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FROM A_1 TO D_5: TOWARDS A FORCING-RELATED CLASSIFICATION OF RELATIONAL STRUCTURES

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Abstract. We investigate the partial orderings of the form $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$, where $\mathbb{X}$ is a relational structure and $\mathbb{P}(\mathbb{X})$ the set of the domains of its isomorphic substructures. A rough classification of countable binary structures corresponding to the forcing-related properties of the posets of their copies is obtained.

§1. Introduction. The relational structure $\mathbb{X} = \langle \omega, < \rangle$, where $<$ is the natural order on the set ω of natural numbers is a structure having the following extremal property: each ω-sized subset A of ω determines a substructure isomorphic to the whole structure. If instead of $\langle \omega, < \rangle$ we take the integer line $\mathbb{Z} = \langle Z, < \rangle$, then we lose the maximality of the set of isomorphic substructures (the set of positive integers is not a copy of $\mathbb{Z}$). Finally, the minimality of the set of copies is reached by the linear graph $G_{\mathbb{Z}} = \langle Z, \rho \rangle$, where $\rho = \{\langle m, n \rangle : |m - n| = 1\}$, since each proper subset A of Z determines a disconnected graph and, hence, fails to be a copy of the whole graph.

We investigate the posets of the form $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$, where $\mathbb{X}$ is a relational structure and $\mathbb{P}(\mathbb{X})$ the set of the domains of its isomorphic substructures. Although some of our statements are general, the main result of the paper is the diagram on Figure 1, describing an interplay between the properties of a countable binary structure $\mathbb{X}$ and the properties of the corresponding poset $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$. So we obtain a rough classification of countable binary structures concerning the forcing-related properties of the posets of their copies: for the structures from column A (resp. B; D) the corresponding posets are forcing equivalent to the trivial poset (resp. the Cohen forcing, $\langle {}^{<\omega}2, \supset \rangle$; a σ-closed atomless poset) and the wild animals are in cages C_3 and C_4, where the posets of copies are forcing equivalent to the quotients of the form $P(\omega)/\mathcal{I}$, for some co-analytic tall ideal $\mathcal{I}$.

Clearly, such classification depends on the model of set theory in which we work. For example, under the CH all the structures from column D are in the same class (having the posets of copies forcing equivalent to $(P(\omega)/\operatorname{Fin})^+$), but this is not true in the Mathias model. Also the classification is very rough. Namely, it is easy to see that equimorphic structures have forcing equivalent posets

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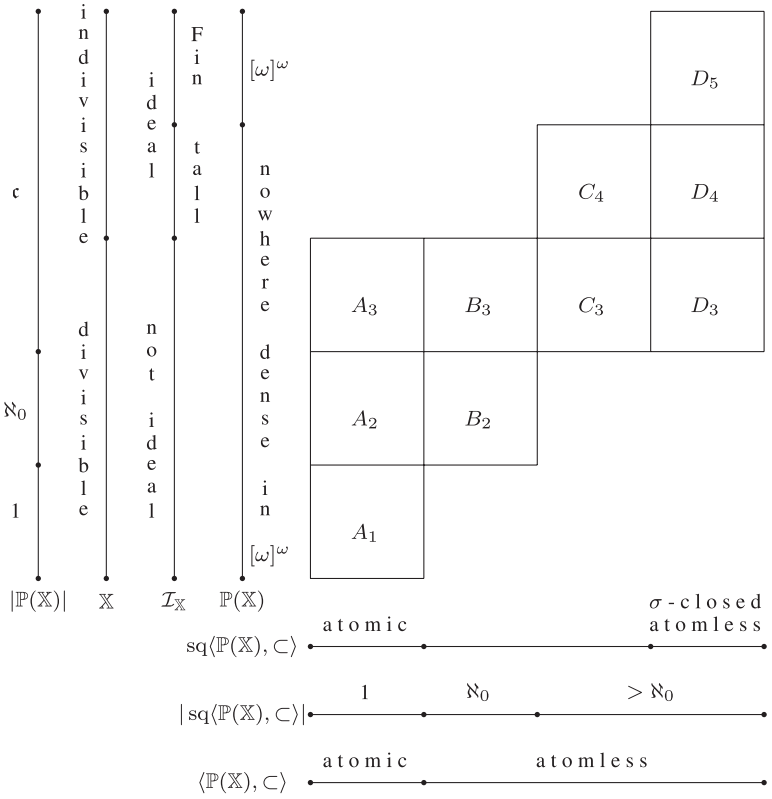

FIGURE 1. Binary relations on countable sets.

of copies [4] and, hence, all countable nonscattered linear orders are equivalent in this sense. Moreover, the class of structures satisfying $\mathbb{P}(\mathbb{X}) = \{X\}$ contains continuum many nonequimorphic structures [7].

A few words on notation. Let $L = \{R_i : i \in I\}$ be a relational language, where $\mathrm{ar}(R_i) = n_i$, $i \in I$. An L-structure $\mathbb{X} = \langle X, \{\rho_i : i \in I\}\rangle$ is called *countable* iff $|X| = \omega$; *binary* iff $L = \{R\}$ and $\mathrm{ar}(R) = 2$. If $A \subset X$, then $\langle A, \{(\rho_i)_A : i \in I\}\rangle$ is a *substructure* of $\mathbb{X}$, where $(\rho_i)_A = \rho_i \cap A^{n_i}$, $i \in I$. If $\mathbb{Y} = \langle Y, \{\sigma_i : i \in I\}\rangle$ is an L-structure too, a mapping $f : X \to Y$ is an *embedding* (we write $\mathbb{X} \hookrightarrow_f \mathbb{Y}$) iff it is an injection and

$$\forall i \in I \ \ \forall \langle x_1, \ldots, x_{n_i}\rangle \in X^{n_i} \ \ (\langle x_1, \ldots, x_{n_i}\rangle \in \rho_i \Leftrightarrow \langle f(x_1), \ldots, f(x_{n_i})\rangle \in \sigma_i).$$

If $\mathbb{X}$ embeds in $\mathbb{Y}$ we write $\mathbb{X} \hookrightarrow \mathbb{Y}$. Let $\mathrm{Emb}(\mathbb{X}, \mathbb{Y}) = \{f : \mathbb{X} \hookrightarrow_f \mathbb{Y}\}$ and $\mathrm{Emb}(\mathbb{X}) = \{f : \mathbb{X} \hookrightarrow_f \mathbb{X}\}$. If, in addition, f is a surjection, it is an *isomorphism* (we write $\mathbb{X} \cong_f \mathbb{Y}$) and the structures $\mathbb{X}$ and $\mathbb{Y}$ are *isomorphic*, in notation $\mathbb{X} \cong \mathbb{Y}$. So we investigate the posets of the form $\langle \mathbb{P}(\mathbb{X}), \subset\rangle$, where $\mathbb{X} = \langle X, \{\rho_i : i \in I\}\rangle$ is a relational structure and

$$\mathbb{P}(\mathbb{X}) = \{A \subset X : \langle A, \{(\rho_i)_A : i \in I\}\rangle \cong \mathbb{X}\} = \{f[X] : f \in \mathrm{Emb}(\mathbb{X})\}.$$

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More generally, if $\mathbb{Y} = \langle Y, \{\sigma_i : i \in I\}\rangle$ is a structure of the same language, let $\mathbb{P}(\mathbb{X}, \mathbb{Y}) = \{B \subset Y : \langle B, \{(\sigma_i)_B : i \in I\}\rangle \cong \mathbb{X}\} = \{f[X] : f \in \mathrm{Emb}(\mathbb{X}, \mathbb{Y})\}$.

§2. Homogeneity and atoms. If $\mathbb{P} = \langle P, \leq\rangle$ is a partial order, $p, q \in \mathbb{P}$ are *compatible* iff there is $r \leq p, q$. Otherwise p and q are *incompatible* and we write $p \perp q$. $p \in P$ is an *atom*, in notation $p \in \mathrm{At}(\mathbb{P})$, iff each $q, r \leq p$ are compatible. $\mathbb{P}$ is called: *atomless* iff $\mathrm{At}(\mathbb{P}) = \emptyset$; *atomic* iff $\mathrm{At}(\mathbb{P})$ is dense in $\mathbb{P}$; *homogeneous* iff it has a largest element and $\mathbb{P} \cong p{\downarrow} = (-\infty, p]_{\mathbb{P}}$, for each $p \in P$. Clearly we have

Fact 2.1. *A homogeneous poset $\mathbb{P} = \langle P, \leq\rangle$ is either atomless or downwards directed and $\mathrm{At}(\mathbb{P}) = P$ in the second case.*

A family $\mathcal{B}$ is an *uniform filter base* on a set X iff (UFB1) $\emptyset \neq \mathcal{B} \subset [X]^{|X|}$; (UFB2) For each $A, B \in \mathcal{B}$ there is $C \in \mathcal{B}$ such that $C \subset A \cap B$.

Theorem 2.2. *Let $\mathbb{X} = \langle X, \{\rho_i : i \in I\}\rangle$ be a relational structure. Then*

(a) *$\langle \mathbb{P}(\mathbb{X}), \subset\rangle$ is a homogeneous poset;*
(b) *$\langle \mathbb{P}(\mathbb{X}), \subset\rangle$ is either atomless or atomic;*
(c) *$\langle \mathbb{P}(\mathbb{X}), \subset\rangle$ is atomless iff it contains two incompatible elements;*
(d) *If $\langle \mathbb{P}(\mathbb{X}), \subset\rangle$ is atomic, then $\mathrm{At}(\mathbb{P}(\mathbb{X})) = \mathbb{P}(\mathbb{X})$ and, moreover, $\mathbb{P}(\mathbb{X})$ is an uniform filter base on X. Also $\bigcap \mathbb{P}(\mathbb{X}) \in \mathbb{P}(\mathbb{X})$ iff $\mathbb{P}(\mathbb{X}) = \{X\}$.*

Proof. (a) Clearly, $1_{\mathbb{P}(\mathbb{X})} = X$. Let $C \in \mathbb{P}(\mathbb{X})$ and $f \in \mathrm{Emb}(\mathbb{X})$, where $C = f[X]$. We show that $\langle \mathbb{P}(\mathbb{X}), \subset\rangle \cong_F \langle (-\infty, C]_{\mathbb{P}(\mathbb{X})}, \subset\rangle$, where the function F is defined by $F(A) = f[A]$, for each $A \in \mathbb{P}(\mathbb{X})$. For $A \in \mathbb{P}(\mathbb{X})$ we have $F(A) \subset C$ and there is $g \in \mathrm{Emb}(\mathbb{X})$ such that $A = g[X]$. Clearly $f \circ g \in \mathrm{Emb}(\mathbb{X})$ and, hence, $F(A) = f[g[X]] \in \mathbb{P}(\mathbb{X})$. Thus $F : \mathbb{P}(\mathbb{X}) \to (-\infty, C]_{\mathbb{P}(\mathbb{X})}$.

Since f is an injection, $f[A] = f[B]$ implies $A = B$, so F is an injection.

Let $\mathbb{P}(\mathbb{X}) \ni B \subset C$. Since $B \subset f[X]$ we have $B = f[f^{-1}[B]]$ and, clearly, $\langle f^{-1}[B], \{(\rho_i)_{f^{-1}[B]} : i \in I\}\rangle \cong_{f|f^{-1}[B]} \langle B, \{(\rho_i)_B : i \in I\}\rangle \cong \mathbb{X}$. Thus $f^{-1}[B] \in \mathbb{P}(\mathbb{X})$ and $B = F(f^{-1}[B])$, so F is a surjection.

Since f is an injection, for $A, B \in \mathbb{P}(\mathbb{X})$ we have $A \subset B \Leftrightarrow f[A] \subset f[B]$. Thus F is an order isomorphism.

(b) Follows from (a) and Fact 2.1.

(c) If $\mathbb{P}(\mathbb{X})$ contains two incompatible elements, then it is not downwards directed and, by Fact 2.1, must be atomless.

(d) Let $\langle \mathbb{P}(\mathbb{X}), \subset\rangle$ be atomic. By Fact 2.1, $\mathrm{At}(\mathbb{P}(\mathbb{X})) = \mathbb{P}(\mathbb{X})$ and $\mathbb{P}(\mathbb{X})$ satisfies (UFB2). Since $X \in \mathbb{P}(\mathbb{X}) \subset [X]^{|X|}$, (UFB1) holds as well. Suppose that $A = \bigcap \mathbb{P}(\mathbb{X}) \in \mathbb{P}(\mathbb{X})$ and $\mathbb{P}(\mathbb{X}) \neq \{X\}$. Then $A \subsetneq X$ and, since $\mathbb{P}(\mathbb{X}) \cong A{\downarrow}$, there is $B \in \mathbb{P}(\mathbb{X})$ such that $B \subsetneq A$. A contradiction. ⊣

§3. The complexity and size. For each relational structure $\mathbb{X}$ we have $\{X\} \subset \mathbb{P}(\mathbb{X}) \subset [X]^{|X|}$ and $\mathbb{P}(\mathbb{X})$ is of size 1 or infinite, because if $f \in \mathrm{Emb}(\mathbb{X})$ and $f[X] \neq X$, then $f^n[X]$, $n \in \mathbb{N}$, is a decreasing sequence of elements of $\mathbb{P}(\mathbb{X})$. Now we show that $|\mathbb{P}(\mathbb{X})| \in \{1, \aleph_0, \mathfrak{c}\}$.

By 2^ω and ω^ω we denote the Cantor cube and the Baire space respectively and $p_k : 2^\omega \to 2$ and $\pi_k : \omega^\omega \to \omega$, $k \in \omega$, will be the corresponding projections. As usual, the mapping $\chi : P(\omega) \to 2^\omega$, where $\chi(A) = \chi_A$, for each $A \subset \omega$, identifies the subsets of ω with their characteristic functions and a set $\mathcal{S} \subset P(\omega)$ is called closed (Borel, analytic ...) iff $\chi[\mathcal{S}]$ is a closed (Borel, analytic ...) set in 2^ω.

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For $\mathcal{S} \subset P(\omega)$ let $\mathcal{S}\uparrow = \{A \subset \omega : \exists S \in \mathcal{S}\ S \subset A\}$ and, for $A \subset 2^\omega$, let $A\uparrow = \{x \in 2^\omega : \exists a \in A\ a \leq x\}$, where $a \leq x$ means that $a(n) \leq x(n)$, for all $n \in \omega$. Instead of $\{a\}\uparrow$ we will write $a\uparrow$.

THEOREM 3.1. *If* $\mathbb{X} = \langle X, \{\rho_i : i \in I\}\rangle$ *is a countable relational structure and* $\mathcal{I}_\mathbb{X} = \{I \subset X : \neg\exists A \in \mathbb{P}(\mathbb{X})\ A \subset I\}$, *then*

(a) $\mathbb{P}(\mathbb{X})$ *is an analytic set*;
(b) $\mathbb{P}(\mathbb{X})\uparrow$ *is an analytic set*;
(c) $\mathcal{I}_\mathbb{X}$ *is a co-analytic set, has the Baire property and contains the ideal* Fin_X.
(d) *The sets* $\mathbb{P}(\mathbb{X})$ *and* $\mathbb{P}(\mathbb{X})\uparrow$ *have the Baire property and size* 1, $\aleph_0$ *or* $\mathfrak{c}$.

PROOF. Without loss of generality, we suppose $X = \omega$. Let $\mathrm{ar}(\rho_i) = n_i$, $i \in I$.
(a) This statement is a folklore but, for completeness, we include its proof.

CLAIM 3.2. $\mathrm{Emb}(\mathbb{X})$ *is a closed set in the Baire space,* ω^ω.

PROOF OF CLAIM 3.2. We show that the set $\omega^\omega \setminus \mathrm{Emb}(\mathbb{X})$ is open. Let $f \in \omega^\omega \setminus \mathrm{Emb}(\mathbb{X})$.

If f is not an injection and $m, n \in \omega$, where $m \neq n$ and $f(m) = f(n) = k$, then $\pi_m^{-1}[\{k\}] \cap \pi_n^{-1}[\{k\}]$ is a neighborhood of f contained in $\omega^\omega \setminus \mathrm{Emb}(\mathbb{X})$.

Otherwise there are $i \in I$ and $m_1, \ldots, m_{n_i} \in \omega$ such that $\langle m_1, \ldots, m_{n_i}\rangle \in \rho_i \not\Leftrightarrow \langle f(m_1), \ldots, f(m_{n_i})\rangle \in \rho_i$. Then $B = \bigcap_{j \leq n_i} \pi_{m_j}^{-1}[\{f(m_j)\}]$ is a neighborhood of f contained in $\omega^\omega \setminus \mathrm{Emb}(\mathbb{X})$. ⊣

CLAIM 3.3. *The mapping* $F : \omega^\omega \to 2^\omega$ *defined by* $F(f) = \chi_{f[\omega]}$ *is a Borel mapping.*

PROOF OF CLAIM 3.3. It is sufficient to show that $F^{-1}[p_n^{-1}[\{j\}]]$ is a Borel set, for each $n \in \omega$ and $j \in 2$. It is easy to see that $F^{-1}[p_n^{-1}[\{1\}]] = \bigcup_{k\in\omega} \pi_k^{-1}[\{n\}]$ and $F^{-1}[p_n^{-1}[\{0\}]] = \omega^\omega \setminus \bigcup_{k\in\omega} \pi_k^{-1}[\{n\}]$. ⊣

Now, by Claims 3.2 and 3.3, $F[\mathrm{Emb}(\mathbb{X})]$ is an analytic set (see e.g., [1], p. 86) and it is easy to check that $\chi[\mathbb{P}(\mathbb{X})] = F[\mathrm{Emb}(\mathbb{X})]$.

(b) If we regard the set $\mathrm{Emb}(\mathbb{X})$ as a subspace of the Baire space ω^ω, then $\{\pi_k^{-1}[\{n\}] \cap \mathrm{Emb}(\mathbb{X}) : k, n \in \omega\}$ is a subbase for the corresponding topology on $\mathrm{Emb}(\mathbb{X})$ and we have

CLAIM 3.4. $B = \bigcup_{f\in\mathrm{Emb}(\mathbb{X})} \{f\} \times \chi_{f[\omega]}\uparrow$ *is a closed set in the product* $\mathrm{Emb}(\mathbb{X}) \times 2^\omega$.

PROOF OF CLAIM 3.4. Let $\langle f, x\rangle \in (\mathrm{Emb}(\mathbb{X}) \times 2^\omega) \setminus B$. Then $x \notin \chi_{f[\omega]}\uparrow$ and, hence, there is $n_0 \in \omega$ such that $x(n_0) < \chi_{f[\omega]}(n_0)$. Thus, first, $x(n_0) = 0$, which implies $x \in p_{n_0}^{-1}[\{0\}]$ and, second, $\chi_{f[\omega]}(n_0) = 1$, that is $n_0 \in f[\omega]$ so there is $k_0 \in \omega$ satisfying $f(k_0) = n_0$ and, hence, $f \in \pi_{k_0}^{-1}[\{n_0\}]$. Now we have $\langle f, x\rangle \in O = (\pi_{k_0}^{-1}[\{n_0\}] \cap \mathrm{Emb}(\mathbb{X})) \times p_{n_0}^{-1}[\{0\}]$ and we show that $O \cap B = \emptyset$. Suppose that $\langle g, y\rangle \in O \cap B$. Then, since $\langle g, y\rangle \in O$, we have $g(k_0) = n_0$ and $y(n_0) = 0$; since $\langle g, y\rangle \in B$ we have $y \geq \chi_{g[\omega]}$, which implies $\forall n \in g[\omega]\ \ y(n) = 1$. So $y(n_0) = 0$ implies $n_0 \notin g[\omega]$, which is not true because $g(k_0) = n_0$. Thus O is a neighborhood of $\langle f, x\rangle$ contained in $(\mathrm{Emb}(\mathbb{X}) \times 2^\omega) \setminus B$ and this set is open. ⊣

CLAIM 3.5. $\chi[\mathbb{P}(\mathbb{X})\uparrow] = \pi_{2^\omega}[B]$, *where* $\pi_{2^\omega} : \mathrm{Emb}(\mathbb{X}) \times 2^\omega \to 2^\omega$ *is the projection*.

PROOF OF CLAIM 3.5. If $x \in \chi[\mathbb{P}(\mathbb{X})\uparrow]$, then there are $C \in \mathbb{P}(\mathbb{X})$ and A such that $C \subset A \subset \omega$ and $x = \chi_A$. Let $f \in \mathrm{Emb}(\mathbb{X})$, where $C = f[\omega]$. Then $f[\omega] \subset A$ implies $x \geq \chi_{f[\omega]}$ and, hence, $\langle f, x\rangle \in B$ and $x = \pi_{2^\omega}(\langle f, x\rangle) \in \pi_{2^\omega}[B]$.

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If $x \in \pi_{2^\omega}[B]$, then there is $f \in \mathrm{Emb}(\mathbb{X})$ such that $x \geq \chi_{f[\omega]}$ and for $A = x^{-1}[\{1\}]$ we have $x = \chi_A \geq \chi_{f[\omega]}$, which implies $\mathbb{P}(\mathbb{X}) \ni f[\omega] \subset A$, that is $A \in \mathbb{P}(\mathbb{X})\uparrow$ and, hence, $x = \chi(A) \in \chi[\mathbb{P}(\mathbb{X})\uparrow]$. ⊣

By Claim 3.2, $\mathrm{Emb}(\mathbb{X})$ is a Polish space so $\mathrm{Emb}(\mathbb{X}) \times 2^\omega$ is a Polish space too. Since the projection π_{2^ω} is continuous, it is a Borel mapping and, by Claim 3.4, $\pi_{2^\omega}[B]$ is an analytic set (see [1], p. 86). By Claim 3.5, the set $\chi[\mathbb{P}(\mathbb{X})\uparrow]$ is analytic.

(c) follows from (b) and the equality $\mathcal{I}_{\mathbb{X}} = P(X) \setminus \mathbb{P}(\mathbb{X})\uparrow$.

(d) By Theorem 2.2(d), $\mathbb{P}(\mathbb{X}) = \{X\}$ or $|\mathbb{P}(\mathbb{X})| \geq \aleph_0$. Now, the statement follows from (a), (b) and known facts about analytic sets (see [1]). ⊣

Concerning the complexity of the ideal $\mathcal{I}_{\mathbb{X}}$ we note that, for example, the ideal $\mathcal{I}_{\mathbb{Q}}$ of scattered subsets of the rational line, $\mathbb{Q}$, (see Section 9) is Π^1_1-complete and, hence, not Borel (apply the rank method and 34.18 of [1]).

§4. The separative quotient. A partial order $\mathbb{P} = \langle P, \leq \rangle$ is called *separative* iff for each $p, q \in P$ satisfying $p \not\leq q$ there is $r \in P$ such that $r \leq p$ and $r \perp q$. The *separative modification* of $\mathbb{P}$ is the separative preorder $\mathrm{sm}(\mathbb{P}) = \langle P, \leq^* \rangle$, where $p \leq^* q$ iff $\forall r \leq p\ \exists s \leq r\ s \leq q$. The *separative quotient* of $\mathbb{P}$ is the separative partial order $\mathrm{sq}(\mathbb{P}) = \langle P/=^*, \trianglelefteq \rangle$, where $p =^* q \Leftrightarrow p \leq^* q \wedge q \leq^* p$ and $[p] \trianglelefteq [q] \Leftrightarrow p \leq^* q$.

If κ is a regular cardinal, a preorder $\mathbb{P} = \langle P, \leq \rangle$ is *κ-closed* iff for each $\gamma < \kappa$ each sequence $\langle p_\alpha : \alpha < \gamma \rangle$ in P, such that $\alpha < \beta \Rightarrow p_\beta \leq p_\alpha$, has a lower bound. ω_1-closed preorders are called *σ-closed* and the following facts are well known.

Fact 4.1. *Let $\mathbb{P}$ be a partial order. Then*

(a) $\mathbb{P}$, $\mathrm{sm}(\mathbb{P})$, *and* $\mathrm{sq}(\mathbb{P})$ *are forcing equivalent forcing notions*;

(b) $\mathbb{P}$ *is atomless iff* $\mathrm{sm}(\mathbb{P})$ *is atomless iff* $\mathrm{sq}(\mathbb{P})$ *is atomless.*

Fact 4.2. *If $\kappa^{<\kappa} = \kappa$, then all atomless separative κ-closed preorders of size κ, are forcing equivalent (e.g., to the tree $\langle {}^{<\kappa}\kappa, \supset \rangle$).*

Theorem 4.3. *Let $\mathbb{X} = \langle X, \{\rho_i : i \in I\} \rangle$ be a relational structure. Then*

(a) $\mathrm{sm}\langle \mathbb{P}(\mathbb{X}), \subset \rangle = \langle \mathbb{P}(\mathbb{X}), \leq^* \rangle$, *where for $A, B \in \mathbb{P}(\mathbb{X})$*

$$A \leq^* B \Leftrightarrow \forall C \in \mathbb{P}(\mathbb{X})\ \ (C \subset A \Rightarrow \exists D \in \mathbb{P}(\mathbb{X})\ \ D \subset C \cap B); \qquad (1)$$

(b) $|\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle| = 1$ *iff* $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ *is atomic*;

(c) $|\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle| \geq \aleph_0$ *iff* $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ *is atomless*;

(d) *If* $|\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle| = \aleph_0$, *then* $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ *is forcing equivalent to the reversed binary tree* $\langle {}^{<\omega}2, \supset \rangle$ (*a forcing notion adding one Cohen real*);

(e) *If CH holds and* $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ *is σ-closed, atomless and of size* $\mathfrak{c}$, *then* $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ *is forcing equivalent to* $(P(\omega)/\mathrm{Fin})^+$.

Proof.

(a) This follows directly from the definition of the separative modification.

(b) If $|\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle| = 1$, then for each $A, B \in \mathbb{P}(\mathbb{X})$ we have $A \leq^* B$ so, by (1), there is $D \in \mathbb{P}(\mathbb{X})$ such that $D \subset A \cap B$. Thus $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is downwards directed and, hence, atomic.
If $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is atomic and $A, B \in \mathbb{P}(\mathbb{X})$, then, by Theorem 2.2(d), for each $C \in \mathbb{P}(\mathbb{X})$ satisfying $C \subset A$ there is $D \in \mathbb{P}(\mathbb{X})$ such that $D \subset C \cap B$. Thus,

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by (1), $A \leq^* B$, for each $A, B \in \mathbb{P}(\mathbb{X})$. Hence $A =^* B$, for each $A, B \in \mathbb{P}(\mathbb{X})$, and, consequently, $|\operatorname{sq}\langle \mathbb{P}(\mathbb{X}), \subset\rangle| = 1$.

(c) The implication "$\Rightarrow$" follows from (b) and Theorem 2.2(b). If the poset $\langle \mathbb{P}(\mathbb{X}), \subset\rangle$ is atomless, then it contains an infinite antichain $\{A_n : n \in \omega\}$. By (a), $A \leq^* B$ implies that A and B are compatible, thus $A_m \neq^* A_n$, for $m \neq n$, which implies that the set $\operatorname{sq}\langle \mathbb{P}(\mathbb{X}), \subset\rangle$ is infinite.

(d) If $|\operatorname{sq}\langle \mathbb{P}(\mathbb{X}), \subset\rangle| = \aleph_0$, then, by (c), the partial order $\langle \mathbb{P}(\mathbb{X}), \subset\rangle$ is atomless and, by Fact 4.1(b), $\operatorname{sq}\langle \mathbb{P}(\mathbb{X}), \subset\rangle$ is atomless as well. By Facts 4.1(a) and 4.2 (for $\kappa = \omega$), $\langle \mathbb{P}(\mathbb{X}), \subset\rangle$ is forcing equivalent to the forcing $\langle {}^{<\omega}\omega, \supset\rangle$ or to $\langle {}^{<\omega}2, \supset\rangle$.

(e) follows from Facts 4.1(a) and 4.2 (for $\kappa = \omega_1$). ⊣

Example 4.4. *$\langle \mathbb{P}(\mathbb{X}), \subset\rangle$ is a separative poset isomorphic to $\langle {}^{<\omega}2, \supset\rangle$. Let $G_{<\omega 2}$ be the digraph $\langle {}^{<\omega}2, \rho\rangle$, where $\rho = \{\langle \varphi, \varphi^\frown i\rangle : \varphi \in {}^{<\omega}2 \wedge i \in 2\}$. For $\varphi \in {}^{<\omega}2$, let $A_\varphi = \{\psi \in {}^{<\omega}2 : \varphi \subset \psi\}$ and let us prove that*

$$\mathbb{P}(G_{<\omega 2}) = \{A_\varphi : \varphi \in {}^{<\omega}2\}. \tag{2}$$

The inclusion "$\supset$" is evident. Conversely, if $A \in \mathbb{P}(G_{<\omega 2})$ and $f : G_{<\omega 2} \hookrightarrow G_{<\omega 2}$, where $A = f[{}^{<\omega}2]$, we show that $A = A_{f(\emptyset)}$.

First, if $f(\varphi) \in A$ and $\operatorname{dom}(\varphi) = n$, then, since $\langle \varphi \restriction k, \varphi \restriction (k+1)\rangle \in \rho$, for $k < n$, we have $\langle f(\varphi \restriction k), f(\varphi \restriction (k+1))\rangle \in \rho$, for $k < n$. But this is an oriented path from $f(\varphi \restriction 0) = f(\emptyset)$ to $f(\varphi \restriction n) = f(\varphi)$, which implies $f(\emptyset) \subset f(\varphi)$, that is $f(\varphi) \in A_{f(\emptyset)}$. Second, by induction we show that $f(\emptyset)^\frown\eta \in A$, for all $\eta \in {}^{<\omega}2$. Let $f(\emptyset)^\frown\eta \in A$. Then $f(\emptyset)^\frown\eta = f(\psi)$, for some $\psi \in {}^{<\omega}2$. Since $\langle \psi, \psi^\frown k\rangle \in \rho$, for $k \in \{0,1\}$, we have $\langle f(\psi), f(\psi^\frown k)\rangle \in \rho$ and, hence, $f(\psi^\frown k) = f(\psi)^\frown j_k = f(\emptyset)^\frown\eta^\frown j_k$, where $j_k \in \{0,1\}$. Since f is an injection we have $j_0 \neq j_1$ and, hence, $f(\emptyset)^\frown\eta^\frown 0$ and $f(\emptyset)^\frown\eta^\frown 1$ are elements of A. So $A = A_{f(\emptyset)}$ and the proof of (2) is finished.

Using (2) it is easy to see that $\langle {}^{<\omega}2, \supset\rangle \cong_F \langle \mathbb{P}(G_{<\omega 2}), \subset\rangle$, where $F(\varphi) = A_\varphi$.

§5. Indivisible structures. Forcing with quotients. A relational structure $\mathbb{X} = \langle X, \{\rho_i : i \in I\}\rangle$ is called *indivisible* iff for each partition $X = A \cup B$ we have $\mathbb{X} \hookrightarrow A$ or $\mathbb{X} \hookrightarrow B$. The aim of this section is to locate indivisible structures in our diagram.

Theorem 5.1. *A relational structure $\mathbb{X}$ is indivisible iff $\mathcal{I}_\mathbb{X}$ is an ideal in $P(X)$.*

Proof. Let $\mathbb{X}$ be a indivisible structure. Clearly, $\emptyset \in \mathcal{I}_\mathbb{X} \not\ni X$ and $I' \subset I \in \mathcal{I}_\mathbb{X}$ implies $I' \in \mathcal{I}_\mathbb{X}$. Suppose that $I \cup J \notin \mathcal{I}_\mathbb{X}$, for some $I, J \in \mathcal{I}_\mathbb{X}$. Then $C \subset I \cup J$, for some $C \in \mathbb{P}(\mathbb{X})$ and $C = (C \cap I) \cup (C \cap (J \setminus I))$. Since $C \cong \mathbb{X}$, C is indivisible and, hence, there is $A \in \mathbb{P}(C) \subset \mathbb{P}(\mathbb{X})$ such that $A \subset C \cap I$ or $A \subset C \cap (J \setminus I)$, which is impossible because $I, J \in \mathcal{I}_\mathbb{X}$. Thus $\mathcal{I}_\mathbb{X}$ is an ideal.

Let $\mathbb{X}$ be a divisible structure and let $X = A \cup B$ be a partition such that $\mathbb{X} \not\hookrightarrow A$ and $\mathbb{X} \not\hookrightarrow B$. Then $A, B \in \mathcal{I}_\mathbb{X}$ and, clearly, $A \cup B \notin \mathcal{I}_\mathbb{X}$. Thus $\mathcal{I}_\mathbb{X}$ is not an ideal. ⊣

Theorem 5.2. *If $\mathbb{X} = \langle X, \{\rho_i : i \in I\}\rangle$ is an indivisible relational structure, then*

(a) $\operatorname{sm}\langle \mathbb{P}(\mathbb{X}), \subset\rangle = \langle \mathbb{P}(\mathbb{X}), \subset_{\mathcal{I}_\mathbb{X}}\rangle$, *where $A \subset_{\mathcal{I}_\mathbb{X}} B \Leftrightarrow A \setminus B \in \mathcal{I}_\mathbb{X}$;*

(b) $\operatorname{sq}\langle \mathbb{P}(\mathbb{X}), \subset\rangle$ *is isomorphic to a dense subset of $\langle (P(X)/{=_{\mathcal{I}_\mathbb{X}}})^+, \leq_{\mathcal{I}_\mathbb{X}}\rangle$.*

Hence the poset $\langle \mathbb{P}(\mathbb{X}), \subset\rangle$ is forcing equivalent to $(P(X)/\mathcal{I}_\mathbb{X})^+$.

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PROOF. (a) Let $A \setminus B \in \mathcal{I}_{\mathbb{X}}$. If $C \in \mathbb{P}(\mathbb{X})$ and $C \subset A$, then $C \setminus B \in \mathcal{I}_{\mathbb{X}}$ and, since $\mathcal{I}_{\mathbb{X}}$ is an ideal and $C \notin \mathcal{I}_{\mathbb{X}}$, we have $C \cap B \notin \mathcal{I}_{\mathbb{X}}$ and, hence, $D \subset C \cap B$, for some $D \in \mathbb{P}(\mathbb{X})$. By (1) (see Theorem 4.3) we have $A \leq^* B$.

If $A \setminus B \notin \mathcal{I}_{\mathbb{X}}$, then $C \subset A \setminus B$, for some $C \in \mathbb{P}(\mathbb{X})$ and $C \cap B = \emptyset$ so, by (1), we have $\neg A \leq^* B$.

(b) By (a) and the definition of the separative quotient, we have $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset\rangle = \langle \mathbb{P}(\mathbb{X})/{=^*}, \trianglelefteq\rangle$, where for $A, B \in \mathbb{P}(\mathbb{X})$,

$$A =^* B \Leftrightarrow A \triangle B \in \mathcal{I}_{\mathbb{X}} \quad \text{and} \quad [A]_{=^*} \trianglelefteq [B]_{=^*} \Leftrightarrow A \setminus B \in \mathcal{I}_{\mathbb{X}}. \tag{3}$$

We show that $\langle \mathbb{P}(\mathbb{X})/{=^*}, \trianglelefteq\rangle \hookrightarrow_f \langle (P(X)/\mathcal{I}_{\mathbb{X}})^+, \leq_{\mathcal{I}_{\mathbb{X}}}\rangle$, where $f([A]_{=^*}) = [A]_{=_{\mathcal{I}_{\mathbb{X}}}}$. By (3) and (a), $[A]_{=^*} = [B]_{=^*}$ iff $A =^* B$ iff $A \triangle B \in \mathcal{I}_{\mathbb{X}}$ iff $A =_{\mathcal{I}_{\mathbb{X}}} B$ iff $[A]_{=_{\mathcal{I}_{\mathbb{X}}}} = [B]_{=_{\mathcal{I}_{\mathbb{X}}}}$ iff $f([A]_{=^*}) = f([B]_{=^*})$ and f is a well defined injection.

f is a strong homomorphism since $[A]_{=^*} \trianglelefteq [B]_{=^*}$ iff $A \setminus B \in \mathcal{I}_{\mathbb{X}}$ iff $[A]_{=_{\mathcal{I}_{\mathbb{X}}}} \leq_{\mathcal{I}_{\mathbb{X}}} [B]_{=_{\mathcal{I}_{\mathbb{X}}}}$ iff $f([A]_{=^*}) \leq_{\mathcal{I}_{\mathbb{X}}} f([B]_{=^*})$.

We prove that $f[\mathbb{P}(\mathbb{X})/{=^*}]$ is a dense subset of $(P(X)/{=_{\mathcal{I}_{\mathbb{X}}}})^+$. If $[S]_{=_{\mathcal{I}_{\mathbb{X}}}} \in (P(X)/{=_{\mathcal{I}_{\mathbb{X}}}})^+$, then $S \notin \mathcal{I}_{\mathbb{X}}$ and there is $A \in \mathbb{P}(\mathbb{X})$ such that $A \subset S$. Hence $A \subset_{\mathcal{I}_{\mathbb{X}}} S$ and $f([A]_{=^*}) = [A]_{=_{\mathcal{I}_{\mathbb{X}}}} \leq_{\mathcal{I}_{\mathbb{X}}} [S]_{=_{\mathcal{I}_{\mathbb{X}}}}$.

By Fact 4.1(a), these three posets are forcing equivalent. ⊣

Confirming a conjecture of Fraïssé, Pouzet proved that each countable indivisible structure $\mathbb{X}$ contains two disjoint copies of itself [8]. Thus, by Theorem 2.2(c), the poset $\langle \mathbb{P}(\mathbb{X}), \subset\rangle$ is atomless and, clearly, contains antichains of size $\aleph_0$. Moreover, we have

THEOREM 5.3. *If $\mathbb{X} = \langle \omega, \{\rho_i : i \in I\}\rangle$ is a countable indivisible structure, then the posets $\langle \mathbb{P}(\mathbb{X}), \subset\rangle$ and $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset\rangle$ are atomless and contain maximal antichains of size $\mathfrak{c}$. Hence $|\mathbb{P}(\mathbb{X})| = |\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset\rangle| = \mathfrak{c}$.*

PROOF. By Theorems 3.1(c) and 5.1, the set $\mathcal{I}_{\mathbb{X}}$ is an ideal having the Baire property and, since the complementation mapping $\chi_A \mapsto \chi_{\omega\setminus A}$ is an homeomorphism from 2^ω to 2^ω, the dual filter $\mathcal{I}_{\mathbb{X}}^*$ has the Baire property as well. Thus, by the well-known characterization of Talagrand [10], there is a partition of ω into finite sets K_n, $n \in \omega$, such that each element of $\mathcal{I}_{\mathbb{X}}^*$ intersects all but finitely many sets K_n, which means that each $I \in \mathcal{I}_{\mathbb{X}}$ contains finitely many sets K_n. So, for each $M \in [\omega]^\omega$ we have $S_M = \bigcup_{n\in M} K_n \notin \mathcal{I}_{\mathbb{X}}$. Let $\{M_\alpha : \alpha < \mathfrak{c}\} \subset [\omega]^\omega$ be an almost disjoint family and, for $\alpha < \mathfrak{c}$, let $A_\alpha \subset S_{M_\alpha}$, where $A_\alpha \in \mathbb{P}(\mathbb{X})$. Then $|A_\alpha \cap A_\beta| < \omega$, for different $\alpha, \beta < \mathfrak{c}$ and, hence $\{A_\alpha : \alpha < \mathfrak{c}\}$ is an antichain in the poset $\langle \mathbb{P}(\mathbb{X}), \subset\rangle$. Suppose that $\alpha \neq \beta$ and $[A]_{=^*} \trianglelefteq [A_\alpha]_{=^*}, [A_\beta]_{=^*}$, for some $A \in \mathbb{P}(\mathbb{X})$. Then, by (3), $A = (A \cap A_\alpha \cap A_\beta) \cup (A \setminus A_\alpha) \cup (A \setminus A_\beta) \in \mathcal{I}_{\mathbb{X}}$, which is not true. Thus $\{[A_\alpha]_{=^*} : \alpha < \mathfrak{c}\}$ is an antichain in the poset $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset\rangle$. ⊣

§6. Embedding-maximal structures. A relational structure $\mathbb{X}$ will be called *embedding-maximal* iff $\mathbb{P}(\mathbb{X}) = [X]^{|X|}$. In this section we characterize countable embedding-maximal structures and obtain more information on the structures which do not have this property. If $\mathbb{P} = \langle P, \leq\rangle$ is a partial order, a set $S \subset P$ is *somewhere dense* in $\mathbb{P}$ iff there is $p \in P$ such that for each $q \leq p$ there is $s \in S$ satisfying $s \leq q$. Otherwise, S is *nowhere dense*.

THEOREM 6.1. *For a countable binary relational structure $\mathbb{X} = \langle \omega, \rho\rangle$, the following conditions are equivalent*:

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(a) $\mathbb{P}(\mathbb{X}) = [\omega]^\omega$;
(b) $\mathbb{P}(\mathbb{X})$ *is a dense set in* $\langle [\omega]^\omega, \subset\rangle$;
(c) $\mathbb{X} = \langle \omega, \rho\rangle$ *is isomorphic to one of the following relational structures*:
  1 *The empty relation*, $\langle \omega, \emptyset\rangle$,
  2 *The complete graph*, $\langle \omega, \omega^2 \setminus \Delta_\omega\rangle$,
  3 *The natural strict linear order on* ω, $\langle \omega, <\rangle$,
  4 *The inverse of the natural strict linear order on* ω, $\langle \omega, <^{-1}\rangle$,
  5 *The diagonal relation*, $\langle \omega, \Delta_\omega\rangle$,
  6 *The full relation*, $\langle \omega, \omega^2\rangle$,
  7 *The natural linear order on* ω, $\langle \omega, \leq\rangle$,
  8 *The inverse of the natural linear order on* ω, $\langle \omega, \leq^{-1}\rangle$;
(d) $\mathbb{P}(\mathbb{X})$ *is a somewhere dense set in* $\langle [\omega]^\omega, \subset\rangle$;
(e) $\mathcal{I}_{\mathbb{X}} = \mathrm{Fin}$.

Then the poset $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset\rangle = (P(\omega)/\mathrm{Fin})^+$ *is atomless and* σ*-closed.*

Proof. The implication (a) $\Rightarrow$ (b) is trivial and it is easy to check (c) $\Rightarrow$ (a).
(b) $\Rightarrow$ (c). Let $\mathbb{P}(\mathbb{X})$ be a dense set in $\langle [\omega]^\omega, \subset\rangle$. ⊣

Claim 6.2. *The relation* ρ *is reflexive or irreflexive.*

Proof of Claim 6.2. If $R = \{x \in \omega : x\rho x\} \in [\omega]^\omega$, then there is $C \subset R$ such that $\langle \omega, \rho\rangle \cong \langle C, \rho_C\rangle$ and, since ρ_C is reflexive, ρ is reflexive as well. Otherwise we have $I = \{x \in \omega : \neg x\rho x\} \in [\omega]^\omega$ and, similarly, ρ must be irreflexive. ⊣

Claim 6.3. *If the relation* ρ *is irreflexive, then the structure* $\langle \omega, \rho\rangle$ *is isomorphic to one of the structures* 1–4 *from* (c).

Proof of Claim 6.3. Clearly, $[\omega]^2 = K_0 \cup K_1 \cup K_2 \cup K_3$, where the sets

$$K_0 = \{\{x, y\} \in [\omega]^2 : \neg x\rho y \wedge \neg y\rho x\},$$
$$K_1 = \{\{x, y\} \in [\omega]^2 : x\rho y \wedge y\rho x\},$$
$$K_2 = \{\{x, y\} \in [\omega]^2 : x\rho y \wedge \neg y\rho x \wedge x < y\},$$
$$K_3 = \{\{x, y\} \in [\omega]^2 : x\rho y \wedge \neg y\rho x \wedge x > y\},$$

are disjoint. By Ramsey's theorem, there are $H \in [\omega]^\omega$ and $i \in \{0, 1, 2, 3\}$ such that $[H]^2 \subset K_i$. Since $\mathbb{P}(\mathbb{X})$ is a dense set in $\langle [\omega]^\omega, \subset\rangle$, there is $C \subset H$ such that

$$\langle \omega, \rho\rangle \cong \langle C, \rho_C\rangle. \tag{4}$$

If $[H]^2 \subset K_0$, then for different $x, y \in C$ we have $\neg x\rho y$ and, since ρ is irreflexive, $\rho_C = \emptyset$. By (4) we have $\rho = \emptyset$.

If $[H]^2 \subset K_1$, then for different $x, y \in C$ we have $x\rho y$ and $y\rho x$. So, since ρ is irreflexive, $\rho_C = C^2 \setminus \Delta_C$, that is the structure $\langle C, \rho_C\rangle$ is a countable complete graph. By (4) we have $\rho = \omega^2 \setminus \Delta_\omega$.

If $[H]^2 \subset K_2$, then for different $x, y \in C$ we have

$$(x\rho y \wedge \neg y\rho x \wedge x < y) \vee (y\rho x \wedge \neg x\rho y \wedge y < x). \tag{5}$$

Let us prove that for each $x, y \in C$

$$x\rho y \Leftrightarrow x < y. \tag{6}$$

If $x = y$, then, since ρ is irreflexive, we have $\neg x\rho y$ and, since $\neg x < y$, (6) is true.

If $x < y$, by (5) we have $x\rho y$ and (6) is true.

If $x > y$, by (5) we have $\neg x\rho y$ and, since $\neg x < y$, (6) is true again.

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Since (6) holds for each $x, y \in C$ we have $\rho_C = <_C$. Clearly $\langle C, <_C\rangle \cong \langle \omega, <\rangle$, which, together with (4), implies $\langle \omega, \rho\rangle \cong \langle \omega, <\rangle$.

If $[H]^2 \subset K_3$, then as in the previous case we show that $\langle \omega, \rho\rangle \cong \langle \omega, <^{-1}\rangle$. ⊣

CLAIM 6.4. *If the relation ρ is reflexive and $\mathbb{Y} = \langle \omega, \rho \setminus \Delta_\omega\rangle$, then*

(i) *$\mathbb{P}(\mathbb{Y})$ is a dense set in $\langle [\omega]^\omega, \subset\rangle$;*
(ii) *The structure $\langle \omega, \rho\rangle$ is isomorphic to one of the structures* 5–8 *from* (*c*).

PROOF OF CLAIM 6.4. (i) Let $A \in [\omega]^\omega$, $C \subset A$ and $\langle \omega, \rho\rangle \cong_f \langle C, \rho_c\rangle$. Then, since f is an isomorphism, we have $\langle x_1, x_2\rangle \in \rho \setminus \Delta_\omega$ iff $\langle x_1, x_2\rangle \in \rho \wedge x_1 \neq x_2$ iff $\langle f(x_1), f(x_2)\rangle \in \rho_C \wedge f(x_1) \neq f(x_2)$ iff $\langle f(x_1), f(x_2)\rangle \in \rho_C \setminus \Delta_\omega = (\rho \setminus \Delta_\omega)_C$. Thus $\langle \omega, \rho \setminus \Delta_\omega\rangle \cong_f \langle C, (\rho \setminus \Delta_\omega)_C\rangle$, which implies $C \in \mathbb{P}(\mathbb{Y})$.

(ii) Since $\rho \setminus \Delta_\omega$ is an irreflexive relation, by (i) and Claim 6.3, the structure $\langle \omega, \rho \setminus \Delta_\omega\rangle$ is isomorphic to one of the structures 1–4. Hence the structure $\langle \omega, \rho\rangle$ is isomorphic to one of the structures 5–8.

(b) $\Leftrightarrow$ (e). Since $\mathcal{I}_\mathbb{X} = P(\omega) \setminus (\mathbb{P}(\mathbb{X})\uparrow)$ we have: $\mathbb{P}(\mathbb{X})$ is a dense set in $\langle [\omega]^\omega, \subset\rangle$ iff $\mathbb{P}(\mathbb{X})\uparrow = [\omega]^\omega$ iff $\mathcal{I}_\mathbb{X} = \mathrm{Fin}$.

(b) $\Rightarrow$ (d) is trivial.

(d) $\Rightarrow$ (b) Let $\mathbb{P}(\mathbb{X})$ be dense below $A \in [\omega]^\omega$. Then there are $C \subset A$ and f such that $\mathbb{X} \cong_f \langle C, \rho_C\rangle$ and, by the assumption,

$$\forall B \in [C]^\omega \ \ \exists D \in \mathbb{P}(\mathbb{X}) \ \ D \subset B. \tag{7}$$

For $S \in [\omega]^\omega$ we have $f[S] \in [C]^\omega$ and, by (7), there is $D \subset f[S]$ such that $\mathbb{X} \cong \langle D, \rho_D\rangle$. Since f is an injection we have $f^{-1}[D] \subset S$; $D \subset f[S]$ implies $f[f^{-1}[D]] = D$ and, since f is an isomorphism, $\langle f^{-1}[D], \rho_{f^{-1}[D]}\rangle \cong_{f|f^{-1}[D]} \langle D, \rho_D\rangle$ and, hence, $f^{-1}[D] \in \mathbb{P}(\mathbb{X})$. Thus $\mathbb{P}(\mathbb{X})$ is a dense set in $\langle [\omega]^\omega, \subset\rangle$. ⊣

COROLLARY 6.5. *If $\mathbb{X} = \langle \omega, \rho\rangle$ is a countable binary relational structure, then*

(a) *$\mathbb{P}(\mathbb{X}) = [\omega]^\omega$ or $\mathbb{P}(\mathbb{X})$ is a nowhere dense set in $\langle [\omega]^\omega, \subset\rangle$;*
(b) *If $\mathbb{X}$ is indivisible, then $\mathcal{I}_\mathbb{X} = \mathrm{Fin}$ or $\mathcal{I}_\mathbb{X}$ is a tall ideal* (*that is, for each $S \in [\omega]^\omega$ there is $I \in \mathcal{I}_\mathbb{X} \cap [S]^\omega$*).

PROOF. (b) If $\mathcal{I}_\mathbb{X} \neq \mathrm{Fin}$, then, by Theorem 6.1, $\mathbb{P}(\mathbb{X})$ is a nowhere dense subset of $[\omega]^\omega$, so for $S \in [\omega]^\omega$ there is $I \in [S]^\omega$ such that $A \subset I$, for no $A \in \mathbb{P}(\mathbb{X})$, which means that $I \in \mathcal{I}_\mathbb{X}$. ⊣

§7. Embeddings of disconnected structures. If $\mathbb{X}_i = \langle X_i, \rho_i\rangle$, $i \in I$, are binary relational structures and $X_i \cap X_j = \emptyset$, for different $i, j \in I$, then the structure $\bigcup_{i\in I} \mathbb{X}_i = \langle \bigcup_{i\in I} X_i, \bigcup_{i\in I} \rho_i\rangle$ will be called the *disjoint union* of the structures $\mathbb{X}_i$, $i \in I$.

If $\langle X, \rho\rangle$ is a binary structure, then the transitive closure ρ_{rst} of the relation $\rho_{rs} = \Delta_X \cup \rho \cup \rho^{-1}$ (given by $x \ \rho_{rst} \ y$ iff there are $n \in \mathbb{N}$ and $z_0 = x, z_1, \ldots, z_n = y$ such that $z_i \ \rho_{rs} \ z_{i+1}$, for each $i < n$) is the minimal equivalence relation on X containing ρ. In the sequel, the relation ρ_{rst} will be denoted by $\sim_\rho$ or $\sim$. Then for $x \in X$ the corresponding element of the quotient $X/\sim$ will be denoted by $[x]_{\sim_\rho}$ or $[x]_\sim$ or only by $[x]$, if the context admits, and called the *component* of $\langle X, \rho\rangle$ containing x. The structure $\langle X, \rho\rangle$ will be called *connected* iff $|X/\sim\ | = 1$. The main result of this section is Theorem 7.5 describing embeddings of disconnected structures and providing several constructions in the sequel.

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LEMMA 7.1. *Let* $\langle X, \rho\rangle = \langle \bigcup_{i\in I} X_i, \bigcup_{i\in I} \rho_i\rangle$ *be a disjoint union of binary structures. Then for each* $i \in I$ *and each* $x \in X_i$ *we have*

(a) $[x] \subset X_i$;
(b) $[x] = X_i$, *if* $\langle X_i, \rho_i\rangle$ *is a connected structure.*

PROOF. (a) Let $y \in [x]$ and $z_0 = x, z_1, \ldots, z_n = y \in X$, where $z_k \ \rho_{rs}\ z_{k+1}$, for each $k < n$. Using induction we show that $z_k \in X_i$, for each $k \leq n$. Suppose that $z_k \in X_i$. Then $z_k \ \rho_{rs}\ z_{k+1}$ and, if $z_k = z_{k+1}$, we are done. If $\langle z_k, z_{k+1}\rangle \in \rho$, there is $j \in I$ such that $\langle z_k, z_{k+1}\rangle \in \rho_j \subset X_j \times X_j$ and, since $z_k \in X_i$, we have $j = i$ and, hence, $z_{k+1} \in X_i$. If $\langle z_k, z_{k+1}\rangle \in \rho^{-1}$, then $\langle z_{k+1}, z_k\rangle \in \rho$ and, similarly, $z_{k+1} \in X_i$ again.

(b) Let $\langle X_i, \rho_i\rangle$ be a connected structure and $y \in X_i$. Then $x \sim_{\rho_i} y$ and, hence, there are $z_0 = x, z_1, \ldots, z_n = y \in X_i$, where for each $k < n$ we have $z_k \ (\rho_i)_{rs}\ z_{k+1}$, that is $z_k = z_{k+1} \vee z_k \ \rho_i\ z_{k+1} \vee z_k \ (\rho_i)^{-1}\ z_{k+1}$, which implies $z_k \ \rho_{rs}\ z_{k+1}$. Thus $y \sim_\rho x$ and, hence, $y \in [x]$. ⊣

PROPOSITION 7.2. *If* $\langle X, \rho\rangle$ *is a binary structure, then* $\langle \bigcup_{x\in X}[x], \bigcup_{x\in X} \rho_{[x]}\rangle$ *is the unique representation of* $\langle X, \rho\rangle$ *as a disjoint union of connected relations.*

PROOF. Clearly $X = \bigcup_{x\in X}[x]$ is a partition of X and $\bigcup_{x\in X} \rho_{[x]} \subset \rho$. If $\langle x, y\rangle \in \rho$, then $x \sim y$, which implies $x, y \in [x]$. Hence $\langle x, y\rangle \in \rho \cap ([x] \times [x]) = \rho_{[x]}$ and we have $\rho = \bigcup_{x\in X} \rho_{[x]}$.

We show that the structures $\langle [x], \rho_{[x]}\rangle$, $x \in X$, are connected. Let $y \in [x]$ and $z_0 = x, z_1, \ldots, z_n = y \in X$, where $z_k \ \rho_{rs}\ z_{k+1}$, for each $k < n$. Using induction we show that

$$\forall k \leq n \ \ z_k \in [x]. \tag{8}$$

Suppose that $z_k \in [x]$. Then $z_k \ \rho_{rs}\ z_{k+1}$ and, if $z_k = z_{k+1}$, we are done. If $\langle z_k, z_{k+1}\rangle \in \rho$, then $x \sim_\rho z_k \sim_\rho z_{k+1}$ and, by the transitivity of $\sim_\rho$, we have $x \sim_\rho z_{k+1}$, that is, $z_{k+1} \in [x]$, and similarly if $\langle z_{k+1}, z_k\rangle \in \rho$.

For each $k < n$ we have $\langle z_k, z_{k+1}\rangle \in \Delta_X \cup \rho \cup \rho^{-1}$ so, by (8), $\langle z_k, z_{k+1}\rangle \in \Delta_{[x]} \cup \rho_{[x]} \cup \rho_{[x]}^{-1} = (\rho_{[x]})_{rs}$. Thus $x \sim_{\rho_{[x]}} y$ and, since the relation $\sim_{\rho_{[x]}}$ is symmetric, $y \sim_{\rho_{[x]}} x$, for each $y \in [x]$. Since the relation $\sim_{\rho_{[x]}}$ is transitive, for each $y, z \in [x]$ we have $y \sim_{\rho_{[x]}} z$ and, hence, $\langle [x], \rho_{[x]}\rangle$ is a connected structure.

For a proof of the uniqueness of the representation, suppose that $\langle X, \rho\rangle = \langle \bigcup_{i\in I} X_i, \bigcup_{i\in I} \rho_i\rangle$ is a disjoint union, where the structures $\langle X_i, \rho_i\rangle$, $i \in I$, are connected. By Lemma 7.1(b), for $i \in I$ and $x \in X_i$ we have $X_i = [x]$ and, hence, $\rho_i = \rho \cap (X_i \times X_i) = \rho \cap ([x] \times [x]) = \rho_{[x]}$. Thus $\langle X_i, \rho_i\rangle = \langle [x], \rho_{[x]}\rangle$. On the other hand, if $x \in X$, then $x \in X_i$, for some $i \in I$, and, similarly, $\langle [x], \rho_{[x]}\rangle = \langle X_i, \rho_i\rangle$. Consequently we have $\{\langle X_i, \rho_i\rangle : i \in I\} = \{\langle [x], \rho_{[x]}\rangle : x \in X\}$. ⊣

PROPOSITION 7.3. *Let* $\langle X, \rho\rangle$ *be a binary relational structure and* $\rho^c = (X \times X) \setminus \rho$ *the complement of* ρ. *Then*

(a) *At least one of the structures* $\langle X, \rho\rangle$ *and* $\langle X, \rho^c\rangle$ *is connected*;
(b) $\mathrm{Emb}\langle X, \rho\rangle = \mathrm{Emb}\langle X, \rho^c\rangle$ *and* $\mathbb{P}\langle X, \rho\rangle = \mathbb{P}\langle X, \rho^c\rangle$.

PROOF. (a) Suppose that the structure $\mathbb{X} = \langle X, \rho\rangle$ is disconnected. Then, by Proposition 7.2, $\mathbb{X}$ is the disjoint union of connected structures $\mathbb{X}_i = \langle X_i, \rho_i\rangle$, $i \in I$, and we show that $\langle X, \rho^c\rangle$ is connected. Let $x, y \in X$. If $x \in X_i$ and $y \in X_j$, where $i \neq j$, then $x \not\sim_\rho y$, which implies $\langle x, y\rangle \notin \rho$, thus $\langle x, y\rangle \in \rho^c$ and, hence, $x \sim_{\rho^c} y$. Otherwise, if $x, y \in X_i$, for some $i \in I$, then we pick $j \in I \setminus \{i\}$ and $z \in X_j$ and, as

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in the previous case, $x \sim_{\rho^c} z$ and $y \sim_{\rho^c} z$ and, since $\sim_{\rho^c}$ is an equivalence relation, $x \sim_{\rho^c} y$ again.

(b) If $f \in \mathrm{Emb}\langle X, \rho\rangle$, then f is an injection and for each $x, y \in X$ we have $\langle x, y\rangle \in \rho \Leftrightarrow \langle f(x), f(y)\rangle \in \rho$, that is $\langle x, y\rangle \in \rho^c \Leftrightarrow \langle f(x), f(y)\rangle \in \rho^c$ and, hence, $f \in \mathrm{Emb}\langle X, \rho^c\rangle$. The another implication has a similar proof. Now $\mathbb{P}\langle X, \rho\rangle = \{f[X] : f \in \mathrm{Emb}\langle X, \rho\rangle\} = \{f[X] : f \in \mathrm{Emb}\langle X, \rho^c\rangle\} = \mathbb{P}\langle X, \rho^c\rangle$. ⊣

LEMMA 7.4. *Let $\langle X, \rho\rangle$ and $\langle Y, \tau\rangle$ be binary structures and $f : X \to Y$ an embedding. Then for each $x_1, x_2, x \in X$*

(a) $x_1 \rho_{rs} x_2 \Leftrightarrow f(x_1) \tau_{rs} f(x_2)$;
(b) $x_1 \sim_\rho x_2 \Rightarrow f(x_1) \sim_\tau f(x_2)$;
(c) $f[[x]] \subset [f(x)]$;
(d) $f \mid [x] : [x] \to f[[x]]$ *is an isomorphism.*

If, in addition, f is an isomorphism, then

(e) $x_1 \sim_\rho x_2 \Leftrightarrow f(x_1) \sim_\tau f(x_2)$;
(f) $f[[x]] = [f(x)]$;
(g) $\langle X, \rho\rangle$ *is connected iff* $\langle Y, \tau\rangle$ *is connected.*

PROOF.

(a) Since f is an injection and a strong homomorphism we have $x_1 \ \rho_{rs} x_2$ iff $x_1 = x_2 \vee x_1 \ \rho \ x_2 \vee x_2 \ \rho \ x_1$ iff $f(x_1) = f(x_2) \vee f(x_1) \ \rho \ f(x_2) \vee f(x_2) \ \rho \ f(x_1)$ iff $f(x_1) \ \tau_{rs} f(x_2)$.
(b) If $x_1 \sim_\rho x_2$, then there are $z_0, z_1, \ldots, z_n \in X$ such that $x_1 = z_0 \ \rho_{rs} \ z_1 \ \rho_{rs} \ldots \rho_{rs} \ z_n = x_2$ and, by (a), $f(x_1) = f(z_0) \ \tau_{rs} \ f(z_1) \ \tau_{rs} \ldots \tau_{rs} \ f(z_n) = f(x_2)$ and, hence, $f(x_1) \sim_\tau f(x_2)$.
(c) If $x' \in [x]$, then $x' \sim_\rho x$ and, by (b), $f(x') \sim_\tau f(x)$ so $f(x') \in [f(x)]$.
(d) Clearly, $f|[x]$ is a bijection. Since f is a strong homomorphism, for $x_1, x_2 \in [x]$ we have $x_1 \ \rho x_2$ iff $f(x_1) \ \tau f(x_2)$ iff $(f|[x])(x_1) \ \tau (f|[x])(x_2)$.
(e) The implication "$\Rightarrow$" is proved in (b). If $f(x_1) \sim_\tau f(x_2)$, then, applying (b) to f^{-1} we obtain $x_1 \sim_\rho x_2$.
(f) The inclusion "$\subset$" is proved in (b). Let $y \in [f(x)]$, that is $y \sim_\tau f(x)$. Since f is a bijection there is $x' \in X$ such that $y = f(x')$ and, by (e), $x' \sim_\rho x$, that is $x' \in [x]$. Hence $y \in f[[x]]$.
(g) follows from (e). ⊣

THEOREM 7.5. *Let $\mathbb{X}_i = \langle X_i, \rho_i\rangle$, $i \in I$, and $\mathbb{Y}_j = \langle Y_j, \sigma_j\rangle$, $j \in J$, be two families of disjoint connected binary structures and $\mathbb{X}$ and $\mathbb{Y}$ their unions. Then*

(a) *$F : \mathbb{X} \hookrightarrow \mathbb{Y}$ iff there are $f : I \to J$ and $g_i : \mathbb{X}_i \hookrightarrow \mathbb{Y}_{f(i)}$, $i \in I$, such that $F = \bigcup_{i \in I} g_i$ and*

$$\forall \{i_1, i_2\} \in [I]^2 \ \forall x_{i_1} \in X_{i_1} \ \forall x_{i_2} \in X_{i_2} \ \neg \ g_{i_1}(x_{i_1}) \ \sigma_{rs} \ g_{i_2}(x_{i_2}). \tag{9}$$

(b) *$C \in \mathbb{P}(\mathbb{X})$ iff there are $f : I \to I$ and $g_i : \mathbb{X}_i \hookrightarrow \mathbb{X}_{f(i)}$, $i \in I$, such that $C = \bigcup_{i \in I} g_i[X_i]$ and*

$$\forall \{i, j\} \in [I]^2 \ \ \forall x \in X_i \ \ \forall y \in X_j \ \ \neg \ g_i(x) \ \rho_{rs} \ g_j(y). \tag{10}$$

PROOF. (a) ($\Rightarrow$) Let $F : \mathbb{X} \hookrightarrow \mathbb{Y}$. By Proposition 7.2, the sets X_i, $i \in I$, are components of $\mathbb{X}$ and Y_j, $j \in I$, are components of $\mathbb{Y}$. By Lemma 7.4(c), for $i \in I$ and $x \in X_i$ we have $F[[x]] \subset [F(x)]$ so there is (unique) $f(i) \in J$, such that $F[X_i] \subset Y_{f(i)}$. By Lemma 7.4(d), $F|X_i : X_i \to F[X_i] \subset Y_{f(i)}$ is an isomorphism

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and, hence, $g_i : \mathbb{X}_i \hookrightarrow \mathbb{Y}_{f(i)}$, where the mapping $g_i : X_i \to Y_{f(i)}$ is given by $g_i(x) = F(x)$. Clearly $f : I \to J$ and $F = \bigcup_{i\in I} g_i$. Suppose that $g_{i_1}(x_{i_1})\ \sigma_{rs}\ g_{i_2}(x_{i_2})$, that is $F(x_{i_1})\ \sigma_{rs}\ F(x_{i_2})$, for some different $i_1, i_2 \in I$ and some $x_{i_1} \in X_{i_1}$ and $x_{i_2} \in X_{i_2}$. Then, by Lemma 7.4(a), $x_{i_1}\ \rho_{rs}\ x_{i_2}$ and, hence, $x_{i_1} \sim_\rho x_{i_2}$, which is not true, because x_{i_1} and x_{i_2} are elements of different components of $\mathbb{X}$.

($\Leftarrow$) Let $F = \bigcup_{i\in I} g_i$, where the functions $f : I \to J$ and $g_i : \mathbb{X}_i \hookrightarrow \mathbb{Y}_{f(i)}$, $i \in I$, satisfy the given conditions.

Let $u, v \in X$, where $u \neq v$. If $u, v \in X_i$ for some $i \in I$ then, since g_i is an injection, we have $F(u) = g_i(u) \neq g_i(v) = F(v)$. Otherwise $u \in X_{i_1}$ and $v \in X_{i_2}$, where $i_1 \neq i_2$ and, by the assumption, $\neg\ g_{i_1}(u)\ \sigma_{rs}\ g_{i_2}(v)$, which implies $g_{i_1}(u) \neq g_{i_2}(v)$ that is $F(u) \neq F(v)$. Thus F is an injection.

In order to prove that F is a strong homomorphism we take $u, v \in X$ and prove

$$u\ \rho\ v \Leftrightarrow F(u)\ \sigma F(v). \tag{11}$$

If $u, v \in X_i$, for some $i \in I$, then we have: $u\ \rho\ v$ iff $u\ \rho_i\ v$ (since $\rho_{X_i} = \rho_i$) iff $g_i(u)\ \sigma_{f(i)}\ g_i(v)$ (because $g_i : \mathbb{X}_i \hookrightarrow \mathbb{Y}_{f(i)}$) iff $g_i(u)\ \sigma\ g_i(v)$ (since $\sigma_{Y_{f(i)}} = \sigma_{f(i)}$) iff $F(u)\ \sigma\ F(v)$ (because $F \restriction X_i = g_i$). So (11) is true.

If $u \in X_{i_1}$ and $v \in X_{i_2}$, where $i_1 \neq i_2$, then $\neg u\ \rho\ v$, because u and v are in different components of X. By the assumption we have $\neg\ g_{i_1}(u)\ \sigma_{rs}\ g_{i_2}(v)$, which implies $\neg\ g_{i_1}(u)\ \sigma\ g_{i_2}(v)$, that is $\neg\ F(u)\ \sigma\ F(v)$. So (11) is true again.

(b) follows from (a) and the fact that $C \in \mathbb{P}(\mathbb{X})$ iff there is $F : \mathbb{X} \hookrightarrow \mathbb{X}$ such that $C = F[X]$. ⊣

§8. Embedding-incomparable components. Two structures $\mathbb{X}$ and $\mathbb{Y}$ will be called *embedding-incomparable* iff $\mathbb{X} \not\hookrightarrow \mathbb{Y}$ and $\mathbb{Y} \not\hookrightarrow \mathbb{X}$. We will use the following fact.

FACT 8.1. *Let $\mathbb{P}, \mathbb{Q}$ and $\mathbb{P}_i$, $i \in I$, be partial orderings. Then,*

(a) *If $\mathbb{P} \cong \mathbb{Q}$, then $\operatorname{sm}\mathbb{P} \cong \operatorname{sm}\mathbb{Q}$ and $\operatorname{sq}\mathbb{P} \cong \operatorname{sq}\mathbb{Q}$;*
(b) $\operatorname{sm}(\prod_{i\in I} \mathbb{P}_i) = \prod_{i\in I} \operatorname{sm}\mathbb{P}_i$;
(c) $\operatorname{sq}(\prod_{i\in I} \mathbb{P}_i) \cong \prod_{i\in I} \operatorname{sq}\mathbb{P}_i$.

THEOREM 8.2. *Let ρ be a binary relation on a set X. If the components $\mathbb{X}_i = \langle X_i, \rho_{X_i}\rangle$, $i \in I$, of a disconnected structure $\mathbb{X} = \langle X, \rho\rangle$ are embedding-incomparable, then*

(a) $\langle \mathbb{P}(\mathbb{X}), \subset\rangle \cong \prod_{i\in I} \langle \mathbb{P}(\mathbb{X}_i), \subset\rangle$;
(b) $\operatorname{sq}\langle \mathbb{P}(\mathbb{X}), \subset\rangle \cong \prod_{i\in I} \operatorname{sq}\langle \mathbb{P}(\mathbb{X}_i), \subset\rangle$.
(c) *$\mathbb{X}$ is a divisible structure.*

PROOF. (a) By Theorem 7.5(b) and since the structures $\mathbb{X}_i$ are embedding-incomparable, $C \in \mathbb{P}(\mathbb{X})$ iff there are embeddings $g_i : \mathbb{X}_i \hookrightarrow \mathbb{X}_i$, $i \in I$, such that $C = \bigcup_{i\in I} g_i[X_i]$ and $\neg\ g_i(x)\ \rho_{rs}\ g_j(y)$, for each different $i, j \in I$, each $x \in X_i$ and each $y \in X_j$. But, since $i \neq j$, $x \in X_i$ and $y \in X_j$ implies $g_i(x) \in X_i$ and $g_j(y) \in X_j$, it is impossible that $g_i(x)\ \rho_{rs}\ g_j(y)$ and, hence, the last condition is implied by the condition that $g_i : \mathbb{X}_i \hookrightarrow \mathbb{X}_i$, for each $i \in I$. Consequently, $\mathbb{P}(\mathbb{X}) = \{\bigcup_{i\in I} C_i : \langle C_i : i \in I\rangle \in \prod_{i\in I} \mathbb{P}(\mathbb{X}_i)\}$ and it is easy to check that the mapping $f : \prod_{i\in I} \langle \mathbb{P}(\mathbb{X}_i) \subset\rangle \to \langle \mathbb{P}(\mathbb{X}), \subset\rangle$ given by $f(\langle C_i : i \in I\rangle) = \bigcup_{i\in I} C_i$ is an isomorphism of posets.

(b) follows from (a) and Fact 8.1(a) and (c).
(c) The partition $X = X_i \cup (X \setminus X_i)$ witnesses that $\mathbb{X}$ is divisible. ⊣

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§9. From $\mathbf{A_1}$ to $\mathbf{D_5}$. In this section we show that the diagram on Figure 1 is correct. The relations between the properties of X and $\mathbb{P}(\mathbb{X})$ are established in the previous sections. Since $|\operatorname{sq}\langle\mathbb{P}(\mathbb{X}),\subset\rangle| \leq |\mathbb{P}(\mathbb{X})|$, the classes B_1, C_1, D_1, C_2, and D_2 are empty and, since $\operatorname{sq}\langle[\omega]^\omega,\subset\rangle = (P(\omega)/\operatorname{Fin})^+$ is a σ-closed atomless poset, the classes A_5, B_5, and C_5 are empty as well. By Theorem 5.3 we have $A_4 = B_4 = \emptyset$ and in the sequel we show that the remaining classes contain some structures. First, the graph $G_{\mathbb{Z}}$ mentioned in the Introduction belongs to A_1 and its restriction to $\mathbb{N}$ to A_2. The class B_2 contains the digraph constructed in Example 4.4 and in the following examples we construct some structures from A_3, B_3, and C_3.

EXAMPLE 9.1. *$\langle\mathbb{P}(\mathbb{X}),\subset\rangle$ collapses $\mathfrak{c}$ to ω and $\mathbb{X}$ is a divisible structure belonging to C_3. Let $\mathbb{X} = \langle X,\rho\rangle = \langle\bigcup_{n\geq 3} G'_n, \bigcup_{n\geq 3}\rho'_n\rangle$, where the sets G'_n, $n \geq 3$, are pairwise disjoint and $\langle G'_n,\rho'_n\rangle \cong \langle G_n,\rho_n\rangle$, where the structure $\langle G_n,\rho_n\rangle$ is the directed graph defined by $G_n = {}^{<\omega}2 \times \{0,1,\ldots,n-1\}$ and*

$$\rho_n = \{\langle\langle\varphi,0\rangle,\langle\varphi^\frown k,0\rangle\rangle : \varphi \in {}^{<\omega}2 \wedge k \in 2\} \cup \\ \{\langle\langle\varphi,i\rangle,\langle\varphi,j\rangle\rangle : \varphi \in {}^{<\omega}2 \wedge \langle i,j\rangle \in \{\langle 0,1\rangle,\langle 1,2\rangle,\ldots,\langle n-1,0\rangle\}\}.$$

Using the obvious fact that two cycle graphs of different size are embedding incomparable we easily prove that for different $m,n \geq 3$ the structures $\langle G_m,\rho_m\rangle$ and $\langle G_n,\rho_n\rangle$ are embedding incomparable as well so, by (a) of Theorem 8.2,

$$\langle\mathbb{P}(\mathbb{X}),\subset\rangle \cong \textstyle\prod_{n\geq 3}\langle\mathbb{P}(\langle G_n,\rho_n\rangle),\subset\rangle. \tag{12}$$

Let $n \geq 3$. Like in Example 4.4, for $\varphi \in {}^{<\omega}2$ let $A_\varphi = \{\psi \in {}^{<\omega}2 : \varphi \subset \psi\}$ and $B_\varphi = A_\varphi \times \{0,1,\ldots,n-1\}$. Let us prove that

$$\mathbb{P}(\langle G_n,\rho_n\rangle) = \{B_\varphi : \varphi \in {}^{<\omega}2\}. \tag{13}$$

The inclusion "$\supset$" is evident. Conversely, let $B \in \mathbb{P}(\langle G_n,\rho_n\rangle)$ and $f : \langle G_n,\rho_n\rangle \hookrightarrow \langle G_n,\rho_n\rangle$, where $B = f[G_n]$. Clearly, $\deg(v) \in \{4,5\}$, for each vertex $v \in {}^{<\omega}2 \times \{0\}$, and $\deg(v) = 2$, otherwise Thus, since f preserves degrees of vertices we have $f[{}^{<\omega}2 \times \{0\}] \subset {}^{<\omega}2 \times \{0\}$ and $f \restriction {}^{<\omega}2 \times \{0\} : {}^{<\omega}2 \times \{0\} \hookrightarrow {}^{<\omega}2 \times \{0\}$. Since the digraph ${}^{<\omega}2 \times \{0\}$ is isomorphic to the digraph $G_{<\omega 2}$, by Example 4.4, there is $\varphi \in {}^{<\omega}2$ such that

$$f[{}^{<\omega}2 \times \{0\}] = A_\varphi \times \{0\}. \tag{14}$$

Now, since each $v \in G_n$ belongs to a unique cycle graph with n vertices and f preserves this property by (14) we have $B = f[G_n] = B_\varphi$ and (13) is proved.

By (13), like in Example 4.4 we prove that $\langle\mathbb{P}(\langle G_n,\rho_n\rangle),\subset\rangle \cong \langle{}^{<\omega}2,\supset\rangle$. Thus, by (13), the poset $\langle\mathbb{P}(\mathbb{X}),\subset\rangle$ is isomorphic to the direct product $\langle{}^{<\omega}2,\supset\rangle^\omega$ of countably many Cohen posets which collapses $\mathfrak{c}$ to ω (see [2], (E4), p. 294). The partition $X = G_3 \cup (X \setminus G_3)$ witnesses that $\mathbb{X}$ is a divisible structure.

EXAMPLE 9.2. *$\langle\mathbb{P}(\mathbb{X}),\subset\rangle$ is an atomic poset of size $\mathfrak{c}$ and $\mathbb{X} \in A_3$. Let $\mathbb{X} = \langle X,\rho\rangle = \langle\bigcup_{n\geq 3} G'_n, \bigcup_{n\geq 3}\rho'_n\rangle$, where the sets G'_n, $n \geq 3$, are pairwise disjoint and $\langle G'_n,\rho'_n\rangle$ is isomorphic to the digraph $\langle G_n,\rho_n\rangle$ given by $G_n = \omega \times \{0,1,\ldots,n-1\}$ and*

$$\rho_n = \{\langle\langle n,0\rangle,\langle n+1,0\rangle\rangle : n \in \omega\} \cup \\ \{\langle\langle n,i\rangle,\langle n,j\rangle\rangle : n \in \omega \wedge \langle i,j\rangle \in \{\langle 0,1\rangle,\langle 1,2\rangle,\ldots,\langle n-1,0\rangle\}\}.$$

As in Example 9.1 we prove that for different $m,n \geq 3$ the structures $\langle G_m,\rho_m\rangle$ and $\langle G_n,\rho_n\rangle$ are embedding incomparable so, by (a) of Theorem 8.2,

$$\langle\mathbb{P}(\mathbb{X}),\subset\rangle \cong \textstyle\prod_{n\geq 3}\langle\mathbb{P}(\langle G_n,\rho_n\rangle),\subset\rangle. \tag{15}$$

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Let $n \geq 3$. Using the arguments from Example 9.1 *we easily prove that*

$$\mathbb{P}(\langle G_n, \rho_n \rangle) = \{B_k : k \in \omega\}, \tag{16}$$

where $B_k = (\omega \setminus k) \times \{0, 1, \ldots, n-1\}$, for $k \in \omega$.

By (16) *we have $\langle \mathbb{P}(\langle G_n, \rho_n \rangle), \subset \rangle \cong \langle \omega, \geq \rangle = \omega^*$. Thus, by* (15), *the poset $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is isomorphic to the direct product $(\omega^*)^\omega$ of countably many copies of ω^* which is an atomic lattice of size $\mathfrak{c}$.*

Example 9.3. *$\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle \cong \langle {}^{<\omega}2, \supset \rangle$ although $|\mathbb{P}(\mathbb{X})| = \mathfrak{c}$, thus $\mathbb{X} \in B_3$. Let $\mathbb{Y} = \langle Y, \rho \rangle$ be the digraph considered in Example* 4.4 *and $\mathbb{Z} = \langle Z, \sigma^c \rangle$, where $\langle Z, \sigma \rangle$ is isomorphic to the digraph from Example* 9.2 *and $Y \cap Z = \emptyset$. Since $\langle Z, \sigma \rangle$ is a disconnected structure, by Proposition* 7.3(*a*) *the structure $\mathbb{Z}$ is connected and, clearly, $\sigma^c = (Z \times Z) \setminus \sigma$ is a reflexive relation, which implies that the structures $\mathbb{Y}$ and $\mathbb{Z}$ are embedding incomparable. Thus, by Theorem* 8.2(*a*), *for the structure $\mathbb{X} = \mathbb{Y} \cup \mathbb{Z}$ we have $\langle \mathbb{P}(\mathbb{X}), \subset \rangle \cong \langle \mathbb{P}(\mathbb{Y}), \subset \rangle \times \langle \mathbb{P}(\mathbb{Z}), \subset \rangle$ and, since by Proposition* 7.3(*b*) *$\mathbb{P}(\mathbb{Z}) = \mathbb{P}(\langle Z, \sigma \rangle)$, we have $|\mathbb{P}(\mathbb{X})| = \mathfrak{c}$.*

By Theorem 8.2(*b*) *we have $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle \cong \mathrm{sq}\langle \mathbb{P}(\mathbb{Y}), \subset \rangle \times \mathrm{sq}\langle \mathbb{P}(\mathbb{Z}), \subset \rangle$. Since $\langle \mathbb{P}(\mathbb{Z}), \subset \rangle$ is an atomic poset, by Theorem* 4.3(*a*) *we have $|\mathrm{sq}\langle \mathbb{P}(\mathbb{Z}), \subset \rangle| = 1$ and, hence, $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle \cong \langle {}^{<\omega}2, \supset \rangle \times 1 \cong \langle {}^{<\omega}2, \supset \rangle$.*

In the sequel, we show that the remaining classes are non-empty and give more information about some basic classes of structures.

Linear orders. A linear order L is *scattered* iff it does not contain a dense suborder or, equivalently, a copy of the rationals, $\mathbb{Q}$. Otherwise L is a *nonscattered* linear order. So, if L is a countable linear order, we have the following cases.

Case 1: L is nonscattered. By [3], for each nonscattered linear order L the poset $\langle \mathbb{P}(L), \subset \rangle$ is forcing equivalent to the two-step iteration $\mathbb{S} * \pi$, where $\mathbb{S}$ is the Sacks forcing and $1_{\mathbb{S}} \Vdash$ "π is a σ-closed forcing". If the equality $\mathrm{sh}(\mathbb{S}) = \aleph_1$ or PFA holds in the ground model, then the second iterand is forcing equivalent to the poset $(P(\omega)/\mathrm{Fin})^+$ of the Sacks extension. So, if L is a countable nonscattered linear order, then forcing by $\langle \mathbb{P}(L), \subset \rangle$ produces reals. In addition, L is indivisible. Namely, if Q is a copy of $\mathbb{Q}$ in L and $L = A_0 \dot{\cup} A_1$, then, since $\mathbb{Q}$ is indivisible, there is $k \in \{0, 1\}$ such that $Q \cap A_k$ contains a copy of $\mathbb{Q}$ and, by the universality of $\mathbb{Q}$, $Q \cap A_k$ contains a copy of L as well. Hence, $L \in C_4$.

Case 2: L is scattered. By [5], for each countable scattered linear order L, the partial ordering $\mathrm{sq}\langle \mathbb{P}(L), \subset \rangle$ is atomless and σ-closed. In particular, if α is a countable ordinal and $\alpha = \omega^{\gamma_n + r_n} s_n + \cdots + \omega^{\gamma_0 + r_0} s_0 + k$ its representation in the Cantor normal form, where $k \in \omega$, $r_i \in \omega$, $s_i \in \mathbb{N}$, $\gamma_i \in \mathrm{Lim} \cup \{1\}$ and $\gamma_n + r_n > \cdots > \gamma_0 + r_0$, then by [6]

$$\mathrm{sq}\langle \mathbb{P}(\alpha), \subset \rangle \cong \prod_{i=0}^{n} \left(\left(\mathrm{rp}^{r_i} \left(P(\omega^{\gamma_i}) / \mathcal{I}_{\omega^{\gamma_i}} \right) \right)^+ \right)^{s_i}, \tag{17}$$

where, for an ordinal β, $\mathcal{I}_\beta = \{C \subset \beta : \beta \not\leftrightarrow C\}$ and, for a poset $\mathbb{P}$, $\mathrm{rp}(\mathbb{P})$ denotes the reduced power $\mathbb{P}^\omega / \equiv_{\mathrm{Fin}}$ and $\mathrm{rp}^{k+1}(\mathbb{P}) = \mathrm{rp}(\mathrm{rp}^k(\mathbb{P}))$. In particular, for $\omega \leq \alpha < \omega^\omega$ we have

$$\mathrm{sq}\left(\mathbb{P}\left(\textstyle\sum_{i=n}^{0} \omega^{1+r_i} s_i \right), \subset \right) \cong \prod_{i=0}^{n} \left(\left(\mathrm{rp}^{r_i} \left(P(\omega) / \mathrm{Fin} \right) \right)^+ \right)^{s_i}. \tag{18}$$

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Thus if L is a scattered linear order, then $L \in D_3 \cup D_4 \cup D_5$ and, for example, $\omega + \omega \in D_3$, $\omega \cdot \omega \in D_4$, and $\omega \in D_5$, since an ordinal $\alpha < \omega_1$ is an indivisible structure iff $\alpha = \omega^\beta$, for some ordinal $\beta > 0$.
So, under the CH, for a countable linear order L the poset $\langle \mathbb{P}(L), \subset \rangle$ is forcing equivalent to $\mathbb{S} * \pi$, where $1_{\mathbb{S}} \Vdash$ "$\pi = (P(\check{\omega})/\operatorname{Fin})^+$", if L is nonscattered; and to $(P(\omega)/\operatorname{Fin})^+$, if L is scattered. But it is consistent that the poset $\langle \mathbb{P}(\omega + \omega), \subset \rangle$ is not forcing equivalent to $(P(\omega)/\operatorname{Fin})^+$: by (18) we have $\operatorname{sq}\langle \mathbb{P}(\omega + \omega), \subset \rangle \cong (P(\omega)/\operatorname{Fin})^+ \times (P(\omega)/\operatorname{Fin})^+$ and, by a result of Shelah and Spinas [9], it is consistent that $(P(\omega)/\operatorname{Fin})^+$ and its square are not forcing equivalent.

Equivalence relations and similar structures. By a more general theorem from [4] we have: If $\mathbb{X}_i = \langle X_i, \rho_{X_i} \rangle$, $i \in I$, are the components of a countable binary structure $\mathbb{X} = \langle X, \rho \rangle$, which is

- either an equivalence relation,
- or a disjoint union of complete graphs,
- or a disjoint union of ordinals $\leq \omega$,

then $\operatorname{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is a σ-closed atomless poset. More precisely, if $N = \{|X_i| : i \in I\}$, $N_{\text{fin}} = N \setminus \{\omega\}$, $I_\kappa = \{i \in I : |X_i| = \kappa\}$, $\kappa \in N$, and $|I_\omega| = \mu$, then the following table describes a forcing equivalent and some cardinal invariants of $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$

$\mathbb{X}$	$\operatorname{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is forcing equivalent to	$\operatorname{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is	ZFC $\vdash \operatorname{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is $\mathfrak{h}$-distributive
$N \in [\mathbb{N}]^{<\omega}$ or $\lvert I\rvert = 1$	$(P(\omega)/\operatorname{Fin})^+$	$\mathfrak{t}$-closed	YES
$0 < \lvert N_{\text{fin}}\rvert, \lvert I_\omega\rvert < \omega$	$((P(\omega)/\operatorname{Fin})^+)^n$	$\mathfrak{t}$-closed	NO
$\lvert I_\omega\rvert < \omega = \lvert N_{\text{fin}}\rvert$	$(P(\Delta)/\mathcal{ED}_{\text{fin}})^+ \times ((P(\omega)/\operatorname{Fin})^+)^\mu$	σ-closed	NO
$\lvert I_\omega\rvert = \omega$	$(P(\omega \times \omega)/(\operatorname{Fin} \times \operatorname{Fin}))^+$	σ-closed, not ω_2-closed	NO

where $\Delta = \{\langle m, n \rangle \in \mathbb{N} \times \mathbb{N} : n \leq m\}$ and the ideal $\mathcal{ED}_{\text{fin}}$ in $P(\Delta)$ is defined by $\mathcal{ED}_{\text{fin}} = \{S \subset \Delta : \exists r \in \mathbb{N}\ \forall m \in \mathbb{N}\ |S \cap (\{m\} \times \{1, 2, \ldots, m\})| \leq r\}$.

The structure $\mathbb{X}$ is indivisible iff $N \in [\mathbb{N}]^\omega$ or $N = \{1\}$ or $|I| = 1$ or $|I_\omega| = \omega$.

Thus if $\mathbb{X}$ is a countable equivalence relation, then $\mathbb{X} \in D_3 \cup D_4 \cup D_5$ and some examples of such structures are given in the diagram in Figure 2. We remark that, if F_κ denotes the full relation on a set of size κ, the following countable equivalence relations are ultrahomogeneous: $\bigcup_\omega F_n$ (indivisible iff $n = 1$); $\bigcup_n F_\omega$ (indivisible iff $n = 1$) and $\bigcup_\omega F_\omega$ (the ω-homogeneous-universal equivalence relation, indivisible of course).
The same picture is obtained for

- Disconnected countable ultrahomogeneous graphs, which are (by the well-known classification of Lachlan and Woodrow) of the form $\bigcup_m K_n$, where $mn = \omega$ (the disjoint union of m-many complete graphs of size n);
- Countable posets of the form $\bigcup_m L_n$, where $mn = \omega$ (the disjoint union of m-many copies of the ordinal $n \in [1, \omega]$).

We note that the relational structures observed in this section are disconnected but taking their complements we obtain connected structures with the same posets

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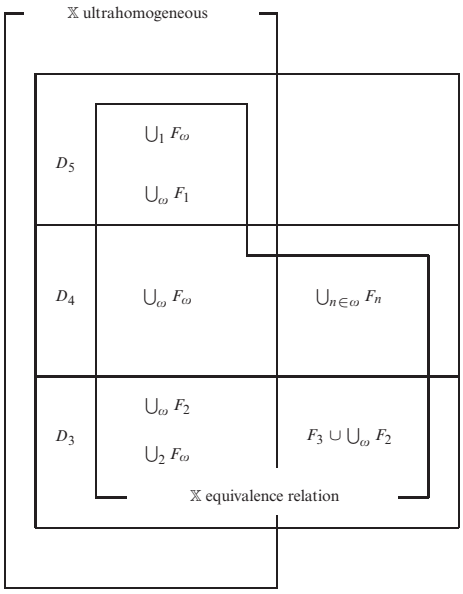

FIGURE 2. Equivalence relations on countable sets.

$\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ and $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$. For example, the complement of $\bigcup_m F_n$ is the graph-theoretic complement of the graph $\bigcup_m K_n$.

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