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A general method for the computation of the canonical form of three-systems of infinitesimal screws

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SUMMARY
A general method for the computation of the canonical form of three-systems of infinitesimal screws is presented. The method is particularly simple when the three-system has a basis that is simultaneously perpendicular and reciprocal. However, it can also handle the special or degenerate cases. The method and the concurrent results are important from the theoretical point of view because of the obvious connection with the classification of screw-systems. Moreover, the results are also important for applied kinematics after recent applications of the canonical form of screw-systems to the analysis and synthesis of manipulators and manipulator substructures.

KEYWORDS: Three-systems; Canonical forms; Orthogonal spaces; Lie algebras.

1. INTRODUCTION
Since its origins, more than a century ago, screw theory has been an important tool in applied mathematics and kinematics. Infinitesimal screws provide a simple and insightful representation of the velocity state of rigid bodies subjected to one- and multi-dimensional motions. Furthermore, once it was shown that screw algebra is isomorphic to the Lie algebra, $e(3)$, of the Euclidean group, $E(3)$, a wealth of powerful and newly obtained results from modern differential geometry and Lie group theory met a host of results obtained after several decades of dedicated work of many well respected geometers.

In this present work, the point of view and nomenclature of the Lie algebra, $e(3)$, of the Euclidean group, $E(3)$, will be adopted. The main reason behind this decision is that a previous work by the authors,[1] written using that point of view, provides most of the necessary groundwork for the present contribution.

The computation of the canonical form of screw systems is a long standing problem. In his treatise,[2] Ball completed an exhaustive examination of the two-system via the cylindroid. Further, this analysis led to the canonical form of two-systems.

The canonical form of three-systems was studied by Hunt,[3] who, in his treatise originally published in 1978, described a method for the computation of the centre of a general three-system. Nayak[4] and Nayak and Roth[5] described a method for the computation of the canonical form of a general three-system. The method is based in the computation of the centre of the three-system, and then, the determination of the perpendicular directions. Nayak and Roth's method was employed by Stanišić and Pennock[6] in their analysis of manipulator substructures.

It is important to note that all these methods for the computation of the canonical form of three-systems dealt only with the general case, and they cannot be extrapolated to special or degenerate cases.

The computation of the canonical form of screw-systems is, at the same time, an application of a theoretical problem in mathematics, namely the simultaneous diagonalization of a pair of symmetric bilinear forms. Wonenburger[7] obtained necessary and sufficient conditions for a pair of symmetric bilinear forms to have a doubly orthogonal basis.* More recently, Becker[8] extended Wonenburger's results to complex orthogonal spaces.

In this paper, a general method for the computation of the canonical form of three-systems is developed. The method is applicable not only to the general case, but it can be applied to all the degenerate or special cases. The method is based on the very same properties of the matrices that represent the Killing and Klein forms with respect to a given basis.

It should be evident that the results obtained here have an obvious connection with the classification of screw systems.[1,9–12] In fact, the results obtained in this paper show that the classification of three-systems developed by the authors[12] is truly exhaustive. Furthermore, the techniques developed by Stanišić and Pennock opened an interesting application of the computation of the canonical form of screw-systems to the analysis and synthesis of manipulators.[6,13,14]

2. NOMENCLATURE AND PRELIMINARY RESULTS
In this section, the nomenclature and a handful of notions about orthogonal spaces and screw systems will be presented, none of this material is original. However, it is presented here to provide a proper foundation for the original results developed later.

Definition 1 (Lie algebra $e(3)$). Let $e(3)$ be the set of elements, called screws, of the form $\$ = (\mathbf{w}; \mathbf{v}_0)$, where $\mathbf{w}$

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* The term "basis", in the linear algebra context, means a linearly independent set of vectors that generates the space.

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is the angular velocity of a rigid body, and $\mathbf{v}_0$ the velocity of the point, fixed in the rigid body, that is instantaneously coincident with an arbitrarily point O, fixed in the reference frame; together with the following operations

$$\forall \$_1 = (\mathbf{w}_1; \mathbf{v}_{01}), \quad \$_2 = (\mathbf{w}_2; \mathbf{v}_{02}) \in e(3), \quad \text{and} \quad \forall \lambda \in \mathbb{R}$$

(i) **Addition**

$$\$_1 + \$_2 = (\mathbf{w}_1; \mathbf{v}_{01}) + (\mathbf{w}_2; \mathbf{v}_{02}) = (\mathbf{w}_1 + \mathbf{w}_2; \mathbf{v}_{01} + \mathbf{v}_{02}), \quad (1a)$$

(ii) **Scalar Multiplication**

$$\lambda \$_1 = \lambda(\mathbf{w}_1; \mathbf{v}_{01}) = (\lambda \mathbf{w}_1; \lambda \mathbf{v}_{01}), \quad (1b)$$

(iii) **Lie Product**

$$[\$_1 \quad \$_2] = [(\mathbf{w}_1; \mathbf{v}_{01})(\mathbf{w}_2; \mathbf{v}_{02})] = (\mathbf{w}_1 \times \mathbf{w}_2; \mathbf{w}_1 \times \mathbf{v}_{02} - \mathbf{w}_2 \times \mathbf{v}_{01}), \quad (1c)$$

where + and × and the juxtaposition stand for the usual vector addition, vector product (or cross product), and scalar multiplication of three-dimensional vector algebra, respectively. Then, the set together with the operations form a Lie algebra.

Further, the Lie algebra $e(3)$ is endowed with a pair of symmetrical bilinear forms or inner products, that provides $e(3)$ with a twofold orthogonal space structure.

Definition 2 (Killing and Klein forms). Let $e(3)$ be the Lie algebra as indicated by definition 1. Then, there are two symmetrical bilinear forms in $e(3)$, the Killing form

$$\text{Ki}: e(3) \times e(3) \to \mathbb{R} \quad \text{Ki}(\$_1, \$_2) = \text{Ki}((\mathbf{w}_1; \mathbf{v}_{01}), (\mathbf{w}_2; \mathbf{v}_{02})) = \mathbf{w}_1 \cdot \mathbf{w}_2, \quad (2)$$

and the Klein form

$$\text{Kl}: e(3) \times e(3) \to \mathbb{R} \quad \text{Kl}(\$_1, \$_2) = \text{Kl}((\mathbf{w}_1; \mathbf{v}_{01}), (\mathbf{w}_2; \mathbf{v}_{02})) = \mathbf{w}_1 \cdot \mathbf{v}_{02} + \mathbf{w}_2 \cdot \mathbf{v}_{01} \quad (3)$$

where · stands for the usual three-dimensional symmetrical bilinear form or dot product. Hence, $e(3)$ forms a finite-dimensional, $\dim e(3) = 6$, orthogonal space with respect to both, the Killing and Klein forms.

The Killing form is degenerate and positive semidefinite, and the Klein form is nondegenerate and indefinite. A more in depth discussion about orthogonal spaces is provided by Porteous,[15] and their significance in infinitesimal kinematics is discussed by Rico and Duffy.[1] Both forms are independent of the point O chosen to obtain the velocity $\mathbf{v}_0$, and of the coordinate system employed to represent the vectors $\mathbf{w}$ and $\mathbf{v}_0$. This pair of symmetric bilinear forms endows the Lie algebra $e(3)$ with a twofold orthogonal space structure. It is precisely the interaction of this twofold orthogonal space structure the subject of the present contribution.

It is necessary to recall a few notions about orthogonal spaces:

Definition 3 (Orthogonal and Orthonormal Bases). Let V be a orthogonal space, and $B = \{\$_1, \$_2, \ldots, \$_n\}$ be a basis of V; then B is said to be an orthogonal basis of V if

$$(\$_i, \$_j) = 0, \quad \forall i, j = 1, 2, \ldots, n \text{ with } i \neq j. \quad (4)$$

Further, an orthogonal basis is said to be orthonormal if, in addition,

$$(\$_i, \$_i) \in \{-1, 0, 1\}, \quad \forall i = 1, 2, \ldots, n, \quad (5)$$

where (,) represents the symmetric bilinear form associated with the orthogonal space.

A basis of $e(3)$ orthogonal with respect to the Killing form is called, for historical reasons, perpendicular, while a basis of $e(3)$ orthogonal with respect to the Klein form is called reciprocal. Further, it has been customary, in kinematics, to define "normalized" bases whose elements satisfy the conditions

$$\text{Ki}(\$_i, \$_i) = \text{Ki}((\mathbf{w}_i; \mathbf{v}_{0i}), (\mathbf{w}_i; \mathbf{v}_{0i})) = \mathbf{w}_i \cdot \mathbf{w}_i = 1, \quad \text{if } \mathbf{w}_i \neq \mathbf{0}, \quad (6a)$$

and

$$\mathbf{v}_{0i} \cdot \mathbf{v}_{0i} = 1, \quad \text{if } \mathbf{w}_i = \mathbf{0}. \quad (6b)$$

It should be noted that these "normalized" bases, are, in general, neither orthonormal with respect to the Killing, nor with respect to the Klein form. Further, condition (6b) involves an arbitrary choice of the unit of length, that it cannot be intrinsically defined. The following result is well known:

Proposition 4. Every finite-dimensional orthogonal space V has an orthonormal basis with respect to the symmetric bilinear form defined in the space.

Proof. A proof is given in Porteous (page 158).[15]

The following definition shows that a screw system is simply a subspace of the Lie algebra $e(3)$.

Definition 5 (Screw System). Let W be a subspace of $e(3)$, denoted $W < e(3)$, such that $\dim W = n$, with $1 \leq n \leq 6$; then W is called an n-system, or n screw system.[1]

Further, since $e(3)$ has a twofold orthogonal space structure, then any screw system $W < e(3)$ has the twofold orthogonal space structure induced by the restriction of the Killing and Klein forms upon W.[15] Therefore, any screw system W will have a basis that is orthogonal with respect to the Klein form; i.e. reciprocal, and another basis that is orthogonal with respect to the Killing form; i.e. perpendicualr. Immediately, the following questions arise:

(i) Under what conditions does W have a basis that is orthogonal with respect to both the Killing and Klein form? i.e. under what conditions does W have a basis that is both reciprocal and perpendicular?

(ii) If W satisfies these conditions, how is it possible to obtain the elements of this perpendicular and reciprocal basis?

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(iii) If W does not satisfy these conditions, what is the "simplest" basis of W; i.e. what is the canonical form of W?

3. THE METHOD AND THE RESULTS

In this section, the answers to the questions posed in Section 2 will be provided. As far as the authors are aware, the results are original. The analyses are restricted to three-systems; nevertheless, the rationale can be applied to subspaces of any other dimension.

Definition 6 (Screw Matrix and its Direction and Moment Matrices). Let $B = \{\$_1, \$_2, \$_3\}$ be any basis of a three-system W; then, the components of the screws in B can be arranged in a matrix S which can be written in block form as

$$S \equiv \begin{bmatrix} \$_1 \\ \$_2 \\ \$_3 \end{bmatrix} = [D \mid M] = \left[\begin{array}{c|c} \mathbf{w}_1 & \mathbf{v}_{01} \\ \mathbf{w}_2 & \mathbf{v}_{02} \\ \mathbf{w}_3 & \mathbf{v}_{03} \end{array}\right], \qquad (7)$$

where the vectors $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, $\mathbf{v}_{01}$, $\mathbf{v}_{02}$, $\mathbf{v}_{03}$ are written in row form. Then, S is called the screw matrix of B, the 3×3 submatrix D is called the direction matrix of B, and the 3×3 submatrix M is called the moment matrix of B.

Definition 7 (Killing and Klein matrices of a basis). Let $B = \{\$_1, \$_2, \$_3\}$ be any basis of a three-system W, the Killing and Klein matrices of B are defined respectively, by

$$\mathrm{Ki}\,(B) \equiv \begin{bmatrix} \mathrm{Ki}\,(\$_1, \$_1) & \mathrm{Ki}\,(\$_1, \$_2) & \mathrm{Ki}\,(\$_1, \$_3) \\ \mathrm{Ki}\,(\$_2, \$_1) & \mathrm{Ki}\,(\$_2, \$_2) & \mathrm{Ki}\,(\$_2, \$_3) \\ \mathrm{Ki}(\$_3, \$_1) & \mathrm{Ki}(\$_3, \$_2) & \mathrm{Ki}(\$_3, \$_3) \end{bmatrix}, \qquad (8)$$

and

$$\mathrm{Kl}\,(B) \equiv \begin{bmatrix} \mathrm{Kl}\,(\$_1, \$_1) & \mathrm{Kl}\,(\$_1, \$_2) & \mathrm{Kl}\,(\$_1, \$_3) \\ \mathrm{Kl}\,(\$_2, \$_1) & \mathrm{Kl}\,(\$_2, \$_2) & \mathrm{Kl}\,(\$_2, \$_3) \\ \mathrm{Kl}\,(\$_3, \$_1) & \mathrm{Kl}\,(\$_3, \$_2) & \mathrm{Kl}\,(\$_3, \$_3) \end{bmatrix}. \qquad (9)$$

Further, since the Killing and Klein forms are symmetrical, the Killing and Klein matrices are also symmetrical.

Proposition 8. Let $B = \{\$_1, \$_2, \$_3\}$ be any basis of a three system W, ad D and M their direction and moment matrices, then

$$\mathrm{Ki}\,(B) = DD^{\mathrm{T}}, \qquad (10)$$

and

$$\mathrm{Kl}\,(B) = MD^{\mathrm{T}} + DM^{\mathrm{T}}. \qquad (11)$$

Proof. By simple computation.

From proposition 4, in Section 2, it is known that any three-system W has a basis orthogonal with respect to the associated symmetric bilinear form. This statement has an equivalent statement, in numerical linear algebra, that indicates that any symmetric matrix can be diagonalized, via an orthogonal matrix, and its eigenvalues are real. The following proposition provides a straightforward method to obtain an orthonormal basis, with respect to the Killing form, of any three-system W.

Proposition 9. Let $B = \{\$_1, \$_2, \$_3\}$ be any basis of a three-system W. Then the basis $B' = \{\$'_1, \$'_2, \$'_3\}$ is an orthonormal basis with respect to the Killing form, provided that

$$\$'_i = (\lambda_{i1}\$_1 + \lambda_{i2}\$_2 + \lambda_{i3}\$) \\ /[\mathrm{Ki}\,(\lambda_{i1}\$_1 + \lambda_{i2}\$_2 + \lambda_{i3}\$_3, \lambda_{i1}\$_1 + \lambda_{i2}\$_2 + \lambda_{i3}\$_3)]^{1/2} \qquad (12a)$$

when $\mathrm{Ki}\,(\lambda_{i1}\$_1 + \lambda_{i2}\$_2 + \lambda_{i3}\$_3,\ \lambda_{i1}\$_1 + \lambda_{i2}\$_2 + \lambda_{i3}\$_3) \neq 0$, and

$$\$'_i = \lambda_{i1}\$_1 + \lambda_{i2}\$_2 + \lambda_{i3}\$_3, \qquad (12b)$$

when $\mathrm{Ki}\,(\lambda_{i1}\$_1 + \lambda_{i2}\$_2 + \lambda_{i3}\$_3,\ \lambda_{i1}\$_1 + \lambda_{i2}\$_2 + \lambda_{i3}\$_3) = 0$, where

$$\lambda_i = [\lambda_{i1}, \lambda_{i2}, \lambda_{i3}] \quad \text{for } i = 1, 2, 3, \qquad (13)$$

are the eigenvectors of $\mathrm{Ki}\,(B)$.

Proof. The elements of the new basis B' are written as

$$\$'_1 = \lambda_{11}\$_1 + \lambda_{12}\$_2 + \lambda_{13}\$_3$$
$$\$'_2 = \lambda_{21}\$_1 + \lambda_{22}\$_2 + \lambda_{23}\$_3$$
$$\$'_3 = \lambda_{31}\$_1 + \lambda_{32}\$_2 + \lambda_{33}\$_3$$

or, in matrix form,

$$S' = \begin{bmatrix} \$'_1 \\ \$'_2 \\ \$'_3 \end{bmatrix} = [D' \mid M'] = [LD\,, LM]$$

$$= L[D \mid M] = L \begin{bmatrix} \$_1 \\ \$_2 \\ \$_3 \end{bmatrix} = LS, \qquad (14)$$

where

$$L \equiv \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{23} & \lambda_{32} & \lambda_{33} \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}. \qquad (15)$$

Then, the Killing matrix for the new basis B' is given by

$$\mathrm{Ki}\,(B') = D'D'^{\mathrm{T}} = (\mathrm{LD})(\mathrm{LD})^{\mathrm{T}} \\ = LDD^{\mathrm{T}}L^{\mathrm{T}} = L\,\mathrm{Ki}\,(B)L^{\mathrm{T}}. \qquad (16)$$

However, it is well known that the eigenvectors of a 3×3 symmetric matrix constitute an orthogonal basis, with respect to the symmetric bilinear form, or dot product, of the usual three-dimensional space. Further, the eigenvectors can be normalized to obtain an orthonormal basis. therefore, it can be always assumed that L is an orthogonal matrix and the columns of $L^{\mathrm{T}} = L^{-1}$, are the normalized eigenvectors of $\mathrm{Ki}\,(B)$. Therefore, the Killing matrix of the new basis B' can be written as

$$\mathrm{Ki}\,(B') = L\,\mathrm{Ki}\,(B)L^{-1}.$$

Then, $\mathrm{Ki}\,(B')$ is a diagonal matrix and its diagonal

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elements are $\text{Ki}(\$_i', \$_i')$.[16] Further, since the Killing form is a positive semi-definite symmetrical bilinear form, then $\text{Ki}(\$_i', \$_i') \geq 0$, $\forall \$_i' \in e(3)$, and the final normalization can be accomplished. Therefore, the matrix $\text{Ki}(B')$ has the form

$$\text{Ki}(B') = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}, \tag{17}$$

where $\sigma_{ii} \in \{0, 1\}$, for $i = 1, 2, 3$.

Corollary 10. The number of diagonal elements, $\sigma_{ii} \neq 0$ in $\text{Ki}(B')$ is equal to the dimension of the space generated by the vectors $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ of the direction part of the screws $\{\$_1, \$_2, \$_3\}$.

Proof: Assume that the dimension of the space generated by $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ is n, where $0 \leq n \leq 3$. Then, define the following "pseudo Killing" form on $\mathbb{R}^3$

$$\begin{gathered} {}_p\text{Ki}: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \\ {}_p\text{Ki}(\mathbf{s}_i, \mathbf{s}_j) = \text{Ki}((\mathbf{s}_i; \mathbf{0}), (\mathbf{s}_j; \mathbf{0})) = \mathbf{s}_i \cdot \mathbf{s}_j, \end{gathered} \tag{18}$$

Then, $\mathbb{R}^3$ is a positive definite orthogonal space under this "pseudo Killing" form. In fact, this is the usual three dimensional space with the usual scalar, or dot product. Hence, if $\dim[\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3] = n$, there exists an orthonormal basis consisting of n elements $\mathbf{s}_i'$ for $i = 1, 2, \ldots, n$, such that

$$\mathbf{s}_i' \cdot \mathbf{s}_i' = \text{Ki}((\mathbf{s}_i'; \mathbf{0}), (\mathbf{s}_i'; \mathbf{0})) = 1.$$

It should be noted that the possibility of

$$\mathbf{s}_i' \cdot \mathbf{s}_i' = 0,$$

is excluded because then $\mathbf{s}_i' = \mathbf{0}$, and it cannot be part of a basis. Moreover, it is important to note that the dimension of the space generated by $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$, has been recognized as an invariant of any screw system.[17]

The following propositions analyze each of the four possible cases:

Proposition 11. Consider a three-system $\{\$_1, \$_2, \$_3\}$; i.e. a three-dimensional vector subspace of $e(3)$, where $\{\$_1, \$_2, \$_3\}$ is one of its bases. Assume that the dimension of the space generated by the direction vectors $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ of $\{\$_1, \$_2, \$_3\}$ is 3. Then the three-system always has a basis that is simultaneously perpendicular and reciprocal; i.e. orthogonal with respect to the Killing and Klein form. Further, the canonical form of the three-system is given by

$$\$_{c1} = (1, 0, 0; h_\alpha, 0, 0). \tag{19a}$$

$$\$_{c2} = (0, 1, 0; 0, h_\beta, 0), \tag{19b}$$

$$\$_{c3} = (0, 0, 1; 0, 0, h_\gamma). \tag{19c}$$

Proof. If the dimension of the space generated by the direction vectors $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ or $\{\$_1, \$_2, \$_3\}$ is 3, then the three-system has a basis $B' = \{\$_1', \$_2', \$_3'\}$ that is orthonormal with respect to the Killing form. Therefore

$$\$_1' = (\mathbf{s}_1'; \mathbf{s}_{01}') \quad \text{with } \mathbf{s}_1' \cdot \mathbf{s}_1' = 1,$$
$$\$_2' = (\mathbf{s}_2'; \mathbf{s}_{02}') \quad \text{with } \mathbf{s}_2' \cdot \mathbf{s}_2' = 1,$$
$$\$_3' = (\mathbf{s}_3'; \mathbf{s}_{03}') \quad \text{with } \mathbf{s}_3' \cdot \mathbf{s}_3' = 1.$$

Further

$$\text{Ki}(\$_1, \$_2) = \mathbf{s}_1' \cdot \mathbf{s}_2' = 0,$$
$$\text{Ki}(\$_1, \$_3) = \mathbf{s}_1' \cdot \mathbf{s}_3' = 0,$$
$$\text{Ki}(\$_2, \$_3) = \mathbf{s}_2' \cdot \mathbf{s}_3' = 0.$$

Hence, $\text{Ki}(B') = I_3$.

Consider now a new basis $B'' = \{\$_1'', \$_2'', \$_3''\}$, whose elements are given by

$$\$_1'' = \gamma_{11}\$_1' + \gamma_{12}\$_2' + \gamma_{13}\$_3', \tag{20a}$$

$$\$_2'' = \gamma_{21}\$_1' + \gamma_{22}\$_2' + \gamma_{23}\$_3', \tag{20b}$$

$$\$_3'' = \gamma_{31}\$_1' + \gamma_{32}\$_2' + \gamma_{33}\$_3'. \tag{20c}$$

Or in matrix form

$$\begin{aligned} S'' = \begin{bmatrix} \$_1'' \\ \$_2'' \\ \$_3'' \end{bmatrix} &= [D'' \mid M''] = [GD' \mid GM'] \\ &= G[D' \mid M'] = G \begin{bmatrix} \$_1' \\ \$_2' \\ \$_3' \end{bmatrix} = GS', \end{aligned} \tag{21}$$

where

$$G = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}. \tag{22}$$

Then, the Killing and Klein matrix of the basis B'' are given by

$$\begin{aligned} \text{Ki}(B'') &= D''D''^{\mathrm{T}} = (GD')(GD')^{\mathrm{T}} = G(D'D'^{\mathrm{T}})G^{\mathrm{T}} \\ &= GK_i(B')G^{\mathrm{T}} = GI_3G^{\mathrm{T}}, \end{aligned} \tag{23a}$$

and

$$\begin{aligned} \text{Kl}(B'') &= M''D''^{\mathrm{T}} + D''M''^{\mathrm{T}} \\ &= (GM')(GD')^{\mathrm{T}} + (GD')(GM')^{\mathrm{T}} \\ &= GM'D'^{\mathrm{T}}G^{\mathrm{T}} + GD'M'^{\mathrm{T}}G^{\mathrm{T}} \\ &= G(M'D'^{\mathrm{T}} + D'M'^{\mathrm{T}})G^{\mathrm{T}} \\ &= G\,\text{Kl}(B')G^{\mathrm{T}}. \end{aligned} \tag{23b}$$

Hence B'' will be perpendicular and reciprocal if, and only if,

$$\text{Ki}(B'') = I_3 = GI_3G^{\mathrm{T}} = GG^{\mathrm{T}}. \tag{24}$$

and, $\text{Kl}(B'')$ must be a diagonal matrix, where

$$\text{Kl}(B'') = G\,\text{Kl}(B')G^{\mathrm{T}}. \tag{25}$$

The condition (24) is satisfied if G is an orthogonal matrix; i.e.

$$G^{\mathrm{T}} = G^{-1}. \tag{26}$$

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Hence, condition (25) is reduced to

$$\mathrm{Kl}\,(B'') = G\,\mathrm{Kl}\,(B')G^{-1}. \quad (27)$$

Therefore, following the same arguments as for proposition 9, G is given by

$$G = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}, \quad (28)$$

where γ_1, γ_2 and γ_3 are the normalized eigenvectors of $\mathrm{Kl}\,(B')$.

Then, the elements of the basis B'' can be written as

$$\$_1'' = (\mathbf{s}_1''; \mathbf{s}_{01}''), \quad \text{with } \mathrm{Ki}\,(\$_1'', \$_1'') = \mathbf{s}_1'' \cdot \mathbf{s}_1'' = 1, \quad (29a)$$

$$\$_2'' = (\mathbf{s}_2''; \mathbf{s}_{02}''), \quad \text{with } \mathrm{Ki}\,(\$_2'', \$_2'') = \mathbf{s}_2'' \cdot \mathbf{s}_2'' = 1, \quad (29b)$$

$$\$_3'' = (\mathbf{s}_3''; \mathbf{s}_{03}''), \quad \text{with } \mathrm{Ki}\,(\$_3'', \$_3'') = \mathbf{s}_3'' \cdot \mathbf{s}_3'' = 1. \quad (29c)$$

Further

$$\mathrm{Ki}\,(\$_1'', \$_2'') = \mathrm{Ki}\,(\$_1'', \$_3'') = \mathrm{Ki}\,(\$_2'', \$_3'') = 0, \quad (30a, b, c)$$

and

$$\mathrm{Kl}\,(\$_1'', \$_2'') = \mathrm{Kl}\,(\$_1'', \$_3'') = \mathrm{Kl}\,(\$_2'', \$_3'') = 0. \quad (30d, e, f)$$

Finally, defining the X, Y, and Z axes of the new coordinate system along the directions $\mathbf{s}_1''$, $\mathbf{s}_2''$, and $\mathbf{s}_3''$, and choosing as the origin O', of the new coordinate system, the common intersection point of the three screws. Then, the canonical form given by equations (19a, b, c) is obtained.

It should be noted that this case produces all the three-systems that belong to the family 1 according with Rico and Duffy's classification.[12] The screw systems that belong to this case are usually called "general".

Definition 12. Let $\{\$_1, \$_2, \$_3\}$ be a three-system such that the dimension of the space generated by the direction vectors $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$, of $\{\$_1, \$_2, \$_3\}$ is 3. Then, the centre of $\{\$_1, \$_2, \$_3\}$ is the unique point where the three lines associated with the three screws meet.

Corollary 13. Let $\{\$_1, \$_2, \$_3\}$ be a three-system such that the dimension of the space generated by the direction vectors $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ of $\{\$_1, \$_2, \$_3\}$ is 3. Further, assume that the elements of the perpendicular, normalized and reciprocal basis $B'' = \{\$_1'', \$_2'', \$_3''\}$ are given by equations (29a–c). Then, the coordinates of the centre of the three-system $\mathbf{r} = (x, y, z)$ appear as the off-diagonal elements of the skewsymmetric matrix, S_s, given by

$$S_s = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} = [M'' - \tfrac{1}{2}\mathrm{Kl}\,(B'')D'']^{\mathrm{T}}D''. \quad (31)$$

Where D'' and M'' are the direction and moment matrices of the basis, B'', which is both perpendicular and reciprocal.

Proof. Firstly, since $\{\$_1, \$_2, \$_3\}$ has a perpendicular and reciprocal basis the existence of the centre is assured. Further, the moment of the lines associated to the screws, with respect to a predetermined origin, is given by

$$S_s[\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3] = S_s[D'']^{\mathrm{T}}, \quad (32)$$

where the moment vectors are written here in column form. However, the same moments can be obtained by subtracting the product of the pitches of the screws by the direction matrix, D'', from the moment matrix, M'', of the basis B'', and then by transposing the resulting matrix; i.e.

$$[M'' - \tfrac{1}{2}\mathrm{Kl}(B'')D'']^{\mathrm{T}}. \quad (33)$$

However D'' is an orthogonal matrix, thus $[D'']^{\mathrm{T}} = [D'']^{-1}$, hence, equating (32) and (33), and postmultiplying both sides of the equation by D'', the desired result is obtained.

Proposition 14. Consider a three-system $\{\$_1, \$_2, \$_3\}$; i.e. a three-dimensional vector subspace of $e(3)$, where $B = \{\$_1, \$_2, \$_3\}$ is one of its bases. Assume that the dimension of the subspace genrated by the direction vectors $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ of $\{\$_1, \$_2, \$_3\}$ is 2, and let $B' = \{\$_1', \$_2', \$_3'\}$ be a perpendicular and normalized basis of the three-system. Then, the three-system has a basis that is simultaneously perpendicular and reciprocal (i.e. orthogonal with respect to the Killing and Klein form) if, and only if,

$$\mathrm{Kl}\,(\$_1', \$_3') = \mathrm{Kl}\,(\$_2', \$_3') = 0. \quad (34a, b)$$

Further, the canonical form of the basis for this three-system is

$$\$_{c1} = (1, 0, 0; h_\alpha, 0, 0), \quad (35a)$$

$$\$_{c2} = (0, 1, 0; 0, h_\beta, 0), \quad (35b)$$

$$\$_{c3} = (0, 0, 0; 0, 0, 1). \quad (35c)$$

Moreover, if conditions (34a, b) are not satisfied, the canonical form of this three-system is

$$\$_{c1} = (1, 0, 0; h_\alpha, 0, 0), \quad (36a)$$

$$\$_{c2} = (1, 0, 0; -h_\alpha, p_{31}^*, 0), \quad (36b)$$

$$\$_{c3} = (0, p_{02}^\#, p_{03}^\#; 0, p_{31}^\#, p_{12}^\#), \quad (36c)$$

with $p_{31}^* p_{02}^\# = 0$.

Proof. If the dimension of the space generated by the direction vectors $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ of the screw system $\{\$_1, \$_2, \$_3\}$ is 2, it is possible to prove that the elements of the perpendicular basis $B' = \{\$_1', \$_2', \$_3'\}$ have the form

$$\$_1' = (\mathbf{s}_1'; \mathbf{s}_{01}') \quad \text{with } \mathbf{s}_1' \cdot \mathbf{s}_1' = 1, \quad (37a)$$

$$\$_2' = (\mathbf{s}_2'; \mathbf{s}_{02}') \quad \text{with } \mathbf{s}_2' \cdot \mathbf{s}_2' = 1, \quad (37b)$$

$$\$_3' = (\mathbf{0}; \mathbf{s}_{03}'), \quad (37c)$$

where

$$\mathrm{Ki}\,(\$_1', \$_2') = \mathbf{s}_1' \cdot \mathbf{s}_2' = 0. \quad (38)$$

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Assume, initially that $\mathbf{s}'_3 \neq \mathbf{0}$. Since $\{\mathbf{s}'_1, \mathbf{s}'_2, \mathbf{s}'_3\}$ is obtained from $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ via a non-singular matrix, then

$$\dim[\mathbf{s}'_1, \mathbf{s}'_2, \mathbf{s}'_3] = \dim[\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3] = 2,$$

Hence

$$\mathbf{s}'_3 = \lambda \mathbf{s}'_1 + \lambda_2 \mathbf{s}'_2, \text{ for some } \lambda_1, \lambda_2 \in \mathbb{R}.$$

However, since B is a perpendicular basis

$$0 = \mathrm{Ki}\,(\mathbf{s}'_1, \mathbf{s}'_3) = \mathrm{Ki}\,(\mathbf{s}'_1, \lambda_1 \mathbf{s}'_1 + \lambda_2 \mathbf{s}'_2) = \lambda_1,$$

and

$$0 = \mathrm{Ki}\,(\mathbf{s}'_2, \mathbf{s}'_3) = \mathrm{Ki}\,(\mathbf{s}'_2, \lambda_1 \mathbf{s}'_1 + \lambda_2 \mathbf{s}'_2) = \lambda_2.$$

Therefore, $\mathbf{s}_3 = \mathbf{0}$. Further,

$$\mathrm{Kl}\,(\$_3, \$_3) = 0.$$

In order to prove that (34a, b) are sufficient and necessary conditions, it is necessary to note that any other perpendicular basis such as $\$''_1$, $\$''_2$ and $\$''_3$ (i.e. orthogonal with respect to the Killing form) is obtained by expressing $\$''_1$, $\$''_2$ and $\$''_3$ as linear combinations of $\$'_1$, $\$'_2$, $\$'_3$ of the form

$$\$''_1 = \lambda_{11}\$'_1 + \lambda_{12}\$'_2 + \lambda_{13}\$'_3, \quad (39a)$$

$$\$''_2 = \lambda_{21}\$'_1 + \lambda_{22}\$'_2 + \lambda_{23}\$'_3, \quad (39b)$$

$$\$''_3 = \qquad\qquad + \lambda_{33}\$'_3, \quad (39c)$$

where

$$\begin{vmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ 0 & 0 & \lambda_{33} \end{vmatrix} = (\lambda_{11}\lambda_{22} - \lambda_{21}\lambda_{12})\lambda_{33} \neq 0. \quad (40)$$

Condition (40) ensures that $\dim[\$''_1, \$''_2, \$''_3] = \dim[\$'_1, \$'_2, \$'_3] = 3$. Further, the submatrix

$$\begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix}$$

must be orthogonal, this requisite ensures that $\mathrm{Ki}\,(\$''_1, \$''_2) = 0$. Finally, the new basis $B'' = \{\$''_1, \$''_2, \$''_3\}$ is reciprocal also if, and only if,

$$\begin{aligned} 0 &= \mathrm{Kl}\,(\$''_1, \$''_2) \\ &= \mathrm{Kl}\,(\lambda_{11}\$'_1 + \lambda_{12}\$'_2 + \lambda_{13}\$'_3, \lambda_{21}\$'_1 + \lambda_{22}\$'_2 + \lambda_{33}\$'_3) \\ &= \lambda_{11}\lambda_{21}\,\mathrm{Kl}\,(\$'_1, \$'_1) + \lambda_{12}\lambda_{22}\,\mathrm{Kl}\,(\$'_2, \$'_2) \\ &\quad + (\lambda_{11}\lambda_{22} + \lambda_{12}\lambda_{21})\,\mathrm{Kl}\,(\$'_1, \$'_2) \\ &\quad + (\lambda_{11}\lambda_{23} + \lambda_{13}\lambda_{21})\,\mathrm{Kl}\,(\$'_1, \$'_3) \\ &\quad + (\lambda_{12}\lambda_{23} + \lambda_{13}\lambda_{22})\,\mathrm{Kl}\,(\$'_2, \$'_3), \end{aligned}$$

$$\begin{aligned} 0 &= \mathrm{Kl}\,(\$''_1, \$''_3) \\ &= \mathrm{Kl}\,(\lambda_{11}\$'_1 + \lambda_{12}\$'_2 + \lambda_{13}\$'_3, \lambda_{33}\$'_3) \\ &= \lambda_{33}[\lambda_{11}\,\mathrm{Kl}\,(\$'_1, \$'_3) + \lambda_{12}\,\mathrm{Kl}\,(\$'_2, \$'_3)], \end{aligned}$$

$$\begin{aligned} 0 &= \mathrm{Kl}\,(\$''_2, \$''_3) \\ &= \mathrm{Kl}\,(\lambda_{21}\$'_1 + \lambda_{22}\$'_2 + \lambda_{23}\$'_3, \lambda_{33}\$'_3) \\ &= \lambda_{33}[\lambda_{21}\,\mathrm{Kl}\,(\$'_1, \$'_3) + \lambda_{22}\,\mathrm{Kl}\,(\$'_2, \$'_3)]. \quad (41c) \end{aligned}$$

Since $\lambda_{33} \neq 0$, equations (41b–c) yield a homogeneous linear system, in the unknowns $\mathrm{Kl}\,(\$'_1, \$'_3)$ and $\mathrm{Kl}\,(\$'_2, \$'_3)$, of the form

$$\begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} \begin{bmatrix} \mathrm{Kl}\,(\$'_1, \$'_3) \\ \mathrm{Kl}\,(\$'_2, \$'_3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (42)$$

However, the coefficient matrix is orthogonal; therefore, the unique solution of the system is

$$\mathrm{Kl}\,(\$'_1, \$'_3) = \mathrm{Kl}\,(\$'_2, \$'_3) = 0.$$

Hence, the equation (41a), that ensures the reciprocity of $\$''_1$ and $\$''_2$, is reduced to

$$\lambda_{11}\lambda_{21}\,\mathrm{Kl}\,(\$'_1, \$'_1) + \lambda_{12}\lambda_{22}\,\mathrm{Kl}\,(\$'_2, \$'_2) + (\lambda_{11}\lambda_{22} + \lambda_{12}\lambda_{21})\,\mathrm{Kl}\,(\$'_1, \$'_2) = (044)$$

It should be noted that the reciprocity of $\$''_1$ and $\$''_2$ depends only on the scalars λ_{11}, λ_{12}, λ_{21}, and λ_{22}, that form the orthogonal submatrix. Then, it is possible to employ the argument given by Rico and Duffy[11] (see pp. 464–466). This argument ensures the existence of linear combination of $\$'_1$, and $\$'_2$ that are perpendicular and reciprocal.

Now, assume that the necessary conditions are satisfied. Then selecting the X and Y axes of a new coordinate system along the directions $\mathbf{s}_1$, and $\mathbf{s}_2$ respectively, and selecting the origin of the new coordinate system O′ as the intersection point of $\$''_1$ and $\$''_2$, the elements of the perpendicular basis $B'' = \{\$''_1, \$''_2, \$''_3\}$, are transformed into

$$\$''_1 = (1, 0, 0; h_\alpha, 0, 0), \quad (45a)$$

$$\$''_2 = (0, 1, 0; 0, h_\beta, 0), \quad (45b)$$

$$\$''_3 = (0, 0, 0; 0, 0, p^{\#}_{12}). \quad (45c)$$

It should be noted that the X and Y components of $\mathbf{s}''_{03}$ are zero; i.e. $p^{\#}_{23} = p^{\#}_{31} = 0$. Otherwise, since

$$\mathrm{Kl}\,(\$''_1, \$''_3) = p^{\#}_{23},$$

and (46a,b)

$$\mathrm{Kl}\,(\$''_2, \$''_3) = p^{\#}_{31},$$

the reciprocity of the basis B'' would be contradicted. Finally, $\$''_3$ can be transformed into the form indicated by equation (35c).

It should be noted that this canonical form includes all the three-systems of the family 3I according to Rico and Duffy's classification.[12]

Assume, now, that conditions (34a, b) are not satisfied; i.e. without loss of generality assume that $\mathrm{Kl}\,(\$'_1, \$'_3) \neq 0$. Then, the perpendicular basis $B' = \{\$'_1, \$'_2, \$'_3\}$, has the form

$$\$'_1 = (\mathbf{s}'_1; \mathbf{s}'_{01}) \quad \text{with } \mathbf{s}'_1 \cdot \mathbf{s}'_1 = 1, \quad (47a)$$

$$\$''_2 = (\mathbf{s}'_2; \mathbf{s}'_{02}) \quad \text{with } \mathbf{s}'_2 \cdot \mathbf{s}'_2 = 1, \quad (47b)$$

$$\$'_3 = (\mathbf{0}; \mathbf{s}'_{03}), \quad (47c)$$

Further

$$\mathrm{Ki}\,(\$'_1, \$'_2) = \mathbf{s}'_1 \cdot \mathbf{s}'_2 = 0,$$

and (48a, b)

$$\mathrm{Kl}\,(\$'_1, \$'_3) = \mathbf{s}'_1 \cdot \mathbf{s}'_{03} \neq 0.$$

Then consider

$$\$''_2 = \$'_1 - \frac{\mathrm{Kl}\,(\$'_1, \$'_1)}{\mathrm{Kl}\,(\$'_1, \$'_3)}\$'_3 \quad \text{if } \mathrm{Kl}\,(\$'_1, \$'_1) \neq 0, \quad (49a)$$

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with $\$_1'' = \$_1'$, and $\$_3'' = \$_2'$, or consider

$$\$_1'' = \$_1' + \$_3', \quad \text{and} \quad \$_2'' = \$_1' - \$_3', \quad \text{if } \mathrm{Kl}\,(\$_1', \$_1') = 0, \tag{49b}$$

with $\$_3'' = \$_2'$.

Then, the new basis has the form

$$\$_1'' = (\mathbf{s}_1''; \mathbf{s}_{01}'') \quad \text{with } \mathbf{s}_1'' \cdot \mathbf{s}_1'' = 1, \tag{50a}$$

$$\$_2'' = (\mathbf{s}_1''; \mathbf{s}_{02}''), \tag{50b}$$

$$\$_3'' = (\mathbf{s}_3''; \mathbf{s}_{03}''), \quad \text{with } \mathbf{s}_3'' \cdot \mathbf{s}_{03}'' = 1, \tag{50c}$$

Further

$$\mathrm{Ki}\,(\$_1'', \$_3'') = \mathrm{Ki}\,(\$_2'', \$_3'') = 0,$$

and (51a, b, c)

$$\mathrm{Kl}\,(\$_1'', \$_2'') = 0.$$

Choosing a new origin along the line defined by the screw $\$_1''$, choosing the X axis along the direction $\mathbf{s}_1''$, and choosing the Y axis so that the Z-component of $\mathbf{s}_{02}''$ be zero, the basis B'' is given by

$$\$_1'' = (1, 0, 0; h_\alpha, 0, 0), \tag{52a}$$

$$\$_2'' = (1, 0, 0; -h_\alpha, p_{31}^*, 0), \tag{52b}$$

$$\$_3'' = (0, p_{02}^\#, p_{03}^\#; p_{23}^\#, p_{31}^\#, p_{12}^\#). \tag{52c}$$

However, since

$$\$_1'' - \$_2'' = (0, 0, 0; 2h_\alpha, -p_{31}^*, 0), \tag{53}$$

$\$_3''$ can be replaced by a new screw $\$_3'''$ of the form

$$\$_3''' = (0, p_{02}^\#, p_{03}^\#; 0, p_{31}^\#, p_{12}^\#). \tag{54}$$

Furthermore, choosing the new origin at the point, along the X axis, where $\$_3'''$ meets the X axis, then $\$_3'''$ is reduced to

$$\$_3'' = (0, p_{02}^\#, p_{03}^\#; 0, 0, 0). \tag{55}$$

It should be noted that $\mathrm{Kl}\,(\$_1'', \$_3'') = 0$. Nevertheless, $\$_2''$ and $\$_3''$ form a perpendicular two-system that it is not reciprocal, because

$$\mathrm{Kl}\,(\$_2'', \$_3'') = p_{31}^* p_{02}^\#.$$

However, it is possible to employ the argument used to obtain equations (47a–c), to produce a perpendicular and reciprocal basis for this two system. Further, since $\$_1''$ is parallel to $\$_2''$, and the Killing and Klein forms are invariant, after a proper relabeling of the axes, the canonical form given by (36a–c) is obtained.

It should be noted that this canonical form includes all the three systems of the family 4II of Rico and Duffy's classification.[12]

Proposition 15. Consider a three system $\{\$_1, \$_2, \$_3\}$; i.e. a three-dimensional vector subspace of $e(3)$, where $B = \{\$_1, \$_2, \$_3\}$ is one of its bases. Assume that the dimension of the subspace generated by the direction vectors $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ of $\{\$_1, \$_2, \$_3\}$ is 1, and let $B' = \{\$_1', \$_2', \$_3'\}$ be a perpendicular and normalized basis of the three-system. Then, the three-system has a basis that is simultaneously perpendicular and reciprocal (i.e. orthogonal with respect to the Killing and Klein form) if, and only if,

$$\mathrm{Kl}\,(\$_1', \$_2') = \mathrm{Kl}\,(\$_1', \$_3') = 0. \tag{56a, b}$$

Further, the canonical form of the basis for this three-system is

$$\$_{c1} = (1, 0, 0; h_\alpha, 0, 0), \tag{57a}$$

$$\$_{c2} = (0, 0, 0; 0, 1, 0), \tag{57b}$$

$$\$_{c3} = (0, 0, 0; 0, 0, 1). \tag{57c}$$

Moreover, if condition (56a, b) are not satisfied, the canonical form of this three-system is

$$\$_{c1} = (1, 0, 0; h_\alpha, 0, 0), \tag{58a}$$

$$\$_{c2} = (1, 0, 0; -h_\alpha, p_{31}^*, 0), \tag{58b}$$

$$\$_{c3} = (0, 0, 0; 0, p_{31}^\#, p_{12}^\#). \tag{58c}$$

Proof. If the dimension of the space generated by the direction vectors $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ of the screw system $\{\$_1, \$_2, \$_3\}$ is 1, it is possible to prove that the elements of the perpendicular basis $B' = \{\$_1', \$_2', \$_3'\}$ have the form

$$\$_1' = (\mathbf{s}_1'; \mathbf{s}_{01}'), \quad \text{with } \mathbf{s}_1' \cdot \mathbf{s}_1' = 1, \tag{59a}$$

$$\$_2' = (\mathbf{0}; \mathbf{s}_{02}'), \tag{59b}$$

$$\$_3' = (\mathbf{0}; \mathbf{s}_{03}'). \tag{59c}$$

Assume, initially and without loss of generality, that $\mathbf{s}_2' \neq \mathbf{0}$. Since $\{\mathbf{s}_1', \mathbf{s}_2', \mathbf{s}_3'\}$ is obtained from $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ via a non-singular matrix, then

$$\dim\,[\mathbf{s}_1', \mathbf{s}_2', \mathbf{s}_3'] = \dim\,[\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3] = 1.$$

Hence

$$\mathbf{s}_2' = \lambda \mathbf{s}_1', \quad \text{for some } \lambda \in \mathbb{R}.$$

However, since B' is a perpendicular basis,

$$0 = \mathrm{Ki}\,(\mathbf{s}_2', \mathbf{s}_1') = \mathrm{Ki}\,(\lambda \mathbf{s}_1'; \mathbf{s}_1')$$
$$= \lambda\, \mathrm{Ki}\,(\mathbf{s}_1', \mathbf{s}_1') = \lambda.$$

Therefore, $\mathbf{s}_2' = \mathbf{0}$. A similar reasoning applies to $\mathbf{s}_3'$.

Further, it is easy to note that

$$\mathrm{Kl}\,(\$_2', \$_3') = \mathrm{Kl}\,(\$_2', \$_2') = \mathrm{Kl}\,(\$_3', \$_3') = 0. \tag{60a, b, c}$$

In order to prove that (56a, b) are sufficient and necessary conditions for the three-system to have a reciprocal and perpendicular basis, it is necessary to note that any other perpendicular basis (i.e. orthogonal with respect to the Killing form) is obtained by expressing $\$_1''$, $\$_2''$ and $\$_3''$ as linear combinations of $\$_1'$, $\$_2'$ and $\$_3'$,

$$\$_1'' = \lambda_{11}\$_1' + \lambda_{12}\$_2' + \lambda_{13}\$_3', \tag{61a}$$

$$\$_2'' = \qquad \lambda_{22}\$_2' + \lambda_{23}\$_3', \tag{61b}$$

$$\$_3'' = \qquad \lambda_{32}\$_2' + \lambda_{33}\$_3', \tag{61c}$$

where

$$\begin{vmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ 0 & \lambda_{22} & \lambda_{23} \\ 0 & \lambda_{32} & \lambda_{33} \end{vmatrix} = \lambda_{11}(\lambda_{22}\lambda_{33} - \lambda_{32}\lambda_{23}) \neq 0; \tag{62}$$

condition (62) ensures that $\dim\,[\$_1'', \$_2'', \$_3''] = \dim\,[\$_1, \$_2, \$_3] = 3$.

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Finally, the new basis $B'' = \{\$''_1, \$''_2, \$''_3\}$ is reciprocal also if, and only if,

$$\mathrm{Kl}\,(\$''_1, \$''_2) = \lambda_{11}\lambda_{22}\,\mathrm{Kl}\,(\$'_1, \$'_2) + \lambda_{11}\lambda_{23}\,\mathrm{Kl}\,(\$'_1, \$'_3) = 0, \qquad (63a)$$

and

$$\mathrm{Kl}\,(\$''_1, \$''_3) = \lambda_{11}\lambda_{32}\,\mathrm{Kl}\,(\$'_1, \$'_2) + \lambda_{11}\lambda_{33}\,\mathrm{Kl}\,(\$'_1, \$'_3) = 0. \qquad (63b)$$

Since by condition (62), $\lambda_{11} \neq 0$, the system can be reduced to

$$\begin{bmatrix} \lambda_{22} & \lambda_{23} \\ \lambda_{32} & \lambda_{33} \end{bmatrix} \begin{bmatrix} \mathrm{Kl}\,(\$'_1, \$'_2) \\ \mathrm{Kl}\,(\$'_1, \$'_3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad (64)$$

However, the same condition (62), indicates that the coefficient matrix is non-singular. therefore, the only possibility is that

$$\mathrm{Kl}\,(\$'_1, \$'_2) = \mathrm{Kl}\,(\$'_1, \$'_3) = 0. \qquad (56a, b\ \mathrm{Rep.})$$

Now, assume that the necessary conditions are satisfied. Then, choosing the X-axis of a new coordinate system along the direction $\mathbf{s}''_1$, and the origin of the new coordinate system as an arbitrary point along the axis of the screw $\$''_1$,[11] the elements of the perpendicular basis B'' are transformed into

$$\$''_1 = (1, 0, 0; h_\alpha, 0, 0), \qquad (65a)$$
$$\$''_2 = (0, 0, 0; 0, p^*_{31}, p^*_{12}), \qquad (65b)$$
$$\$''_3 = (0, 0, 0; 0, p^\#_{31}, p^\#_{12}). \qquad (65c)$$

It should be noted that $p^*_{23} = p^\#_{23} = 0$. This is because $\mathrm{Kl}\,(\$''_1, \$''_2) = p^*_{23}$, and $\mathrm{Kl}\,(\$''_1, \$''_3) = p^\#_{23}$, and, otherwise, the reciprocity conditions for the basis B'' would be contradicted.

Finally, the screws $\$''_2$ and $\$''_3$ can be substituted by the following linear combinations

$$\$_{c2} = (0, 0, 0; 0, 1, 0), \qquad (66a)$$

and

$$\$_{c3} = (0, 0, 0; 0, 0, 1). \qquad (66b)$$

Hence, the canonical form given by equations (58a–c) is obtained. It should be noted that the canonical form given by (58a–c) includes all the three-systems of the family 5 in Rico and Duffy's classification.[12]

Assume, now, that conditions (56a, b) are not satisfied, then the perpendicular basis $B' = \{\$'_1, \$'_2, \$'_3\}$ is given by equations (59a–c). However, assume that

$$\mathrm{Kl}\,(\$'_1, \$'_2) \neq 0. \qquad (67)$$

Then consider

$$\$''_2 = \$'_1 - \frac{\mathrm{Kl}\,(\$'_1, \$'_2)}{\mathrm{Kl}\,(\$'_1, \$'_3)}\$'_3 \quad \text{if } \mathrm{Kl}\,(\$'_1, \$'_1) \neq 0,$$

with $\$''_1 = \$'_1$ and $\$''_3 = \$'_3$, or consider

$$\$''_1 = \$'_1 + \$'_2 \quad \text{and} \quad \$_2 = \$'_1 - \$'_2, \qquad (69a, b)$$

if $\mathrm{Kl}\,(\$'_1, \$'_1) = 0$, with $\$''_3 = \$'_3$.

Then, the new basis has the form

$$\$''_1 = (\mathbf{s}''_1; \mathbf{s}''_{01}) \quad \text{with } \mathbf{s}''_1 \cdot \mathbf{s}''_1 = 1, \qquad (70a)$$
$$\mathbf{s}''_2 = (\mathbf{s}''_1; \mathbf{s}''_{02}), \qquad (70b)$$
$$\$''_3 = (\mathbf{0}; \mathbf{s}''_{03}), \qquad (70c)$$

and the basis satisfies the condition

$$\mathrm{Ki}\,(\$''_1, \$''_2) = 0. \qquad (71)$$

Choosing a new origin along the line defined by the screw $\$''_1$, choosing the X axis along the direction $\mathbf{s}''_1$, and choosing the Y axis so that the Z-component of $\mathbf{s}''_{02}$ be zero, the basis B'' is

$$\$''_1 = (1, 0, 0; h_\alpha, 0, 0), \qquad (72a)$$
$$\$''_2 = (1, 0, 0; -h_\alpha, p^*_{31}, 0), \qquad (72b)$$
$$\$''_3 = (0, 0, 0; p^\#_{23}, p^\#_{31}, p^\#_{12}). \qquad (72c)$$

However $\$''_3$ can be substituted by

$$\$'''_3 = \$''_3 - p^\#_{23}(\$''_1 - \$''_2)/(2h_\alpha).$$

and the canonical form given by equations (58a–c) is obtained.

It should be noted that this canonical form includes all the three systems of the family 4I of Rico and Duffy's classification.[12]

Proposition 16. Consider a three system $\{\$_1, \$_2, \$_3\}$; i.e. a three-dimensional vector subspace of $e(3)$, where $B = \{\$_1, \$_2, \$_3\}$ is one of its bases. Assume that the dimension of the subspace generated by the direction vectors $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ of $\{\$_1, \$_2, \$_3\}$ is 0. Then, the canonical form of this three-system is

$$\$_{c1} = (0, 0, 0; 1, 0, 0), \qquad (73a)$$
$$\$_{c3} = (0, 0, 0; 0, 1, 0), \qquad (73b)$$
$$\$_{c3} = (0, 0, 0; 0, 0, 1). \qquad (73c)$$

Further, this basis is simultaneously orthonormal with respect to the Killing and orthogonal with respect to the Klein form.

Proof. If the dimension of the subspace generated by the direction vector $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ of $\{\$_1, \$_2, \$_3\}$ is 0, then

$$\mathbf{s}_1 = \mathbf{s}_2 = \mathbf{s}_3 = \mathbf{0}. \qquad (74a, b, c)$$

Hence, the elements of the basis B are of the form

$$\$'_1 = (0, 0, 0; p_{23}, p_{31}, p_{12}), \qquad (75a)$$
$$\$_2 = (0, 0, 0; p^*_{23}, p^*_{31}, p^*_{12}), \qquad (75b)$$
$$\$_c = (0, 0, 0; p^\#_{23}, p^\#_{31}, p^\#_{12}). \qquad (75c)$$

Furthermore, since the dimension of the subspace is 3, the rank of the moment submatrix is 3 and the submatrix is non-singular. Therefore, the submatrix can be transformed into the identity matrix by using elementary row operations; i.e. it is possible to find the basis given by (73a–c).

It should be noted that this case corresponds to the unique three-system of the family 6 in Rico and Duffy's classification.[12]

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4. FINAL REMARKS

The computation of the canonical form of screw systems in general, and three-systems in particular, are of importance in robotics because, as shown by Stanišić and Pennock[6,13,14], it might produce more efficient computational schemes for the velocity and acceleration analyses of manipulators or subassemblies. More specifically, Stanišić and Pennock showed that finding the normal form of the three-system formed by the arm-subassembly of a wrist-partitioned six-degree of freedom manipulator leads to a highly efficient computational approach for the inverse velocity and acceleration analysis of the manipulator. Further, any comprehensive classification, analysis and synthesis of the different topologies of manipulators must have as a fundamental prerequisite efficient computational schemes for determining the canonical form of the screw systems formed by the kinematic pairs of the manipulator. Only after the determination of the canonical form of a pair of screw systems, it is possible to decide if the manipulator topologies are equivalent and to weight the advantages or disadvantages of the different topologies.

5. CONCLUSION

A general method for the computation of the canonical form of three-systems has been presented. The method deals not only with the general case, but the method also handles the special and degenerate cases. Furthermore, the method shows conclusively that the classification of three-systems previously published by the authors is truly exhaustive. A MapleV listing and an example are available, upon request, from the first author.

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