} \in \hat{F}$ *such that* $\eta_\epsilon(\hat{x}) \ge \eta$ *and* $L_\epsilon(\hat{x}) \le L$*. If* $n \ge n_1(L)$ *and* $r \le \eta$*:*
(1) $f_{\hat{x}_n}^{-n}(B_{x_n}(r/4)) \subset B_{x_0}(re^{-n\lambda_2 + 3n\epsilon})$ *and* $f_{\hat{x}}^{-n}(B_{x_0}(r/4)) \subset B_{x_{-n}}(re^{-n\lambda_2 + 3n\epsilon})$*; and*
(2) $B_{x_0}(re^{-n\lambda_1 - 4n\epsilon}) \subset f_{\hat{x}_n}^{-n}(B_{x_n}((r/4)e^{-2n\epsilon}))$.

2.2. *Oseledec–Poincaré coordinates.* Let ν be a dilating measure with exponents $\lambda_1 > \lambda_2$. Let $\epsilon > 0$ and let us apply Theorem 2.1. For every $\hat{x} \in \hat{F}$, we denote by $(Z_{\hat{x}}^\epsilon, W_{\hat{x}}^\epsilon)$ the coordinates of $\xi_{\hat{x}}^\epsilon$. If the exponents do not resonate, then the commutative diagram given by Theorem 2.1 implies that

$$Z_{\hat{x}_{-n}}^\epsilon \circ f_{\hat{x}}^{-n} = \alpha_{n,\hat{x}} \times Z_{\hat{x}}^\epsilon, \quad W_{\hat{x}_{-n}}^\epsilon \circ f_{\hat{x}}^{-n} = \beta_{n,\hat{x}} \times W_{\hat{x}}^\epsilon. \tag{2.1}$$

Hence, $f_{\hat{x}}^{-n}$ multiplies the first coordinate by $e^{-n\lambda_1 \pm n\epsilon}$ and the second coordinate by $e^{-n\lambda_2 \pm n\epsilon}$. The second property in (2.1) also holds in the resonant case $\lambda_1 \in \{k\lambda_2, k \ge 2\}$. We shall name the collection of local holomorphic coordinates

$$(Z, W)_\epsilon := (Z_{\hat{x}}^\epsilon, W_{\hat{x}}^\epsilon)_{\hat{x} \in \hat{F}}$$

Oseledec–Poincaré coordinates for (f, ν). If S is a $(1, 1)$ positive closed current, we define

$$\underline{d_{S,Z}}(\hat{x}) := \liminf_{r \to 0} \frac{\log[S \wedge ((i/2)\, dZ_{\hat{x}}^\epsilon \wedge d\overline{Z_{\hat{x}}^\epsilon})(B_x(r))]}{\log r},$$

with similar definitions for $\underline{d_{S,W}}(\hat{x})$, $\overline{d_{S,Z}}(\hat{x})$, $\overline{d_{S,W}}(\hat{x})$. Using (2.1) and the fact that the Green current T is f-invariant, we obtain the following proposition.

PROPOSITION 2.4. *Let* ν *be a dilating measure of exponents* $\lambda_1 > \lambda_2$*. Let* $(Z, W)_\epsilon$ *be Oseledec–Poincaré coordinates for* (f, ν)*. Then there exists an* $\hat{f}$*-invariant Borel set* $\hat{\Lambda}_T \subset \hat{F}$ *of* $\hat{\nu}$*-measure one such that:*
(1) $\hat{x} \mapsto \overline{d_{T,W}}(\hat{x})$ *and* $\hat{x} \mapsto \underline{d_{T,W}}(\hat{x})$ *are* $\hat{f}$*-invariant on* $\hat{\Lambda}_T$*; and*
(2) $\hat{x} \mapsto \overline{d_{T,Z}}(\hat{x})$ *and* $\hat{x} \mapsto \underline{d_{T,Z}}(\hat{x})$ *are* $\hat{f}$*-invariant on* $\hat{\Lambda}_T$ *if* $\lambda_1 \notin \{k\lambda_2, k \ge 2\}$*.*

In particular, if the exponents do not resonate, these functions are constant $\hat{\nu}$*-almost everywhere. We shall denote them by*

$$\overline{d_{T,Z}}(\nu), \quad \underline{d_{T,Z}}(\nu), \quad \overline{d_{T,W}}(\nu), \quad \underline{d_{T,W}}(\nu).$$

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Proof. We prove the invariance of $\overline{d_{T,W}}(\hat{x})$; the same arguments hold for the other functions. For every $z \in \mathbb{P}^2 \setminus \mathcal{C}_f$, we denote

$$a(z) := \tfrac{1}{2}\|(D_z f)^{-1}\|^{-1}, \quad \gamma(z) := \min\{a(z)\|f\|^{-1}_{\mathcal{C}^2,\mathbb{P}^2}, 1\}.$$

Then [**7**, Lemme 2] asserts that f is injective on $B_z(\gamma(z))$ and

$$\text{for all } r \in [0, \gamma(z)], \quad B_{f(z)}(a(z)r) \subset f(B_z(r)).$$

Let $\hat{x} \in \hat{F}$. Since $x_n \notin \mathcal{C}_f$ for every $n \in \mathbb{Z}$, we obtain, for every $r \leq \gamma(x_0)$,

$$T \wedge \left(\frac{i}{2}\, dW^{\epsilon}_{\hat{f}(\hat{x})} \wedge d\overline{W^{\epsilon}_{\hat{f}(\hat{x})}}\right)[B_{f(x_0)}(a(x_0)r)] \leq T \wedge \left(\frac{i}{2}\, dW^{\epsilon}_{\hat{f}(\hat{x})} \wedge d\overline{W^{\epsilon}_{\hat{f}(\hat{x})}}\right)[f(B_{x_0}(r))]. \tag{2.2}$$

Since f is injective on $B_{x_0}(r)$, we can change the variables to get

$$T \wedge \left(\frac{i}{2}\, dW^{\epsilon}_{\hat{f}(\hat{x})} \wedge d\overline{W^{\epsilon}_{\hat{f}(\hat{x})}}\right)[f(B_{x_0}(r))] = \int_{B_{x_0}(r)} f^*T \wedge f^*\left(\frac{i}{2}\, dW^{\epsilon}_{\hat{f}(\hat{x})} \wedge d\overline{W^{\epsilon}_{\hat{f}(\hat{x})}}\right). \tag{2.3}$$

Now let us recall that

$$f^*T = dT \quad \text{and} \quad f^*\left(\frac{i}{2}\, dW^{\epsilon}_{\hat{f}(\hat{x})} \wedge d\overline{W^{\epsilon}_{\hat{f}(\hat{x})}}\right) = |c(\hat{x})|^2 \frac{i}{2}\, dW^{\epsilon}_{\hat{x}} \wedge d\overline{W^{\epsilon}_{\hat{x}}}, \tag{2.4}$$

where the second equality comes from (2.1) by setting $c(\hat{x})^{-1} := \beta_{1,\hat{f}(\hat{x})}$; it is valid near x_0 according to Remark 2.2. By combining (2.2), (2.3) and (2.4) we deduce that

$$T \wedge \left(\frac{i}{2}\, dW^{\epsilon}_{\hat{f}(\hat{x})} \wedge d\overline{W^{\epsilon}_{\hat{f}(\hat{x})}}\right)[B_{f(x_0)}(a(x_0)r)] \leq d|c(\hat{x})|^2 T \wedge \left(\frac{i}{2}\, dW^{\epsilon}_{\hat{x}} \wedge d\overline{W^{\epsilon}_{\hat{x}}}\right)[B_{x_0}(r)]$$

for every r small enough. Taking the logarithm and dividing by $\log(a(x_0)r) < 0$, we get $\overline{d_{T,W}}(\hat{f}(\hat{x})) \geq \overline{d_{T,W}}(\hat{x})$ by taking limits. Since $\hat{\nu}$ is ergodic, the function $\overline{d_{T,W}}(\hat{x})$ is constant on a Borel set $\hat{\Lambda}_T$ of $\hat{\nu}$-measure one (see [**28**, Ch. 1.5]). One can replace it by $\bigcap_{n\in\mathbb{Z}} \hat{f}^n(\hat{\Lambda}_T)$ to obtain an invariant set. □

Similar arguments yield the following Proposition for the trace measure.

PROPOSITION 2.5. *If ν is an ergodic measure, the functions $x \mapsto \underline{d_T}(x)$ and $x \mapsto \overline{d_T}(x)$ are invariant, and hence ν-almost everywhere constant. We denote them by $\underline{d_T}(\nu)$ and $\overline{d_T}(\nu)$.*

Proof. The arguments follow the proof of Proposition 2.4. In this case, we study the measure $T \wedge \omega$, and we replace the second equality in (2.4) by $f^*\omega \leq \rho(x_0)\omega$ on $B_{x_0}(\gamma(x_0))$, where $\rho(x_0) > 0$ is a large enough positive constant. □

2.3. *Pullback of the Fubini–Study form ω.* Let ν be a dilating measure of exponents $\lambda_1 > \lambda_2$. Let $(Z, W)_\epsilon$ be Oseledec–Poincaré coordinates for (f, ν). Let $n_2(\beta)$ be the smallest integer n such that $e^{-n\epsilon} \leq \beta^{-1}$.

PROPOSITION 2.6. *Let $\hat{x} \in \hat{F}$ such that $\eta_\epsilon(\hat{x}) \geq \eta$ and $\beta_\epsilon(\hat{x}) \leq \beta$. If $n \geq \max\{n_2(\beta), n_\epsilon(\hat{x}_n)\}$ and if $r \leq \eta$, then, on $f^{-n}_{\hat{x}_n}(B_{x_n}(re^{-n\epsilon}))$:*

(1) $(f^n)^*\omega \geq e^{-6n\epsilon+2n\lambda_1}((i/2)\, dZ^{\epsilon}_{\hat{x}} \wedge d\overline{Z^{\epsilon}_{\hat{x}}})$ *if the exponents do not resonate; and*

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(2) $(f^n)^*\omega \geq e^{-6n\epsilon+2n\lambda_2}((i/2)\, dW_{\hat{x}}^{\epsilon} \wedge d\overline{W_{\hat{x}}^{\epsilon}})$.

The proof follows from the next two lemmas. First, observe that Theorem 2.1 gives

$$(f^n)^*\omega = (\xi_{\hat{x}}^{\epsilon})^*((R_{n,\hat{x}_n})^{-1})^*((\xi_{\hat{x}_n})^{-1})^*\omega$$

on $f_{\hat{x}_n}^{-n}(B_{x_n}(re^{-n\epsilon}))$. Let $\omega_0 := (i/2)\, dz \wedge d\overline{z} + (i/2)\, dw \wedge d\overline{w}$ be the standard (1, 1)-form on $\mathbb{D}^2$.

LEMMA 2.7. *Let $\hat{x} \in \hat{F}$ such that $\eta_\epsilon(\hat{x}) \geq \eta$ and $\beta_\epsilon(\hat{x}) \leq \beta$. For every $n \geq n_2(\beta)$ and $r \leq \eta$, we have, on $\xi_{\hat{x}_n}^{\epsilon}(B_{x_n}(re^{-n\epsilon}))$,*

$$e^{-4n\epsilon}\omega_0 \leq ((\xi_{\hat{x}_n}^{\epsilon})^{-1})^*\omega \leq 4\omega_0.$$

Proof. For every $p = (z, w)$ and $p' = (z', w')$ in $\xi_{\hat{x}_n}^{\epsilon}(B_{x_n}(re^{-n\epsilon}))$,

$$e^{-n\epsilon}\beta^{-1}|p - p'| \leq d((\xi_{\hat{x}_n}^{\epsilon})^{-1}(p), (\xi_{\hat{x}_n}^{\epsilon})^{-1}(p')) \leq 2|p - p'|.$$

This implies that, for every $n \geq n_2(\beta)$ and $(z, w) \in \xi_{\hat{x}_n}^{\epsilon}(B_{x_n}(re^{-n\epsilon}))$,

$$\text{for all } u \in \mathbb{C}^2, \quad e^{-2n\epsilon}|u| \leq |D_{(z,w)}(\xi_{\hat{x}_n}^{\epsilon})^{-1}(u)| \leq 2|u|.$$

This provides the desired estimates. □

LEMMA 2.8. *Let $\hat{x} \in \hat{F}$. If $n \geq n_\epsilon(\hat{x}_n)$, then:*
(1) $((R_{n,\hat{x}_n})^{-1})^*\omega_0 \geq e^{2(n\lambda_1 - n\epsilon)}(i/2)\, dz \wedge d\overline{z}$ *if the exponents do not resonate; and*
(2) $((R_{n,\hat{x}_n})^{-1})^*\omega_0 \geq e^{2(n\lambda_2 - n\epsilon)}(i/2)\, dw \wedge d\overline{w}$.

Proof. We use the fact that the linear part of $R_{n,\hat{x}_n}$ is diagonal with coefficients $e^{-n\epsilon - n\lambda_1} \leq |\alpha_{n,\hat{x}_n}| \leq e^{n\epsilon - n\lambda_1}$ and $e^{-n\epsilon - n\lambda_2} \leq |\beta_{n,\hat{x}_n}| \leq e^{n\epsilon - n\lambda_2}$ (see Theorem 2.1) and the fact that the (1, 1)-forms $(i/2)\, dz \wedge d\overline{z}$ and $(i/2)\, dw \wedge d\overline{w}$ are positive. □

To end the proof of Proposition 2.6, we observe that, for every $\hat{x} \in \hat{F}$,

$$(\xi_{\hat{x}}^{\epsilon})^*\left(\frac{i}{2}\, dz \wedge d\overline{z}\right) = \frac{i}{2}\, dZ_{\hat{x}}^{\epsilon} \wedge d\overline{Z_{\hat{x}}^{\epsilon}}, \quad (\xi_{\hat{x}}^{\epsilon})^*\left(\frac{i}{2}\, dw \wedge d\overline{w}\right) = \frac{i}{2}\, dW_{\hat{x}}^{\epsilon} \wedge d\overline{W_{\hat{x}}^{\epsilon}},$$

which follows from the definitions of $Z_{\hat{x}}^{\epsilon}$ and $W_{\hat{x}}^{\epsilon}$.

3. *Upper bounds for the directional dimensions of T*

Let ν be a dilating measure such that $\mathrm{Supp}(\nu) \subset \mathrm{Supp}(\mu)$ and whose exponents $\lambda_1 > \lambda_2$ do not resonate. Let $\epsilon > 0$ and let $(Z, W)_\epsilon$ be Oseledec–Poincaré coordinates for (f, ν). In this section, we establish Theorem 1.2 stated in the introduction: that is,

$$\underline{d_{T,Z}}(\nu) \leq 2\frac{\lambda_1}{\lambda_2} + \frac{\log d}{\lambda_2} + O(\epsilon) \quad \text{and} \quad \underline{d_{T,W}}(\nu) \leq 2 + \frac{\log d}{\lambda_2} + O(\epsilon).$$

The proof relies on the Monge–Ampère equation $\mu = T \wedge T$ and on a study of the Jacobians of $T \wedge (i/2)\, dZ_{\hat{x}}^{\epsilon} \wedge d\overline{Z_{\hat{x}}^{\epsilon}}$ and $T \wedge (i/2)\, dW_{\hat{x}}^{\epsilon} \wedge d\overline{W_{\hat{x}}^{\epsilon}}$ with respect to f.

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3.1. *Dimensions of the Green current on the equilibrium measure.* The following Proposition is proved in §3.3.

PROPOSITION 3.1. *Let f be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$ and let T be its Green current. Let $x \in \operatorname{Supp} \mu$ and let Z be a local holomorphic coordinate (submersion) in a neighbourhood V of x. Then $T \wedge ((i/2)\, dZ \wedge d\overline{Z})$ is not the zero measure on V.*

This implies the following proposition.

PROPOSITION 3.2. *Let ν be a dilating measure with exponents $\lambda_1 > \lambda_2$ whose support is contained in the support of μ. Let $\epsilon > 0$ and let $(Z, W)_\epsilon$ be Oseledec–Poincaré coordinates for (f, ν). Then, for every $0 < r < \eta_\epsilon(\hat{x})$,*

$$\left[T \wedge \frac{i}{2}\, dZ^\epsilon_{\hat{x}} \wedge d\overline{Z^\epsilon_{\hat{x}}}\right](B_x(r)) > 0 \quad \textit{and} \quad \left[T \wedge \frac{i}{2}\, dW^\epsilon_{\hat{x}} \wedge d\overline{W^\epsilon_{\hat{x}}}\right](B_x(r)) > 0.$$

In particular, for every $\delta > 0$, there exist $m_0 \geq 1$, $L_0 \geq 1$ and $q_0 \geq 1$ such that

$$\hat{\Omega}_\epsilon := \left\{ \eta_\epsilon(\hat{x}) \geq \frac{1}{4m_0},\ L(\hat{x}) \leq L_0,\quad \begin{matrix} \left[T \wedge \frac{i}{2}\, dZ^\epsilon_{\hat{x}} \wedge d\overline{Z^\epsilon_{\hat{x}}}\right]\left(B_x\left(\frac{1}{4m_0}\right)\right) \geq \frac{1}{q_0} \\ \left[T \wedge \frac{i}{2}\, dW^\epsilon_{\hat{x}} \wedge d\overline{W^\epsilon_{\hat{x}}}\right]\left(B_x\left(\frac{1}{4m_0}\right)\right) \geq \frac{1}{q_0} \end{matrix} \right\}$$

satisfies $\hat{\nu}(\hat{\Omega}_\epsilon) \geq 1 - \delta$.

Proof. The first part immediately follows from Proposition 3.1. To prove the second part, let $m_0 \geq 1$ and $L_0 \geq 1$ be such that $\hat{\nu}\{\eta_\epsilon \geq (1/4m_0)\} \cap \{L \leq L_0\} \geq 1 - \delta/2$. Then we choose q_0 large enough so that $\hat{\nu}(\hat{\Omega}_\epsilon) \geq 1 - \delta$. □

For every $n \geq 1$, we define

$$\hat{\Omega}^n_\epsilon := \hat{\Omega}_\epsilon \cap \hat{f}^{-n}(\hat{\Omega}_\epsilon).$$

Since $\hat{\nu}$ is invariant,

$$\hat{\nu}(\hat{\Omega}^n_\epsilon) \geq 1 - 2\delta. \tag{3.1}$$

The following proposition is crucial for proving Theorems 1.1 and 1.2. L_0 is defined in Proposition 3.2 and $n_1(L_0) \geq 1$ is defined before Lemma 2.3.

PROPOSITION 3.3. *Let ν be a dilating measure with exponents $\lambda_1 > \lambda_2$ and such that* $\operatorname{Supp}(\nu) \subset \operatorname{Supp}(\mu)$. *For every $n \geq n_1(L_0)$ and $\hat{x} \in \hat{\Omega}^n_\epsilon$,*

$$\left[T \wedge \frac{i}{2}\, dZ^\epsilon_{\hat{x}} \wedge d\overline{Z^\epsilon_{\hat{x}}}\right]\left(B_x\left(\frac{1}{m_0} e^{-n\lambda_2 + 3n\epsilon}\right)\right) \geq \frac{1}{d^n} e^{-2n\lambda_1 - 2n\epsilon} \frac{1}{q_0} \quad \textit{if } \lambda_1 \notin \{k\lambda_2, k \geq 2\},$$

$$\left[T \wedge \frac{i}{2}\, dW^\epsilon_{\hat{x}} \wedge d\overline{W^\epsilon_{\hat{x}}}\right]\left(B_x\left(\frac{1}{m_0} e^{-n\lambda_2 + 3n\epsilon}\right)\right) \geq \frac{1}{d^n} e^{-2n\lambda_2 - 2n\epsilon} \frac{1}{q_0}.$$

Proof. Let $\hat{x} \in \hat{\Omega}^n_\epsilon$ and let

$$E_n := f^{-n}_{\hat{x}_n}\left(B_{x_n}\left(\frac{1}{4m_0}\right)\right).$$

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The inverse branch $f_{\hat{x}_n}^{-n}$ is well defined on $B_{x_n}(1/4m_0)$ since $\hat{x}_n \in \hat{\Omega}_\epsilon$. Let g_n be the restriction of f^n on E_n. By using $f_{\hat{x}_n}^{-n} \circ g_n = \mathrm{Id}_{E_n}$ and $T = (1/d^n) g_n^* T$ on E_n, we obtain

$$T \wedge \frac{i}{2}\, dZ_{\hat{x}}^\epsilon \wedge d\overline{Z_{\hat{x}}^\epsilon} = \frac{1}{d^n} g_n^* T \wedge g_n^* (f_{\hat{x}_n}^{-n})^* \bigg(\frac{i}{2}\, dZ_{\hat{x}}^\epsilon \wedge d\overline{Z_{\hat{x}}^\epsilon}\bigg)$$
$$= \frac{1}{d^n} g_n^* \bigg[T \wedge \frac{i}{2} (dZ_{\hat{x}}^\epsilon \circ (f_{\hat{x}_n}^{-n})) \wedge d(\overline{Z_{\hat{x}}^\epsilon \circ (f_{\hat{x}_n}^{-n})})\bigg]$$

on the open subset E_n. Now we use (2.1) to write $Z_{\hat{x}}^\epsilon \circ (f_{\hat{x}_n}^{-n}) = \alpha_{n,\hat{x}_n} Z_{\hat{x}_n}$. Since $|\alpha_{n,\hat{x}_n}|^2 \geq e^{-2n\lambda_1 - 2n\epsilon}$, we get, on E_n,

$$T \wedge \frac{i}{2}\, dZ_{\hat{x}}^\epsilon \wedge d\overline{Z_{\hat{x}}^\epsilon} \geq \frac{1}{d^n} e^{-2n\lambda_1 - 2n\epsilon} g_n^* \bigg[T \wedge \frac{i}{2}\, dZ_{\hat{x}_n}^\epsilon \wedge d\overline{Z_{\hat{x}_n}^\epsilon}\bigg]. \tag{3.2}$$

Now we bound from above the left-hand side and we bound from below the right-hand side (applied to E_n). Using Lemma 2.3 with $r = 1/m_0$ and $n \geq n_1(L_0)$, we obtain $E_n \subset B_x((1/m_0)e^{-n\lambda_2 + 3n\epsilon})$, and hence

$$T \wedge \frac{i}{2}\, dZ_{\hat{x}}^\epsilon \wedge d\overline{Z_{\hat{x}}^\epsilon}\bigg(B_x\bigg(\frac{1}{m_0} e^{-n\lambda_2 + 3n\epsilon}\bigg)\bigg) \geq T \wedge \frac{i}{2}\, dZ_{\hat{x}}^\epsilon \wedge d\overline{Z_{\hat{x}}^\epsilon}(E_n). \tag{3.3}$$

For the right-hand side, since g_n is injective on E_n and $g_n(E_n) = B_{x_n}(1/4m_0)$, we get

$$g_n^* \bigg[T \wedge \frac{i}{2} (dZ_{\hat{x}_n}^\epsilon \wedge d\overline{Z_{\hat{x}_n}^\epsilon})\bigg](E_n) = \bigg[T \wedge \frac{i}{2} (dZ_{\hat{x}_n}^\epsilon \wedge (\overline{dZ_{\hat{x}_n}^\epsilon}))\bigg]\bigg(B_{x_n}\bigg(\frac{1}{4m_0}\bigg)\bigg) \geq \frac{1}{q_0}, \tag{3.4}$$

where the inequality comes from $\hat{x}_n \in \hat{\Omega}_\epsilon$. By combining (3.2), (3.3) and (3.4), we obtain

$$\bigg[T \wedge \frac{i}{2}\, dZ_{\hat{x}}^\epsilon \wedge d\overline{Z_{\hat{x}}^\epsilon}\bigg]\bigg(B_x\bigg(\frac{1}{m_0} e^{-n\lambda_2 + 3n\epsilon}\bigg)\bigg) \geq \frac{1}{d^n} e^{-2n\lambda_1 - 2n\epsilon} \frac{1}{q_0}.$$

We use $W_{\hat{x}}^\epsilon \circ (f_{\hat{x}_n}^{-n}) = \beta_{n,\hat{x}} W_{\hat{x}_n}$ and $|\beta_{n,\hat{x}}|^2 \geq e^{-2n\lambda_2 - 2n\epsilon}$ to prove the other estimate. □

3.2. *Proof of Theorem 1.2.* We take the notation of §3.1. Let

$$\hat{\Theta}_\epsilon := \limsup_{n\in\mathbb{N}} \hat{\Omega}_\epsilon \cap \hat{f}^{-n}(\hat{\Omega}_\epsilon) = \limsup_{n\in\mathbb{N}} \hat{\Omega}_\epsilon^n.$$

We have $\hat{\nu}(\hat{\Theta}_\epsilon) \geq 1 - 2\delta$ according to (3.1). Let $\hat{x} \in \hat{\Theta}_\epsilon$. There exists an increasing sequence of integers $(l_p)_p$ such that

$$\hat{x} \in \hat{\Omega}_\epsilon \cap \hat{f}^{-l_p}(\hat{\Omega}_\epsilon) = \hat{\Omega}_\epsilon^{l_p}.$$

Proposition 3.3 then asserts that, for p large enough,

$$\bigg[T \wedge \frac{i}{2}\, dZ_{\hat{x}}^\epsilon \wedge d\overline{Z_{\hat{x}}^\epsilon}\bigg]\bigg(B_x\bigg(\frac{1}{m_0} e^{-l_p\lambda_2 + 3l_p\epsilon}\bigg)\bigg) \geq \frac{1}{d^{l_p}} e^{-2l_p\lambda_1 - 2l_p\epsilon} \frac{1}{q_0}.$$

If p is also large enough so that $e^{l_p\epsilon} \geq 1/m_0$ and $1/q_0 \geq e^{-l_p\epsilon}$, we obtain, with $r_p := e^{-l_p(\lambda_2 - 4\epsilon)}$,

$$\bigg[T \wedge \frac{i}{2}\, dZ_{\hat{x}}^\epsilon \wedge d\overline{Z_{\hat{x}}^\epsilon}\bigg](B_x(r_p)) \geq e^{-l_p(\log d + 2\lambda_1 + 3\epsilon)} = r_p^{(\log d + 2\lambda_1 + 3\epsilon)/(\lambda_2 - 4\epsilon)}.$$

Since $(r_p)_p$ tends to zero and $\hat{\nu}(\hat{\Theta}_\epsilon) > 0$, we get

$$\underline{d_{T,Z}}(\nu) \leq \frac{\log d + 2\lambda_1 + 3\epsilon}{\lambda_2 - 4\epsilon} =: 2\frac{\lambda_1}{\lambda_2} + \frac{\log d}{\lambda_2} + O(\epsilon).$$

We prove that

$$\underline{d_{T,W}}(\nu) \leq \frac{\log d + 2\lambda_2 + 3\epsilon}{\lambda_2 - 4\epsilon} =: 2 + \frac{\log d}{\lambda_2} + O(\epsilon)$$

in a similar way.

3.3. *Monge–Ampère mass.* We prove Proposition 3.1. Let $x \in \operatorname{Supp} \mu$. Let V be a neighbourhood of x and let $Z : V \to \mathbb{C}$ be a holomorphic coordinate (submersion) on V. We want to prove that the positive measure $T \wedge (i/2)\, dZ \wedge d\overline{Z}$ is not the zero measure on V. With no loss of generality, we can assume that $x = (0, 0)$, $V = \mathbb{D}(2) \times \mathbb{D}(2)$ and $Z(z, w) = z$. Let also $T = 2i\partial\overline{\partial}G$ on V, where G is a continuous plurisubharmonic function (see §1.2). We denote $\sigma_z(u) := (z, u)$.

LEMMA 3.4. *If $(T \wedge (i/2)\, dZ \wedge d\overline{Z})(\mathbb{D}(2) \times \mathbb{D}(2)) = 0$, then $G \circ \sigma_z$ is harmonic on $\mathbb{D}$ for every $z \in \mathbb{D}$.*

Proof. Let $z_0 \in \mathbb{D}$ and let $\varphi \in C_0^\infty(\mathbb{D})$ be a non-negative test function. Let $\psi \in C_0^\infty(\mathbb{D}^2)$ non-negative be such that $\psi \circ \sigma_{z_0} = \varphi$ on $\mathbb{D}$. According to Proposition A.4,

$$\bigg(T \wedge \frac{i}{2}\, dZ \wedge d\overline{Z}\bigg)(\psi)$$
$$= \int_{z\in\mathbb{D}} \bigg(\int_{w\in\mathbb{D}} (G \circ \sigma_z)(w) \times \Delta(\psi \circ \sigma_z)(w) \;\, d\operatorname{Leb}(w)\bigg)\; d\operatorname{Leb}(z),$$

which is equal to zero by our assumption. Since G is plurisubharmonic, the measurable function

$$z \mapsto \int_{w\in\mathbb{D}} (G \circ \sigma_z)(w) \times \Delta(\psi \circ \sigma_z)(w) \;\, d\operatorname{Leb}(w)$$

is non-negative, and hence there exists $A \subset \mathbb{D}$ such that $\operatorname{Leb}(A) = \operatorname{Leb}(\mathbb{D})$ and

$$\text{for all } z \in A, \quad \int_{w\in\mathbb{D}} (G \circ \sigma_z)(w) \times \Delta(\psi \circ \sigma_z)(w) \;\, d\operatorname{Leb}(w) = 0. \tag{3.5}$$

Let us extend this property to every $z \in \mathbb{D}$. Since A is a dense subset of $\mathbb{D}$, there exists a sequence $(z_n)_{n\geq 1}$ of points in A that converges to z. Using (3.5), we get

$$\text{for all } n \geq 1, \quad \int_{w\in\mathbb{D}} (G \circ \sigma_{z_n})(w) \times \Delta(\psi \circ \sigma_{z_n})(w) \;\, d\operatorname{Leb}(w) = 0. \tag{3.6}$$

Since G is continuous on $\overline{\mathbb{D}^2}$ and ψ is smooth on $\overline{\mathbb{D}^2}$, G and $2i\partial\overline{\partial}\psi$ are uniformly continuous on $\overline{\mathbb{D}^2}$. This implies that $G \circ \sigma_{z_n}$ uniformly converges to $G \circ \sigma_z$ on $\mathbb{D}$ and that $\Delta(\psi \circ \sigma_{z_n})$ uniformly converges to $\Delta(\psi \circ \sigma_z)$ on $\mathbb{D}$. Taking the limits in (3.6), we get

$$\text{for all } z \in \mathbb{D}, \quad \int_{w\in\mathbb{D}} (G \circ \sigma_z)(w) \times \Delta(\psi \circ \sigma_z)(w) \;\, d\operatorname{Leb}(w) = 0.$$

In particular, using $\psi \circ \sigma_{z_0} = \varphi$, we obtain

$$\int_{w\in\mathbb{D}} (G \circ \sigma_{z_0})(w) \times \Delta\varphi(w) \;\, d\operatorname{Leb}(w) = 0.$$

This holds for every non-negative $\varphi \in C_0^\infty(\mathbb{D})$, and hence $G \circ \sigma_{z_0}$ is harmonic on $\mathbb{D}$. □

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Now we use the following result (see [**6**, Lemme IV.1.1] and [**27**, §A.10]).

THEOREM 3.5. (Briend) *Let* G *be a continuous plurisubharmonic function on* $\mathbb{D}(2) \times \mathbb{D}(2)$. *Let* E *be the set of points* $p \in \mathbb{D}(\frac{1}{4}) \times \mathbb{D}(\frac{1}{4})$ *such that there exists a holomorphic disc* $\sigma_p : \mathbb{D} \to \mathbb{D}(2) \times \mathbb{D}(2)$ *satisfying:*

(1) *the boundary of* σ_p *is outside* $\mathbb{D}(\frac{1}{2}) \times \mathbb{D}(\frac{1}{2})$*; and*

(2) $G \circ \sigma_p$ *is harmonic* $\mathbb{D}$.

Then $(2i\partial\overline{\partial}G \wedge 2i\partial\overline{\partial}G)(E) = 0$.

In our situation, we have $\mathbb{D}(\frac{1}{4}) \times \mathbb{D}(\frac{1}{4}) = E$ by taking for σ_p the discs $\sigma_z : \mathbb{D} \to \mathbb{D} \times \mathbb{D},\ u \mapsto (z, u)$. Indeed, the boundary of σ_z is contained in $\{z\} \times \partial\mathbb{D}$ and $G \circ \sigma_z$ is harmonic on $\mathbb{D}$ according to Lemma 3.4. Theorem 3.5 then gives

$$(2i\partial\overline{\partial}G \wedge 2i\partial\overline{\partial}G)(\mathbb{D}(\tfrac{1}{4}) \times \mathbb{D}(\tfrac{1}{4})) = 0,$$

which contradicts $x = 0 \in \operatorname{Supp} \mu = \operatorname{Supp}(2i\partial\overline{\partial}G \wedge 2i\partial\overline{\partial}G)$.

4. *Lower bounds for the directional dimensions of* T *and* S

We prove Theorem 1.3 and Corollary 1.5. Let ν be a dilating measure with exponents $\lambda_1 > \lambda_2$, and let $(Z, W)_\epsilon$ be Oseledec–Poincaré coordinates for (f, ν). Theorems 4.6 and 4.7 provide lower bounds for directional dimensions depending on $\underline{d_\nu}$, Theorem 1.3 follows from the lower bound (1.4) on $\underline{d_\nu}$. We shall use arguments of [**13**]. Precisely, we shall replace for the lower bound $(f^n)^*\omega \geq e^{2n\lambda_2}e^{-n\epsilon}\omega$ obtained in [**13**] by slicing arguments by the two lower bounds

$$(f^n)^*\omega \geq e^{2n\lambda_1}e^{-6n\epsilon}\frac{i}{2}\,dZ \wedge d\overline{Z} \quad \text{and} \quad (f^n)^*\omega \geq e^{2n\lambda_2}e^{-6n\epsilon}\frac{i}{2}\,dW \wedge d\overline{W}$$

coming from Proposition 2.6. In §§4.1 and 4.2 we construct a set $\hat{\Lambda}_\epsilon \subset \hat{X}$ of uniformizations satisfying $\hat{\nu}(\hat{\Lambda}_\epsilon) \geq 1 - \delta/2$. Section 4.3 deals with separated sets.

4.1. *Dynamical balls.*

The dynamical distance is defined by $d_n(x, y) := \max_{0\leq k\leq n} d(f^k(x), f^k(y))$. We denote by $B_n(x, r)$ the ball centred at x and of radius r for the distance d_n.

LEMMA 4.1. *Let* $\delta > 0$ *and* $R > 0$. *There exist* $\eta_1 < R$, $n_3 \geq 1$ *and* $C \subset \mathbb{P}^2$ *such that* $\nu(C) \geq 1 - \delta/8$ *and which satisfy the following properties. For every* $x \in C$ *and* $n \geq n_3$,

$$\nu(B_n(x, \eta_1/8)) \geq e^{-nh_\nu - \epsilon n},$$

$$\text{for all } r \leq \eta_1, \quad \nu(B_n(x, 5r)) \leq \nu(B_n(x, 5\eta_1)) \leq e^{-nh_\nu + \epsilon n}.$$

Proof. The Brin–Katok theorem [**9**] ensures that there exists $C_1 \subset \mathbb{P}^2$ of full ν-measure such that, for every $x \in C_1$,

$$\lim_{r\to 0}\Big(\liminf_{n\to+\infty}\frac{-1}{n}\log \nu(B_n(x, r))\Big) = \lim_{r\to 0}\Big(\limsup_{n\to+\infty}\frac{-1}{n}\log \nu(B_n(x, r))\Big) = h_\nu.$$

Hence, for every $x \in C_1$, there exists $\eta_1(x) \leq R$ such that $r \leq \eta_1(x)$ implies that

$$\liminf_{n\to+\infty}\frac{-1}{n}\log(\nu(B_n(x, 5r))) \geq h_\nu - \epsilon/2 \quad \text{and}$$

$$\limsup_{n\to+\infty}\frac{-1}{n}\log(\nu(B_n(x, r/8))) \leq h_\nu + \epsilon/2.$$

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Let $\eta_1 > 0$ small enough such that $C_2 := \{x \in C_1\ ,\ \eta_1 \leq \eta_1(x) \leq R\}$ satisfies $\nu(C_2) \geq 1 - \delta/16$. For every $x \in C_2$, there exists $n_3(x)$ such that $n \geq n_3(x)$ implies that

$$\nu(B_n(x, \eta_1/8)) \geq e^{-nh_\nu - \epsilon n},$$
$$\text{for all } r \leq \eta_1, \quad \nu(B_n(x, 5r)) \leq \nu(B_n(x, 5\eta_1)) \leq e^{-nh_\nu + \epsilon n}.$$

Let $n_3 \geq 1$ such that $C := C_2 \cap \{x \in C_1, n_3(x) \leq n_3\}$ satisfies $\nu(C) \geq 1 - \delta/8$. □

For every $L > 0$, let m_L be the smallest integer m such that $Le^{-m(\lambda_2+\epsilon)} \leq 1$ and let $n_4(L)$ be the smallest integer n larger than m_L such that $e^{-n\epsilon} \leq M^{-m_L}$, where $M := \max\{\|Df\|_{\infty,\mathbb{P}^2}, 1\}$. The set $\hat{F}$ is defined in Theorem 2.1.

LEMMA 4.2. *Let $\hat{x} \in \hat{F}$ such that $\eta_\epsilon(\hat{x}) \geq \eta$ and $L_\epsilon(\hat{x}) \leq L$. For every $n \geq n_4(L)$ and every $r \leq \eta$,*

$$f_{\hat{x}_n}^{-n}(B_{x_n}(re^{-2n\epsilon})) \subset B_n(x_0, r).$$

Proof. Let us observe that, for every $0 \leq k \leq n$, f^k is injective on $f_{\hat{x}_n}^{-n}(B_{x_n}(re^{-2n\epsilon}))$ and that $f^k f_{\hat{x}_n}^{-n} = f_{\hat{x}_n}^{-n+k}$. By setting $p = n - k$, it suffices to show that

$$\text{for all } p \in [\![0, n]\!] \quad f_{\hat{x}_n}^{-p}(B_{x_n}(re^{-2n\epsilon})) \subset B_{x_{n-p}}(r). \tag{4.1}$$

To simplify, let us set $m := m_L$ and $n_4 := n_4(L)$. We immediately have $n_4 \geq m$ and also

$$\text{for all } n \geq n_4, \quad \text{for all } p \in [\![0, n]\!],\ f_{\hat{x}_n}^{-p}(B_{x_n}(re^{-2n\epsilon})) \subset f_{\hat{x}_n}^{-p}\left(B_{x_n}\left(\frac{r}{M^m}e^{-n\epsilon}\right)\right). \tag{4.2}$$

To verify (4.1), we shall consider separately the cases $p \leq m$ and $p > m$. We know that, for every p, Lip $f_{\hat{x}_n}^{-p} \leq L(\hat{x}_n)e^{-p\lambda_2+p\epsilon} \leq Le^{n\epsilon}e^{-p\lambda_2+p\epsilon}$ on $B_{x_n}(\eta_\epsilon(\hat{x}_n))$, which contains $B_{x_n}(\eta e^{-n\epsilon})$. Hence, for every $n \geq n_4 \geq m$, $p \in [\![m, n]\!]$ and $r \leq \eta$,

$$f_{\hat{x}_n}^{-p}(B_{x_n}(re^{-n\epsilon})) \subset B_{x_{n-p}}(re^{-n\epsilon}Le^{n\epsilon}e^{-p\lambda_2+p\epsilon}) = B_{x_{n-p}}(rLe^{-p\lambda_2+p\epsilon}) \subset B_{x_{n-p}}(r).$$

Since $M^m \geq 1$, this implies that, for every $n \geq n_4 \geq m$, $p \in [\![m, n]\!]$ and $r \leq \eta$,

$$f_{\hat{x}_n}^{-p}\left(B_{x_n}\left(\frac{r}{M^m}e^{-n\epsilon}\right)\right) \subset B_{x_{n-p}}\left(\frac{r}{M^m}\right). \tag{4.3}$$

Thus, by using (4.2) and $M^m \geq 1$,

$$\text{for all } p \in [\![m, n]\!], \quad f_{\hat{x}_n}^{-p}(B_{x_n}(re^{-2n\epsilon})) \subset B_{x_{n-p}}(r). \tag{4.4}$$

We have proved (4.1) for $p \in [\![m, n]\!]$. Let us show this inclusion for $p \in [\![0, m]\!]$. For every $p \in [\![0, m]\!]$, let us set $p = m - p'$, where $p' \in [\![0, m]\!]$. Then

$$f_{\hat{x}_n}^{-p}\left(B_{x_n}\left(\frac{r}{M^m}e^{-n\epsilon}\right)\right) = f^{p'}\left(f_{\hat{x}_n}^{-m}\left(B_{x_n}\left(\frac{r}{M^m}e^{-n\epsilon}\right)\right)\right) \subset f^{p'}\left(B_{x_{n-m}}\left(\frac{r}{M^m}\right)\right),$$

where the inclusion comes from (4.3) with $p = m$. Using $e^{-n\epsilon} \leq 1/M^m$ for the left-hand side and $\|Df^{p'}\|_{\infty,\mathbb{P}^2} \leq M^{p'}$ for the right-hand side, we get

$$\text{for all } p \in [\![0, m]\!], \quad f_{\hat{x}_n}^{-p}(B_{x_n}(re^{-2n\epsilon})) \subset B_{x_{n-m+p'}}\left(\frac{r}{M^m}M^{p'}\right) \subset B_{x_{n-p}}(r). \tag{4.5}$$

We finally obtain (4.1) by combining (4.4) and (4.5). □

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4.2. *Uniformizations.* In this section, we introduce uniformizations for the functions of Theorem 2.1 and for the radii of balls in relation to their ν-measure or their directional measures. This will allow us to consider those quantities as constant on Borel sets of $\hat{\nu}$-measure close to one.

1. *Control of the functions n_ϵ, ρ_ϵ, L_ϵ, η_ϵ, β_ϵ of Theorem 2.1*

We recall that $\hat{F}$ provided by Theorem 2.1 satisfies $\hat{\nu}(\hat{F}) = 1$. Let $n_0 \geq 1$, $\rho_0 > 0$, $L_0 \geq 1$, $\eta_0 > 0$, $\beta_0 \geq 1$ such that

$$\Lambda^{(1)} := \{\hat{x} \in \hat{F}, n_\epsilon(\hat{x}) \leq n_0,\ \ \rho_\epsilon(\hat{x}) \geq \rho_0, L_\epsilon(\hat{x}) \leq L_0,\ \ \eta_\epsilon(\hat{x}) \geq \eta_0,\ \ \beta_\epsilon(\hat{x}) \leq \beta_0\}$$

satisfies $\hat{\nu}(\Lambda^{(1)}) \geq 1 - \delta/8$.

2. *Uniformization of the directional dimensions*

Let S be a positive closed current on $\mathbb{P}^2$ whose support contains the support of ν. Let $r_1 > 0$ such that

$$\Lambda^{(2)} := \left\{ \hat{x} \in \hat{F}, \forall r \leq r_1, \quad \begin{array}{c} r^{\overline{d_{S,Z}}(\hat{x})+\epsilon} \leq \left(S \wedge \left(\frac{i}{2}\, dZ^\epsilon_{\hat{x}} \wedge d\overline{Z^\epsilon_{\hat{x}}}\right)\right)(B_{x_0}(r)) \leq r^{\underline{d_{S,Z}}(\hat{x})-\epsilon} \\ \left(S \wedge \left(\frac{i}{2}\, dW^\epsilon_{\hat{x}} \wedge d\overline{W^\epsilon_{\hat{x}}}\right)\right)(B_{x_0}(r)) \leq r^{\underline{d_{S,W}}(\hat{x})-\epsilon} \end{array} \right\}$$

satisfies $\hat{\nu}(\Lambda^{(2)}) \geq 1 - \delta/8$.

3. *Uniformization of the dimension of the measure*

Let $r_2 > 0$ such that

$$D := \{x \in \mathbb{P}^2, \forall r \leq r_2, \nu(B_x(r)) \leq r^{\underline{d_\nu}-\epsilon}\}$$

satisfies $\nu(D) \geq 1 - \delta/8$. We set $\Lambda^{(3)} := \pi_0^{-1}(D) \cap \hat{F}$.

4. *Measure of dynamical balls, definition of η_1*

We apply Lemma 4.1 with $R := \min\{\eta_0, r_1, r_2\}$. There exist $\eta_1 \leq R$, $n_3 \geq 1$ and $C \subset \mathbb{P}^2$ such that $\nu(C) \geq 1 - \delta/8$ and, for every $x \in C$ and $n \geq n_3$,

$$\nu(B_n(x, \eta_1/8)) \geq e^{-nh_\nu - \epsilon n},$$
$$\text{for all } r \leq \eta_1,\ \nu(B_n(x, 5r)) \leq \nu(B_n(x, 5\eta_1)) \leq e^{-nh_\nu + \epsilon n}.$$

We denote $\Lambda^{(4)} := \pi_0^{-1}(C) \cap \hat{F}$.

5. *Definitions of $\hat{\Lambda}_\epsilon$ and N_ϵ*

We set

$$\hat{\Lambda}_\epsilon := \Lambda^{(1)} \cap \Lambda^{(2)} \cap \Lambda^{(3)} \cap \Lambda^{(4)}.$$

We have $\hat{\nu}(\hat{\Lambda}_\epsilon) \geq 1 - \delta/2$. We recall that $n_1(L)$, $n_2(\beta)$ and $n_4(L)$ are defined before Lemma 2.3, Proposition 2.6 and Lemma 4.2. Let n_5 be the smallest integer n such that $e^{-n\epsilon} \leq 1/2$ and $2e^{-n(\lambda_1+\epsilon)} < 1$. We set

$$N_\epsilon := \max\{n_0, n_1(L_0), n_2(\beta_0), n_3, n_4(L_0), n_5\}.$$

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6. *Definition of* $\hat{\Delta}_\epsilon^n$

For all $n \geq N_\epsilon$, we define

$$\hat{\Delta}_\epsilon^n := \hat{F} \cap \hat{f}^{-n}\{n_\epsilon(\hat{x}) \leq n\} = \{\hat{x} \in \hat{F}, n_\epsilon(\hat{x}_n) \leq n\}.$$

Since $\hat{\nu}$ is $\hat{f}$-invariant and $\Lambda^{(1)} \subset \{n_\epsilon(\hat{x}) \leq n\}$ for every $n \geq N_\epsilon$, we have $\hat{\nu}(\hat{\Delta}_\epsilon^n) \geq \hat{\nu}(\Lambda^{(1)}) \geq 1 - \delta/8$. Hence

$$\text{for all } n \geq N_\epsilon, \quad \hat{\nu}(\hat{\Lambda}_\epsilon \cap \hat{\Delta}_\epsilon^n) \geq 1 - \delta.$$

4.3. *Separated sets.* A subset $\{x_1, \ldots, x_N\} \subset \mathbb{P}^2$ is r-separated if $d(x_i, x_j) \geq r$ for every $i \neq j$. For $A \subset \mathbb{P}^2$, a subset $\{x_1, \ldots, x_N\} \subset A$ is maximal r-separated with respect to A if it is r-separated and if, for every $y \in A$, there exists $i \in \{1, \ldots, N\}$ such that $d(y, x_i) < r$. We use similar definitions for the distance d_n, in which case the subsets are called (n, r) separated.

LEMMA 4.3. *Let $A \subset \pi_0(\hat{\Lambda}_\epsilon)$ such that $\nu(A) > 0$ and let $c \in]0, 1]$. Let $n \geq N_\epsilon$ and let $\{x_1, \ldots, x_{N_n}\} \subset A$ be a maximal $(n, c\eta_1)$-separated with respect to A. Then:*

(1) *$A \subset \bigcup_{i=1}^{N_n} B_n(x_i, c\eta_1)$ and $B_n(x_i, c\eta_1/2) \cap B_n(x_j, c\eta_1/2) = \emptyset$ for every $i \neq j$;*

(2) *$\nu(B_n(x_i, c\eta_1)) \leq e^{-nh_\nu + n\epsilon}$ and $e^{-nh_\nu - n\epsilon} \leq \nu(B_n(x_i, c\eta_1))$ if $c \geq 1/8$; and*

(3) *$N_n \geq \nu(A)e^{nh_\nu - n\epsilon}$.*

Proof. Item 1 comes from the definitions, and item 2 comes from §4 of §4.2. They imply that $\nu(A) \leq \sum_{i=1}^{N_n} \nu(B_n(x_i, c\eta_1)) \leq N_n e^{-nh_\nu + n\epsilon}$, which gives item 3. □

The next lemmas give an amout of concentrated separation. We take the arguments of de Thélin–Vigny [**13**, §7]. By applying Lemma 4.3 with $c = 1/4$, we obtain $\nu(B_n(x_i, \eta_1/4)) \geq e^{-nh_\nu - n\epsilon}$ for every x_i in a maximal $(n, \eta_1/4)$-separated subset of A. Lemma 4.4 allows us to select a large number of x_i such that $\nu(B_n(x_i, \eta_1/4) \cap A) \geq e^{-nh_\nu - 2n\epsilon}$ and which are (n, η_1)-separated, and then Lemma 4.5 deals with these x_i. Let n_δ be the smallest n such that $e^{-n\epsilon} \leq \delta/2$.

LEMMA 4.4. *Let $A \subset \pi_0(\hat{\Lambda}_\epsilon)$ such that $\nu(A) \geq \delta$. For every $n \geq \max\{N_\epsilon, n_\delta\}$, there exist a $(n, \eta_1/4)$-separated subset $\{x_1, \ldots, x_{N_{n,2}}\}$ of A and $N_{n,3} \in [\![1, N_{n,2}]\!]$ such that:*

(1) *for $i \neq j$ in $[\![1, N_{n,2}]\!]$, $B_n(x_i, \eta_1/8) \cap B_n(x_j, \eta_1/8) = \emptyset$;*

(2) *for $i \in [\![1, N_{n,2}]\!]$, $\nu(B_n(x_i, \eta_1/4) \cap A) \geq e^{-nh_\nu - 2\epsilon n}$, where $N_{n,2} \geq \nu(A)e^{nh_\nu - 2n\epsilon}$;*

(3) *for $i \in [\![1, N_{n,2}]\!]$ and $c \geq 1/8$, $e^{-nh_\nu - n\epsilon} \leq \nu(B_n(x_i, c\eta_1))$; and*

(4) *for $i \neq j$ in $[\![1, N_{n,3}]\!]$, $B_n(x_i, \eta_1/2) \cap B_n(x_j, \eta_1/2) = \emptyset$, where $N_{n,3} \geq \nu(A)e^{nh_\nu - 4n\epsilon}$.*

Proof. Let us apply Lemma 4.3 with $c = 1/4$ and $n \geq \max\{N_\epsilon, n_\delta\}$. There exists a maximal $(n, \eta_1/4)$-separated subset $\{x_1, \ldots, x_{N_{n,1}}\}$ of A satisfying:

- $A \subset \bigcup_{i=1}^{N_{n,1}} B_n(x_i, \eta_1/4)$ and $B_n(x_i, \eta_1/8) \cap B_n(x_j, \eta_1/8) = \emptyset$ for every $i \neq j$;
- $e^{-nh_\nu - n\epsilon} \leq \nu(B_n(x_i, \eta_1/8))$; and
- $N_{n,1} \geq \nu(A)e^{nh_\nu - n\epsilon}$.

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Let us set $I := \{1 \le i \le N_{n,1},\ \nu(B_n(x_i, \eta_1/4) \cap A) \ge e^{-nh_\nu - 2\epsilon n}\}$. Let $N_{n,2}$ be the cardinality of I, and assume that I consists of the first $N_{n,2}$ elements of $[\![1, N_{n,1}]\!]$. We want to bound $N_{n,2}$ from below. We know that $A \subset \bigcup_{i=1}^{N_{n,1}} B_n(x_i, \eta_1/4)$, and hence

$$\nu(A) \le \sum_{i=1}^{N_{n,2}} \nu(B_n(x_i, \eta_1/4) \cap A) + \sum_{i=N_{n,2}+1}^{N_{n,1}} \nu(B_n(x_i, \eta_1/4) \cap A).$$

If $i \notin [\![1, N_{n,2}]\!]$, we have $\nu(B_n(x_i, \eta_1/4) \cap A) < e^{-nh_\nu - 2\epsilon n}$ by the definition of I. Otherwise, $\nu(B_n(x_i, \eta_1/4)) \le e^{-nh_\nu + \epsilon n}$ since $x_i \in C$. This implies that

$$\nu(A) \le N_{n,2} e^{-nh_\nu + \epsilon n} + (N_{n,1} - N_{n,2}) e^{-nh_\nu - 2\epsilon n}. \tag{4.6}$$

Let us give an upper bound for $N_{n,1}$. Since the balls $B_n(x_i, \eta_1/8)$ are pairwise disjoint and since $\nu(B_n(x_i, \eta_1/8)) \ge e^{-nh_\nu - \epsilon n}$, we get $e^{nh_\nu + \epsilon n} \ge N_{n,1} \ge N_{n,1} - N_{n,2}$. Combining this and (4.6), we obtain

$$\nu(A) \le N_{n,2} e^{-nh_\nu + \epsilon n} + e^{-\epsilon n}.$$

Since $n \ge n_\delta$, we have $e^{-n\epsilon} \le \delta/2 \le \nu(A)/2$, and hence $N_{n,2} \ge \nu(A) e^{nh_\nu - \epsilon n}/2$. We deduce that $N_{n,2} \ge \nu(A) e^{nh_\nu - 2\epsilon n}$ because $n \ge N_\epsilon \ge n_5$. Let us verify the last item. Let E_n be a maximal (n, η_1)-separated subset of $\{x_1, \ldots, x_{N_{n,2}}\}$. By reordering, we can assume that $E_n = \{x_1, \ldots, x_{N_{n,3}}\}$. For every $i \in [\![1, N_{n,3}]\!]$, let m_i be the cardinality of $\{j \in [\![1, N_{n,2}]\!], x_j \in B_n(x_i, \eta_1)\}$, and let $m_{i_0} := \max m_i$, which is simply denoted by m in the subsequent work. We have $N_{n,2} \le m N_{n,3}$. Let us verify that $m \le e^{2n\epsilon}$, which implies that $N_{n,3} \ge N_{n,2} e^{-2n\epsilon}$, as desired. For that purpose, let $\{x'_1, \ldots, x'_m\} := B_n(x_{i_0}, \eta_1) \cap \{x_1, \ldots, x_{N_{n,2}}\}$. We obviously have $\bigcup_{i=1}^m B_n(x'_i, \eta_1/8) \subset B_n(x_{i_0}, 2\eta_1)$. Since the dynamical balls $B_n(x'_i, \eta_1/8)$ are pairwise disjoint, we finally get $m e^{-nh_\nu - n\epsilon} \le e^{-nh_\nu + n\epsilon}$ by considering the ν-measures and using $x'_i, x_{i_0} \in C$ (see §4 of §4.2). □

Now, for $1 \le i \le N_{n,3}$, we put in $B_n(x_i, \eta_1/2)$ a lot of balls centred in $B_n(x_i, \eta_1/4) \cap A$.

LEMMA 4.5. *Let $A \subset \pi_0(\hat{\Lambda}_\epsilon)$ such that $\nu(A) > 0$. Let $x \in A$ and let $n \ge N_\epsilon$ such that*

$$\nu(B_n(x, \eta_1/4) \cap A) \ge e^{-nh_\nu - 2n\epsilon}.$$

Let $\{y_1, \ldots, y_{M_n}\}$ be a maximal $2\eta_1 e^{-n\lambda_1 - 4n\epsilon}$-separated subset in $B_n(x, \eta_1/4) \cap A$.

(1) *For every $i \neq j$, $B(y_i, \eta_1 e^{-n\lambda_1 - 4n\epsilon}) \cap B(y_j, \eta_1 e^{-n\lambda_1 - 4n\epsilon}) = \emptyset$.*

(2) $B_n(x, \eta_1/4) \cap A \subset \bigcup_{i=1}^{M_n} B(y_i, 2\eta_1 e^{-n\lambda_1 - 4n\epsilon})$.

(3) $B(y_i, \eta_1 e^{-n\lambda_1 - 4n\epsilon}) \subset B_n(x, \eta_1/2)$.

(4) $M_n \ge e^{-nh_\nu - 2n\epsilon}((1/2\eta_1) e^{n\lambda_1 + 4n\epsilon})^{\underline{d_\nu} - \epsilon}$.

Proof. Items 1 and 2 come from the definitions. Lemmas 2.3 and 4.2 successively give

$$B(y_i, \eta_1 e^{-n\lambda_1 - 4n\epsilon}) \subset f^{-n}_{\hat{y}_{i,n}}(B_{y_{i,n}}(\eta_1 e^{-2n\epsilon}/4) \subset B_n(y_i, \eta_1/4)).$$

Since $y_i \in B_n(x, \eta_1/4)$, we get $B_n(y_i, \eta_1/4) \subset B_n(x, \eta_1/2)$, and item 3 follows. Item 2 implies that

$$\nu(B_n(x, \eta_1/4) \cap A) \le \sum_{i=1}^{M_n} \nu(B(y_i, 2\eta_1 e^{-n\lambda_1 - 4n\epsilon})). \tag{4.7}$$

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By our assumption, the left-hand side of (4.7) is larger than $e^{-nh_\nu-2n\epsilon}$. For the right-hand side, since $n \geq N_\epsilon \geq n_5$, we have $2\eta_1 e^{-n\lambda_1-4n\epsilon} < \eta_1 \leq r_2$ by the definition of n_5 and η_1 (given in §§5 and 4 of §4.2). Then, by using $y_i \in A \subset \pi_0(\hat{\Lambda}_\epsilon) \subset D$, we get, by the definition of D (given in §3 of §4.2),

$$\nu(B(y_i, 2\eta_1 e^{-n\lambda_1-4n\epsilon})) \leq (2\eta_1 e^{-n\lambda_1-4n\epsilon})^{\underline{d_\nu}-\epsilon}.$$

We finally deduce from (4.7) that $e^{-nh_\nu-2n\epsilon} \leq M_n (2\eta_1 e^{-n\lambda_1-4n\epsilon})^{\underline{d_\nu}-\epsilon}$, as desired. □

4.4. *Lower bounds for the upper dimensions $\overline{d_{T,Z}}$ and $\overline{d_{T,W}}$.*

THEOREM 4.6. *Let ν be a dilating measure whose exponents $\lambda_1 > \lambda_2$ do not resonate. Let $\epsilon > 0$ and let $(Z, W)_\epsilon$ be Oseledec–Poincaré coordinates for (f, ν). Then*

$$\overline{d_{T,Z}}(\nu) \geq 2 + \underline{d_\nu} - \frac{\log d}{\lambda_1} - O(\epsilon),$$
$$\overline{d_{T,W}}(\nu) \geq 2\frac{\lambda_2}{\lambda_1} + \underline{d_\nu} - \frac{\log d}{\lambda_1} - O(\epsilon).$$

Proof. We recall that $\overline{d_{T,Z}}(\nu) := \overline{d_{T,Z}}(\hat{x})$ and $\overline{d_{T,W}}(\nu) := \overline{d_{T,W}}(\hat{x})$ for $\hat{\nu}$-almost every $\hat{x}$ (see Proposition 2.4). Let us set $\overline{d_{T,Z}} := \overline{d_{T,Z}}(\nu)$. We are going to show that

$$(\lambda_1 + 4\epsilon)(\overline{d_{T,Z}} - \underline{d_\nu} + 2\epsilon) + 12\epsilon \geq 2\lambda_1 - \log d. \tag{4.8}$$

This yields, as desired,

$$\overline{d_{T,Z}} - \underline{d_\nu} \geq \frac{2\lambda_1}{\lambda_1 + 4\epsilon} - \frac{\log d + 12\epsilon}{\lambda_1 + 4\epsilon} - 2\epsilon = 2 - \frac{\log d}{\lambda_1} - O(\epsilon). \tag{4.9}$$

Let $\delta > 0$. Let $\hat{\Lambda}_\epsilon$, N_ϵ and $\hat{\Delta}_\epsilon^n$ be given by §4.2. For every $n \geq N_\epsilon$, we set $A_n := \pi_0(\hat{\Lambda}_\epsilon \cap \hat{\Delta}_\epsilon^n)$, which satisfies $\nu(A_n) \geq 1 - \delta$. Lemma 4.4 provides $\{x_1, \ldots, x_{N_{n,3}}\} \subset A_n$ satisfying $N_{n,3} \geq \nu(A_n) e^{nh_\nu - 4n\epsilon}$, $\nu(B_n(x_i, \eta_1/4) \cap A_n) \geq e^{-nh_\nu - 2\epsilon n}$ and $B_n(x_i, \eta_1/2) \cap B_n(x_j, \eta_1/2) = \emptyset$ for $i \neq j$. Then, for every x_i, we set a maximal $2\eta_1 e^{-n\lambda_1-4n\epsilon}$-separated subset $\{y_1^i, \ldots, y_{M_n}^i\}$ in $B_n(x_i, \eta_1/4) \cap A_n$ given by Lemma 4.5. Let us note that $B(y_j^i, \eta_1 e^{-n\lambda_1-4n\epsilon}) \subset B_n(x_i, \eta_1/2)$. Finally, for every i, we choose $\hat{x}_i \in \hat{\Lambda}_\epsilon \cap \hat{\Delta}_\epsilon^n$ such that $\pi_0(\hat{x}_i) = x_i$ and, for every j, we choose $\hat{y}_j^i \in \hat{\Lambda}_\epsilon \cap \hat{\Delta}_\epsilon^n$ such that $\pi_0(\hat{y}_j^i) = y_j^i$.

Now we use $d^n = \int_{\mathbb{P}^2} (f^n)_* T \wedge \omega$ provided by Proposition A.3 to get

$$d^n \geq \sum_{i=1}^{N_{n,3}} \sum_{j=1}^{M_n} \int_{\mathbb{P}^2} (f^n)_* (1_{B(y_j^i, \eta_1 e^{-n\lambda_1-4n\epsilon})} T) \wedge \omega = \sum_{i=1}^{N_{n,3}} \sum_{j=1}^{M_n} \int_{B(y_j^i, \eta_1 e^{-n\lambda_1-4n\epsilon})} T \wedge (f^n)^* \omega.$$

Lemma 2.3 with $\hat{y}_j^i \in \hat{\Lambda}_\epsilon$ implies that

$$B(y_j^i, \eta_1 e^{-n\lambda_1-4n\epsilon}) \subset f_{\hat{y}_{j,n}^i}^{-n}\left(B\left(y_{j,n}^i, \frac{\eta_1}{4} e^{-2n\epsilon}\right)\right). \tag{4.10}$$

Since $\hat{y}_j^i \in \hat{\Delta}_\epsilon^n$, we can apply Proposition 2.6 to bound $(f^n)^*\omega$ from below: i.e.,

$$d^n \geq \sum_{i=1}^{N_{n,3}} \sum_{j=1}^{M_n} e^{2n\lambda_1 - 6n\epsilon} \left(T \wedge \frac{i}{2} dZ_{\hat{y}_j^i}^\epsilon \wedge d\overline{Z_{\hat{y}_j^i}^\epsilon}\right)(B_{y_j^i}(\eta_1 e^{-n\lambda_1-4n\epsilon})). \tag{4.11}$$

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Now $\hat{y}^i_j \in \hat{\Lambda}_\epsilon \subset \Lambda^{(2)}$ and $n \geq N_\epsilon$, and hence

$$d^n \geq \sum_{i=1}^{N_{n,3}} \sum_{j=1}^{M_n} e^{2n\lambda_1 - 6n\epsilon} (\eta_1 e^{-n\lambda_1 - 4n\epsilon})^{\overline{d_{T,Z}}+\epsilon}.$$

Finally, we use the lower bounds for $N_{n,3}$ (Lemma 4.4) and M_n (Lemma 4.5), and $\nu(A_n) \geq 1 - \delta$. We obtain, for every $n \geq \max\{N_\epsilon, n_{1-\delta}\}$,

$$d^n \geq (1-\delta) e^{nh_\nu - 4n\epsilon} \cdot e^{-nh_\nu - 2n\epsilon} \bigg(\frac{1}{2\eta_1} e^{n\lambda_1 + 4n\epsilon}\bigg)^{\underline{d_\nu} - \epsilon} \cdot e^{2n\lambda_1 - 6n\epsilon} (\eta_1 e^{-n\lambda_1 - 4n\epsilon})^{\overline{d_{T,Z}}+\epsilon}.$$

Let us note that the entropy h_ν disappears for the benefit of $\underline{d_\nu}$, and we get

$$d^n \geq c\, e^{-12n\epsilon} (e^{n\lambda_1 + 4n\epsilon})^{\underline{d_\nu} - \overline{d_{T,Z}} - 2\epsilon} e^{2n\lambda_1}, \tag{4.12}$$

where $c := (1-\delta)\eta_1^{\overline{d_{T,Z}}+\epsilon}/(2\eta_1)^{\underline{d_\nu}-\epsilon}$. We obtain (4.8) by looking at the exponential growth rates. Similarly, we can prove that

$$(\lambda_1 + 4\epsilon)(\overline{d_{T,W}} - \underline{d_\nu} + 2\epsilon) + 12\epsilon \geq 2\lambda_2 - \log d$$

by using Proposition 2.6 again to bound $(f^n)^*\omega$ from below. □

4.5. *Lower bounds for the upper dimensions* $\overline{d_{S,Z}}$ *and* $\overline{d_{S,W}}$. Let S be a (1, 1) closed positive current of $\mathbb{P}^2$. If S does not satisfy $f^*S = dS$, the directional dimensions may be not $\hat{\nu}$-almost everywhere constant (see Proposition 2.4). In this case, in the manner of de Thélin–Vigny [**13**], we take on an adapted definition and obtain the following result. It implies Theorem 1.3 by using the lower bound (1.4).

THEOREM 4.7. *Let* S *be a* (1, 1) *closed positive current of* $\mathbb{P}^2$. *Let* ν *be a dilating measure whose exponents* $\lambda_1 > \lambda_2$ *do not resonate. We assume that* $\mathrm{Supp}(\nu) \subset \mathrm{Supp}\, S$. *Let* $\epsilon > 0$ *and let* $(Z, W)_\epsilon$ *be Oseledec–Poincaré coordinates for* (f, ν). *For every* $\hat{\Lambda} \subset \hat{F}$ *such that* $\hat{\nu}(\hat{\Lambda}) > 0$, *we set*

$$\overline{d_{S,Z}}(\hat{\Lambda}) := \sup_{\hat{x}\in\hat{\Lambda}} \overline{d_{S,Z}}(\hat{x}), \quad \overline{d_{S,W}}(\hat{\Lambda}) := \sup_{\hat{x}\in\hat{\Lambda}} \overline{d_{S,W}}(\hat{x}).$$

Then

$$\overline{d_{S,Z}}(\hat{\Lambda}) \geq 2 + \underline{d_\nu} - \frac{\log d}{\lambda_1} - O(\epsilon),$$
$$\overline{d_{S,W}}(\hat{\Lambda}) \geq 2\frac{\lambda_2}{\lambda_1} + \underline{d_\nu} - \frac{\log d}{\lambda_1} - O(\epsilon).$$

Proof. Let $2\delta := \hat{\nu}(\hat{\Lambda})$. We construct $\hat{\Lambda}_\epsilon$, N_ϵ and $\hat{\Delta}^n_\epsilon$ for the current S as in §4.2. We have $\hat{\nu}(\hat{\Lambda}_\epsilon \cap \hat{\Delta}^n_\epsilon) \geq 1 - \delta$ for every $n \geq N_\epsilon$, and thus $\hat{\nu}(\hat{\Lambda} \cap \hat{\Lambda}_\epsilon \cap \hat{\Delta}^n_\epsilon) \geq \delta > 0$. We follow the arguments of Theorem 4.6. Let $n \geq \max\{N_\epsilon, n_\delta\}$ and $A_n := \pi_0(\hat{\Lambda} \cap \hat{\Lambda}_\epsilon \cap \hat{\Delta}^n_\epsilon)$. Lemma 4.4 provides $\{x_1, \ldots, x_{N_{n,3}}\} \subset A_n$ satisfying $N_{n,3} \geq \nu(A_n) e^{nh_\nu - 4n\epsilon}$, $\nu(B_n(x_i, \eta_1/4) \cap A_n) \geq e^{-nh_\nu - 2\epsilon n}$ and $B_n(x_i, \eta_1/2) \cap B_n(x_j, \eta_1/2) = \emptyset$ for $i \neq j$. Then, for every x_i, let $\{y^i_1, \ldots, y^i_{M_n}\}$ be a maximal $2\eta_1 e^{-n\lambda_1 - 4n\epsilon}$-separated subset in

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$B_n(x_i, \eta_1/4) \cap A_n$ given by Lemma 4.5. We have the inclusions $B(y^i_j, \eta_1 e^{-n\lambda_1 - 4n\epsilon}) \subset B_n(x_i, \eta_1/2)$. Finally, for every i, j let $\hat{x}_i \in \hat{\Lambda}_\epsilon \cap \hat{\Delta}^n_\epsilon$ such that $\pi_0(\hat{x}_i) = x_i$, and let $\hat{y}^i_j \in \hat{\Lambda}_\epsilon \cap \hat{\Delta}^n_\epsilon$ such that $\pi_0(\hat{y}^i_j) = y^i_j$.

Using $d^n = \int_{\mathbb{P}^2} (f^n)_* S \wedge \omega$, we deduce, as in (4.11), that

$$d^n \geq \sum_{i=1}^{N_{n,3}} \sum_{j=1}^{M_n} e^{2n\lambda_1 - 6n\epsilon} \bigg(S \wedge \frac{i}{2}\, dZ^\epsilon_{\hat{y}^i_j} \wedge d\overline{Z^\epsilon_{\hat{y}^i_j}} \bigg) (B_{y^i_j}(\eta_1 e^{-n\lambda_1 - 4n\epsilon})).$$

Since $\hat{y}^i_j \in \hat{\Lambda}_\epsilon$ and $n \geq N_\epsilon$,

$$d^n \geq \sum_{i=1}^{N_{n,3}} \sum_{j=1}^{M_n} e^{2n\lambda_1 - 6n\epsilon} (\eta_1 e^{-n\lambda_1 - 4n\epsilon})^{\overline{d_{S,Z}}(\hat{y}^i_j) + \epsilon}.$$

Then we use the inequality $\overline{d_{S,Z}}(\hat{y}^i_j) \leq \overline{d_{S,Z}}(\hat{\Lambda})$ and conclude as in the proof of Theorem 4.6. The lower bound concerning W is proved in a similar way. □

4.6. *Proof of Corollary 1.5.* We recall that $\overline{d_T}(x)$ is μ-almost everywhere constant and denoted by $\overline{d_T}(\mu)$ (see Proposition 2.5). According to Proposition A.1, $\mu \ll T \wedge \omega$ implies that

$$\overline{d_T}(\mu) \leq \overline{d_\mu}. \tag{4.13}$$

Let us analyze these quantities. On the one hand, Proposition A.2 yields $\overline{d_T}(\mu) = \min\{\overline{d_{T,Z}}(x), \overline{d_{T,W}}(x)\}$ for μ-almost every $x \in \mathbb{P}^2$ and for all holomorphic coordinates (Z, W) near x. On the other hand, since $\underline{d_\mu} = \overline{d_\mu}$, we have $\underline{d_\mu} = \overline{d_\mu} = \dim_H(\mu)$. Observe also that $\lambda_2 = \frac{1}{2} \log d$, since $\mu \ll T \wedge \omega$ (see [**19**]). Hence $\dim_H(\mu) = 2 + (\log d)/\lambda_1$ by (1.6). Then one deduces from (4.13) that

$$\min\{\overline{d_{T,Z}}(x), \overline{d_{T,W}}(x)\} \leq 2 + \frac{\log d}{\lambda_1}. \tag{4.14}$$

Now we use Theorem 4.6. If $(Z, W)_\epsilon$ are Oseledec–Poincaré coordinates for (f, μ),

$$\overline{d_{T,Z}}(\mu) \geq 2 + \underline{d_\mu} - \frac{\log d}{\lambda_1} - O(\epsilon),$$
$$\overline{d_{T,W}}(\mu) \geq 2\frac{\lambda_2}{\lambda_1} + \underline{d_\mu} - \frac{\log d}{\lambda_1} - O(\epsilon).$$

Let us note that $\lambda_2 = \frac{1}{2} \log d$ implies that $\underline{d_\mu} - (\log d)/\lambda_1 \geq 2$ by (1.4). We deduce that $\overline{d_{T,Z}}(\mu) \geq 4 - O(\epsilon)$ and that $\overline{d_{T,W}}(\mu) \geq 2 + (\log d)/\lambda_1 - O(\epsilon)$. It remains to show that $\overline{d_{T,W}}(\mu) \leq 2 + (\log d)/\lambda_1$. Let ϵ be small enough such that $4 - O(\epsilon) > 2 + (\log d)/\lambda_1$. This implies that $\overline{d_{T,Z}}(\mu) > 2 + (\log d)/\lambda_1$, and thus the minimum in (4.14) is attained for W.

5. *Proof of Theorem 1.1*

Let ν be a dilating measure with exponents $\lambda_1 \geq \lambda_2$ whose support is contained in the support of μ. If $\lambda_1 = \lambda_2$, then $\overline{d_\nu} = \underline{d_\nu} = 2(\log d)/\lambda_1$ by (1.1), and Theorem 1.1 is true. So we can assume that $\lambda_1 > \lambda_2$. The proof of Theorem 1.1 will rely on the estimates of Proposition 3.3, on the arguments of Theorem 4.6 and on a rescaling of the time n with respect to the exponents. A crucial point is that Proposition 3.3 provides estimates for the directional measures of T in terms of λ_1, λ_2, d; we shall use them in (5.6). The dimension $\underline{d_\nu}$ will appear in (5.3).

Let $\epsilon > 0$. The set $\hat{\Omega}_\epsilon^n$ has been defined in §3.1; it satisfies $\hat{\nu}(\hat{\Omega}_\epsilon^n) \geq 1 - 2\delta$ for every $n \geq 1$. Let $\hat{\Lambda}_\epsilon$, $\hat{\Delta}_\epsilon^n$ be the sets defined in §4.2. We have $\hat{\nu}(\hat{\Lambda}_\epsilon \cap \hat{\Delta}_\epsilon^n \cap \hat{\Omega}_\epsilon^n) \geq 1 - 3\delta$ for every $n \geq N_\epsilon$. Now let K_n be the unique integer in $]\alpha_n, \alpha_n + 1]$, where $\alpha_n := (1/(\lambda_2 - 3\epsilon))(-\log(\eta_1 m_0) + n(\lambda_1 + 4\epsilon))$. It satisfies

$$\eta_1 e^{-n\lambda_1 - 4n\epsilon} e^{-\lambda_2 + 3\epsilon} \leq \frac{1}{m_0} e^{-K_n\lambda_2 + 3K_n\epsilon} < \eta_1 e^{-n\lambda_1 - 4n\epsilon}. \tag{5.1}$$

These inequalities will be useful for (5.5). Let us have in mind that $K_n \simeq n\lambda_1/\lambda_2 \geq n \geq N_\epsilon$. Let $\{x_1, \ldots, x_{N_{n,3}}\}$ be a $(n, \eta_1/4)$-separated subset of $A_n := \pi_0(\hat{\Lambda}_\epsilon \cap \hat{\Delta}_\epsilon^n \cap \hat{\Omega}_\epsilon^{K_n})$ provided by Lemma 4.4. For every $n \geq \max\{N_\epsilon, n_{1-3\delta}\}$,

$$N_{n,3} \geq \nu(A_n) e^{nh_\nu - 4n\epsilon} \geq (1 - 3\delta) e^{nh_\nu - 4n\epsilon}. \tag{5.2}$$

Then, for every x_i, let $\{y_1^i, \ldots, y_{M_n}^i\}$ be a maximal $2\eta_1 e^{-n\lambda_1 - 4n\epsilon}$-separated subset of $B_n(x_i, \eta_1/4) \cap A_n$ provided by Lemma 4.5. The cardinality of this set satisfies

$$M_n \geq e^{-nh_\nu - 2n\epsilon} \left(\frac{1}{2\eta_1} e^{n\lambda_1 + 4n\epsilon} \right)^{\underline{d_\nu} - \epsilon}. \tag{5.3}$$

For every $j \in \{1, \ldots, M_n\}$, we set $\hat{y}_j^i \in \hat{\Lambda}_\epsilon \cap \hat{\Delta}_\epsilon^n \cap \hat{\Omega}_\epsilon^{K_n}$ such that $y_j^i = \pi_0(\hat{y}_j^i)$. By following the proof of Theorem 4.6 until (4.11), we get

$$d^n \geq \sum_{i=1}^{N_{n,3}} \sum_{j=1}^{M_n} e^{2n\lambda_1 - 6n\epsilon} \left[T \wedge \left(\frac{i}{2}\, dZ^\epsilon_{\hat{y}_j^i} \wedge d\overline{Z^\epsilon_{\hat{y}_j^i}} \right) \right] (B_{y_j^i}(\eta_1 e^{-n\lambda_1 - 4n\epsilon})). \tag{5.4}$$

Now, according to (5.1),

$$B_{y_j^i}(\eta_1 e^{-n\lambda_1 - 4n\epsilon}) \supset B_{y_j^i}\left(\frac{1}{m_0} e^{-K_n\lambda_2 + 3K_n\epsilon} \right). \tag{5.5}$$

Since $\hat{y}_j^i \in \hat{\Omega}_\epsilon^{K_n}$, Proposition 3.3 implies that, for every n satisfying $n \geq N_\epsilon$ and $K_n \geq N_\epsilon$,

$$\left[T \wedge \left(\frac{i}{2}\, dZ^\epsilon_{\hat{y}_{i,j}} \wedge d\overline{Z^\epsilon_{\hat{y}_{i,j}}} \right) \right] (B_{y_j^i}(\eta_1 e^{-n\lambda_1 - 4n\epsilon})) \geq \frac{1}{d^{K_n}} e^{-2K_n\lambda_1 - 2K_n\epsilon} \frac{1}{q_0}. \tag{5.6}$$

We infer from (5.4) that, for every n satisfying $n \geq N_\epsilon$ and $K_n \geq N_\epsilon$,

$$d^n \geq N_{n,3} \cdot M_n \cdot e^{2n\lambda_1 - 6n\epsilon} \frac{1}{d^{K_n}} e^{-2K_n\lambda_1 - 2K_n\epsilon} \frac{1}{q_0}. \tag{5.7}$$

By using the upper bounds for $N_{n,3}$ and M_n given by (5.2) and (5.3),

$$d^{n+K_n} \geq (1 - 3\delta) e^{nh_\nu - 4n\epsilon} \cdot e^{-nh_\nu - 2n\epsilon} \left(\frac{1}{2\eta_1} e^{n\lambda_1 + 4n\epsilon} \right)^{\underline{d_\nu} - \epsilon} \cdot e^{2n\lambda_1 - 6n\epsilon} e^{-2K_n\lambda_1 - 2K_n\epsilon} \frac{1}{q_0}.$$

If $C_1(\epsilon) := (1 - 3\delta)/q_0(2\eta_1)^{\underline{d_\nu} - \epsilon}$, we get

$$\log d + \frac{K_n}{n} \log d \geq \frac{1}{n} \log C_1(\epsilon) - 12\epsilon + (\lambda_1 + 4\epsilon)(\underline{d_\nu} - \epsilon) + 2\lambda_1 - 2\frac{K_n}{n}(\lambda_1 + \epsilon).$$

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By using the lower and upper bounds defining K_n just before (5.1), we obtain

$$\log d + \frac{\lambda_1 + 4\epsilon}{\lambda_2 - 3\epsilon} \log d \geq \frac{1}{n} \log C_2(\epsilon) - 12\epsilon + (\lambda_1 + 4\epsilon)(\underline{d_\nu} - \epsilon) + 2\lambda_1 - 2\frac{\lambda_1 + 4\epsilon}{\lambda_2 - 3\epsilon}(\lambda_1 + \epsilon),$$

where $C_2(\epsilon)$ is another constant. Letting n tend to $+\infty$ and then ϵ to zero, we get

$$\underline{d_\nu} \leq \frac{\log d}{\lambda_1} + \frac{\log d}{\lambda_2} + 2\bigg(\frac{\lambda_1}{\lambda_2} - 1\bigg).$$

To obtain the other upper bound, we use the analogue of (5.4) for W. Applying Proposition 3.3 with respect to W, we obtain, instead of (5.7),

$$d^n \geq N_{n,3} \cdot M_n \cdot e^{2n\lambda_2 - 6n\epsilon} \frac{1}{d^{K_n}} e^{-2K_n\lambda_2 - 2K_n\epsilon} \frac{1}{q_0}.$$

Then we get

$$\underline{d_\nu} \leq \frac{\log d}{\lambda_1} + \frac{\log d}{\lambda_2} + 2\bigg(1 - \frac{\lambda_2}{\lambda_1}\bigg),$$

which completes the proof of Theorem 1.1.

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A. *Appendix.*

A.1. *Dimension of measures.*

PROPOSITION A.1. *Let ν_1 and ν_2 be two probability measures on $\mathbb{P}^2$ such that $\nu_1 \ll \nu_2$. Then, for ν_1-almost every $x \in \mathbb{P}^2$,*

$$\underline{d_{\nu_1}}(x) \geq \underline{d_{\nu_2}}(x) \quad \textit{and} \quad \overline{d_{\nu_1}}(x) \geq \overline{d_{\nu_2}}(x).$$

Proof. Let $\varphi \in L^1(\nu_2)$ such that $\nu_1(A) = \int_A \varphi \, d\nu_2$ for every Borel set A of $\mathbb{P}^2$. Using the dominated convergence theorem,

$$\lim_{M \to +\infty} \int_{\mathbb{P}^2} 1_{\{\varphi \leq M\}} \varphi \, d\nu_2 = \int_{\mathbb{P}^2} \varphi \, d\nu_2 = 1.$$

For every $n \geq 1$, we let M_n satisfy $\int_{\mathbb{P}^2} 1_{\{\varphi \leq M_n\}} \varphi \, d\nu_2 \geq 1 - 1/n$. By the Lebesgue density theorem, for ν_1-almost every x in $\{\varphi \leq M_n\}$,

$$\lim_{r \to 0} \frac{\nu_1(B_x(r) \cap \{\varphi \leq M_n\})}{\nu_1(B_x(r))} = 1.$$

Then, for every r small enough,

$$\frac{1}{2}\nu_1(B_x(r)) \leq \nu_1(B_x(r) \cap \{\varphi \leq M_n\}) \leq M_n \int_{B_x(r)} d\nu_2.$$

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Thus $\nu_1(B_x(r)) \le 2M_n\nu_2(B_x(r))$. Taking limits when r tends to zero, we deduce that

$$\underline{d_{\nu_1}}(x) \ge \underline{d_{\nu_2}}(x) \quad \text{and} \quad \overline{d_{\nu_1}}(x) \ge \overline{d_{\nu_2}}(x)$$

for ν_1-almost every $x \in \{\varphi \le M_n\}$. We end with $\nu_1(\bigcup_{n\in\mathbb{N}}\{\varphi \le M_n\}) = 1$. □

PROPOSITION A.2. *Let S be a $(1, 1)$-closed positive current on $\mathbb{P}^2$. Let $x \in \mathbb{P}^2$ and let (Z, W) be holomorphic coordinates near x. Then*

$$\underline{d_S}(x) = \min\{\underline{d_{S,Z}}(x), \underline{d_{S,W}}(x)\}, \quad \overline{d_S}(x) = \min\{\overline{d_{S,Z}}(x), \overline{d_{T,W}}(x)\}.$$

Proof. Let us set $\sigma_{S,Z} = S \wedge ((i/2)\, dZ \wedge d\overline{Z})$ and $\sigma_{S,W} = S \wedge (i/2\, dW \wedge d\overline{W})$. There exists $c > 0$ such that $(1/c)(\sigma_{S,Z} + \sigma_{S,W}) \le \sigma_S \le c(\sigma_{S,Z} + \sigma_{S,W})$ on a neighbourhood of x (see [**14**, Ch. III, §3]). We deduce that, for every r small enough,

$$\frac{1}{c}\max[\sigma_{S,Z}(B_x(r)), \sigma_{S,W}(B_x(r))] \le \sigma_S(B_x(r)) \le 2c\max[\sigma_{S,Z}(B_x(r)), \sigma_{S,W}(B_x(r))].$$

We finish by observing that the local dimension of the maximum of two measures is equal to the minimum of the local dimensions, since one divides by $\log r$ which is negative. □

A.2. *Cohomology and slices.* We refer to Dinh–Sibony's book [**18**, §§1.2 and A.3].

PROPOSITION A.3. *Let S be a $(1, 1)$-closed positive current of $\mathbb{P}^2$ of mass one. Let ω be the Fubini–Study form on $\mathbb{P}^2$ and let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be an endomorphism of degree d. Then*

$$\int_{\mathbb{P}^2} (f^n)_* S \wedge \omega = \int_{\mathbb{P}^2} S \wedge (f^n)^*\omega = d^n.$$

Proof. The first equality comes from the definition of duality. We verify the second one. By using $f^*\omega = d \cdot \omega + 2i\partial\overline{\partial}u$, where u is a smooth function on $\mathbb{P}^2$, we obtain by induction that

$$(f^n)^*\omega = d^n\omega + 2i\partial\overline{\partial}v_n,$$

where $v_n := (d^{n-1}u + \cdots + du \circ f^{n-2} + u \circ f^{n-1})$. Since S is a closed current of mass one, we have $\int_{\mathbb{P}^2} S \wedge 2i\partial\overline{\partial}v_n = 0$ and $\int_{\mathbb{P}^2} S \wedge d^n\omega = d^n$. □

PROPOSITION A.4. *Let G be a continuous plurisubharmonic function on $\mathbb{D}^2$ and let $S = 2i\partial\overline{\partial}G$. Let (Z, W) be the coordinates on $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$ and let $\phi \in C_0^\infty(\mathbb{D}^2)$. Then*

$$\begin{aligned} S \wedge \frac{i}{2}\, dZ \wedge d\overline{Z}(\phi) &= \int_{z\in\mathbb{D}} \left(\int_{w\in\mathbb{D}} G_z(w) \times \Delta\phi_z(w)\ d\,\mathrm{Leb}(w)\right) d\,\mathrm{Leb}(z) \\ &= \int_{z\in\mathbb{D}} (\sigma_z^* S)(\phi_z) d\,\mathrm{Leb}(z), \end{aligned}$$

where $\sigma_z : u \mapsto (z, u)$, $G_z := G \circ \sigma_z$ and $\phi_z := \phi \circ \sigma_z$.

Proof. By definition,

$$S \wedge \frac{i}{2}\, dZ \wedge d\overline{Z}(\phi) = 2i\partial\overline{\partial}G\left(\phi\frac{i}{2}\, dZ \wedge d\overline{Z}\right) = \int_{\mathbb{D}^2} G \cdot 2i\partial\overline{\partial}\left(\phi\frac{i}{2}\, dZ \wedge d\overline{Z}\right).$$

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The computation

$$2i\partial\overline{\partial}\left(\phi\frac{i}{2}\,dZ\wedge d\overline{Z}\right)=4\left(\frac{\partial^2\phi}{\partial w\partial\overline{w}}\right)\frac{i}{2}\,dW\wedge d\overline{W}\wedge\frac{i}{2}\,dZ\wedge d\overline{Z}$$
$$=4\left(\frac{\partial^2\phi}{\partial w\partial\overline{w}}\right)d\,\mathrm{Leb}(z,\,w)$$

allows to write

$$S\wedge\frac{i}{2}\,dZ\wedge d\overline{Z}(\phi)=\int_{(z,w)\in\mathbb{D}^2}G(z,\,w)\times 4\frac{\partial^2\phi}{\partial w\partial\overline{w}}(z,\,w)\,d\,\mathrm{Leb}(z,\,w)$$
$$=\int_{z\in\mathbb{D}}\left(\int_{w\in\mathbb{D}}G_z(w)\times\Delta\phi_z(w)\,d\,\mathrm{Leb}(w)\right)d\,\mathrm{Leb}(z),$$

where the second equality comes from Fubini's theorem. Finally, the quantity in brackets is equal to $(\Delta G_z)(\phi_z)=(\sigma_z^*S)(\phi_z)$. □

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