Adv. Appl. Prob. **49,** 1037–1066 (2017)
*doi:*10.1017/apr.2017.33

UNIFORM TAIL APPROXIMATION OF HOMOGENOUS FUNCTIONALS OF GAUSSIAN FIELDS

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Abstract

Let $X(t)$, $t \in \mathbb{R}^d$, be a centered Gaussian random field with continuous trajectories and set $\xi_u(t) = X(f(u)t)$, $t \in \mathbb{R}^d$, with f some positive function. Using classical results we can establish the tail asymptotics of $\mathbb{P}\{\Gamma(\xi_u) > u\}$ as $u \to \infty$ with $\Gamma(\xi_u) = \sup_{t\in[0,T]^d} \xi_u(t)$, $T > 0$, by requiring that $f(u)$ tends to 0 as $u \to \infty$ with speed controlled by the local behavior of the correlation function of X. Recent research shows that for applications, more general functionals than the supremum should be considered and the Gaussian field can depend also on some additional parameter $\tau_u \in K$ say $\xi_{u,\tau_u}(t)$, $t \in \mathbb{R}^d$. In this paper we derive uniform approximations of $\mathbb{P}\{\Gamma(\xi_{u,\tau_u}) > u\}$ with respect to τ_u, in some index set K_u as $u \to \infty$. Our main result has important theoretical implications; two applications are already included in Dębicki *et al.* (2016), (2017). In this paper we present three additional applications. First we derive uniform upper bounds for the probability of double maxima. Second, we extend the Piterbarg–Prisyazhnyuk theorem to some large classes of homogeneous functionals of centered Gaussian fields ξ_u. Finally, we show the finiteness of generalized Piterbarg constants.

Keywords: Fractional Brownian motion; supremum of Gaussian random fields; stationary process; double maxima; uniform double-sum method; generalized Piterbarg constant

2010 Mathematics Subject Classification: Primary 60G15
Secondary 60G70

1. Introduction

Let $(X(t),\ t \geq 0)$ be a centered stationary Gaussian process with continuous trajectories, unit variance, and correlation function r satisfying, for some $\alpha \in (0, 2]$,

$$1 - r(t) \sim |t|^\alpha \quad \text{as } t \to 0 \quad \text{and} \quad r(t) < 1 \quad \text{for all } t > 0.$$

We write '$\sim$' for asymptotic equivalence when the argument tends to 0 or ∞.

In the seminal paper of Pickands [24], the author established that, for any T positive and $q(u) = u^{-2/\alpha}$,

$$\mathbb{P}\Big\{\sup_{t\in[0,T]} X(t) > u\Big\} \sim T\mathcal{H}_\alpha \frac{\mathbb{P}\{X(0) > u\}}{q(u)} \quad \text{as } u \to \infty, \tag{1.1}$$

Received 18 July 2016; revision received 19 May 2017.
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where $\mathcal{H}_\alpha$ is the *Pickands constant* defined by

$$\mathcal{H}_\alpha = \lim_{T\to\infty} \frac{1}{T}\mathcal{H}_\alpha[0, T] \in (0, \infty),$$

with $\mathcal{H}_\alpha[0, T] = \mathbb{E}\{\sup_{t\in[0,T]} \exp(\sqrt{2}B_\alpha(t) - t^\alpha)\}$ and B_α a standard fractional Brownian motion with Hurst index $\alpha/2$; see the recent contributions [6], [10], [12], [19], and [20] for the main properties of Pickands and related constants.

While the original proof of Pickands utilizes a discretization approach, in [25] and [26] the asymptotics (1.1) were derived by establishing first the exact asymptotics on the short interval $[0, q(u)T]$, namely (see, e.g. Lemma 6.1 of [26])

$$\mathbb{P}\Big\{\sup_{t\in[0,q(u)T]} X(t) > u\Big\} \sim \mathcal{H}_\alpha[0, T]\mathbb{P}\{X(0) > u\} \quad \text{as } u \to \infty, \tag{1.2}$$

and then using the *double-sum method*. A completely independent proof for the stationary case, based on the notion of a *sojourn time*, was derived by Berman (see [3] and [4]).

In this paper we develop the *uniform double-sum method*. Originally introduced by Piterbarg for the nonstationary case, see, e.g. [26], the *double-sum method* is a powerful tool in the derivation of the exact asymptotics of the tail distribution of the supremum for nonstationary Gaussian processes (and fields). With no loss of generality, for a given centered Gaussian process $(Y(t),\ t \in [0, S])$ with continuous trajectories, the crucial steps of this method are:

- an application of the Slepian inequality allowing for the uniform approximation as $u \to \infty$ (uniformity is with respect to $k \le N(u)$) of the summands of

$$\mathbb{P}\Big\{\sup_{t\in[kTq(u),\,(k+1)Tq(u)]} Y(t) > u\Big\}$$

  by $\mathbb{P}\{\sup_{t\in[0,Tq(u)]} X^\epsilon(t) > u_k\} =: p(u_k)$ for an appropriately chosen stationary process $(X^\varepsilon,\ \varepsilon > 0)$;

- the uniform approximation for $k \le N(u)$ of $p(u_k)$ as $u \to \infty$;

- obtaining uniformly tight upper bounds for the probability of the double supremum

$$\mathbb{P}\Big\{\sup_{t\in[kTq(u),\,(k+1)Tq(u)]} Y(t) > u,\ \sup_{t\in[lTq(u),\,(l+1)Tq(u)]} Y(t) > u\Big\} \quad \text{for } k, l \in \mathcal{A}_u, \tag{1.3}$$

  where the set $\mathcal{A}_u$ is suitably chosen.

The deep contribution of [18] showed that while dealing with the supremum of Gaussian processes on the half-line, it is convenient to replace the Slepian inequality by a uniform version of the tail asymptotics of threshold-dependent Gaussian processes. Omitting technical details, Dieker [18] derived the exact asymptotics and a uniform upper bound of

$$\mathbb{P}\Big\{\sup_{t\in[0,T]} \xi_{u,\tau_u}(t) > g_{u,\tau_u}\Big\} \quad \text{as } u \to \infty,$$

with respect to $\tau_u \in K_u$, for ξ_{u,τ_u} being centered Gaussian processes indexed by u and τ_u; see also Lemma 5.1 of [8]. This uniform counterpart of (1.2) is crucial when the processes X_{u,τ_u} are parameterized by u and τ_u.

Recent contributions have shown the strong need for the analysis of the distributional properties of more general continuous functionals rather than the supremum, e.g.

$$\sup_{t\in[0,T]}\inf_{s\in[0,S]} X(s+f(u)t), \qquad S>0,$$

see [11] and [13], or $\inf_{s\in\mathcal{A}_u}\sup_{t\in\mathcal{B}_u} Y(s,t)$, see [7] and [8].

The lack of Slepian-type results for general continuous functionals Γ can be overcome by the derivation of uniform approximations with respect to τ_u of the tail distribution of $\Gamma(\xi_{u,\tau_u})$ as $u\to\infty$. Therefore, our principal goal in this paper is to derive uniform approximations for the tails of homogeneous continuous functionals Γ of general Gaussian random fields. Specifically, we shall consider Γ defined on $C(E)$, the space of continuous functions on E with $E\subset\mathbb{R}^d$, $d\ge 1$, a compact set containing the origin. In Theorem 2.1 we derive the following uniform asymptotics:

$$\lim_{u\to\infty}\sup_{\tau_u\in K_u}\left|\frac{\mathbb{P}\{\Gamma(\xi_{u,\tau_u})>g_{u,\tau_u}\}}{\Psi(g_{u,\tau_u})}-C\right|=0, \tag{1.4}$$

where $\xi_{u,\tau_u}(t)$, $t\in E$, $\tau_u\in K_u$, is a centered Gaussian random field, C is a positive finite constant, and Ψ denotes the survival function of an $N(0,1)$ random variable. This result allows us to derive the counterparts of (1.1) for a class of homogeneous functionals of centered Gaussian fields satisfying some weak asymptotic conditions. Additionally, in Section 3.1 we derive a uniform upper bound for the double maxima for general Gaussian fields parameterized by u and τ_u. This extends and unifies the known upper bounds for (1.3).

The paper is organized as follows. The main results and related discussions are presented in Section 2. We dedicate Section 3 to applications. We present the proofs of all the results in Section 4, postponing some technical calculations to Appendix A.

2. Main result

We begin this section with the motivation behind the investigation of distributional properties of functionals of threshold-dependent Gaussian random fields. For this purpose we focus on the supremum of noncentered Gaussian processes. Then we introduce the class of functionals that are of interest and provide the main result of this paper; see Theorem 2.1.

Numerous authors e.g. [17], [18], [21], and [22], have developed techniques for the approximation, as $u\to\infty$, of the so-called ruin probability

$$p(u)=\mathbb{P}\Big\{\sup_{t\in\mathcal{T}}(X(t)-ct)>u\Big\},$$

where X is a centered continuous Gaussian process, $c>0$ is some constant, and $\mathcal{T}=[0,\infty)$ or $\mathcal{T}=[0,T]$, $T>0$. Originally, the *double-sum method* was designed to handle the supremum of centered Gaussian processes. For our case, this method is still applicable under the following modifications. First we rewrite the original problem in the form of a centered, threshold-dependent family of Gaussian processes $Z_u(t)=X(t)/(u+ct)$, $u>0$, as follows:

$$p(u)=\mathbb{P}\Big\{\sup_{t\in\mathcal{T}}Z_u(t)>1\Big\}.$$

Then, we check that, for suitably chosen $w(u)$ and $N(u)$,

$$p(u)\sim\mathbb{P}\Big\{\text{there exists } |k|\le N(u)\colon \sup_{t\in[0,w(u)S]} Z_u(t+kSw(u))>1\Big\}$$

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$$\sim \sum_{|k|\le N(u)} \mathbb{P}\Big\{ \sup_{t\in[0,S]} Y_{u,k}(t) > v_k(u) \Big\}$$
$$=: \sum_{|k|\le N(u)} p_k(u) \quad \text{as } u\to\infty \text{ and } S\to\infty, \tag{2.1}$$

where

$$Y_{u,k}(t) = Z_u(w(u)t + w(u)kS)v_k(u), \qquad v_k(u) = \inf_{t\in[0,S]} \frac{1}{\sqrt{\operatorname{var}(Z_u(w(u)t + w(u)kS))}}.$$

Finally, since usually $\lim_{u\to\infty} N(u) = \infty$ then in order to determine the asymptotics of $p(u)$ it is necessary to derive the asymptotics of $p_k(u)$, as $u\to\infty$, uniformly for $|k|\le N(u)$.

In this section we consider a more general situation focusing on the validity of (1.4) for centered Gaussian random fields.

Next let $E\subset\mathbb{R}^d$ be a compact set including the origin and write $C(E)$ for the set of real-valued continuous functions defined on E. Let $\Gamma\colon C(E)\to\mathbb{R}$ be a real-valued continuous functional satisfying:

(F1) there exists $c>0$ such that $\Gamma(f)\le c\sup_{t\in E} f(t)$ for any $f\in C(E)$;

(F2) $\Gamma(af+b) = a\Gamma(f)+b$ for any $f\in C(E)$ and $a>0,\ b\in\mathbb{R}$.

Note that (F1) and (F2) cover the following important examples:

$$\Gamma = \sup,\quad \inf,\quad a\sup + (1-a)\inf, \qquad a\in\mathbb{R}.$$

We shall consider a family of centered Gaussian random fields ξ_{u,τ_u} given by

$$\xi_{u,\tau_u}(t) = \frac{Z_{u,\tau_u}(t)}{1+h_{u,\tau_u}(t)}, \qquad t\in E,\ \tau_u\in K_u,$$

with Z_{u,τ_u} a centered Gaussian random field with unit variance and continuous trajectories, and $h_{u,\tau_u}\in C_0(E)$, where $C_0(E)$ is the Banach space of all continuous functions f on E such that $f(0)=0$ is equipped with the sup-norm. In order to avoid trivialities, the thresholds g_{u,τ_u} will be chosen such that

$$\lim_{u\to\infty} \mathbb{P}\{\Gamma(\xi_{u,\tau_u}) > g_{u,\tau_u}\} = 0.$$

In order to derive the asymptotics of $\mathbb{P}\{\Gamma(\xi_{u,\tau_u}) > g_{u,\tau_u}\}$ as $u\to\infty$, we shall first condition on $\xi_{u,\tau_u}(0) = g_{u,\tau_u} - w/g_{u,\tau_u}$, yielding

$$\mathbb{P}\{\Gamma(\xi_{u,\tau_u}) > g_{u,\tau_u}\} = \frac{\exp(-g_{u,\tau_u}^2/2)}{\sqrt{2\pi}\, g_{u,\tau_u}} \int_{\mathbb{R}} \exp\Big(w - \frac{w^2}{2g_{u,\tau_u}^2}\Big)\mathbb{P}\{\Gamma(\chi_{u,\tau_u}) > w\}\,\mathrm{d}w,$$

where

$$\chi_{u,\tau_u}(t) = g_{u,\tau_u}(\xi_{u,\tau_u}(t) - g_{u,\tau_u}) + w \,\Big|\, \Big(\xi_{u,\tau_u}(0) = g_{u,\tau_u} - \frac{w}{g_{u,\tau_u}}\Big).$$

Note that

$$\chi_{u,\tau_u}(t) \overset{\mathrm{D}}{=} \frac{g_{u,\tau_u}}{1+h_{u,\tau_u}(t)}(Z_{u,\tau_u}(t) - r_{u,\tau_u}(t,0)Z_{u,\tau_u}(0)) + \mathbb{E}\{\chi_{u,\tau_u}(t)\}, \qquad t\in E,$$

where '$\overset{\mathrm{D}}{=}$' denotes equality of distribution.

Next, we shall impose the following assumptions (see [8, Lemma 5.1] and [18, Lemma 2]) to ensure the weak convergence of $\{\chi_{u,\tau_u}(t),\ t\in E\}$ as $u\to\infty$.

(C0) The positive constants g_{u,τ_u} are such that $\lim_{u\to\infty}\inf_{\tau_u\in K_u} g_{u,\tau_u}=\infty$.

(C1) There exists $h\in C_0(E)$ such that

$$\lim_{u\to\infty}\sup_{\tau_u\in K_u,\,t\in E}|g_{u,\tau_u}^2 h_{u,\tau_u}(t)-h(t)|=0.$$

(C2) There exists $\theta_{u,\tau_u}(s,t)$ such that

$$\lim_{u\to\infty}\sup_{\tau_u\in K_u}\sup_{s\neq t\in E}\left|g_{u,\tau_u}^2\frac{\mathrm{var}(Z_{u,\tau_u}(t)-Z_{u,\tau_u}(s))}{2\theta_{u,\tau_u}(s,t)}-1\right|=0$$

and, for some centered Gaussian random field $\eta(t)$, $t\in\mathbb{R}^d$ with continuous trajectories and $\eta(0)=0$,

$$\lim_{u\to\infty}\sup_{\tau_u\in K_u}|\theta_{u,\tau_u}(s,t)-\mathrm{var}(\eta(t)-\eta(s))|=0\quad\text{for all } s,t\in E. \tag{2.2}$$

(C3) There exists $a>0$ such that

$$\limsup_{u\to\infty}\sup_{\tau_u\in K_u}\sup_{s\neq t,s,\,t\in E}\frac{\theta_{u,\tau_u}(s,t)}{\sum_{i=1}^d|s_i-t_i|^a}<\infty, \tag{2.3}$$

$$\lim_{\epsilon\downarrow 0}\limsup_{u\to\infty}\sup_{\tau_u\in K_u}\sup_{\|t-s\|<\epsilon,s,\,t\in E} g_{u,\tau_u}^2\mathbb{E}\{[Z_{u,\tau_u}(t)-Z_{u,\tau_u}(s)]Z_{u,\tau_u}(0)\}=0. \tag{2.4}$$

If X is a centered Gaussian process with stationary increments satisfying Assumptions AI–AII of [8], then $Y_{u,k}(t)$, $t\in[0,S]$, $|k|\le N(u)$, in (2.1) satisfies (C0)–(C3); see also [18].

The intuitive explanation behind these assumptions is as follows: (C1) and (2.4) in (C3) are used to guarantee the uniform convergence of the function $\mathbb{E}\{\chi_{u,\tau_u}(t)\}$ for $t\in E$ as $u\to\infty$. Utilizing further (C2), the convergence of finite-dimensional distributions of $\chi_{u,\tau_u}(t)$, $t\in E$, to those of $\eta(t)$, $t\in E$, can be shown. Moreover, the tightness follows by (2.3) in (C3).

Given $h\in C_0(E)$ and the functional Γ satisfying (F1) and (F2), for η introduced in (C2) we define a new constant

$$\mathcal{H}_{\eta,h}^{\Gamma}(E):=\mathbb{E}\{\mathrm{e}^{\Gamma(\eta^h)}\},\qquad \eta^h(t):=\sqrt{2}\eta(t)-\mathrm{var}(\eta(t))-h(t),$$

which is finite by (F1). For notational simplicity, we set

$$\mathcal{H}_\eta(E)=\mathcal{H}_{\eta,0}^{\sup}(E).$$

We present next the main result of this section. Recall that Ψ stands for the survival function of an $N(0,1)$ random variable.

Theorem 2.1. *Under assumptions (C0)–(C3) and (F1) and (F2), if, further,* $\mathbb{P}\{\Gamma(\xi_{u,\tau_u})>g_{u,\tau_u}\}>0$ *for all* $\tau_u\in K_u$ *and all large* u*, then*

$$\lim_{u\to\infty}\sup_{\tau_u\in K_u}\left|\frac{\mathbb{P}\{\Gamma(\xi_{u,\tau_u})>g_{u,\tau_u}\}}{\Psi(g_{u,\tau_u})}-\mathcal{H}_{\eta,h}^{\Gamma}(E)\right|=0. \tag{2.5}$$

Remark 2.1. (i) Under the assumptions of Theorem 2.1, we have

$$\limsup_{u\to\infty}\sup_{\tau_u\in K_u}\frac{\mathbb{P}\{\Gamma(\xi_{u,\tau_u})>g_{u,\tau_u}\}}{\Psi(g_{u,\tau_u})}<\infty,$$

which coincides with the results of Lemma 5.1 of [8] and extends Lemma 2 of [18].

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(ii) Condition (C2) and (2.4) in (C3) are equivalent to (C2) and

$$\lim_{u\to\infty} \sup_{t\in E,\, \tau_u\in K_u} |g_{u,\tau_u}^2 \operatorname{var}(Z_{u,\tau_u}(t) - Z_{u,\tau_u}(0)) - 2\operatorname{var}(\eta(t))| = 0. \tag{2.6}$$

(iii) Condition (C2) can be formulated also for the degenerated case $\eta(t) = 0$, $t \in \mathbb{R}^d$, almost surely. The claim of Theorem 2.1 holds also for such η.

Next we give a simplified version of Theorem 2.1. Instead of (C2) and (C3), we assume that

$$\lim_{u\to\infty} \sup_{\tau_u\in K_u} \sup_{s\neq t, s,\, t\in E} \left| g_{u,\tau_u}^2 \frac{\operatorname{var}(Z_{u,\tau_u}(t) - Z_{u,\tau_u}(s))}{2\sum_{i=1}^d c_i\sigma_i^2(q_i(u)|s_i - t_i|)/\sigma_i^2(q_i(u))} - 1 \right| = 0, \tag{2.7}$$

where $q_i(u)$, $i = 1, \dots, d$, are some functions of u with $q_i(u) > 0$ for large enough u and $\lim_{u\to\infty} q_i(u) = \varphi_i \in [0, \infty]$ with

$$\varphi_i = \begin{cases} 0, & 1 \le i \le d_1, \\ (0,\infty), & d_1 + 1 \le i \le d_2, \\ \infty, & d_2 + 1 \le i \le d, \end{cases}$$

and $c_i \ge 0$, $1 \le i \le d$. Moreover, σ_i, $1 \le i \le d$, are regularly varying at 0 with indices $\alpha_{i,0}/2 \in (0, 1]$, respectively, and $\sigma_i(0) = 0$, $\sigma_i(t) > 0$, $t > 0$, $1 \le i \le d$; σ_i, $d_2 + 1 \le i \le d$, are bounded on any compact interval and regularly varying at ∞ with indices $\alpha_{i,\infty}/2 \in (0, 1]$, respectively; $\sigma_i^2(t)$, $d_1 + 1 \le i \le d_2$, are continuous and nonnegative definite, implying that there exist centered Gaussian processes η_i, $d_1 + 1 \le i \le d_2$, with a continuous sample path and stationary increments such that $\operatorname{var}(\eta_i(t)) := \sigma_i^2(t)$, $d_1 + 1 \le i \le d_2$. We refer the reader to, e.g. [17], [18], [21], and [22], where particular examples of Gaussian processes that satisfy the above regularity assumptions were investigated; see also [23] for a characterization of such processes in terms of max-stable stationary processes.

Proposition 2.1. *Suppose that (C0) and (C1) and (F1) and (F2) hold. If (2.7) holds with* $\sum_{i=1}^d c_i > 0$ *and* $\mathbb{P}\{\Gamma(\xi_{u,\tau_u}) > g_{u,\tau_u}\} > 0$ *for all* $\tau_u \in K_u$ *and all large u, then (2.5) holds with*

$$\eta(t) = \sum_{i=1}^{d_1} \sqrt{c_i} B_{\alpha_{i,0}}(t_i) + \sum_{i=d_1+1}^{d_2} \sqrt{c_i} \frac{\eta_i(\varphi_i t_i)}{\sigma_i(\varphi_i)} + \sum_{i=d_2+1}^{d} \sqrt{c_i} B_{\alpha_{i,\infty}}(t_i), \tag{2.8}$$

where $B_{\alpha_{i,0}}$, $1 \le i \le d_1$, η_i, $d_1 + 1 \le i \le d_2$, *and* $B_{\alpha_{i,\infty}}$, $d_2 + 1 \le i \le d$, *are mutually independent.*

Remark 2.2. (i) Condition (2.7) is satisfied by a large class of important processes previously investigated in the literature; see, e.g. [8], [14], [17], [18], and [21].

(ii) Under the assumptions of Theorem 2.1,

$$\lim_{u\to\infty} \sup_{\tau_u\in K_u} \left| \frac{\mathbb{P}\{\Gamma_i(\xi_{u,\tau_u}) > u,\ i = 1, \dots, d\}}{\Psi(g_{u,\tau_u})} - \mathcal{H}_{\eta,h}^{\Gamma_1,\dots,\Gamma_d} \right| = 0, \tag{2.9}$$

with Γ_i, $i \le d$, continuous functionals satisfying (F1), (F2), and

$$\mathcal{H}_{\eta,h}^{\Gamma_1,\dots,\Gamma_d} = \int_{\mathbb{R}} \mathrm{e}^w \mathbb{P}\{\Gamma_i(\eta^h) > w,\ i = 1 \dots d\}\, \mathrm{d}w \in (0, \infty).$$

Moreover, (2.9) holds also in the case that η is degenerated, i.e. $\eta(t) = 0$, $t \in \mathbb{R}^d$, almost surely.

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Finally, we present below a version of Theorem 2.1 under slightly different and more explicit assumptions. We keep the same notation as in Theorem 2.1 and, moreover, let $\sigma^2_{u,\tau_u}(t) := \operatorname{var}(\xi_{u,\tau_u}(t))$.

(D1) Condition (C0) holds for g_{u,τ_u} and $\sigma_{u,\tau_u}(0) = 1$ for all $\tau_u \in K_u$ and all $u > 0$, and there exists some $h \in C_0(E)$ such that

$$\lim_{u\to\infty} \sup_{t\in E,\ \tau_u\in K_u} |g^2_{u,\tau_u}(1 - \sigma_{u,\tau_u}(t)) - h(t)| = 0.$$

(D2) There exists a centered Gaussian random field $\eta(t)$, $t \in \mathbb{R}^d$, with continuous sample paths $\eta(0) = 0$ such that, for any $s, t \in E$, and $\tau_u \in K_u$,

$$\lim_{u\to\infty} \sup_{\tau_u\in K_u} |g^2_{u,\tau_u} \operatorname{var}(\xi_{u,\tau_u}(t) - \xi_{u,\tau_u}(s)) - 2\operatorname{var}(\eta(t) - \eta(s))| = 0,$$
$$\lim_{u\to\infty} \sup_{t\in E,\ \tau_u\in K_u} |g^2_{u,\tau_u} \operatorname{var}(\xi_{u,\tau_u}(t) - \xi_{u,\tau_u}(0)) - 2\operatorname{var}(\eta(t))| = 0.$$

(D3) There exist positive constants G, ν, and u_0 such that, for any $u > u_0$,

$$\sup_{\tau_u\in K_u} g^2_{u,\tau_u} \operatorname{var}(\xi_{u,\tau_u}(t) - \xi_{u,\tau_u}(s)) \le G\|t - s\|^\nu$$

holds for all $s, t \in E$.

Theorem 2.2. *If (D1)–(D3) and (F1) and (F2) are satisfied then (2.5) holds.*

3. Applications

3.1. Upper bounds for the double supremum

Uniform bounds for the tail distribution of bivariate maxima of Gaussian processes play a key role in the double-sum technique of Piterbarg; see, e.g. [26] and [27]. More precisely, of interest is to find an optimal upper bound for

$$D(\lambda_1, \lambda_2, \mathcal{E}_1, \mathcal{E}_2, u) := \mathbb{P}\Big\{\sup_{t\in\lambda_1+\mathcal{E}_1} X_u(t) > m_{\lambda_1}(u),\ \sup_{t\in\lambda_2+\mathcal{E}_2} X_u(t) > m_{\lambda_2}(u)\Big\},$$

which is valid for all large u with the λ_i and $\mathcal{E}_i$ controlled by E_u by requiring that $\lambda_i + \mathcal{E}_i \subset E_u$, with E_u a compact subset of $\mathbb{R}^d$. Further, the thresholds $m_{\lambda_1}(u)$, $m_{\lambda_2}(u)$ are assumed to satisfy

$$\lim_{u\to\infty} m(u) = \infty, \qquad \lim_{u\to\infty} \sup_{\lambda_i+\mathcal{E}_i\subset E_u} \left|\frac{m_{\lambda_i}(u)}{m(u)} - 1\right| = 0, \quad i = 1, 2, \tag{3.1}$$

for some positive function m.

Set below $F(A, B) = \inf_{s\in A,\, t\in B} \|s - t\|$ with A, B two nonempty subsets of $\mathbb{R}^d$ and $\|\cdot\|$ the Euclidean norm. Let $\mathbb{K} = \{(\lambda_1, \lambda_2) \colon \lambda_i + \mathcal{E}_i \subset E_u,\ i = 1, 2\}$.

Theorem 3.1. *Let $X_u(t)$, $t \in E_u \subset \mathbb{R}^d$, be a family of centered Gaussian random fields with continuous trajectories, variance 1, and correlation function r_u. Suppose that there exist positive constants S_1, $\mathcal{C}_1$, $\mathcal{C}_2$, β, and $\alpha \in (0, 2]$ such that, for sufficiently large u,*

$$m^2(u)(1 - r_u(s, t)) \ge \mathcal{C}_1\|s - t\|^\beta, \qquad \|s - t\| \ge S_1, \qquad s, t \in E_u, \tag{3.2}$$
$$m^2(u)(1 - r_u(s, t)) \le \mathcal{C}_2\|s - t\|^\alpha, \qquad s, t \in E_u,\ s - t \in [-1, 1]^d. \tag{3.3}$$

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Moreover, there exists $\delta > 0$ *such that, for large enough* u,

$$r_u(s,t) > \delta - 1, \qquad s,t \in E_u. \tag{3.4}$$

Further, if (3.1) holds then there exists $\mathcal{C} > 0$ *such that, for all large enough* u,

$$\sup_{\substack{(\lambda_1,\lambda_2)\in\mathbb{K}\\ \mathcal{E}_i\subset[0,S_2]^d,\ \mathcal{E}_i\neq\varnothing,\ i=1,2}} \frac{\exp(\mathcal{C}_1 F^{\beta}(\lambda_1+\mathcal{E}_1,\lambda_2+\mathcal{E}_2)/8) D(\lambda_1,\lambda_2,\mathcal{E}_1,\mathcal{E}_2,u)}{S_2^{2d}\Psi(m_{\lambda_1,\lambda_2}(u))} \le \mathcal{C}, \tag{3.5}$$

with $S_2 > 1$, $m_{\lambda_1,\lambda_2}(u) = \min(m_{\lambda_1}(u), m_{\lambda_2}(u))$*, and* $\mathcal{C}$ *a positive constant independent of* S_2, u.

Next assume that $\kappa_i(t) > 0,\ t > 0,\ 1 \le i \le 2d$, are some nonnegative locally bounded functions and define

$$g_u(s,t) = \sum_{i=1}^{d} \frac{\kappa_i(q_i(u)|s_i - t_i|)}{\kappa_i(q_i(u))} \quad \text{and} \quad \widetilde{g}_u(s,t) = \sum_{i=1}^{d} \frac{\kappa_{i+d}(q_{i+d}(u)|s_i - t_i|)}{\kappa_{i+d}(q_{i+d}(u))}.$$

Further, let $q_i(u) > 0,\ u > 0$, be such that

$$\lim_{u\to\infty} q_i(u) = \varphi_i \in [0,\infty], \qquad 1 \le i \le 2d.$$

Corollary 3.1. *Let* $X_u(t),\ t \in E_u$*, be centered Gaussian random fields with continuous trajectories, variance 1, and correlation function* r_u *satisfying (3.4). Assume further that (3.1) holds. Further, if for sufficiently large* u,

$$\mathcal{C}_3 g_u(s,t) \le m^2(u)(1 - r_u(s,t)) \le \mathcal{C}_4 \widetilde{g}_u(s,t), \qquad s,t \in E_u, \tag{3.6}$$

with $\mathcal{C}_3, \mathcal{C}_4 > 0$*, and* $\kappa_i,\ 1 \le i \le 2d$*, being regularly varying both at* 0 *and at* ∞ *with indices* $\alpha_{i,0} > 0$ *and* $\alpha_{i,\infty} > 0$*, respectively, then there exists* $\mathcal{C} > 0$ *such that, for large enough* u*, (3.5) holds with* $\beta = \frac{1}{2}\min_{i=1,\dots,2d}\min(\alpha_{i,0},\alpha_{i,\infty},2)$ *and* $\mathcal{C}_1$ *a fixed positive constant.*

Corollary 3.2. *Let* $X_u(t),\ t \in E_u \subset \mathbb{R}^d$*, be centered Gaussian random fields with continuous trajectories, variance 1, and correlation function* r_u *satisfying (3.4) and (3.6) with* $\varphi_i = 0,\ 1 \le i \le 2d$*, and* $\kappa_i,\ 1 \le i \le 2d$*, being regularly varying at* 0 *with indices* $\alpha_{i,0} > 0$*. Further, if (3.1) and*

$$\limsup_{u\to\infty} \sup_{s,t\in E_u} \max_{i=1,\dots,2d} q_i(u)|s_i - t_i| < \infty$$

hold, then there exist positive constants $\mathcal{C}, \mathcal{C}_1$ *such that, for large enough* u*, (3.5) holds with* $\beta = \frac{1}{2}\min(2, \min_{i=1,\dots,2d}\alpha_{i,0})$.

Remark 3.1. (i) Under the assumptions of Theorem 3.1, using the idea of [16] and [28], since for $\gamma \in (0,1)$,

$$D(\lambda_1,\lambda_2,\mathcal{E}_1,\mathcal{E}_2,u) \le \mathbb{P}\Big\{\sup_{s\in\lambda_1+\mathcal{E}_1,\ t\in\lambda_2+\mathcal{E}_2} (\gamma X_u(s) + (1-\gamma)X_u(t)) > m_{\lambda_1,\lambda_2,\gamma}(u)\Big\},$$

with $m_{\lambda_1,\lambda_2,\gamma}(u) = \gamma m_{\lambda_1}(u) + (1-\gamma)m_{\lambda_2}(u)$, then in some cases (3.5) can be improved by setting $4\gamma(1-\gamma)\mathcal{C}_1$ instead of $\mathcal{C}_1$, and $m_{\lambda_1,\lambda_2,\gamma}(u)$ instead of $m_{\lambda_1,\lambda_2}(u)$, respectively.

(ii) A particular example is $\kappa_i(x) = x^{\alpha_i},\ \alpha_i \in (0,2]$. For such a case, the result of Corollary 3.2 yields the claim of Lemma 9.14 of [27]; see also Lemma 6.3 of [26].

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3.2. Tail approximation of $\Gamma_{E_u}(X_u)$

In many applications the tail asymptotics of general functionals of Gaussian random fields X_u indexed by thresholds $u > 0$ are of interest. In this section we present an application of Theorem 2.1 concerned with the tail asymptotics of $\Gamma_{E_u}(X_u)$, where

$$E_u := \left(\prod_{i=1}^{d}[a_i(u), b_i(u)]\right) \times E$$

is also parameterized by u, with E a compact subset of $\mathbb{R}^n$, $n \in \mathbb{N}$. Without loss of generality, we assume that $0 \in E$. The functional Γ_{E_u} is defined as follows.

Let $\Gamma^*\colon C(E) \to \mathbb{R}$ be a real-valued continuous functional satisfying (F1) and (F2) with $c = 1$ in (F1). For any compact set $A \subset \mathbb{R}^d$, define

$$\Gamma_{A\times E}(f) = \sup_{s\in A} \Gamma^*(f(s,t)), \qquad f \in C(A \times E).$$

It follows that $\Gamma_{A\times E}$ is a continuous functional and satisfies (F1) and (F2) with $c = 1$ in (F1). Examples of Γ^* are

$$\Gamma^* = \sup, \quad \inf, \quad a\sup + (1-a)\inf, \qquad a \le 1.$$

We shall consider $X_u(s, t)$, $(s, t) \in E_u$, a family of centered continuous Gaussian random fields with variance function $\sigma_u(s, t)$ and correlation function $r_u(s, t, s', t')$ satisfying, as $u \to \infty$,

$$\sigma_u(0,0) = 1, \qquad 1 - \sigma_u(s,0) \sim \sum_{i=1}^{d} \frac{|s_i|^{\beta_i}}{g_i(u)}, \qquad s \in \prod_{i=1}^{d}[a_i(u), b_i(u)], \tag{3.7}$$

and

$$\lim_{u\to\infty} \sup_{s\in\prod_{i=1}^{d}[a_i(u),\, b_i(u)],\, t\neq 0,\, t\in E} \left| \frac{1-\sigma_u(s,t)/\sigma_u(s,0)}{\sum_{i=d+1}^{d+n} |t_i|^{\beta_i}/g_i(u)} - 1 \right| = 0, \tag{3.8}$$

where $\beta_i > 0$ and $g_i(u)$ is a function of u satisfying $\lim_{u\to\infty} g_i(u) = \infty$ for $1 \le i \le d+n$. Moreover, there exists $m(u)$ such that $\lim_{u\to\infty} m(u) = \infty$ and

$$\begin{aligned}\lim_{u\to\infty} \sup_{(s,t),\,(s',t')\in E_u,\,(s,t)\neq(s',t')} &\Bigg|\Bigg\{ m^2(u)(1 - r_u(s,t,s',t')) \\ &\times \Bigg[\sum_{i=1}^{d} \frac{c_i\sigma_i^2(q_i(u)|s_i - s_i'|)}{\sigma_i^2(q_i(u))} \\ &\quad + \sum_{i=d+1}^{d+n} \frac{c_i\sigma_i^2(q_i(u)|t_i - t_i'|)}{\sigma_i^2(q_i(u))}\Bigg]^{-1}\Bigg\} - 1\Bigg| \\ = 0, \end{aligned} \tag{3.9}$$

where $c_i > 0$, $q_i(u) > 0$, $\lim_{u\to\infty} q_i(u) = \varphi_i \in [0,\infty]$, $1 \le i \le d+n$, and σ_i are the variance functions of η_i's, centered continuous Gaussian processes with stationary increments and $\eta_i(0) = 0$, satisfying further the following assumptions:

(A1) $\sigma_i^2(t)$ is regularly varying at ∞ with index $2\alpha_{i,\infty} \in (0, 2)$ and is continuously differentiable over $(0,\infty)$ with the first derivative $\dot\sigma_i^2(t)$ being ultimately monotone at ∞;

(A2) $\sigma_i^2(t)$ is regularly varying at 0 with index $2\alpha_{i,0} \in (0, 2]$.

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Moreover, we shall assume that

$$\lim_{u\to\infty}\frac{|a_i(u)|^{\beta_i}}{g_i(u)}=\lim_{u\to\infty}\frac{|b_i(u)|^{\beta_i}}{g_i(u)}=0, \qquad 1\le i\le d+n.$$

Let

$$V_{\varphi_i}(t_i)=\begin{cases}\sqrt{c_i}\,B_{\alpha_{i,0}}(t_i), & \varphi_i=0,\\ \dfrac{\sqrt{c_i}}{\sigma_i(\varphi_i)}\eta_i(\varphi_i t_i), & \varphi_i\in(0,\infty),\\ \sqrt{c_i}\,B_{\alpha_{i,\infty}}(t_i), & \varphi_i=\infty,\end{cases} \qquad 1\le i\le d+n. \tag{3.10}$$

In the sequel, we shall denote

$$\mathcal{P}_\eta^h(E)=\mathcal{H}_{\eta,h}^{\sup}(E), \qquad \mathcal{H}_\eta(E)=\mathcal{H}_{\eta,0}^{\sup}(E),$$

and set

$$\mathcal{P}_\eta^h=\lim_{S\to\infty}\mathcal{P}_\eta^h([0,S]), \qquad \widehat{\mathcal{P}}_\eta^h=\lim_{S\to\infty}\mathcal{P}_\eta^h([-S,S]), \qquad \mathcal{H}_\eta=\lim_{S\to\infty}S^{-1}\mathcal{H}_\eta([0,S]),$$

if the limits exist. We refer the reader to [9], [14], and [26] for the properties of Piterbarg constants $\mathcal{P}_\eta^h$ and Pickands constants $\mathcal{H}_\eta$. Next, suppose that

$$\lim_{u\to\infty}\frac{m^2(u)}{g_i(u)}=\gamma_i\in[0,\infty]$$

and, for all large u, $\mathbb{P}\{\Gamma_{E_u}(X_u)>m(u)\}>0$.

Theorem 3.2. *Let $X_u(s,t)$, $(s,t)\in E_u\subset\mathbb{R}^{d+n}$, be a family of centered Gaussian random fields with continuous trajectories satisfying (3.7)–(3.9) and*

$$\gamma_i=\begin{cases}0 & \text{if } 1\le i\le d_1,\\ \infty & \text{if } d_2+1\le i\le d,\end{cases}$$

$$\gamma_i\in(0,\infty), \quad d_1+1\le i\le d_2, \qquad \gamma_i\in[0,\infty), \quad d+1\le i\le d+n.$$

If, further, for $1\le i\le d_1$,

$$\lim_{u\to\infty}\frac{(m(u))^{2/\beta_i}a_i(u)}{(g_i(u))^{1/\beta_i}}=y_{i,1}, \qquad \lim_{u\to\infty}\frac{(m(u))^{2/\beta_i}b_i(u)}{(g_i(u))^{1/\beta_i}}=y_{i,2},$$
$$\lim_{u\to\infty}\frac{(m(u))^{2/\beta_i}(a_i^2(u)+b_i^2(u))}{(g_i(u))^{2/\beta_i}}=0,$$

with $-\infty\le y_{i,1}<y_{i,2}\le\infty$, for $d_1+1\le i\le d_2$, $a_i(u)\le 0\le b_i(u)$, $\lim_{u\to\infty}a_i(u)=a_i\in[-\infty,0]$, $\lim_{u\to\infty}b_i(u)=b_i\in[0,\infty]$, and $a_i(u)\le 0\le b_i(u)$ for $d_2+1\le i\le d$, then

$$\mathbb{P}\{\Gamma_{E_u}(X_u)>m(u)\}\sim\prod_{i=1}^{d_1}\mathcal{H}_{V_{\varphi_i}}\prod_{i=d_1+1}^{d_2}\mathcal{P}_{V_{\varphi_i}}^{h_i}[a_i,b_i]\mathcal{H}_{\widetilde{V}_\varphi,\widetilde{h}^{\Gamma^*}}(E)$$
$$\times\prod_{i=1}^{d_1}\int_{y_{i,1}}^{y_{i,2}}\mathrm{e}^{-|s|^{\beta_i}}\,\mathrm{d}s\prod_{i=1}^{d_1}\left(\frac{g_i(u)}{m^2(u)}\right)^{1/\beta_i}\Psi(m(u)), \tag{3.11}$$

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where

$$\widetilde{V}_{\varphi}(t) = \sum_{i=1}^{n} V_{\varphi_{d+i}}(t_i), \qquad \widetilde{h}(t) = \sum_{i=1}^{n} \gamma_{d+i}|t_i|^{\beta_{d+i}},$$
$$h_i(s_i) = \gamma_i |s_i|^{\beta_i}, \qquad d_1 + 1 \le i \le d_2. \tag{3.12}$$

Remark 3.2. Theorem 3.2 extends and unifies both the previous findings of [17], [18], [21], [22], and, in particular, Theorem 8.2 of [26].

3.3. Generalized Piterbarg constants

Let $(X(t),\ t \in \mathbb{R})$ be a centered Gaussian process with stationary increments and continuous trajectories. Suppose that the variance function $\sigma^2(t) = \operatorname{var}(X(t))$ is strictly positive for all $t \neq 0$ and $\sigma(0) = 0$. Define

$$\mathcal{P}_X^b([0, S], [0, T]) = \mathbb{E}\Big\{ \sup_{t\in[0,T]} \inf_{s\in[0,S]} \exp(\sqrt{2}X(t-s) - (1+b)\sigma^2(|t-s|)) \Big\},$$

where b, S, and T are positive constants. In the special case that $X = B_\alpha$ is a fractional Brownian motion with Hurst index $\alpha/2 \in (0, 1]$, the generalized Piterbarg constant

$$\mathcal{P}_{B_\alpha}^b(S) = \lim_{T\to\infty} \mathcal{P}_{B_\alpha}^b([0, S], [0, T]) \in (0, \infty)$$

determines the asymptotics of the Parisian ruin of the corresponding risk model; see [13]. Note that the classical Piterbarg constant corresponds to the $S = 0$ case. Our next result shows that $\mathcal{P}_X^b(S) \in (0, \infty)$ for a general Gaussian process with stationary increments.

Proposition 3.1. *If* $(X(t),\ t \in \mathbb{R})$ *is a centered Gaussian process with stationary increments and variance function satisfying (A1) with regularly varying index* $2\alpha_\infty \in (0, 2]$*, and (A2) with regularly varying index* $2\alpha_0 \in (0, 2)$*, then for any* b, S *positive, we have*

$$\lim_{T\to\infty} \mathcal{P}_X^b([0, S], [0, T]) \in (0, \infty).$$

4. Proofs

Hereafter, by $\mathbb{Q}, \mathbb{Q}_i,\ i = 1, 2, \dots,$ we denote positive constants which may differ from line to line.

Proof of Theorem 2.1. Since we assume that $\mathbb{P}\{\Gamma(\xi_{u,\tau_u}) > g_{u,\tau_u}\} > 0$ for all large u and any $\tau_u \in K_u$, then by conditioning

$$\begin{aligned}
\mathbb{P}\{\Gamma(\xi_{u,\tau_u}) > g_{u,\tau_u}\} &= \int_{\mathbb{R}} \mathbb{P}\{\Gamma(\xi_{u,\tau_u}) > g_{u,\tau_u} \mid \xi_{u,\tau_u}(0) = x\} \frac{\exp(-x^2/2)}{\sqrt{2\pi}}\, \mathrm{d}x \\
&= \frac{\exp(-g_{u,\tau_u}^2/2)}{\sqrt{2\pi}\, g_{u,\tau_u}} \int_{\mathbb{R}} \exp\bigg(w - \frac{w^2}{2g_{u,\tau_u}^2}\bigg) \mathbb{P}\{\Gamma(\chi_{u,\tau_u}) > w\}\, \mathrm{d}w \\
&=: \frac{\exp(-g_{u,\tau_u}^2/2)}{\sqrt{2\pi}\, g_{u,\tau_u}} \mathcal{I}_{u,\tau_u},
\end{aligned}$$

with $\mathcal{I}_{u,\tau_u} > 0$ for all large u and

$$\chi_{u,\tau_u}(t) = (\zeta_{u,\tau_u}(t) \mid \zeta_{u,\tau_u}(0) = 0), \qquad \zeta_{u,\tau_u}(t) = g_{u,\tau_u}(\xi_{u,\tau_u}(t) - g_{u,\tau_u}) + w.$$

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Hence, the proof follows by showing that $\mathcal{H}^{\Gamma}_{\eta,h}(E)$ is finite and

$$\lim_{u\to\infty}\sup_{\tau_u\in K_u}|\mathcal{I}_{u,\tau_u}-\mathcal{H}^{\Gamma}_{\eta,h}(E)|=0. \tag{4.1}$$

Weak convergence of $\Gamma(\chi_{u,\tau_u})$. We have $\chi_{u,\tau_u}(0)=0$ almost surely. Setting $r_{u,\tau_u}(s,t)=\operatorname{corr}(Z_{u,\tau_u}(s),Z_{u,\tau_u}(t))$, we may write

$$\chi_{u,\tau_u}(t)\overset{\mathrm{D}}{=}\frac{g_{u,\tau_u}}{1+h_{u,\tau_u}(t)}(Z_{u,\tau_u}(t)-r_{u,\tau_u}(t,0)Z_{u,\tau_u}(0))+\mathbb{E}\{\chi_{u,\tau_u}(t)\},\qquad t\in E,$$

where '$\overset{\mathrm{D}}{=}$' denotes equality of the finite-dimensional distributions. Since

$$(1+h_{u,\tau_u}(t))\mathbb{E}\{\chi_{u,\tau_u}(t)\}=-g^2_{u,\tau_u}(1-r_{u,\tau_u}(t,0))-g^2_{u,\tau_u}h_{u,\tau_u}(t)+w(1-r_{u,\tau_u}(t,0)+h_{u,\tau_u}(t))$$

by (C1), (C3) for some arbitrary M positive, uniformly with respect to $t\in E$, $\tau_u\in K_u$, $w\in[-M,M]$,

$$(1+h_{u,\tau_u}(t))\mathbb{E}\{\chi_{u,\tau_u}(t)\}\to-(\sigma^2_\eta(t)+h(t)),\qquad u\to\infty, \tag{4.2}$$

and also, for any $s,t\in E$ uniformly with respect to $\tau_u\in K_u$, $w\in[-M,M]$,

$$\begin{aligned}&\operatorname{var}((1+h_{u,\tau_u}(t))\chi_{u,\tau_u}(t)-(1+h_{u,\tau_u}(s))\chi_{u,\tau_u}(s))\\&\quad=g^2_{u,\tau_u}[\mathbb{E}\{(Z_{u,\tau_u}(t)-Z_{u,\tau_u}(s))^2\}-(\mathbb{E}\{Z_{u,\tau_u}(0)[Z_{u,\tau_u}(t)-Z_{u,\tau_u}(s)]\})^2]\\&\quad\to 2\operatorname{var}(\eta(t)-\eta(s)),\qquad u\to\infty.\end{aligned} \tag{4.3}$$

Consequently, by Lemma 4.1 of [29], the finite-dimensional distributions of

$$(1+h_{u,\tau_u}(t))\chi_{u,\tau_u}(t),\qquad t\in E,$$

converge to those of $\eta^h(t)$, $t\in E$, as $u\to\infty$ uniformly for $\tau_u\in K_u$, $w\in[-M,M]$, where $M>0$ is fixed (recall that $\eta^h(t)=\sqrt{2}\eta(t)-\operatorname{var}(\eta(t))-h(t)$).

Condition (C3) together with the uniform convergence in (4.2) guarantee that Proposition 9.7 of [27] can be applied to yield the uniform tightness of $(1+h_{u,\tau_u}(t))\chi_{u,\tau_u}(t)$, $t\in E$, and, thus, $\{(1+h_{u,\tau_u}(t))\chi_{u,\tau_u}(t), t\in E\}$ weakly converges to $\{\eta^h(t), t\in E\}$ as $u\to\infty$, uniformly with respect to $\tau_u\in K_u$. Further, since

$$\lim_{u\to\infty}\sup_{t\in E,\,\tau_u\in K_u}h_{u,\tau_u}(t)=0,$$

then $\{\chi_{u,\tau_u}(t), t\in E\}$ converges weakly to $\{\eta^h(t), t\in E\}$ as $u\to\infty$, uniformly with respect to $\tau_u\in K_u$.

Consequently, since we assume that Γ is a continuous functional, by the continuous mapping theorem, $\Gamma(\chi_{u,\tau_u})$ converges in distribution to $\Gamma(\eta^h)$ as $u\to\infty$, uniformly with respect to $\tau_u\in K_u$.

Convergence of (4.1). Denote $\mathbb{A}=\{w\colon \mathbb{P}\{\Gamma(\eta^h)>w\}$ is discontinuous at $w\}$ then $\mathbb{A}$ is a countable set with measure 0. Hence, for any $w\in\mathbb{R}\setminus\mathbb{A}$,

$$\lim_{u\to\infty}\sup_{\tau_u\in K_u}|\mathbb{P}\{\Gamma(\chi_{u,\tau_u})>w\}-\mathbb{P}\{\Gamma(\eta^h)>w\}|=0$$

and, by (C0),

$$\lim_{u\to\infty}\sup_{\tau_u\in K_u,\,w\in[-M,M]}\mathrm{e}^w\bigg[1-\exp\bigg(-\frac{w^2}{2g^2_{u,\tau_u}}\bigg)\bigg]\le\frac{\mathrm{e}^M M^2}{2\liminf_{u\to\infty}\inf_{\tau_u\in K_u}g^2_{u,\tau_u}}\to 0,$$

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$u \to \infty$, implying that

$$\begin{aligned}
&\lim_{u\to\infty} \sup_{\tau_u \in K_u} \left| \int_{-M}^{M} \left[\exp\left(w - \frac{w^2}{2g_{u,\tau_u}^2} \right) \mathbb{P}\{\Gamma(\chi_{u,\tau_u}) > w\} - \mathrm{e}^{w} \mathbb{P}\{\Gamma(\eta^h) > w\} \right] \mathrm{d}w \right| \\
&\quad \le \lim_{u\to\infty} \sup_{\tau_u \in K_u} \int_{-M}^{M} \mathrm{e}^{w} \left(1 - \exp\left(-\frac{w^2}{2g_{u,\tau_u}^2} \right) \right) \mathbb{P}\{\Gamma(\eta^h) > w\} \, \mathrm{d}w \\
&\qquad + \lim_{u\to\infty} \sup_{\tau_u \in K_u} \left| \int_{-M}^{M} \left[\exp\left(w - \frac{w^2}{2g_{u,\tau_u}^2} \right) (\mathbb{P}\{\Gamma(\chi_{u,\tau_u}) > w\} - \mathbb{P}\{\Gamma(\eta^h) > w\}) \right] \mathrm{d}w \right| \\
&\quad \le \mathrm{e}^{M} \lim_{u\to\infty} \int_{-M}^{M} \sup_{\tau_u \in K_u} |\mathbb{P}\{\Gamma(\chi_{u,\tau_u}) > w\} - \mathbb{P}\{\Gamma(\eta^h) > w\}| \, \mathrm{d}w \\
&\quad = 0.
\end{aligned}$$

Using (4.2) for $\delta \in (0, 1/c)$, $|w| > M$ with sufficiently large M, and all large u, we have

$$\sup_{\tau_u \in K_u,\, t \in E} (1 + h_{u,\tau_u}(t)) \mathbb{E}\{\chi_{u,\tau_u}(t)\} \le \delta |w|.$$

Moreover, in view of (4.3) and (2.3) in (C3), we have, for sufficiently large u,

$$\begin{aligned}
\operatorname{var}((1 + h_{u,\tau_u}(t))\chi_{u,\tau_u}(t) - (1 + h_{u,\tau_u}(s))\chi_{u,\tau_u}(s)) &\le g_{u,\tau_u}^2 \mathbb{E}\{(Z_{u,\tau_u}(t) - Z_{u,\tau_u}(s))^2\} \\
&\le \mathbb{Q} \sum_{i=1}^{d} |s_i - t_i|^a.
\end{aligned}$$

Consequently, by the Piterbarg inequality (see, e.g. Theorem 8.1 of [26]), we obtain, for some $\varepsilon \in (0, 1)$, $\delta \in (0, 1/c)$ with c given in (F1), and all large u,

$$\begin{aligned}
&\int_{|w|>M} \exp\left(w - \frac{w^2}{2g_{u,\tau_u}^2} \right) \mathbb{P}\{\Gamma(\chi_{u,\tau_u}) > w\} \, \mathrm{d}w \\
&\quad \le \int_{|w|>M} \mathrm{e}^{w} \mathbb{P}\Big\{ c \sup_{t\in E} (1 + h_{u,\tau_u}(t))(\chi_{u,\tau_u}(t) - \mathbb{E}\{\chi_{u,\tau_u}(t)\}) \\
&\quad > w - c \sup_{t \in E,\, \tau_u \in K_u} (1 + h_{u,\tau_u}(t)) \mathbb{E}\{\chi_{u,\tau_u}(t)\} \Big\} \, \mathrm{d}w \\
&\quad \le \mathrm{e}^{-M} + \int_{M}^{\infty} \mathrm{e}^{w} \Psi\left((1 - \varepsilon)\left(\frac{1}{c} - \delta \right) w \right) \mathrm{d}w \\
&\quad =: A(M) \\
&\quad \to 0, \qquad M \to \infty.
\end{aligned}$$

Moreover, by the Borell–TIS inequality (see, e.g. [1])

$$\begin{aligned}
&\int_{|w|>M} \mathrm{e}^{w} \mathbb{P}\{\Gamma(\eta^h) > w\} \, \mathrm{d}w \\
&\quad \le \int_{|w|>M} \mathrm{e}^{w} \mathbb{P}\Big\{ c \sup_{t\in E} \eta^h(t) > w \Big\} \, \mathrm{d}w \\
&\quad \le \mathrm{e}^{-M} + \int_{M}^{\infty} \mathrm{e}^{w} \mathbb{P}\Big\{ \sqrt{2} c \sup_{t\in E} \eta(t) > w - c \sup_{t\in E} (\operatorname{var}(\eta(t)) + h(t)) \Big\} \, \mathrm{d}w
\end{aligned}$$

$$\begin{aligned}&\leq \mathrm{e}^{-M} + \int_M^\infty \exp\biggl(w - \frac{(w-a)^2}{2\sup_{t\in E}\operatorname{var}(\sqrt{2}c\eta(t))}\biggr)\,\mathrm{d}w\\ &=: B(M)\\ &\to 0, \qquad M\to\infty,\end{aligned}$$

with $a = \sqrt{2}c\mathbb{E}\{\sup_{t\in E}\eta(t)\} - c\sup_{t\in E}(\operatorname{var}(\eta(t)) + h(t)) < \infty$. Hence, (4.1) follows from

$$\begin{aligned}&\sup_{\tau_u\in K_u} |\mathcal{I}_{u,\tau_u} - \mathcal{H}_{\eta,h}^{\Gamma}(E)|\\ &\quad\leq \sup_{\tau_u\in K_u}\biggl|\int_{-M}^{M}\biggl[\exp\biggl(w - \frac{w^2}{2g_{u,\tau_u}^2}\biggr)\mathbb{P}\{\Gamma(\chi_{u,\tau_u}) > w\} - \mathrm{e}^w\mathbb{P}\{\Gamma(\eta^h) > w\}\biggr]\,\mathrm{d}w\biggr|\\ &\qquad + A(M) + B(M)\\ &\quad\to 0, \qquad u\to\infty,\ M\to\infty,\end{aligned}$$

establishing the proof. □

Proof of Proposition 2.1. It follows from Remark 2.1(ii) that it suffices to prove (2.2), (2.3), and (2.6). Without loss of generality, in the following derivation we assume that $c_i > 0$, $1 \leq i \leq d$. By (2.7), we have

$$\theta_{u,\tau_u}(s,t) = \sum_{i=1}^{d} \frac{c_i\sigma_i^2(q_i(u)|s_i - t_i|)}{\sigma_i^2(q_i(u))}, \qquad (s,t)\in E.$$

By the uniform convergence theorem (UCT) for regularly varying functions (see [5]), (2.2) holds with η defined in (2.8). Next we verify (2.3). In the case of $0 < \beta < \min(\min_{1\leq i\leq d}\alpha_{i,0}, \min_{d_2+1\leq i\leq d}\alpha_{i,\infty})$, we have

$$\sum_{i=1}^{d} \frac{c_i\sigma_i^2(q_i(u)|s_i - t_i|)}{\sigma_i^2(q_i(u))} = \sum_{i=1}^{d} c_i\frac{f_i(q_i(u)|s_i - t_i|)}{f_i(q_i(u))}|s_i - t_i|^{\beta/2},$$

with $f_i(t) = \sigma_i^2(t)/t^{\beta/2}$, $t > 0$. Note that f_i is regularly varying at 0 with index $\alpha_{i,0} - \beta/2 > 0$ for $1 \leq i \leq d$ and, for $d_2 + 1 \leq i \leq d$, f_i is regularly varying at ∞ with index $\alpha_{i,\infty} - \beta/2 > 0$. By the UCT, for any $M > 0$, we have

$$\lim_{u\to\infty}\max_{i=1,\dots,d_1}\sup_{0<|s_i-t_i|\leq M}\biggl|\frac{f_i(q_i(u)|s_i - t_i|)}{f_i(q_i(u))} - |s_i - t_i|^{\alpha_{i,0}-\beta/2}\biggr| = 0.$$

Using the fact that f_i is bounded on compact intervals for $d_2 + 1 \leq i \leq d$, again by the UCT, for any $M > 0$,

$$\lim_{u\to\infty}\max_{i=d_2+1,\dots,d}\sup_{0<|s_i-t_i|\leq M}\biggl|\frac{f_i(q_i(u)|s_i - t_i|)}{f_i(q_i(u))} - |s_i - t_i|^{\alpha_{i,\infty}-\beta/2}\biggr| = 0.$$

Moreover, since f_i is regularly varying at 0 with index $\alpha_{i,0} - \beta > 0$ and $\varphi_i \in (0,\infty)$, $d_1 + 1 \leq i \leq d_2$, then, for any $M > 0$ and large enough u,

$$\max_{d_1+1\leq i\leq d_2}\sup_{0<|s_i-t_i|\leq M}\frac{f_i(q_i(u)|s_i - t_i|)}{f_i(q_i(u))} < \infty.$$

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Thus, we conclude that, for large enough u,

$$\sum_{i=1}^{d} \frac{c_i \sigma_i^2(q_i(u)|s_i - t_i|)}{\sigma_i^2(q_i(u))} \le \mathbb{Q} \sum_{i=1}^{d} |s_i - t_i|^{\beta/2}, \qquad s, t \in E,$$

which confirms (2.3). We are now left to prove (2.6). In light of (2.7) and the UCT, we have

$$\begin{aligned}
&\lim_{u\to\infty} \sup_{t\in E\setminus\{0\},\, \tau_u\in K_u} |g_{u,\tau_u}^2 \operatorname{var}(Z_{u,\tau_u}(t) - Z_{u,\tau_u}(0)) - 2\operatorname{var}(\eta(t))| \\
&\quad \le \lim_{u\to\infty} \sup_{t\in E\setminus\{0\}, \tau_u\in K_u} \left| \frac{g_{u,\tau_u}^2 \operatorname{var}(Z_{u,\tau_u}(t) - Z_{u,\tau_u}(0))}{2\theta_{u,\tau_u}(0,t)} - 1 \right| |2\theta_{u,\tau_u}(0,t)| \\
&\qquad + \lim_{u\to\infty} \sup_{t\in E,\, \tau_u\in K_u} |2\theta_{u,\tau_u}(0,t) - 2\operatorname{var}(\eta(t))| \\
&\quad = 0,
\end{aligned}$$

which implies that (2.6) holds. This completes the proof. □

Proof of Theorem 2.2. We check that (C0)–(C3) hold. Clearly, (C0) is satisfied by the assumptions. We observe that

$$\xi_{u,\tau_u}(t) = \frac{\overline{\xi}_{u,\tau_u}(t)}{1 + h_{u,\tau_u}(t)}, \qquad t \in E,\ \tau_u \in K_u,$$

with

$$\overline{\xi}_{u,\tau_u}(t) = \frac{\xi_{u,\tau_u}(t)}{\sigma_{u,\tau_u}(t)}, \qquad h_{u,\tau_u}(t) = \frac{1 - \sigma_{u,\tau_u}(t)}{\sigma_{u,\tau_u}(t)},$$

which, together with (D1), immediately implies that (C1) is valid. Next, for $u > 0$,

$$\theta_{u,\tau_u}(s,t) = \frac{g_{u,\tau_u}^2}{2} \operatorname{var}(\overline{\xi}_{u,\tau_u}(t) - \overline{\xi}_{u,\tau_u}(s)).$$

Direct calculations yield

$$\theta_{u,\tau_u}(s,t) = I_{1,u,\tau_u}(s,t) + I_{2,u,\tau_u}(s,t) + I_{3,u,\tau_u}(s,t), \qquad s, t \in E,$$

where

$$I_{1,u,\tau_u}(s,t) = \frac{g_{u,\tau_u}^2}{2} \frac{\operatorname{var}(\xi_{u,\tau_u}(t) - \xi_{u,\tau_u}(s))}{\sigma_{u,\tau_u}^2(t)}, \qquad I_{2,u,\tau_u}(s,t) = \frac{g_{u,\tau_u}^2}{2} \frac{(\sigma_{u,\tau_u}(t) - \sigma_{u,\tau_u}(s))^2}{\sigma_{u,\tau_u}^2(t)},$$

$$I_{3,u,\tau_u}(s,t) = g_{u,\tau_u}^2 \frac{\sigma_{u,\tau_u}(t) - \sigma_{u,\tau_u}(s)}{\sigma_{u,\tau_u}^2(t)\sigma_{u,\tau_u}(s)} \mathbb{E}\{(\xi_{u,\tau_u}(s) - \xi_{u,\tau_u}(t))\xi_{u,\tau_u}(s)\}.$$

It follows from (D1) that

$$\lim_{u\to\infty} \sup_{s,t\in E,\, \tau_u\in K_u} I_{2,u,\tau_u}(s,t) \le \lim_{u\to\infty} \sup_{s,t\in E,\, \tau_u\in K_u} g_{u,\tau_u}^2 \frac{(\sigma_{u,\tau_u}(t) - 1)^2 + (1 - \sigma_{u,\tau_u}(s))^2}{\sigma_{u,\tau_u}^2(t)} = 0.$$

Further, by (D1) and (D2),

$$\lim_{u\to\infty} \sup_{\tau_u\in K_u} |I_{1,u,\tau_u}(s,t) - \operatorname{var}(\eta(t) - \eta(s))| = 0, \qquad s, t \in E,$$

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and

$$\lim_{u\to\infty}\sup_{\tau_u\in K_u}|I_{3,u,\tau_u}(s,t)| \le \lim_{u\to\infty}\sup_{\tau_u\in K_u} g^2_{u,\tau_u}\frac{|\sigma_{u,\tau_u}(t)-\sigma_{u,\tau_u}(s)|}{\sigma^2_{u,\tau_u}(t)}\sqrt{\operatorname{var}(\xi_{u,\tau_u}(s)-\xi_{u,\tau_u}(t))}$$
$$= 0, \qquad s,t\in E.$$

Thus, we confirm that (C2) holds. Moreover, by (D3) and the fact that

$$(\sigma_{u,\tau_u}(t)-\sigma_{u,\tau_u}(s))^2 \le \operatorname{var}(\xi_{u,\tau_u}(t)-\xi_{u,\tau_u}(s)),$$

we obtain

$$\lim_{u\to\infty}\sup_{\tau_u\in K_u}\sup_{s\ne t,\, s,\, t\in E}\frac{\theta_{u,\tau_u}(s,t)}{\|t-s\|^{\nu}} \le \mathbb{Q}\lim_{u\to\infty}\sup_{\tau_u\in K_u}\sup_{s\ne t,\, s,\, t\in E}\frac{g^2_{u,\tau_u}\operatorname{var}(\xi_{u,\tau_u}(t)-\xi_{u,\tau_u}(s))}{\|t-s\|^{\nu}} < \infty.$$

Using again (D1) and (D2), we obtain

$$\lim_{u\to\infty}\sup_{t\in E,\,\tau_u\in K_u}|I_{1,u,\tau_u}(0,t)-\operatorname{var}(\eta(t))| = 0,$$
$$\lim_{u\to\infty}\sup_{t\in E,\,\tau_u\in K_u} I_{2,u,\tau_u}(0,t)=0, \qquad \lim_{u\to\infty}\sup_{t\in E,\,\tau_u\in K_u}|I_{3,u,\tau_u}(0,t)|=0,$$

which imply

$$\lim_{u\to\infty}\sup_{t\in E,\,\tau_u\in K_u}|\theta_{u,\tau_u}(0,t)-\operatorname{var}(\eta(t))| = 0.$$

Hence, (C3) is satisfied using (2.6) instead of (2.4). In view of Remark 2.1, the proof is completed. □

Proof of Theorem 3.1. Recall that $F(A,B)=\inf_{s\in A,\,t\in B}\|s-t\|$ with A, B two nonempty subsets of $\mathbb{R}^d$ and $\|\cdot\|$ the Euclidean norm. Clearly, for any positive u,

$$\mathbb{P}\Big\{\sup_{t\in\lambda_1+\mathcal{E}_1}X_u(t)>m_{\lambda_1}(u),\ \sup_{t\in\lambda_2+\mathcal{E}_2}X_u(t)>m_{\lambda_2}(u)\Big\}$$
$$\le \mathbb{P}\Big\{\sup_{s\in\lambda_1+\mathcal{E}_1,\,t\in\lambda_2+\mathcal{E}_2}(X_u(s)+X_u(t))>2m_{\lambda_1,\lambda_2}(u)\Big\},$$

where $m_{\lambda_1,\lambda_2}(u)=\min(m_{\lambda_1}(u),m_{\lambda_2}(u))$. By (3.2) and (3.4), we have, for sufficiently large u and $F(\lambda_1+\mathcal{E}_1,\lambda_2+\mathcal{E}_2)>S_1$, with large enough S_1,

$$2\delta\le\operatorname{var}(X_u(s)+X_u(t)) = 4-2(1-r_u(s,t)) \le 4-\frac{2\mathcal{C}_1F^{\beta}(\lambda_1+\mathcal{E}_1,\lambda_2+\mathcal{E}_2)}{m^2(u)}.$$

Moreover, by (3.3) and the above inequality,

$$1-\operatorname{corr}(X_u(s)+X_u(t),X_u(s')+X_u(t'))$$
$$\le \frac{\operatorname{var}(X_u(s)+X_u(t)-X_u(s')-X_u(t'))}{2\sqrt{\operatorname{var}(X_u(s)+X_u(t))}\sqrt{\operatorname{var}(X_u(s')+X_u(t'))}}$$
$$\le \delta^{-1}(1-r_u(s,s')+1-r_u(t,t'))$$
$$\le \mathcal{C}_2\frac{\delta^{-1}d^{\alpha/2}}{m^2(u)}\sum_{i=1}^{d}(|s_i-s_i'|^{\alpha}+|t_i-t_i'|^{\alpha})$$

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holds for $s, t, s', t' \in [0,1]^d$. Let $X_u^*(s,t)$, $s,t \in \mathbb{R}^d$, $u > 0$, be a family of centered Gaussian random fields with unit variance and correlation satisfying

$$r_u(s,t) = \exp\bigg(-\frac{2\delta^{-1}d^{\alpha/2}\mathcal{C}_2}{m^2(u)}\sum_{i=1}^{d}(|s_i|^\alpha + |t_i|^\alpha)\bigg), \qquad s,t \in \mathbb{R}^d,$$

and further let

$$m_{u,\lambda_1,\lambda_2,\mathcal{E}_1,\mathcal{E}_2} := \frac{2m_{\lambda_1,\lambda_2}(u)}{\sqrt{4 - 2\mathcal{C}_1 F^\beta(\lambda_1+\mathcal{E}_1, \lambda_2+\mathcal{E}_2)/m^2(u)}}, \qquad I_{i_1,\ldots,i_d} = \prod_{j=1}^{d}[i_j, i_j+1].$$

For all large u, we have

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{s\in\lambda_1+\mathcal{E}_1,\, t\in\lambda_2+\mathcal{E}_2}(X_u(s)+X_u(t)) > 2m_{\lambda_1,\lambda_2}(u)\Big\}\\
&\quad\le \mathbb{P}\Big\{\sup_{s\in\lambda_1+\mathcal{E}_1,\, t\in\lambda_2+\mathcal{E}_2}\overline{X_u(s)+X_u(t)} > m_{u,\lambda_1,\lambda_2,\mathcal{E}_1,\mathcal{E}_2}\Big\}\\
&\quad\le \mathbb{P}\Big\{\sup_{s\in\lambda_1+[0,S_2]^d,\, t\in\lambda_2+[0,S_2]^d}\overline{X_u(s)+X_u(t)} > m_{u,\lambda_1,\lambda_2,\mathcal{E}_1,\mathcal{E}_2}\Big\}\\
&\quad\le \sum_{i_1,i_2,\ldots,i_d,i_1',i_2',\ldots,i_d'=0}^{[S_2]}\mathbb{P}\Big\{\sup_{s\in\lambda_1+I_{i_1,\ldots,i_d},\, t\in\lambda_2+I_{i_1',\ldots,i_d'}}\overline{X_u(s)+X_u(t)} > m_{u,\lambda_1,\lambda_2,\mathcal{E}_1,\mathcal{E}_2}\Big\}\\
&\quad\le \sum_{i_1,i_2,\ldots,i_d,i_1',i_2',\ldots,i_d'=0}^{[S_2]}\mathbb{P}\Big\{\sup_{s\in\lambda_1+I_{i_1,\ldots,i_d},\, t\in\lambda_2+I_{i_1',\ldots,i_d'}}X_u^*(s,t) > m_{u,\lambda_1,\lambda_2,\mathcal{E}_1,\mathcal{E}_2}\Big\}\\
&\quad= (S_2+1)^{2d}\mathbb{P}\Big\{\sup_{s,t\in[0,1]^d}X_u^*(s,t) > m_{u,\lambda_1,\lambda_2,\mathcal{E}_1,\mathcal{E}_2}\Big\},
\end{aligned}\tag{4.4}$$

where we used the Slepian inequality (see, e.g. [1] and [2]) to derive (4.4). Hence, in order to complete the proof, we need to apply Proposition 2.1 to the family of Gaussian random fields $\{X_u^*(s,t),\ (s,t)\in[0,1]^{2d}\}$. Let

$$K_u = \{(\lambda_1,\lambda_2), \lambda_i + \mathcal{E}_i \subset E_u,\ i = 1,2\}.$$

Note that

$$\begin{aligned}
&\lim_{u\to\infty}\sup_{(\lambda_1,\lambda_2)\in K_u}\sup_{\substack{(s,t)\neq(s',t'),\\(s,t),(s',t')\in[0,1]^{2d}}}\left|\frac{(m_{u,\lambda_1,\lambda_2,\mathcal{E}_1,\mathcal{E}_2})^2\operatorname{var}(X_u^*(s,t)-X_u^*(s',t'))}{2\sum_{i=1}^d 2\delta^{-1}d^{\alpha/2}\mathcal{C}_2(\sum_{i=1}^d|s_i-s_i'|^\alpha + \sum_{i=1}^d|t_i-t_i'|^\alpha)} - 1\right|\\
&\quad= 0.
\end{aligned}$$

Since conditions (C0) and (C1) are clearly satisfied, then Proposition 2.1 implies that

$$\lim_{u\to\infty}\sup_{(\lambda_1,\lambda_2)\in K_u}\left|\frac{1}{\Psi(m_{u,\lambda_1,\lambda_2,\mathcal{E}_1,\mathcal{E}_2})}\mathbb{P}\Big\{\sup_{s,t\in[0,1]^{2d}}X_u^*(s,t) > m_{u,\lambda_1,\lambda_2,\mathcal{E}_1,\mathcal{E}_2}\Big\} - \mathcal{H}_\eta([0,1]^{2d})\right| = 0,$$

where

$$\eta(s,t) = \sum_{i=1}^{d}\sqrt{2\delta^{-1}d^{\alpha/2}\mathcal{C}_2}B_\alpha^{(i)}(s_i) + \sum_{i=d+1}^{2d}\sqrt{2\delta^{-1}d^{\alpha/2}\mathcal{C}_2}B_\alpha^{(i)}(t_{i-d}),$$

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with $B_\alpha^{(i)}$, $1 \le i \le 2d$, an independent fractional Brownian motions with index α. Thus, we establish the claim for $F(\lambda_1+\mathcal{E}_1, \lambda_2+\mathcal{E}_2) > S_1$. For $F(\lambda_1+\mathcal{E}_1, \lambda_2+\mathcal{E}_2) \le S_1$, we have

$$\mathbb{P}\Big\{\sup_{s\in\lambda_1+\mathcal{E}_1} X_u(s) > m_{\lambda_1}(u), \sup_{t\in\lambda_2+\mathcal{E}_2} X_u(t) > m_{\lambda_2}(u)\Big\} \le \mathbb{P}\Big\{\sup_{t\in\lambda_1+[-S_1,S_2+S_1]^d} X_u(t) > m_{\lambda_1,\lambda_2}(u)\Big\}.$$

By (3.3) and the Slepian inequality

$$\begin{aligned}\mathbb{P}\Big\{\sup_{s\in\lambda_1+[-S_1,S_2+S_1]^d} X_u(s) > m_{\lambda_1,\lambda_2}(u)\Big\} &\le (S_2+2S_1+1)^d\mathbb{P}\Big\{\sup_{s\in[0,1]^d} X_u^*(\delta^{1/\alpha}s,0,\dots,0) > m_{\lambda_1,\lambda_2}(u)\Big\}\\ &\sim (S_2+2S_1+1)^d\mathcal{H}_\lambda([0,1]^d)\Psi(m_{\lambda_1,\lambda_2}(u)), \qquad u\to\infty,\end{aligned}$$

with $\lambda(s) = \sqrt{\delta}\eta(s,0,\dots,0)$. This completes the proof. □

Proof of Corollary 3.1. Let $\beta = \frac{1}{2}\min_{i=1,\dots,2d}\min(\alpha_{i,0},\alpha_{i,\infty},2)$ and $f_i(t) = \kappa_i(t)/t^\beta$. Clearly, the f_i are regularly varying at 0 with index $\alpha_{i,0}-\beta > 0$ and regularly varying at ∞ with index $\alpha_{i,\infty}-\beta > 0$. With this notation, we have

$$\frac{\kappa_i(q_i(u)|s_i-t_i|)}{\kappa_i(q_i(u))} = \frac{f_i(q_i(u)|s_i-t_i|)}{f_i(q_i(u))}|s_i-t_i|^\beta, \qquad s_i\ne t_i,\ i=1,\dots,2d. \tag{4.5}$$

Next we focus on $f_i(q_i(u)|s_i-t_i|)/f_i(q_i(u))$. We consider the upper bound and lower bound.

Lower bound. For $\varphi_i = 0$ we define $g_i(t) = 1/f_i(1/t)$. Then g_i is both regularly varying at 0 with index $\alpha_{i,\infty}-\beta > 0$ and regularly varying at ∞ with index $\alpha_{i,0}-\beta > 0$. By the assumption on the κ_i, further, the g_i are bounded over any compact interval and, by the UCT,

$$\lim_{u\to\infty}\sup_{|s_i-t_i|\ge1}\bigg|\frac{g_i(1/q_i(u)|s_i-t_i|)}{g_i(1/q_i(u))} - \bigg(\frac{1}{|s_i-t_i|}\bigg)^{\alpha_{i,0}-\beta}\bigg| = 0$$

implying that, for large enough u,

$$\frac{g_i(1/q_i(u)|s_i-t_i|)}{g_i(1/q_i(u))} \le 2, \qquad \frac{1}{|s_i-t_i|} \le 1.$$

Consequently, for sufficiently large u,

$$\frac{f_i(q_i(u)|s_i-t_i|)}{f_i(q_i(u))} = \frac{g_i(1/q_i(u))}{g_i(1/q_i(u)|s_i-t_i|)} \ge \frac{1}{2}, \qquad |s_i-t_i|\ge1.$$

Next, if $\varphi_i\in(0,\infty)$ then by the fact that $\lim_{t\to\infty}f_i(t)=\infty$, there exists $S_1>0$ and M_i' such that, for sufficiently large u,

$$\frac{f_i(q_i(u)|s_i-t_i|)}{f_i(q_i(u))} > M_i', \qquad |s_i-t_i| > S_1.$$

For $\varphi = \infty$, Potter's theorem (see, e.g. [5, Theorem 1.5.6]) implies that, for any $0 < \epsilon < \alpha_{i,\infty} - \beta$, there exist $M_i'' > 0$ and $S_1' > 1$ such that, for sufficiently large u,

$$\frac{f_i(q_i(u)|s_i - t_i|)}{f_i(q_i(u))} > M_i''|s_i - t_i|^{\alpha_{i,\infty}-\beta-\epsilon} \geq M_1'', \qquad |s_i - t_i| > S_1'.$$

Consequently, there exist $S > 1$ and $M > 0$ such that, for sufficiently large u,

$$\frac{\kappa_i(q_i(u)|s_i - t_i|)}{\kappa_i(q_i(u))} \geq M|s_i - t_i|^{\beta}, \qquad |s_i - t_i| > S, \qquad i = 1, \dots, d.$$

Further, for large enough u,

$$g_u(s, t) \geq d^{-\beta/2} M\|s - t\|^{\beta}, \qquad \|s - t\| > \sqrt{d}S. \tag{4.6}$$

Upper bound. If $\varphi_i \in \{0, \infty\}$ then using again the UCT, we have

$$\sup_{|s_i - t_i| \leq 1} \frac{f_i(q_i(u)|s_i - t_i|)}{f_i(q_i(u))} \leq C$$

is valid for all large enough u and some constant C. Further, since f_i is locally bounded then the above also holds if $\varphi_i \in (0, \infty)$. This implies that, for some $M' > 0$,

$$\widetilde{g}_u(s, t) \leq M' \sum_{i=1}^{d} |s_i - t_i|^{\beta} \leq dM'\|s - t\|^{\beta}, \qquad s - t \in [-1, 1]^d,$$

which combined with (4.6) and Theorem 3.1 establishes the claim. □

Proof of Corollary 3.2. The claim follows straightforwardly using the arguments of Corollary 3.1 for the $\varphi_i = 0$ case. □

Proof of Theorem 3.1. Without loss of generality, we assume that $a_i = -\infty$, $b_i = \infty$ for $d_1 + 1 \leq i \leq d_2$. In what follows, set

$$I_k = \prod_{i=1}^{d_1} [k_i S, (k_i + 1)S], \qquad k = (k_1, \dots, k_{d_1}),$$

$$J_l = \prod_{i=d_1+1}^{d_2} [l_i S, (l_i + 1)S] \times \prod_{i=d_2+1}^{d} [l_i T, (l_i + 1)T], \qquad l = (l_{d_1+1}, \dots, l_d),$$

$$J^* = \prod_{i=d_1+1}^{d_2} [-S, S] \times \prod_{i=d_2+1}^{d} [-T, T], \qquad \widetilde{J} = \prod_{i=d_1+1}^{d_2} [-S, S] \times \{0\}, \qquad 0 \in \mathbb{R}^{d-d_2}.$$

Further, define

$$I_k^* = I_k \times J^* \times E, \qquad \widetilde{I}_k = I_k \times \widetilde{J} \times E, \qquad I_{k,l} = I_k \times J_l \times E,$$

$$K_u^{\pm} = \left\{k, \frac{a_i(u)}{S} \mp 1 \leq k_i \leq \frac{b_i(u)}{S} \pm 1, \ 1 \leq i \leq d_1\right\},$$

and

$$L_u = \left\{ l,\ \frac{a_i(u)}{S} - 1 \le l_i \le \frac{b_i(u)}{S} + 1,\ d_1 + 1 \le i \le d_2, \right.$$
$$\left. \frac{a_i(u)}{T} - 1 \le l_i \le \frac{b_i(u)}{T} + 1,\ d_2 + 1 \le i \le d,\ J_l \nsubseteq J^* \right\}.$$

For some $\varepsilon \in (-1, 1)$ and $u > 0$, set

$$\Theta_\epsilon(u) := \prod_{i=1}^{d_1} \int_{y_{i,1}}^{y_{i,2}} \exp(-(1-\epsilon)|s|^{\beta_i})\, \mathrm{d}s \prod_{i=1}^{d_1} \left(\frac{g_i(u)}{m^2(u)} \right)^{1/\beta_i} \Psi(m(u)).$$

Observe that

$$X_u(s,t) = \frac{\sigma_u(s,t)\overline{X}_u(s,t)}{\sigma_u(0,0)}, \qquad \frac{\sigma_u(0,0)}{\sigma_u(s,t)} = \frac{\sigma_u(0,0)}{\sigma_u(s,0)} \frac{\sigma_u(s,0)}{\sigma_u(s,t)}.$$

Using (3.7) and (3.8), there exist $e_{u,1}(s)$ and $e_{u,2}(s,t)$ such that, as $u \to \infty$,

$$\sup_{s \in \prod_{i=1}^{d} [a_i(u), b_i(u)]} |e_{u,1}(s)| = o(1), \qquad \sup_{(s,t) \in E_u} |e_{u,2}(s,t)| = o(1),$$

and

$$\frac{\sigma_u(0,0)}{\sigma_u(s,0)} = 1 + (1 + e_{u,1}(s)) \sum_{i=1}^{d} \frac{|s_i|^{\beta_i}}{g_i(u)}, \qquad s \in \prod_{i=1}^{d} [a_i(u), b_i(u)],$$
$$\frac{\sigma_u(s,0)}{\sigma_u(s,t)} = 1 + (1 + e_{u,2}(s,t)) \sum_{i=d+1}^{d+n} \frac{|t_i|^{\beta_i}}{g_i(u)}, \qquad (s,t) \in E_u.$$

Note that, by (F2) for Γ^*,

$$\Gamma_{E_u}(X_u(s,t)) = \sup_{s \in \prod_{i=1}^{d} [a_i(u),\, b_i(u)]} \Gamma^*(X_u(s,t))$$
$$= \sup_{s \in \prod_{i=1}^{d} [a_i(u),\, b_i(u)]} \sigma_u(s,0)\Gamma^* \left(\overline{X}_u(s,t) \frac{\sigma_u(s,t)}{\sigma_u(s,0)} \right).$$

Thus, by (F2) for Γ^*, and the property of the sup functional, we have, for $0 < \epsilon < \frac{1}{2}$ and sufficiently large u,

$$\mathbb{P}\{\Gamma_{E_u}(X_u^{+\epsilon}) > m(u)\} \le \mathbb{P}\{\Gamma_{E_u}(X_u) > m(u)\} \le \mathbb{P}\{\Gamma_{E_u}(X_u^{-\epsilon,y}) > m(u)\}, \tag{4.7}$$

where, for $(s,t) \in E_u$,

$$X_u^{-\epsilon,y}(s,t)$$
$$= \frac{\overline{X}_u(s,t)}{(1 + \sum_{i=1}^{d_1} (1-\epsilon)[|s_i|^{\beta_i}/g_i(u)])(1 + \sum_{i=d_1+1}^{d_2} (1-\epsilon)[|s_i|^{\beta_i}/g_i(u)] + \sum_{i=d_2+1}^{d} y[|s_i|^{\beta_i}/m^2(u)])}$$
$$\times \frac{1}{(1 + (1 + e_{u,2}(s,t)) \sum_{i=d+1}^{d+n} [|t_i|^{\beta_i}/g_i(u)])},$$

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and

$$X_u^{+\epsilon}(s,t) = \overline{X}_u(s,t)\bigg[\bigg(1+\sum_{i=1}^{d_1}(1+\epsilon)\frac{|s_i|^{\beta_i}}{g_i(u)}\bigg)\bigg(1+\sum_{i=d_1+1}^{d}(1+\epsilon)\frac{|s_i|^{\beta_i}}{g_i(u)}\bigg) \times\bigg(1+(1+e_{u,2}(s,t))\sum_{i=d+1}^{d+n}\frac{|t_i|^{\beta_i}}{g_i(u)}\bigg)\bigg]^{-1}.$$

Upper bound. By the property of the sup functional, we have

$$\begin{aligned}\mathbb{P}\{\Gamma_{E_u}(X_u^{-\epsilon,y}) > m(u)\} &\le \sum_{k\in K_u^+}\mathbb{P}\{\Gamma_{I_k^*}(X_u^{-\epsilon,y}) > m(u)\} + \sum_{(k,l)\in K_u^+\times L_u}\mathbb{P}\{\Gamma_{I_{k,l}}(X_u^{-\epsilon,y}) > m(u)\}\\ &\le \sum_{k\in K_u^+}\mathbb{P}\{\Gamma_{I_0^*}(\xi_{u,k}) > m_{u,k}\} + \sum_{(k,l)\in K_u^+\times L_u}\mathbb{P}\{\Gamma_{I_{0,0}}(\xi_{u,k,l}) > m_{u,k,l}\},\end{aligned} \tag{4.8}$$

where

$$\xi_{u,k}(s,t) = \overline{X}_u(s+kS,t)\bigg[\bigg(1+\sum_{i=d_1+1}^{d_2}(1-\epsilon)\frac{|s_i|^{\beta_i}}{g_i(u)}+\sum_{i=d_2+1}^{d}y\frac{|s_i|^{\beta_i}}{m^2(u)}\bigg) \times\bigg(1+(1+e_{u,2}(s,t))\sum_{i=d+1}^{d+n}\frac{|t_i|^{\beta_i}}{g_i(u)}\bigg)\bigg]^{-1}, \qquad (s,t)\in I_0^*,$$

$$\xi_{u,k,l}(s,t) = \frac{\overline{X}_u(s+(k,l)(S,T),t)}{1+(1+e_{u,2}(s,t))\sum_{i=d+1}^{d+n}|t_i|^{\beta_i}/g_i(u)}, \qquad (s,t)\in I_{0,0},$$

$$m_{u,k} = m(u)\bigg(1+\sum_{i=1}^{d_1}(1-\epsilon)\frac{|k_i^*S|^{\beta_i}}{g_i(u)}\bigg),$$

$$m_{u,k,l} = m(u)\bigg(1+\sum_{i=1}^{d_1}(1-\epsilon)\frac{|k_i^*S|^{\beta_i}}{g_i(u)}+\sum_{i=d_1+1}^{d_2}(1-2\epsilon)\frac{|l_i^*S|^{\beta_i}}{g_i(u)} + \sum_{i=d_2+1}^{d}\frac{y}{2(|l_i^*S|^{\beta_i}/m^2(u))}\bigg),$$

with $kS = (k_1S,\dots,k_{d_1}S,0,\dots,0)\in\mathbb{R}^d$ and

$$(k,l)(S,T) = (k_1S,\dots,k_{d_1}S,l_{d_1+1}S,\dots,l_{d_2}S,l_{d_2+1}T,l_dT)\in\mathbb{R}^d,$$
$$k_i^* = \min(|k_i|,|k_i+1|),\quad 1\le i\le d_1,\qquad l_i^* = \min(|l_i|,|l_i+1|),\quad d_1+1\le i\le d.$$

In order to apply Proposition 2.1, by (3.9), set

$$\theta_{u,k}(s,t,s',t') = \sum_{i=1}^{d}\frac{c_i\sigma_i^2(q_i(u)|s_i-s_i'|)}{\sigma_i^2(q_i(u))} + \sum_{i=d+1}^{d+n}\frac{c_i\sigma_i^2(q_i(u)|t_i-t_i'|)}{\sigma_i^2(q_i(u))}, \qquad (s,t),\ (s',t')\in I_0^*,$$

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and

$$h_{u,k}(s,t) = \bigg(\sum_{i=d_1+1}^{d_2} (1-\epsilon) \frac{|s_i|^{\beta_i}}{g_i(u)} + \sum_{i=d_2+1}^{d} y \frac{|s_i|^{\beta_i}}{m^2(u)} + \sum_{i=d+1}^{d+n} \frac{|t_i|^{\beta_i}}{g_i(u)} \bigg)(1+o(1)), \qquad (s,t) \in I_0^*,$$

$$g_{u,k} = m_{u,k}, \quad K_u = K_u^+, \quad E = I_0^*.$$

First we note that condition (C0) holds straightforwardly. One can easily check that (C1) holds with

$$h_\epsilon(s,t) = \sum_{i=d_1+1}^{d_2} (1-\epsilon)\gamma_i |s_i|^{\beta_i} + \sum_{i=d_2+1}^{d} y|s_i|^{\beta_i} + \sum_{i=d+1}^{d+n} \gamma_i |t_i|^{\beta_i}, \qquad (s,t) \in I_0^*. \tag{4.9}$$

Thus, in view of (A1) and (A2) and by Proposition 2.1, we have

$$\lim_{u\to\infty} \sup_{k \in K_u^+} \left| \frac{\mathbb{P}\{\Gamma_{I_0^*}(\xi_{u,k}) > m_{u,k}\}}{\Psi(m_{u,k})} - \mathcal{H}^{\Gamma}_{V_\varphi,h_\epsilon}(I_0^*) \right| = 0, \tag{4.10}$$

with h_ϵ defined in (4.9) and $V_\varphi(s,t) = \sum_{i=1}^{d} V_{\varphi_i}(s_i) + \sum_{i=1}^{n} V_{\varphi_{d+i}}(t_i)$ with V_{φ_i} defined in (3.10). Similarly, we have

$$\lim_{u\to\infty} \sup_{(k,l)\in K_u^+ \times L_u} \left| \frac{\mathbb{P}\{\Gamma_{I_{0,0}}(\xi_{u,k,l}) > m_{u,k,l}\}}{\Psi(m_{u,k,l})} - \mathcal{H}^{\Gamma}_{V_\varphi,\widetilde{h}}(I_{0,0}) \right| = 0, \tag{4.11}$$

with $\widetilde{h}(s,t) = \sum_{i=1}^{n} \gamma_{i+d} |t_i|^{\beta_{i+d}}$. Further, as $u \to \infty$,

$$\begin{aligned} \sum_{k\in K_u^+} \mathbb{P}\{\Gamma_{I_0^*}(\xi_{u,k}) > m_k(u)\} &\sim \mathcal{H}^{\Gamma}_{V_\varphi,h_\epsilon}(I_0^*) \sum_{k\in K_u^+} \Psi(m_{u,k}) \\ &\sim \mathcal{H}^{\Gamma}_{V_\varphi,h_\epsilon}(I_0^*)\Psi(m(u)) \sum_{k\in K_u^+} \exp\bigg(-\sum_{i=1}^{d_1}(1-\epsilon)m^2(u)\frac{|k_i^* S|^{\beta_i}}{g_i(u)}\bigg) \\ &\sim S^{-d_1}\mathcal{H}^{\Gamma}_{V_\varphi,h_\epsilon}(I_0^*)\Theta_\epsilon(u) \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} &\sum_{(k,l)\in K_u^+ \times L_u} \mathbb{P}\{\Gamma_{I_{0,0}}(\xi_{u,k,l}) > m_{u,k,l}\} \\ &\quad \sim \mathcal{H}^{\Gamma}_{V_\varphi,\widetilde{h}}(I_{0,0}) \sum_{(k,l)\in K_u^+ \times L_u} \Psi(m_{u,k,l}) \\ &\quad \le \mathcal{H}^{\Gamma}_{V_\varphi,\widetilde{h}}(I_{0,0}) \sum_{k\in K_u^+} \Psi(m_{u,k}) \sum_{l\in L_u} \exp\bigg(-m^2(u)\bigg(\sum_{i=d_1+1}^{d_2} (1-2\epsilon)\frac{|l_i^* S|^{\beta_i}}{g_i(u)} \\ &\qquad + \sum_{i=d_2+1}^{d} \frac{y}{2(|l_i^* T|^{\beta_i}/m^2(u))}\bigg)\bigg)(1+o(1)) \end{aligned}$$

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$$\le \mathcal{H}^{\Gamma}_{V_\varphi,\widetilde{h}}(I_{0,0}) \sum_{k\in K_u^+} \Psi(m_{u,k}) \sum_{l\in L_u} \exp\biggl(-\sum_{i=d_1+1}^{d_2} (1-2\epsilon)\gamma_i |l_i^* S|^{\beta_i} - \sum_{i=d_2+1}^{d} \frac{y}{2|l_i^* T|^{\beta_i}}\biggr)(1+o(1))$$

$$\le S^{-d_1} \mathcal{H}^{\Gamma}_{V_\varphi,\widetilde{h}}(I_{0,0}) \biggl(\sum_{i=d_1+1}^{d_2} \mathrm{e}^{-\mathbb{Q}S^{\beta_i}} + \sum_{i=d_2+1}^{d} \mathrm{e}^{-y\mathbb{Q}T^{\beta_i}}\biggr)\Theta_\epsilon(u)(1+o(1)). \tag{4.13}$$

Lower bound. By the property of the sup functional and Bonferroni inequality, we obtain

$$\mathbb{P}\{\Gamma_{E_u}(X_u^{+\epsilon}) > m(u)\} \ge \sum_{k\in K_u^-} \mathbb{P}\{\Gamma_{\widetilde{I}_k}(X_u^{+\epsilon}) > m(u)\} - \sum_{k,q\in K_u^-,\, k\neq q} \mathbb{P}\{\Gamma_{\widetilde{I}_k}(X_u^{+\epsilon}) > m(u), \Gamma_{\widetilde{I}_q}(X_u^{+\epsilon}) > m(u)\}. \tag{4.14}$$

Similarly, as in (4.12), we have

$$\sum_{k\in K_u^-} \mathbb{P}\{\Gamma_{\widetilde{I}_k}(X_u^{+\epsilon}) > m(u)\} \sim S^{-d_1} \mathcal{H}^{\Gamma}_{V_\varphi,h_\epsilon^*}(\widetilde{I}_0)\Theta_{-\epsilon}(u), \tag{4.15}$$

with $h_\epsilon^*(s,t) = \sum_{i=d_1+1}^{d_2}(1+\epsilon)\gamma_i|s_i|^{\beta_i} + \sum_{i=1}^{n}\gamma_{i+d}|t_i|^{\beta_{i+d}}$, $(s,t)\in\widetilde{I}_0$. Finally, we focus on the double-sum term. From (F1), it follows that

$$\begin{aligned}
&\sum_{k,q\in K_u^-,\, k\neq q} \mathbb{P}\{\Gamma_{\widetilde{I}_k}(X_u^{+\epsilon}) > m(u), \Gamma_{\widetilde{I}_k}(X_u^{+\epsilon}) > m(u)\} \\
&\quad\le \sum_{k,q\in K_u^-,\, k\neq q} \mathbb{P}\Bigl\{\sup_{(s,t)\in\widetilde{I}_k} X_u^{+\epsilon}(s,t) > m(u),\ \sup_{(s,t)\in\widetilde{I}_q} X_u^{+\epsilon}(s,t) > m(u)\Bigr\} \\
&\quad\le \sum_{k,q\in K_u^-,\, k\neq q} \mathbb{P}\Bigl\{\sup_{(s,t)\in\widetilde{I}_k} \overline{X}_u(s,t) > m_{u,k},\ \sup_{(s,t)\in\widetilde{I}_q} \overline{X}_u(s,t) > m_{u,q}\Bigr\}.
\end{aligned}$$

Let, for $u>0$,

$$\mathcal{T}_1 = \{(k,q), k,q\in K_u^-, k\neq q, \widetilde{I}_k\cap\widetilde{I}_q \neq \varnothing\}, \qquad \mathcal{T}_2 = \{(k,q), k,q\in K_u^-, \widetilde{I}_k\cap\widetilde{I}_q = \varnothing\}.$$

Without loss of generality, we assume that $q_1 = k_1+1$, $S>1$. Then $\widetilde{I}_k = \widetilde{I}_k' \cup \widetilde{I}_k''$ with

$$\widetilde{I}_k' = [k_1 S, (k_1+1)S - \sqrt{S}] \times \prod_{i=2}^{d_1}[k_i S,\ (k_i+1)S] \times \widetilde{J} \times E,$$

$$\widetilde{I}_k'' = [(k_1+1)S - \sqrt{S}, (k_1+1)S] \times \prod_{i=2}^{d_1}[k_i S,\ (k_i+1)S] \times \widetilde{J} \times E.$$

Consequently,

$$\begin{aligned}
&\mathbb{P}\Bigl\{\sup_{(s,t)\in\widetilde{I}_k} \overline{X}_u(s,t) > m_{u,k},\ \sup_{(s,t)\in\widetilde{I}_q} \overline{X}_u(s,t) > m_{u,q}\Bigr\} \\
&\quad\le \mathbb{P}\Bigl\{\sup_{(s,t)\in\widetilde{I}_k'} \overline{X}_u(s,t) > m_{u,k},\ \sup_{(s,t)\in\widetilde{I}_q} \overline{X}_u(s,t) > m_{u,q}\Bigr\} + \mathbb{P}\Bigl\{\sup_{(s,t)\in\widetilde{I}_k''} \overline{X}_u(s,t) > m_{u,k}\Bigr\}.
\end{aligned}$$

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Similarly, as in (4.10), we have

$$\lim_{u\to\infty}\sup_{k\in K_u^-}\left|\frac{\mathbb{P}\{\sup_{(s,t)\in\widetilde{I}_k''}\overline{X}_u(s,t)>m_{u,k}\}}{\Psi(m_{u,k})}-\mathcal{H}^{\sup}_{V_\varphi,h_\epsilon^*}(\widehat{I}_0)\right|=0,$$

with $\widehat{I}_0=[0,\sqrt{S}]\times[0,S]^{d_1-1}\times\widetilde{J}\times E$.

Let $\beta=\min(\min_{i=1,\dots,d+n}(\alpha_{i,0}),\min_{i=1,\dots,d+n}(\alpha_{i,\infty}))$. By (3.9) and Corollary 3.1, there exist $\mathcal{C}>0$ and $\mathcal{C}_1>0$ such that

$$\mathbb{P}\Big\{\sup_{(s,t)\in\widetilde{I}_k'}\overline{X}_u(s,t)>m_{u,k},\ \sup_{(s,t)\in\widetilde{I}_q}\overline{X}_u(s,t)>m_{u,q}\Big\}\\ \le\mathcal{C}(S+|E|+1)^{2(d_2+n)}\,\mathrm{e}^{-\mathcal{C}_1S^{\beta/2}}\Psi(m^*_{u,k,q})$$

and, for $(k,q)\in\mathcal{T}_2$,

$$\mathbb{P}\Big\{\sup_{(s,t)\in\widetilde{I}_k}\overline{X}_u(s,t)>m_{u,k},\ \sup_{(s,t)\in\widetilde{I}_q}\overline{X}_u(s,t)>m_{u,q}\Big\}\\ \le\mathcal{C}(S+|E|+1)^{2(d_2+n)}\,\mathrm{e}^{-\mathcal{C}_1F^\beta(\widetilde{I}_k,\widetilde{I}_q)}\Psi(m^*_{u,k,q}),$$

with $m^*_{u,k,q}=\min(m_{u,k},m_{u,q})$. Since each $\widetilde{I}_k$ has at most 3^{d_1} neighbors then, for S and sufficiently large u,

$$\begin{aligned}\sum_{(k,q)\in\mathcal{T}_1}&\mathbb{P}\Big\{\sup_{(s,t)\in\widetilde{I}_k}\overline{X}_u(s,t)>m_{u,k},\ \sup_{(s,t)\in\widetilde{I}_q}\overline{X}_u(s,t)>m_{u,q}\Big\}\\ &\le 3^d\sum_{k\in K_u^-}\mathcal{H}^{\sup}_{V_\varphi,h_\epsilon^*}(\widehat{I}_0)\Psi(m_{u,k})+\sum_{(k,q)\in\mathcal{T}_1}\mathcal{C}(S+|E|+1)^{2(d_2+n)}\,\mathrm{e}^{-\mathcal{C}_1S^{\beta/2}}\Psi(m^*_{u,k,q})\\ &\le\mathbb{Q}\sum_{k\in K_u^-}\bigg(\mathcal{H}^{\sup}_{V_\varphi,h_\epsilon^*}(\widehat{I}_0)+\exp\bigg(-\frac{\mathcal{C}_1S^{\beta/2}}{2}\bigg)\bigg)\Psi(m_{u,k})\\ &\le\mathbb{Q}S^{-d_1}\bigg(\mathcal{H}^{\sup}_{V_\varphi,h_\epsilon^*}(\widehat{I}_0)+\exp\bigg(-\frac{\mathcal{C}_1S^{\beta/2}}{2}\bigg)\bigg)\Theta_\epsilon(u).\end{aligned}\tag{4.16}$$

Moreover, for all large u,

$$\begin{aligned}\sum_{(k,q)\in\mathcal{T}_2}&\mathbb{P}\Big\{\sup_{(s,t)\in\widetilde{I}_k}\overline{X}_u(s,t)>m_{u,k},\ \sup_{(s,t)\in\widetilde{I}_q}\overline{X}_u(s,t)>m_{u,q}\Big\}\\ &\le\sum_{(k,q)\in\mathcal{T}_2}\mathcal{C}(S+|E|+1)^{2(d_2+n)}\,\mathrm{e}^{-\mathcal{C}_1F^\beta(\widetilde{I}_k,\widetilde{I}_q)}\Psi(m_{u,k,q})\\ &\le\sum_{k\in K_u^-}\Psi(m_{u,k})\mathbb{Q}S^{\mathbb{Q}_1}\sum_{q\ne0}\exp\bigg(-\mathcal{C}_1\bigg(S^2\sum_{i=1}^{d_1}q_i^2\bigg)^{\beta/2}\bigg)\\ &\le\mathbb{Q}S^{\mathbb{Q}_1}\,\mathrm{e}^{-\mathbb{Q}_2S^\beta}\Theta_\epsilon(u).\end{aligned}\tag{4.17}$$

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Substituting (4.8)–(4.17) into (4.7) and dividing each term by $\Theta_0(u)$, we have, with $\epsilon \to 0$,

$$
\begin{aligned}
&S^{-d_1}\mathcal{H}^{\Gamma}_{V_\varphi,\,h_0^*}(\widetilde{I}_0) - \mathbb{Q}S^{-d_1}(\mathcal{H}^{\sup}_{V_\varphi,\,h_0^*}(\widehat{I}_0) + \mathrm{e}^{-\mathcal{C}_1 S^{\beta/2}/2}) - \mathbb{Q}S^{\mathbb{Q}_1}\mathrm{e}^{-\mathbb{Q}_2 S^{\beta}} \\
&\quad\le \liminf_{u\to\infty}\frac{\mathbb{P}\{\Gamma_{E_u}(X_u) > m(u)\}}{\Theta_0(u)} \\
&\quad\le \lim_{T\to 0}\lim_{y\to\infty}\limsup_{u\to\infty}\frac{\mathbb{P}\{\Gamma_{E_u}(X_u) > m(u)\}}{\Theta_0(u)} \\
&\quad\le \lim_{T\to 0} S^{-d_1}\mathcal{H}^{\Gamma}_{V_\varphi,h_0}(I_0^*) + \lim_{T\to 0}\lim_{y\to\infty} S^{-d_1}\mathcal{H}^{\Gamma}_{V_\varphi,\widetilde{h}}(I_0^*)\bigg(\sum_{i=d_1+1}^{d_2}\mathrm{e}^{-\mathbb{Q}S^{\beta_i}} + \sum_{i=d_2+1}^{d}\mathrm{e}^{-y\mathbb{Q}T^{\beta_i}}\bigg) \\
&\quad= S^{-d_1}\mathcal{H}^{\Gamma}_{V_\varphi,h_0^*}(\widetilde{I}_0)\bigg(1 + \sum_{i=d_1+1}^{d_2}\mathrm{e}^{-\mathbb{Q}S^{\beta_i}}\bigg).
\end{aligned}
\tag{4.18}
$$

Note further that

$$
\mathcal{H}^{\sup}_{V_\varphi,h_0^*}(\widehat{I}_0) = \mathcal{H}_{V_{\varphi_1}}([0,\sqrt{S}])\prod_{i=2}^{d_1}\mathcal{H}_{V_{\varphi_i}}[0,S]\prod_{i=d_1+1}^{d_2}\mathcal{P}^{h_i}_{V_{\varphi_i}}[0,S]\mathcal{H}_{\widetilde{V}^{\Gamma^*}_{\varphi,\widetilde{h}}(E)},
$$

$$
\mathcal{H}^{\Gamma}_{V_\varphi,h_0^*}(\widetilde{I}_0) = \prod_{i=1}^{d_1}\mathcal{H}_{V_{\varphi_i}}[0,S]\prod_{i=d_1+1}^{d_2}\mathcal{P}^{h_i}_{V_{\varphi_i}}[0,S]\mathcal{H}_{\widetilde{V}^{\Gamma^*}_{\varphi,\widetilde{h}}(E)},
$$

with V_{φ_i}, $\widetilde{V}_\varphi$, and $\widetilde{h}$ defined in (3.10) and (3.12). Further, using the fact that (see, e.g. Theorem 3.1 of [17])

$$
\lim_{S\to\infty}\frac{\mathcal{H}_{V_{\varphi_i}}[0,S]}{S} = \mathcal{H}_{V_{\varphi_i}} \in (0,\infty), \qquad 1 \le i \le d_1,
$$

and letting $S \to \infty$ on the left-hand side of (4.18), we have

$$
\prod_{i=1}^{d_1}\mathcal{H}_{V_{\varphi_i}}\prod_{i=d_1+1}^{d_2}\lim_{S\to\infty}\mathcal{P}^{h_i}_{V_{\varphi_i}}[-S,S]\mathcal{H}^{\Gamma^*}_{\widetilde{V}_{\varphi,\widetilde{h}}}(E) \le S^{-d_1}\mathcal{H}^{\Gamma}_{V_\varphi,h_0^*}(\widetilde{I}_0)\bigg(1 + \sum_{i=d_1+1}^{d_2}\mathrm{e}^{-\mathbb{Q}S^{\beta_i}}\bigg) < \infty.
$$

Thus, we conclude that

$$
\lim_{S\to\infty}\mathcal{P}^{h_i}_{V_{\varphi_i}}[-S,S] \in (0,\infty), \qquad d_1 + 1 \le i \le d_2,
$$

which establishes the claim by letting $S \to \infty$ on both sides of (4.18). For the other cases of $a_i, b_i, d_1 + 1 \le i \le d_2$, the proof is similar as above. □

Proof of Proposition 3.1. We have, for any S, T positive,

$$
0 < \mathcal{P}^{b}_{X}([0,S],[0,T]) \le \mathcal{P}^{b\sigma^2(t)}_{X}[0,T].
$$

In order to complete the proof, it suffices to prove that $\lim_{T\to\infty}\mathcal{P}^{b\sigma^2(t)}_{X}[0,T] < \infty$. For this purpose, define, for any $S > 0$, $u > 1$,

$$
Y_u(t) = \frac{\overline{X}(u(t+1))}{1 + b\sigma^2(ut)/2\sigma^2(u)}, \quad t \in [0, u^{-1}\ln u].
$$

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Note that

$$1-\operatorname{corr}(X(ut),X(us)) = \frac{\sigma^2(u|t-s|)-(\sigma(ut)-\sigma(us))^2}{2\sigma(ut)\sigma(us)} = \frac{\sigma^2(u|t-s|)-(u\dot\sigma(u\theta)(t-s))^2}{2\sigma(ut)\sigma(us)},$$

with $\theta\in[s,t]$. By (A1) and Theorem 1.7.2 of [5], it follows that

$$\lim_{u\to\infty}\frac{u\dot\sigma(u)}{\sigma(u)}=\alpha_\infty.$$

If we set $f(t)=t^2/\sigma^2(t)$ then by Lemma 5.2 of [8], it follows that f is bounded over any compact set and regularly varying at ∞ with index $2-2\alpha_\infty>0$. Consequently, the UCT implies that, for any $S>0$,

$$\lim_{u\to\infty}\sup_{t\in(0,S]}\left|\frac{f(ut)}{f(u)}-|t|^{2-2\alpha_\infty}\right|=0$$

and, therefore, as $u\to\infty$,

$$\begin{aligned}1-\operatorname{corr}(X(ut),X(us)) &\sim \frac{\sigma^2(u|t-s|)}{2\sigma(ut)\sigma(us)}\left(1-\frac{\alpha_\infty^2}{\theta^2}\frac{\sigma^2(u\theta)(t-s)^2}{\sigma^2(u|t-s|)}\right)\\ &= \frac{\sigma^2(u|t-s|)}{2\sigma(ut)\sigma(us)}\left(1-\alpha_\infty^2\frac{f(u|t-s|)}{f(u\theta)}\right)\\ &\sim \frac{\sigma^2(u|t-s|)}{2\sigma^2(u)} \qquad (4.19)\end{aligned}$$

for $s,t\in[1,1+u^{-1}\ln u]$. Further, let

$$I_k(u)=[ku^{-1}S,u^{-1}(k+1)S],\qquad 0\le k\le N(u),$$

with $N(u):=[S^{-1}\ln u]+1$. It follows that, for sufficiently large S,

$$p_0(u)\le\mathbb{P}\Big\{\sup_{t\in[0,u^{-1}\ln u]}Y_u(t)>\sqrt{2}\sigma(u)\Big\}\le p_0(u)+\sum_{k=1}^{N(u)}p_k(u), \qquad (4.20)$$

where

$$p_0(u)=\mathbb{P}\Big\{\sup_{t\in I_0(u)}Y_u(t)>\sqrt{2}\sigma(u)\Big\},$$

$$p_k(u)=\mathbb{P}\left\{\sup_{t\in I_k(u)}\overline{X}(u(t+1))>\sqrt{2}\sigma(u)\left(1+\frac{b\sigma^2(kS)}{4\sigma^2(u)}\right)\right\},\qquad k\ge1.$$

In order to apply Theorem 2.1, in view of (4.19) we set (using the notation of Theorem 2.1)

$$K_u=\{k\colon 0\le k\le N(u)\},\qquad E=[0,S],$$

$$g_{u,k}=\sqrt{2}\sigma(u)\left(1+\frac{b\sigma^2(kS)}{4\sigma^2(u)}\right),\qquad k\in K_u,$$

$$Z_{u,k}(t)=\overline{X}(u(u^{-1}kS+u^{-1}t+1)),\qquad k\in K_u,$$

$$\theta_{u,k}(s,t)=g_{u,k}^2\frac{\sigma^2(|t-s|)}{2\sigma^2(u)},\qquad s,t\in E,\ k\in K_u,$$

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and

$$h_{u,0}(t)=\frac{b\sigma^2(t)}{2\sigma^2(u)},\quad t\in E,\qquad h_{u,k}=0,\quad k\in K_u\setminus\{0\},\qquad \eta=X.$$

Conditions (C0) and (C2) are obviously fulfilled. Condition (C1) is also satisfied with

$$g_{u,0}^2 h_{u,0}(t)\to b\sigma^2(t),\qquad u\to\infty,$$

uniformly with respect to $t\in E$ and

$$g_{u,k}^2 h_{u,k}(t)=0,\qquad t\in E,\ k\in K_u\backslash\{0\},\qquad u>0.$$

Next we shall verify (C3). Clearly, by (A2), for sufficiently large u,

$$\theta_{u,k}(s,t)=g_{u,k}^2\frac{\sigma^2(|t-s|)}{2\sigma^2(u)}\le 2\sigma^2(|t-s|)\le Q|t-s|^{\alpha_0},\qquad s,t\in E,\ k\in K_u.$$

Moreover, by (4.19),

$$\begin{aligned}
&\sup_{k\in K_u}\sup_{\|t-s\|<\epsilon,s,\,t\in E} g_{u,k}^2\mathbb{E}\{[Z_{u,k}(t)-Z_{u,\tau}(s)]Z_{u,k}(0)\}\\
&\quad\le \sup_{k\in K_u}\sup_{\|t-s\|<\epsilon,s,\,t\in E} g_{u,k}^2\left(\frac{\sigma^2(t)}{2\sigma^2(u)}(1+o(1))-\frac{\sigma^2(s)}{2\sigma^2(u)}(1+o(1))\right)\\
&\quad\le \sup_{k\in K_u}\sup_{\|t-s\|<\epsilon,s,\,t\in E}\frac{g_{u,k}^2}{2\sigma^2(u)}(|\sigma^2(t)-\sigma^2(s)|+o(1))\\
&\quad\to 0,\qquad u\to\infty,\ \epsilon\downarrow 0.
\end{aligned}$$

Thus, (C3) is satisfied. Therefore, in light of Theorem 2.1, we have

$$\lim_{u\to\infty}\frac{p_0(u)}{\Psi(\sqrt{2}\sigma(u))}=\mathcal{P}_X^{b\sigma^2(t)}[0,S]$$

and

$$\lim_{u\to\infty}\sup_{k\in K_u/\{0\}}\left|\frac{p_k(u)}{\Psi(\sqrt{2}\sigma(u)(1+b\sigma^2(kS)/4\sigma^2(u)))}-\mathcal{H}_X[0,S]\right|=0.$$

Dividing (4.20) by $\Psi(\sqrt{2}\sigma(u))$, letting $u\to\infty$, and by (A1), we have, for sufficiently large S_1,

$$\begin{aligned}
\mathcal{P}_X^{b\sigma^2(t)}[0,S]&\le \mathcal{P}_X^{b\sigma^2(t)}[0,S_1]+\mathcal{H}_X[0,S_1]\sum_{k=1}^{\infty}\mathrm{e}^{-b\sigma^2(kS_1)/2}\\
&\le \mathcal{P}_X^{b\sigma^2(t)}[0,S_1]+\mathcal{H}_X[0,S_1]\sum_{k=1}^{\infty}\mathrm{e}^{-Q_1(kS_1)^{\alpha_\infty}}\\
&\le \mathcal{P}_X^{b\sigma^2(t)}[0,S_1]+\mathcal{H}_X[0,S_1]\,\mathrm{e}^{-Q_2S_1^{\alpha_\infty}}.
\end{aligned}$$

Next, letting $S\to\infty$, we arrive at

$$\lim_{S\to\infty}\mathcal{P}_X^{b\sigma^2(t)}[0,S]\le \mathcal{P}_X^{b\sigma^2(t)}[0,S_1]+\mathcal{H}_X[0,S_1]\,\mathrm{e}^{-Q_2S_1^{\alpha_\infty}}<\infty,$$

establishing the claim. □

https://doi.org/10.1017/apr.2017.33 Published online by Cambridge University Press

Appendix A

Proof of Remark 2.1(ii). First we suppose that (C2) and (2.4) hold. Our aim is to prove (2.6). By (2.4), the continuity of $\sigma_\eta^2(t),\ t \in E$, and the compactness of E, for any $c > 0$, there exists a constant $\epsilon := \epsilon_c > 0$ such that

$$\limsup_{u\to\infty} \sup_{\tau_u \in K_u} \sup_{\|t-s\|<\epsilon,\, s,t\in E} |g_{u,\tau_u}^2 \operatorname{var}(b_u(s)) - g_{u,\tau_u}^2 \operatorname{var}(b_u(t))| < \frac{c}{3},$$

with $b_u(t) = Z_{u,\tau_u}(t) - Z_{u,\tau_u}(0)$ and, further,

$$\sup_{\|t-s\|<\epsilon,\, s,t\in E} |\sigma_\eta^2(t) - \sigma_\eta^2(s)| < \frac{c}{3}.$$

By the compactness of E, we can find $E_c \subset E$ which has a finite number of elements such that, for any $t \in E$,

$$O_\epsilon(t) \cap E_c \neq \varnothing, \qquad O_\epsilon(t) := \{s \in \mathbb{R}^d : \|t-s\| < \epsilon\}.$$

For any $t \in E$ with $t' \in O_\epsilon(t) \cap E_c$,

$$\begin{aligned} |g_{u,\tau_u}^2 \operatorname{var}(b_u(t)) - 2\sigma_\eta^2(t)| &\le |g_{u,\tau_u}^2 \operatorname{var}(b_u(t)) - g_{u,\tau_u}^2 \operatorname{var}(b_u(t'))| \\ &\quad + 2|\sigma_\eta^2(t) - \sigma_\eta^2(t')| + |g_{u,\tau_u}^2 \operatorname{var}(b_u(t')) - 2\sigma_\eta^2(t')|. \end{aligned}$$

From (C2), it follows that

$$\lim_{u\to\infty} \sup_{\tau_u\in K_u} |g_{u,\tau_u}^2 \operatorname{var}(b_u(t)) - 2\sigma_\eta^2(t)| = 0, \qquad t \in E.$$

Consequently, we have

$$\begin{aligned} &\limsup_{u\to\infty} \sup_{\tau_u\in K_u} \sup_{t\in E} |g_{u,\tau_u}^2 \operatorname{var}(b_u(t) - 2\sigma_\eta^2(t)| \\ &\quad \le \limsup_{u\to\infty} \sup_{\tau_u\in K_u} \sup_{\|t-s\|<\epsilon,\, s,t\in E} |g_{u,\tau_u}^2 \operatorname{var}(b_u(s)) - g_{u,\tau_u}^2 \operatorname{var}(b_u(t))| \\ &\qquad + 2 \sup_{\|t-s\|<\epsilon,\, s,t\in E} |\sigma_\eta^2(t) - \sigma_\eta^2(s)| + \limsup_{u\to\infty} \sup_{\tau_u\in K_u} \sup_{t\in E_c} |g_{u,\tau_u}^2 \operatorname{var}(b_u(t)) - 2\sigma_\eta^2(t)| \\ &\quad \le c. \end{aligned}$$

Hence, letting c go to 0 yields (2.6).

Next, supposing that (C2) and (2.6) hold, we prove (2.4). By the continuity of $\sigma_\eta^2(t),\ t \in E$, and the compactness of E, for any $c > 0$, there exists a constant $\epsilon > 0$ such that

$$\sup_{\|t-s\|<\epsilon,\, s,t\in E} |\sigma_\eta^2(t) - \sigma_\eta^2(s)| < \frac{c}{3}.$$

For any $s, t \in E$,

$$\begin{aligned} &|g_{u,\tau_u}^2 \operatorname{var}(b_u(s)) - g_{u,\tau_u}^2 \operatorname{var}(b_u(t))| \\ &\quad \le |g_{u,\tau_u}^2 \operatorname{var}(b_u(s)) - 2\sigma_\eta^2(s)| + 2|\sigma_\eta^2(s) - \sigma_\eta^2(t)| + |2\sigma_\eta^2(t) - g_{u,\tau_u}^2 \operatorname{var}(b_u(t))|. \end{aligned}$$

Consequently, by (2.6),

$$\begin{aligned}
&\limsup_{u\to\infty}\sup_{\tau_u\in K_u}\sup_{\|t-s\|<\epsilon,\, s,t\in E}|g^2_{u,\tau_u}\operatorname{var}(b_u(s))-g^2_{u,\tau_u}\operatorname{var}(b_u(t))|\\
&\quad\le 2\limsup_{u\to\infty}\sup_{\tau_u\in K_u}\sup_{t\in E}|g^2_{u,\tau_u}\operatorname{var}(b_u(t))-2\sigma^2_\eta(t)|+2\sup_{\|t-s\|<\epsilon,\, s,t\in E}|\sigma^2_\eta(t)-\sigma^2_\eta(s)|\\
&\quad\le c.
\end{aligned}$$

Letting $c\to 0$, the above establishes (2.4), which completes the proof. □

Acknowledgements

We would like to thank the anonymous referees for their useful comments leading to significant improvements in the paper. Thanks to support from the Swiss National Science Foundation (grant number 200021-166274). KD acknowledges partial support from the National Science Center (grant number 2015/17/B/ST1/01102) over the period 2016–2019.

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