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SOME RESULTS ABOUT MIT ORDER AND IMIT CLASS OF LIFE DISTRIBUTIONS

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We investigate some new properties of mean inactivity time (MIT) order and increasing MIT (IMIT) class of life distributions. The preservation property of MIT order under increasing and concave transformations, reversed preservation properties of MIT order, and IMIT class of life distributions under the taking of maximum are developed. Based on the residual life at a random time and the excess lifetime in a renewal process, stochastic comparisons of both IMIT and decreasing mean residual life distributions are conducted as well.

1. INTRODUCTION

Assume X and Y are two random variables with distribution functions F and G, respectively, and denote by $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$ their respective reliability functions. Let $X_t = X - t | X > t$ be the residual life of X at time $t \geq 0$ and $X_{(t)} = t - X | X \leq t$ be the inactivity time (IT) at time $t > 0$; their corresponding reliability functions can be represented as

$$P(X_t > x) = P(X - t > x | X > t) = \frac{\bar{F}(x+t)}{\bar{F}(t)}, \qquad x, t \geq 0,$$

and

$$P(X_{(t)} > x) = P(t - X > x | X \leq t) = \frac{F(t-x)}{F(t)}, \qquad 0 \leq x < t.$$

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The mean inactivity time (MIT) is thus

$$m_X(t) = \mathrm{E}X_{(t)} = \mathrm{E}(t - X | X \le t) = \frac{\int_0^t F(x)\,dx}{F(t)}, \qquad t > 0.$$

X or its distribution F is said to be of *increasing mean inactivity time* (IMIT) if $m_X(t)$ is nondecreasing in $t > 0$.

Having played important roles in survival analysis, reliability theory, maintenance polices, and many other areas of applied probability, both IT and IMIT life distribution received much attention during this decade (see, e.g., Ruiz and Navarro [19], Block, Savits, and Singh [4], Li and Lu [14], Li and Zuo [15], and Ahmad, Kayid, and Pellerey [1]).

In the theory of reliability, X is often regarded as the total life of a component; it is of interest to study $X_Y = X - Y | X > Y$, the residual life of X with a random age Y (RLRT) (Stoyan [22]), and $X_{(Y)} = Y - X | X \le Y$, the inactivity time of X at a random time Y (ITRT). The distribution function of X_Y is

$$P(X_Y \le x) = P(X - Y \le x | X > Y)$$
$$= \frac{\int_0^\infty [F(y + x) - F(y)]\,dG(y)}{\int_0^\infty G(y)\,dF(y)} \quad \text{for any } x \ge 0. \qquad \textbf{(1)}$$

The RLRT represents the actual working time of the standby unit if X is regarded as the total random life of a warm standby unit with its age Y, and the idle time of the server in a $GI/G/1$ queuing system can also be expressed as a RLRT (see Marshall [16]). For more research conclusions about stochastic comparisons of RLRT and ITRT, refer to Yue and Cao [23] and Li and Zuo [15].

In the current investigation, we further focus on MIT order and IMIT class of life distributions. Section 2 builds some new properties of MIT order and IMIT class of life distributions. Section 3 investigates the preservation property of MIT order under increasing and concave transformations, reversed preservation properties of MIT order, and IMIT class of life distributions under the taking of maximum. Section 4 develops stochastic comparison of a residual life at a random time with certain aging properties. Finally, based on the excess lifetime in a renewal process, the stochastic comparison of IMIT life distribution is conducted in Section 5 as well.

Throughout this article, the term *increasing* is used instead of monotone nondecreasing and the term *decreasing* is used instead of monotone nonincreasing.

We assume that the random variables under consideration have zero as the common left end point of their supports, and the expectation is assumed to be finite when used.

2. DEFINITIONS AND BASIC PROPERTIES

In this section, we first recall some definitions that will be used in the sequel; then we discuss some basic properties about IMIT class and MIT order.

DEFINITION 2.1:

- *(a) X is said to be smaller than Y in the hazard rate order (denoted by $X \le_{\mathrm{hr}} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in x for which the ratio is well defined.*
- *(b) X is said to be smaller than Y in the mean residual life order (denoted by $X \le_{\mathrm{mrl}} Y$) if $\mathrm{E}X_t \le \mathrm{E}Y_t$ for all $t \ge 0$ for which the expectations exist.*
- *(c) X is said to be smaller than Y in the mean inactivity time order (denoted by $X \le_{\mathrm{MIT}} Y$) if $\mathrm{E}X_{(t)} \ge \mathrm{E}Y_{(t)}$ for all $t > 0$ for which the expectations exist.*
- *(d) X is said to be smaller than Y in the increasing convex order (denoted by $X \le_{\mathrm{icx}} Y$) if $\mathrm{E}h(X) \le \mathrm{E}h(Y)$ for all increasing and convex h.*

For a more comprehensive discussion of the properties as well as other details of those stochastic orderings, readers is referred to Shaked and Shanthikumar [20], Müller and Stoyan [17], Ahmad et al. [1], and Kayid and Ahmad [9].

The following aging properties are closely related to our main theme. For further details on them, refer to Barlow and Proschan [2], Müller and Stoyan [17], and Nanda, Singh, Misra, and Paul [18].

DEFINITION 2.2:

- *(a) X is of increasing failure rate (IFR) if X_t is stochastically decreasing in $t \ge 0$.*
- *(b) X is of decreasing mean residual life (DMRL) if $\mathrm{E}X_t$ is decreasing in $t \ge 0$.*
- *(c) X is of increasing mean inactivity time (IMIT) if $m_X(t)$ is increasing in $t > 0$.*

Now let us turn to the following basic properties.

PROPOSITION 2.3: *X is IMIT if and only if $X \le_{\mathrm{MIT}} X + Y$ for any Y independent of X.*

PROOF: *Necessity*: If $m_X(t)$ increases in $t > 0$, then, by Fubini's theorem, we have, for any $t > 0$,

$$m_{X+Y}(t) = \frac{\int_0^t \int_0^x F(x-u)\, dG(u)\, dx}{\int_0^t F(t-u)\, dG(u)}$$

$$= \frac{\int_0^t \int_u^t F(x-u)\, dx\, dG(u)}{\int_0^t F(t-u)\, dG(u)}$$

$$= \frac{1}{\int_0^t F(t-u)\, dG(u)} \int_0^t \int_0^{t-u} F(x)\, dx\, dG(u)$$

$$= \frac{1}{\int_0^t F(t-u)\, dG(u)} \int_0^t F(t-u) m_X(t-u)\, dG(u)$$

$$\le \frac{1}{\int_0^t F(t-u)\, dG(u)} \int_0^t F(t-u) m_X(t)\, dG(u)$$

$$= m_X(t).$$

Thus, $X \le_{\mathrm{MIT}} X + Y$.

Sufficiency: In view of the fact that

$$m_{X+s}(t) = m_X(t-s) \quad \text{for any } t > s \ge 0,$$

the desired result follows immediately by putting $Y = s \ge 0$. ■

According to Theorem 1.D.8 of Shaked and Shanthikumar [20], X is of decreasing mean residual life (DMRL) if and only if $X_t \le_{\mathrm{mrl}} X$ for any $t \ge 0$. One may wonder whether IMIT property of X is also equivalent to $X_t \le_{\mathrm{MIT}} X$ for any $t \ge 0$. Proposition 2.4 shows that $X_t \le_{\mathrm{MIT}} X$ for any $t \ge 0$ implies that X is of IMIT; However, Example 2.5 states that the inverse is not valid.

Proposition 2.4: *If $X_t \le_{\mathrm{MIT}} X$ for any $t \ge 0$, then X is of IMIT.*

Proof: For any $t \ge 0$ and $s > x \ge 0$,

$$P((X_t)_{(s)} > x) = \frac{P(X_t < s-x)}{P(X_t < s)} = \frac{F(s+t-x) - F(t)}{F(s+t) - F(t)}.$$

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For any $t \geq 0$ and $s > 0$, $X_t \leq_{\mathrm{MIT}} X$ is equivalent to

$$\int_0^s \frac{F(t+x) - F(t)}{F(t+s) - F(t)}\, dx \geq \int_0^s \frac{F(x)}{F(s)}\, dx. \tag{2}$$

Then for any $t \geq 0$ and $s > 0$,

$$\frac{\int_0^s F(t+x)\, dx}{\int_0^s F(x)\, dx} - \frac{F(t+s)}{F(s)} \geq \frac{sF(t)}{\int_0^s F(x)\, dx} - \frac{F(t)}{F(s)} \geq 0.$$

Also,

$$\frac{\int_0^s F(t+x)\, dx}{\int_0^s F(x)\, dx} \geq \frac{F(t+s)}{F(s)}.$$

Equivalently, for any $t \geq 0$ and $s > 0$,

$$\frac{\int_0^{t+s} F(x)\, dx}{F(t+s)} \geq \frac{\int_0^s F(t+x)\, dx}{F(t+s)} \geq \frac{\int_0^s F(x)\, dx}{F(s)},$$

which asserts that $m_X(s)$ is increasing in $s > 0$; that is, X is of IMIT. ■

Example 2.5: For a random life X with distribution function

$$F(x) = \begin{cases} x^{1/2} & 0 \leq x \leq 1 \\ 1, & x > 1, \end{cases}$$

$$m_X(s) = \frac{\int_0^s F(x)\, dx}{F(s)} = \begin{cases} \dfrac{2s}{3} & 0 \leq s \leq 1 \\ s - \dfrac{1}{3} & s > 1, \end{cases}$$

Thus, X is of IMIT.

Put $s = t = \frac{1}{2}$; it can be easily found that

$$m_X(s) = \frac{\int_0^s F(x)\, dx}{F(s)} = \frac{1}{3},$$

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$$m_{X_t}(s) = \frac{\int_0^s [F(t+x) - F(t)]\,dx}{F(t+s) - F(t)} = \frac{8 - 5\sqrt{2}}{6(2-\sqrt{2})}.$$

The inequality $\frac{1}{3} > (8 - 5\sqrt{2})/(6(2-\sqrt{2}))$ tells that $X_t \leq_{\rm MIT} X$ is not true.

3. PRESERVATION PROPERTIES

This section will develop some preservation properties as well as reversed preservation properties of MIT order and IMIT class under both monotonic transformations and the taking of maximum, respectively.

THEOREM 3.1: *Assume that ϕ is strictly increasing and concave; $\phi(0) = 0$. If $X \leq_{\rm MIT} Y$, then $\phi(X) \leq_{\rm MIT} \phi(Y)$.*

PROOF: Without loss of generality, assume that ϕ is differentiable. $X \leq_{\rm MIT} Y$ implies that for any $t > 0$,

$$\int_0^{\phi^{-1}(t)} \left[\frac{F(x)}{F(\phi^{-1}(t))} - \frac{G(x)}{G(\phi^{-1}(t))}\right] dx \geq 0.$$

Since $\phi'(t)$ is nonnegative and decreasing, by Lemma 7.1(b) of Barlow and Proschan [2] it holds that

$$\int_0^{\phi^{-1}(t)} \phi'(x)\left[\frac{F(x)}{F(\phi^{-1}(t))} - \frac{G(x)}{G(\phi^{-1}(t))}\right] dx \geq 0 \quad \text{for any } t > 0.$$

Equivalently,

$$\frac{\int_0^{\phi^{-1}(t)} F(x)\phi'(x)\,dx}{F(\phi^{-1}(t))} \geq \frac{\int_0^{\phi^{-1}(t)} G(x)\phi'(x)\,dx}{G(\phi^{-1}(t))};$$

that is, for any $t > 0$,

$$\frac{\int_0^{t} F(\phi^{-1}(x))\,dx}{F(\phi^{-1}(t))} \geq \frac{\int_0^{t} G(\phi^{-1}(x))\,dx}{G(\phi^{-1}(t))}.$$

Note that

$$P(\phi(X) \leq x) = F(\phi^{-1}(x)), \quad x \geq 0;$$

we have for any $t > 0$ that

$$\frac{\int_0^t P(\phi(X) \le x)\,dx}{P(\phi(X) \le t)} \ge \frac{\int_0^t P(\phi(Y) \le x)\,dx}{P(\phi(Y) \le t)},$$

which tells that $\phi(X) \le_{\mathrm{MIT}} \phi(Y)$. ■

THEOREM 3.2: *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be independent and identically distributed (i.i.d.) copies of X and Y, respectively. If* $\max\{X_1, \ldots, X_n\} \le_{\mathrm{MIT}} \max\{Y_1, \ldots, Y_n\}$, *then $X \le_{\mathrm{MIT}} Y$.*

PROOF: $\max\{X_1, \ldots, X_n\} \le_{\mathrm{MIT}} \max\{Y_1, \ldots, Y_n\}$ implies that

$$\int_0^t \frac{F^n(x)}{F^n(t)}\,dx \ge \int_0^t \frac{G^n(x)}{G^n(t)}\,dx \quad \text{for any } t > 0;$$

that is,

$$\int_0^t [G^n(t)F^n(x) - F^n(t)G^n(x)]\,dx \ge 0.$$

Since, for any $t > 0$,

$$h(x) = \left[\sum_{i=1}^{n} [G^{n-i}(t)F^{n-i}(x)][F^{i-1}(t)G^{i-1}(x)]\right]^{-1}$$

is nonnegative and decreasing in $x \ge 0$, by Lemma 7.1(b) of Barlow and Proschan [2] we have

$$\int_0^t [G(t)F(x) - F(t)G(x)]\,dx = \int_0^t [G^n(t)F^n(x) - F^n(t)G^n(x)]h(x)\,dx \ge 0,$$

which states that $X \le_{\mathrm{MIT}} Y$. ■

Remark: Theorem 3.2 in fact also holds for general variables. Since $X \le_{\mathrm{MIT}} Y$ if and only if $-X \ge_{\mathrm{mrl}} -Y$ of Ahmad et al. [1] and

$$\max\{-X_1, \ldots, -X_n\} = -\min\{X_1, \ldots, X_n\},$$

by Theorem 3.2, we immediately reach the following corollary.

COROLLARY 3.3: *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be i.i.d. copies of X and Y, respectively. If* $\min\{X_1, \ldots, X_n\} \ge_{\mathrm{mrl}} \min\{Y_1, \ldots, Y_n\}$, *then $X \ge_{\mathrm{mrl}} Y$.*

Based on Corollary 3.3 and Lemma 3.4, Theorem 3.5 presents the reversed preservation property of IMIT under the taking of maximum.

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Lemma 3.4: *X is of IMIT if and only if* $X_{(s)} \leq_{\mathrm{mrl}} X_{(t)}$ *for all* $0 < s \leq t$.

Proof: It can be easily established; hence, the proof is omitted. ■

Theorem 3.5: *Let* $X, X_1, \ldots, X_n$ *be i.i.d. random variables. If* $\max\{X_1, \ldots, X_n\}$ *is of IMIT, then X is also of IMIT.*

Proof: For any $s > x \geq 0$,

$$P((\max\{X_1, \ldots, X_n\})_{(s)} > x) = \frac{F^n(s-x)}{F^n(s)} = P(\min\{(X_1)_{(s)}, \ldots, (X_n)_{(s)}\} > x);$$

that is, for all $s \geq 0$,

$$(\max\{X_1, \ldots, X_n\})_{(s)} \stackrel{\mathrm{st}}{=} \min\{(X_1)_{(s)}, \ldots, (X_n)_{(s)}\}.$$

$\max\{X_1, \ldots, X_n\}$ is of IMIT and by Lemma 3.4, it holds that

$$(\max\{X_1, \ldots, X_n\})_{(s)} \leq_{\mathrm{mrl}} (\max\{X_1, \ldots, X_n\})_{(t)}, \quad \forall\, 0 < s \leq t.$$

Thus,

$$\min\{(X_1)_{(s)}, \ldots, (X_n)_{(s)}\} \leq_{\mathrm{mrl}} \min\{(X_1)_{(t)}, \ldots, (X_n)_{(t)}\}, \quad \forall\, 0 < s \leq t.$$

Now it follows from Corollary 3.3 that

$$X_{(s)} \leq_{\mathrm{mrl}} X_{(t)} \quad \text{for any } 0 < s \leq t.$$

By Lemma 3.4 again, X is of IMIT. ■

4. MEAN RESIDUAL LIFE AT A RANDOM TIME

Stochastic comparison under certain conditions on the concerned total life and random time as well as preservation properties of decreasing reversed hazard rate (DRHR) and IMIT have been conducted by Yue and Cao [23] and Li and Zuo [15]. This section reports some results from further comparisons of RLRT.

Yue and Cao [24, Thm. 3(a)] proved that X is $\mathrm{NBU}_{\mathrm{L_t}}$ (new better than used in Laplace transform order) if and only if X_Y is smaller than X in Laplace transform order for all Y independent of X. Recently, Li [11, Thm. 4.1(i)] showed that X is $\mathrm{NBU}_{\mathrm{Mg}}$ (new better than used in moment generating function order) if and only if X_Y is smaller than X in the moment generating function order for all Y independent of X. The following theorem gives a parallel result about the MIT order.

Theorem 4.1: $X_Y \leq_{\mathrm{MIT}} X$ *for any Y that is independent of X if and only if* $X_t \leq_{\mathrm{MIT}} X$ *for all* $t \geq 0$.

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PROOF: *Sufficiency*: Since $X_t \leq_{\text{MIT}} X$ for all $t \geq 0$, it follows from (2) that

$$\frac{\int_0^\infty \int_0^s [F(t+x) - F(t)]\, dx\, dG(t)}{\int_0^\infty [F(t+s) - F(t)]\, dG(t)} \geq \frac{\int_0^\infty \left[\frac{F(t+s) - F(t)}{F(s)} \int_0^s F(x)\, dx \right] dG(t)}{\int_0^\infty [F(t+s) - F(t)]\, dG(t)} = \frac{\int_0^s F(x)\, dx}{F(s)} \quad \text{for any } s > 0.$$

In view of (1), this shows $X_Y \leq_{\text{MIT}} X$.

Necessity: Suppose $X_Y \leq_{\text{MIT}} X$ holds for any nonnegative random variable Y. Then $X_t \leq_{\text{MIT}} X$ for all $t \geq 0$ follows by taking Y as a degenerate variable. ∎

Recall that for two subsets Θ and Γ of the real line, a nonnegative function h defined on $\Theta \times \Gamma$ is said to be *totally positive of order* 2 (TP2) on $\Theta \times \Gamma$ if

$$h(x,y)h(x',y') \geq h(x,y')h(x',y)$$

whenever $x \leq x'$ and $y \leq y'$, and $x, x' \in \Theta$ and $y, y' \in \Gamma$. The following lemma, which is due to Joag-Dev, Kochar, and Proschan [8], will be utilized to derive Theorem 4.3.

LEMMA 4.2: *Let $\psi(x,y)$ be any TP2 function (not necessarily a reliability function) in $x \in \Theta$ and $y \in \Gamma$ and $F_i(x)$ be a distribution function for each i. Denote*

$$H_i(y) = \int_\Theta \psi(x,y)\, dF_i(x).$$

If $\bar{F}_i(x)$ is TP2 in $i \in \{1,2\}$ and $x \in \Theta$ and if $\psi(x,y)$ is increasing in x for every y, then $H_i(y)$ is TP2 in $y \in \Gamma$ and $i \in \{1,2\}$.

Proof of the next result is based on the idea of Theorem 4.2 of Gao, Belzunce, Hu, and Pellerey [7].

THEOREM 4.3: *Assume that Z is independent of X and Y. If $X \leq_{\text{hr}} Y$ and Z is of IMIT, then $X_Z \leq_{\text{mrl}} Y_Z$.*

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PROOF: Denote by F_1, F_2, and G distribution functions of X, Y, and Z, respectively. Since Z is of IMIT, we have, for all $y \geq 0$ and $\Delta > 0$,

$$\frac{\int_0^{y+\Delta} G(u)\,du}{G(y+\Delta)} \geq \frac{\int_0^{y} G(u)\,du}{G(y)}$$

and

$$\left(\frac{\int_0^{y+\Delta} G(u)\,du}{\int_0^{y} G(u)\,du}\right)' = \frac{G(y+\Delta)}{\int_0^{y} G(u)\,du} - \frac{G(y)\int_0^{y+\Delta} G(u)\,du}{\left(\int_0^{y} G(u)\,du\right)^2} \leq 0;$$

that is,

$$\frac{\int_0^{y+\Delta} G(u)\,du}{\int_0^{y} G(u)\,du}$$

is decreasing in $y \geq 0$. Then

$$\frac{\int_0^{y_2-t_2} G(u)\,du}{\int_0^{y_1-t_2} G(u)\,du} \geq \frac{\int_0^{y_2-t_1} G(u)\,du}{\int_0^{y_1-t_1} G(u)\,du}$$

for all $0 < t_1 \leq t_2 < y_1 \leq y_2$.

Denote

$$\psi(y,t) = \begin{cases} \int_0^{y-t} G(u)\,du, & y \geq t \\ 0, & y < t. \end{cases}$$

The last inequality states that

$$\psi(y_1,t_1)\psi(y_2,t_2) \geq \psi(y_1,t_2)\psi(y_2,t_1) \tag{3}$$

holds for all $(t_1,t_2,y_1,y_2) \in S = \{(t_1,t_2,y_1,y_2) : 0 < t_1 \leq t_2 < y_1 \leq y_2\}$.

It can be easily verified that (3) is also valid for those $(t_1, t_2, y_1, y_2) \in \{(t_1, t_2, y_1, y_2) : 0 < t_1 \le t_2,\ 0 < y_1 \le y_2\} - S$. Thus, $\psi(y,t)$ is TP2 in $(y,t) \in (0,\infty) \times (0,\infty)$.

For $i = 1,2$, let

$$H_i(t) = \frac{\int_0^\infty \psi(y,t)\, dF_i(y)}{\int_0^\infty G(y)\, dF_i(y)}.$$

$X \le_{\mathrm{hr}} Y$ implies that $\bar{F}_i(x)$ is TP2 in $(i,x) \in \{1,2\} \times (0,\infty)$ and $\psi(y,t)$ is increasing in y for each fixed t. From Lemma 4.2 it follows that $H_i(t)$ is TP2 in $(i,t) \in \{1,2\} \times (0,\infty)$; that is,

$$\begin{aligned}
\frac{H_2(t)}{H_1(t)} &= \frac{\int_0^\infty \psi(y,t)\, dF_2(y)}{\int_0^\infty \psi(y,t)\, dF_1(y)} \times \frac{\int_0^\infty G(y)\, dF_1(y)}{\int_0^\infty G(y)\, dF_2(y)} \\
&= \frac{\int_t^\infty \int_t^y G(y-u)\, du\, dF_2(y)}{\int_t^\infty \int_t^y G(y-u)\, du\, dF_1(y)} \times \frac{\int_0^\infty G(y)\, dF_1(y)}{\int_0^\infty G(y)\, dF_2(y)} \\
&= \frac{\int_t^\infty \bar{F}_{Y_Z}(u)\, du}{\int_t^\infty \bar{F}_{X_Z}(u)\, du}
\end{aligned}$$

is increasing in $t \ge 0$, which is equivalent to

$$\frac{\int_t^\infty \bar{F}_{X_Z}(u)\, du}{\bar{F}_{X_Z}(t)} \le \frac{\int_t^\infty \bar{F}_{Y_Z}(u)\, du}{\bar{F}_{Y_Z}(t)} \quad \text{for any } t \ge 0.$$

Hence, $X_Z \le_{\mathrm{mrl}} Y_Z$. ■

Remark: Assume X, Y, and Z are three independent nonnegative random variables. Li and Zuo [15, Thm. 5] showed that if $X \le_{\mathrm{hr}} Y$ and Z is of IMIT, then $Z_{(X)} \le_{\mathrm{icx}} Z_{(Y)}$. In view of the fact that $X_Z = Z_{(X)}$, it is also valid, under the condition of Theorem 4.3, that $Z_{(X)} \le_{\mathrm{mrl}} Z_{(Y)}$. According to Theorem 3.A.13 of Shaked and Shanthikumar [20], $X \le_{\mathrm{mrl}} Y$ implies $X \le_{\mathrm{icx}} Y$ and we in fact develop a stronger conclusion here.

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As a direct application of Theorem 4.3, we can get a simple and brief proof of Theorem 6 about DMRL in Li and Zuo [15].

Corollary 4.4: *Assume X and Z are independent. If X is of IFR and Z is of IMIT, then X_Z is of DMRL.*

Proof: According to Theorem 1.B.19 of Shaked and Shanthikumar [20], X is of IFR if and only if $X_t \le_{\rm hr} X$ for all $t \ge 0$. By Theorem 4.3, $(X_t)_Z \le_{\rm mrl} X_Z$ for all $t \ge 0$. Note that $(X_Z)_t \stackrel{\rm st}{=} (X_t)_Z$ for all $t \ge 0$; it holds that $(X_Z)_t \le_{\rm mrl} X_Z$ for all $t \ge 0$. Now, from Theorem 1.D.8 of Shaked and Shanthikumar [20], it follows immediately that X_Z is of DMRL. ■

5. STOCHASTIC COMPARISON OF RENEWAL EXCESS LIFETIME

Consider a renewal process with i.i.d. interarrival times X_i with common distribution F. For $k = 1,2,\ldots$, let $S_k = \sum_{i=1}^{k} X_i$ be the time of the kth arrival and $S_0 = 0$. The renewal counting process $N(t) = \sup\{n : S_n \le t\}$ gives the number of arrivals until time $t \ge 0$, and the excess lifetime $\gamma(t) = S_{N(t)+1} - t$ at time $t \ge 0$ is the time elapsed from the time t to the first arrival after time t. Of course, it is obvious that $\gamma(0) \stackrel{\rm st}{=} X_1$. $M(t) = \mathrm{E}N(t)$ is called as renewal function, which satisfies the well-known fundamental renewal equation

$$M(t) = F(t) + \int_0^t F(t-y)\,dM(y), \qquad t \ge 0. \tag{4}$$

Brown [5] is among the first to survey the stochastic monotonicity of the excess lifetime; afterward, Shaked and Zhu [21] had some further discussions on this line of research. Subsequently, many authors devoted themselves to investigate the behavior of the renewal excess lifetime of a renewal process with interarrivals having certain aging properties, such as NBU, NBUC, NBU(2), NBUE, NBU_{Lt} and NBU_{Mg}, and so forth. Their results in fact assert that various NBU interarrivals can reduce to the corresponding NBU properties of the excess lifetime. For more details, readers are referred to Li [11], Li and Kochar [12], Belzunce, Ortega, and Ruiz [3], Li, Li, and Jing [13], Chen [6], and Barlow and Proschan [2]. In this section, we will investigate the behavior of the excess lifetime of a renewal process with IMIT interarrivals.

Theorem 5.1: *If $\gamma(t)$ is decreasing in the MIT order in $t \ge 0$, then X is of IMIT.*

Proof: According to Karlin and Taylor [10, p. 193], for any $t \ge 0$ and $x \ge 0$,

$$P(\gamma(t) > x) = \bar{F}(t+x) + \int_0^t P(\gamma(t-y) > x)\,dF(y).$$

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Then

$$P(\gamma(t) \le x) = F(t+x) - F(t) + \int_0^t P(\gamma(t-y) \le x)\, dF(y), \tag{5}$$

for any $t \ge 0$ and $x \ge 0$.

Since $\gamma(t)$ is decreasing in t in the sense of MIT order, we have, for all $s > 0$ and $t \ge 0$,

$$\begin{aligned}
&\int_0^s P(\gamma(t) \le x)\, dx \\
&\quad = \int_0^s \left[F(t+x) - F(t) + \int_0^t P(\gamma(t-y) \le x)\, dF(y) \right] dx \\
&\quad = \int_0^s [F(t+x) - F(t)]\, dx + \int_0^t \int_0^s P(\gamma(t-y) \le x)\, dx\, dF(y) \\
&\quad = \int_0^s [F(t+x) - F(t)]\, dx + \int_0^t m_{\gamma(t-y)}(s) P(\gamma(t-y) \le s)\, dF(y) \\
&\quad \le \int_0^s [F(t+x) - F(t)]\, dx + \int_0^t m_{\gamma(t)}(s) P(\gamma(t-y) \le s)\, dF(y) \\
&\quad = \int_0^s [F(t+x) - F(t)]\, dx + m_{\gamma(t)}(s)[P(\gamma(t) \le s) - F(t+s) + F(t)],
\end{aligned}$$

where the last equality is due to (5).

Note that

$$m_{\gamma(t)}(s) P(\gamma(t) \le s) = \int_0^s P(\gamma(t) \le x)\, dx;$$

it holds that

$$m_{\gamma(t)}(s)[F(t+s) - F(t)] \le \int_0^s [F(t+x) - F(t)]\, dx.$$

On the other hand,

$$\gamma(t) \le_{\mathrm{MIT}} \gamma(0) \stackrel{\mathrm{st}}{=} X \quad \text{for all } t \ge 0;$$

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it holds that for all $t \geq 0$ and $s > 0$,

$$\frac{F(t+s) - F(t)}{F(s)} \int_0^s F(x)\,dx \leq \int_0^s [F(t+x) - F(t)]\,dx,$$

which is equivalent to (2); that is, $X_t \leq_{\mathrm{MIT}} X$ for all $t \geq 0$. From Proposition 2.4, the desired result follows immediately. ■

THEOREM 5.2: *If $X_t \leq_{\mathrm{MIT}} X$ for all $t \geq 0$, then $\gamma(t) \leq_{\mathrm{MIT}} \gamma(0)$ for all $t \geq 0$.*

PROOF: According to Barlow and Proschan [2], it holds that

$$P(\gamma(t) > x) = \bar{F}(t+x) + \int_0^t \bar{F}(t+x-u)\,dM(u) \quad \text{for all } t \geq 0 \text{ and } x \geq 0.$$

Thus,

$$P(\gamma(t) \leq x) = F(t+x) + \int_0^t F(t+x-u)\,dM(u) - M(t) \quad \text{for all } t \geq 0 \text{ and } x \geq 0.$$

$X_t \leq_{\mathrm{MIT}} X$, by (2), for any $t \geq 0$ and $s > 0$,

$$\int_0^s [F(t+x) - F(t)]\,dx \geq [F(t+s) - F(t)] \frac{\int_0^s F(x)\,dx}{F(s)}.$$

Now based on (4) and inequality (6), we have, for any $t \geq 0$ and $s > 0$,

$$\begin{aligned}
&\int_0^s P(\gamma(t) \leq x)\,dx \\
&\quad = \int_0^s \left[F(t+x) + \int_0^t F(t+x-u)\,dM(u) - F(t) - \int_0^t F(t-u)\,dM(u) \right] dx \\
&\quad = \int_0^s [F(t+x) - F(t)]\,dx + \int_0^s \int_0^t [F(t-u+x) - F(t-u)]\,dM(u)\,dx \\
&\quad = \int_0^s [F(t+x) - F(t)]\,dx + \int_0^t \int_0^s [F(t-u+x) - F(t-u)]\,dx\,dM(u) \\
&\quad \geq \int_0^s [F(t+x) - F(t)]\,dx \\
&\qquad + \int_0^t \left[\frac{F(t-u+s) - F(t-u)}{F(s)} \int_0^s F(x)\,dx \right] dM(u)
\end{aligned}$$

$$= \int_0^s [F(t+x) - F(t)]\,dx + \frac{\int_0^s F(x)\,dx}{F(s)} \int_0^t [F(t-u+s) - F(t-u)]\,dM(u)$$

$$= \int_0^s [F(t+x) - F(t)]\,dx + \frac{\int_0^s F(x)\,dx}{F(s)} [P(\gamma(t) \le s) - F(t+s) + F(t)]$$

$$\ge [F(t+s) - F(t)] \frac{\int_0^s F(x)\,dx}{F(s)}$$

$$+ [P(\gamma(t) \le s) - F(t+s) + F(t)] \frac{\int_0^s F(x)\,dx}{F(s)}$$

$$= \frac{\int_0^s F(x)\,dx}{F(s)} P(\gamma(t) \le s).$$

Hence, it holds that for all $t \ge 0$ and $s > 0$,

$$\frac{\int_0^s P(\gamma(t) \le x)\,dx}{P(\gamma(t) \le s)} \ge \frac{\int_0^s F(x)\,dx}{F(s)}.$$

That is, $\gamma(t) \le_{\text{MIT}} \gamma(0)$, for all $t \ge 0$. ■

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