Proceedings of the Royal Society of Edinburgh, **143A**, 929–955, *2013*

Conical functions of purely imaginary order and argument

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(MS received 6 November 2011; accepted 1 August 2012)

Associated Legendre functions are studied for the case where the degree is in conical form $-\frac{1}{2}+\mathrm{i}\tau$ (τ real), and the order $\mathrm{i}\mu$ and argument $\mathrm{i}x$ are purely imaginary (μ and x real). Conical functions in this form have applications to Fourier expansions of the eigenfunctions on a closed geodesic. Real-valued numerically satisfactory solutions are introduced which are continuous for all real x. Uniform asymptotic approximations and expansions are then derived for the cases where one or both of μ and τ are large; these results (which involve elementary, Airy, Bessel and parabolic cylinder functions) are uniformly valid for unbounded x.

1. Introduction and definition of solutions

We consider solutions of the associated Legendre equation

$$(1-z^2)\frac{\mathrm{d}^2y}{\mathrm{d}z^2}-2z\frac{\mathrm{d}y}{\mathrm{d}z}+\left(n(n+1)-\frac{m^2}{1-z^2}\right)y=0, \tag{1.1}$$

for the case where the degree n is complex, of the form $n=-\frac{1}{2}+\mathrm{i}\tau$ with τ real. Solutions for the degree in this form are usually referred as conical functions. Conical functions have applications in problems involving the solution of Laplace's equation, for instance, when it is expressed in toroidal coordinates (see [14, ch. 7, § 4]). They also appear in the kernels of Mehler–Fock transforms (see, for example, [1] and [14, ch. 7]).

In this study we focus on conical functions that have both the argument z and order m purely imaginary. The main motivation for a study of these functions is that they appear naturally in the Fourier expansions of the eigenfunctions on a closed geodesic (see [5, 12, 13]). In the study of analytic properties of Laplacian eigenfunction on hyperbolic surfaces that are non-compact and of finite volume, if one expands in rectangular coordinates, one gets modified Bessel functions of the second kind with purely imaginary order. However, if one chooses geodesic polar coordinates, then one gets conical functions for the expansion of the form studied here.

Although understanding the behaviour of the eigenfunctions on a geodesic is a classic problem regarding hyperbolic surfaces, very few results beyond those that also hold for general Riemann surfaces are known. There are conjectures expected to be true only for hyperbolic surfaces, as opposed to Riemann surfaces; see, for

https://doi.org/10.1017/S0308210511001582 Published online by Cambridge University Press

instance, [13, conjecture B]. In this context, the asymptotics of Legendre functions of imaginary order could aid in quantifying the restricted L^2 norm on the closed geodesic of the L^2 normalized eigenfunction.

Returning to the differential equation (1.1), standard solutions are given by $P_n^{-m}(z)$ and $\boldsymbol{Q}_n^m(z)$, where

$$\boldsymbol{Q}_n^m(z) = \frac{\mathrm{e}^{-m\pi\mathrm{i}}}{\Gamma(n+m+1)} Q_n^m(z). \tag{1.2}$$

$P_n^{-m}(z)$ and $\boldsymbol{Q}_n^m(z)$ are real for z, m and n real, provided that $z > 1$. For complex z they are analytic in the plane having a cut along $(-\infty, 1]$, and their principal branches form a numerically satisfactory pair (in the sense of [6]) in the half-plane $\operatorname{Re} z \geqslant 0$, for $\operatorname{Re} n \geqslant -\frac{1}{2}$ and $\operatorname{Re} \mu \geqslant 0$. See [10, ch. 5] for more details.

Since we are considering a purely imaginary argument, let us write $z = \mathrm{i}x$ (x real). We then observe that $P_n^{-m}(\mathrm{i}x)$ and $\boldsymbol{Q}_n^m(\mathrm{i}x)$ are complex-valued functions, and, moreover, the principal branches of both suffer from being discontinuous at $x = 0$. Thus, we first define a continuous solution as follows. Firstly, let $\boldsymbol{Q}_n^m(1+(z-1)\mathrm{e}^{2\pi\mathrm{i}})$ denote the branch obtained from the principal branch of $\boldsymbol{Q}_n^m(z)$ by encircling the branch point 1 (but not the branch point -1) once in the positive sense. Then, using

$$\boldsymbol{Q}_n^m(1+(z-1)\mathrm{e}^{2\pi\mathrm{i}}) = \mathrm{e}^{-m\pi\mathrm{i}}\boldsymbol{Q}_n^m(z) - \frac{\pi\mathrm{i}}{\Gamma(n-m+1)} P_n^{-m}(z) \tag{1.3}$$

(see [11, § 14.24]), we define $\tilde{\boldsymbol{Q}}_n^m(z)$ to be the analytic continuation of the principal branch of $\boldsymbol{Q}_n^m(z)$ from the upper half-plane across the cut along $-\infty < z \leqslant 1$. Specifically, we define

$$\tilde{\boldsymbol{Q}}_n^m(z) = \begin{cases} \boldsymbol{Q}_n^m(z), & \operatorname{Im} z \geqslant 0, \\ \mathrm{e}^{-m\pi\mathrm{i}}\boldsymbol{Q}_n^m(z) - \dfrac{\pi\mathrm{i}}{\Gamma(n-m+1)} P_n^{-m}(z), & \operatorname{Im} z < 0, \end{cases} \tag{1.4}$$

with principal branches applying for $P_n^{-m}(z)$ and $\boldsymbol{Q}_n^m(z)$ here. Thus, $\tilde{\boldsymbol{Q}}_n^m(z)$ is analytic in the plane having cuts along $-\infty < z \leqslant -1$ and $1 \leqslant z < \infty$, and, in particular, $\tilde{\boldsymbol{Q}}_n^m(\mathrm{i}x)$ is continuous (indeed infinitely differentiable) for $-\infty < x < \infty$.

We next define an even (continuous) solution of (1.1), for the case $z = \mathrm{i}x$, $-\infty < x < \infty$, by

$$e_n^m(x) = \frac{\tilde{\boldsymbol{Q}}_n^m(\mathrm{i}x) + \tilde{\boldsymbol{Q}}_n^m(-\mathrm{i}x)}{2\tilde{\boldsymbol{Q}}_n^m(0)} = (1+x^2)^{-m/2} F(-\tfrac{1}{2}n - \tfrac{1}{2}m, \tfrac{1}{2}n - \tfrac{1}{2}m + \tfrac{1}{2}; \tfrac{1}{2}; -x^2), \tag{1.5}$$

with the property $e_n^m(0) = 1$.

Similarly, an odd solution is defined by

$$\begin{aligned} o_n^m(x) &= \frac{\tilde{\boldsymbol{Q}}_n^m(\mathrm{i}x) - \tilde{\boldsymbol{Q}}_n^m(-\mathrm{i}x)}{2\mathrm{i}\tilde{\boldsymbol{Q}}_n^{m\prime}(0)} \\ &= x(1+x^2)^{-m/2} F(\tfrac{1}{2} - \tfrac{1}{2}n - \tfrac{1}{2}m, \tfrac{1}{2}n - \tfrac{1}{2}m + 1; \tfrac{3}{2}; -x^2), \end{aligned} \tag{1.6}$$

with the property $o_n^{m\prime}(0) = 1$.

The denominators of (1.5) and (1.6) have the explicit representations

$$\tilde{\boldsymbol{Q}}_n^m(0) = -\frac{\mathrm{i}\mathrm{e}^{-n\pi\mathrm{i}/2}\pi}{2^{n+1}\Gamma(\frac{1}{2}n - \frac{1}{2}m + 1)\Gamma(\frac{1}{2}n + \frac{1}{2}m + 1)} \tag{1.7}$$

and

$$\tilde{\boldsymbol{Q}}_n^{m\prime}(0) = \frac{\mathrm{e}^{-n\pi\mathrm{i}/2}\pi}{2^{n}\Gamma(\frac{1}{2}n - \frac{1}{2}m + \frac{1}{2})\Gamma(\frac{1}{2}n + \frac{1}{2}m + \frac{1}{2})}. \tag{1.8}$$

Next, for purely imaginary order, we write $m = \mathrm{i}\mu$ (μ real), and so for $n = -\frac{1}{2} + \mathrm{i}\tau$ (τ real) we have, from (1.5) and (1.6),

$$e^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x) = (1 + x^2)^{-\mathrm{i}\mu/2} F(\tfrac{1}{4} - \tfrac{1}{2}\mathrm{i}\tau - \tfrac{1}{2}\mathrm{i}\mu, \tfrac{1}{4} + \tfrac{1}{2}\mathrm{i}\tau - \tfrac{1}{2}\mathrm{i}\mu; \tfrac{1}{2}; -x^2) \tag{1.9}$$

and

$$o^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x) = x(1 + x^2)^{-\mathrm{i}\mu/2} F(\tfrac{3}{4} - \tfrac{1}{2}\mathrm{i}\tau - \tfrac{1}{2}\mathrm{i}\mu, \tfrac{3}{4} + \tfrac{1}{2}\mathrm{i}\tau - \tfrac{1}{2}\mathrm{i}\mu; \tfrac{3}{2}; -x^2). \tag{1.10}$$

These functions are solutions of (1.1) in the form

$$(1 + x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\tau^2 + \frac{1}{4} - \frac{\mu^2}{1 + x^2}\right) y = 0. \tag{1.11}$$

This equation has coefficients which are all real, and has no finite singularities for $x \in (-\infty, \infty)$. Now, the solution $e^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x)$ has the properties $e^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(0) = 1$ and $e^{\mathrm{i}\mu\prime}_{-(1/2)+\mathrm{i}\tau}(0) = 0$, and hence it is seen by induction that all the coefficients in its Maclaurin series, when derived from (1.11), are real. We conclude that $e^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x)$ *is real for all real* x. Similarly, the odd solution $o^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x)$ *is also real for all real* x.

We remark that if $y(x)$ is a solution of (1.11), then $W(z) = (\cosh(z))^{1/2} y(\sinh(z))$ satisfies the equation

$$\frac{\mathrm{d}^2 W}{\mathrm{d}z^2} + \left(\tau^2 - \frac{\mu^2 + \frac{1}{4}}{\cosh^2(z)}\right) W = 0. \tag{1.12}$$

If we neglect the second term in the parentheses, we observe, at least heuristically, that $W \sim A\cos(\tau z) + B\sin(\tau z)$ as either $\tau \to \infty$ or $z \to \pm\infty$, where A and B are constants. Hence, the even solution of (1.11) has the asymptotic behaviour

$$e^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x) \sim A(1 + x^2)^{-1/4}\cos\{\tau\sinh^{-1}(x)\}, \tag{1.13}$$

as either $\tau \to \infty$ or $x \to \pm\infty$, with μ bounded. Similarly, for $o^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x)$, the cosine in (1.13) is replaced by sine.

From (1.9) and (1.10) we have, as $x \to 0$,

$$e^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x) = 1 + \tfrac{1}{8}(4\mu^2 - 4\tau^2 - 1)x^2 + O(x^2), \tag{1.14}$$

$$o^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x) = x + \tfrac{1}{24}(4\mu^2 - 4\tau^2 - 9)x^3 + O(x^5), \tag{1.15}$$

and, moreover, their Wronskian is given by

$$\mathcal{W}\{e^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x), o^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x)\} = \frac{1}{x^2 + 1}. \tag{1.16}$$

From

$$\boldsymbol{Q}_n^m(z) \sim \frac{\pi^{1/2}}{\Gamma(n+\frac{3}{2})(2z)^{n+1}}, \quad z \to \infty,\ n \neq -\tfrac{3}{2}, -\tfrac{5}{2}, -\tfrac{7}{2}, \dots, \tag{1.17}$$

and

$$\cos(n\pi) P_n^{-m}(z) = -\frac{\boldsymbol{Q}_n^m(z)}{\Gamma(m-n)} + \frac{\boldsymbol{Q}_{-n-1}^m(z)}{\Gamma(m+n+1)} \tag{1.18}$$

(see [10, ch. 5, (12.09), (12.12)]) and the definitions (1.4)–(1.6), we have

$$e_{-(1/2)+\mathrm{i}\tau}^{\mathrm{i}\mu}(x) \sim \operatorname{Re}\left\{ \frac{2\pi^{1/2}\Gamma(\mathrm{i}\tau)|x|^{-1/2+\mathrm{i}\tau}}{\Gamma(\frac{1}{4}+\frac{1}{2}\mathrm{i}\tau+\frac{1}{2}\mathrm{i}\mu)\Gamma(\frac{1}{4}+\frac{1}{2}\mathrm{i}\tau-\frac{1}{2}\mathrm{i}\mu)} \right\} \tag{1.19}$$

and

$$o_{-(1/2)+\mathrm{i}\tau}^{\mathrm{i}\mu}(x) \sim \pm \operatorname{Re}\left\{ \frac{\pi^{1/2}\Gamma(\mathrm{i}\tau)|x|^{-1/2+\mathrm{i}\tau}}{\Gamma(\frac{3}{4}+\frac{1}{2}\mathrm{i}\tau+\frac{1}{2}\mathrm{i}\mu)\Gamma(\frac{3}{4}+\frac{1}{2}\mathrm{i}\tau-\frac{1}{2}\mathrm{i}\mu)} \right\} \tag{1.20}$$

as $x \to \pm\infty$.

We next introduce solutions of (1.11) which are characterized by their behaviour at infinity. Specifically, we define, for any real θ,

$$R_\tau^\mu(\theta, x) = (2/\pi)^{1/2} \mathrm{e}^{-\tau\pi/2} \operatorname{Re}\{\mathrm{e}^{\mathrm{i}\pi/4+\mathrm{i}\theta} 2^{\mathrm{i}\tau} \Gamma(1+\mathrm{i}\tau) \tilde{\boldsymbol{Q}}_{-(1/2)+\mathrm{i}\tau}^{\mathrm{i}\mu}(\mathrm{i}x)\}. \tag{1.21}$$

From (1.4) and (1.7) we observe that this solution has the property

$$R_\tau^\mu(\theta, x) \sim x^{-1/2} \cos\{\tau \ln(x) + \theta\}, \quad x \to \infty. \tag{1.22}$$

We note that the Wronskian

$$\mathcal{W}\{R_\tau^\mu(\theta, x), R_\tau^\mu(\theta - \tfrac{1}{2}\pi, x)\} = \frac{\tau}{x^2+1}, \tag{1.23}$$

and hence $R_\tau^\mu(\theta, x)$ and $R_\tau^\mu(\theta - \frac{1}{2}\pi, x)$ form a numerically satisfactory pair for the interval $[0, \infty)$.

For $x = 0$, we note from (1.7), (1.8) and (1.21) that

$$\begin{aligned} R_\tau^\mu(\theta, 0) &= (2/\pi)^{1/2} \mathrm{e}^{-\tau\pi/2} \operatorname{Re}\{\mathrm{e}^{\mathrm{i}\pi/4+\mathrm{i}\theta} 2^{\mathrm{i}\tau} \Gamma(1+\mathrm{i}\tau) \tilde{\boldsymbol{Q}}_{-(1/2)+\mathrm{i}\tau}^{\mathrm{i}\mu}(0)\} \\ &= \operatorname{Re}\left\{ \frac{\mathrm{e}^{\mathrm{i}\theta}\pi^{1/2}\Gamma(1+\mathrm{i}\tau)}{\Gamma(\frac{3}{4}+\frac{1}{2}\mathrm{i}\tau+\frac{1}{2}\mathrm{i}\mu)\Gamma(\frac{3}{4}+\frac{1}{2}\mathrm{i}\tau-\frac{1}{2}\mathrm{i}\mu)} \right\}, \end{aligned} \tag{1.24}$$

along with

$$\begin{aligned} R_\tau^{\mu\prime}(\theta, 0) &= (2/\pi)^{1/2} \mathrm{e}^{-\tau\pi/2} \operatorname{Re}\{\mathrm{e}^{3\mathrm{i}\pi/4+\mathrm{i}\theta} 2^{\mathrm{i}\tau} \Gamma(1+\mathrm{i}\tau) \tilde{\boldsymbol{Q}}_{-(1/2)+\mathrm{i}\tau}^{\mathrm{i}\mu\prime}(0)\} \\ &= -\operatorname{Re}\left\{ \frac{2\mathrm{e}^{\mathrm{i}\theta}\pi^{1/2}\Gamma(1+\mathrm{i}\tau)}{\Gamma(\frac{1}{4}+\frac{1}{2}\mathrm{i}\tau+\frac{1}{2}\mathrm{i}\mu)\Gamma(\frac{1}{4}+\frac{1}{2}\mathrm{i}\tau-\frac{1}{2}\mathrm{i}\mu)} \right\}. \end{aligned} \tag{1.25}$$

From (1.14), (1.15) and (1.23) the following results are easily verified.

https://doi.org/10.1017/S0308210511001582 Published online by Cambridge University Press

Connection formulae.

$$\tau e^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x) = R^{\mu'}_{\tau}(\theta - \tfrac{1}{2}\pi, 0) R^{\mu}_{\tau}(\theta, x) - R^{\mu'}_{\tau}(\theta, 0) R^{\mu}_{\tau}(\theta - \tfrac{1}{2}\pi, x), \tag{1.26}$$

$$\tau o^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x) = -R^{\mu}_{\tau}(\theta - \tfrac{1}{2}\pi, 0) R^{\mu}_{\tau}(\theta, x) + R^{\mu}_{\tau}(\theta, 0) R^{\mu}_{\tau}(\theta - \tfrac{1}{2}\pi, x), \tag{1.27}$$

$$R^{\mu}_{\tau}(\theta, \pm x) = R^{\mu}_{\tau}(\theta, 0) e^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x) \pm R^{\mu'}_{\tau}(\theta, 0) o^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x), \tag{1.28}$$

and

$$\begin{aligned} \tau R^{\mu}_{\tau}(\theta, -x) = \{R^{\mu}_{\tau}(\theta, 0) R^{\mu'}_{\tau}(\theta - \tfrac{1}{2}\pi, 0) + R^{\mu'}_{\tau}(\theta, 0) R^{\mu}_{\tau}(\theta - \tfrac{1}{2}\pi, 0)\} R^{\mu}_{\tau}(\theta, x) \\ - 2R^{\mu}_{\tau}(\theta, 0) R^{\mu'}_{\tau}(\theta, 0) R^{\mu}_{\tau}(\theta - \tfrac{1}{2}\pi, x). \end{aligned} \tag{1.29}$$

The plan of this paper is as follows. In §§ 2–6 we consider τ large, and obtain asymptotic approximations for the above-defined functions, which when taken together are valid for $0 \leqslant \mu/\tau \leqslant B < \infty$, uniformly for $0 \leqslant x < \infty$. Specifically, in § 2 we treat the case $0 \leqslant \mu/\tau \leqslant 1 - \delta$, $0 < \delta < 1$, and apply Liouville–Green (WKBJ) asymptotic expansions involving elementary functions, which are uniformly valid for $-\infty < x < \infty$. In § 3 the case $1 - \delta \leqslant \mu/\tau \leqslant 1 + \delta$, $0 < \delta < 1$, is tackled, and for this parameter range two turning points of the differential equation can coalesce at $x = 0$. The appropriate theory is given by [9], which furnishes approximations in terms of modified parabolic cylinder functions (see [11, § 12.14]), which also are uniformly valid for $-\infty < x < \infty$. In § 4 we consider $1 + \delta \leqslant \mu/\tau \leqslant B < \infty$ with $0 \leqslant x < \infty$. This time there is one simple turning point in the interval, and classic Airy function expansions are derived (see [10, ch. 11]). These results are unified in § 5, in which $0 \leqslant \mu/\tau \leqslant B < \infty$, but with x restricted to lying in the interval $[\operatorname{Re}\sqrt{B^2 - 1} + \delta, \infty)$, $\delta > 0$. Due to this restriction on x, there are no turning points in the interval, and this allows the construction of simpler Liouville–Green (WKBJ) asymptotic expansions.

In § 6–8 we consider μ large. In § 6 we assume τ is bounded, and apply the theory of [3] to obtain asymptotic expansions involving modified Bessel functions with purely imaginary order. The differential equation in this case is characterized by the dominant term having a simple pole, with solutions oscillatory in its neighbourhood. Asymptotic expansions of a similar form, but with a more complicated transformation of independent variables, are derived in § 7 from the theory in [2], and these are valid for $0 \leqslant \tau/\mu \leqslant 1 - \delta$, $0 < \delta < 1$. In this case the appropriate transformed differential equation has a simple pole and coalescing turning point, with solutions being oscillatory in behaviour in between the two critical points. The expansions in both cases of §§ 6 and 7 are uniformly valid for $0 \leqslant x < \infty$. Finally, in § 8, we consider $0 \leqslant \tau/\mu \leqslant A$, $0 < A < 1$, and obtain simpler expansions (Liouville–Green/WKBJ) by making the restriction $-A^{-1}\sqrt{1 - A^2} + \delta \leqslant x \leqslant A^{-1}\sqrt{1 - A^2} - \delta$, $\delta > 0$: in this interval there is no turning point.

It should be noted that μ large with $1 - \delta \leqslant \tau/\mu \leqslant B < \infty$, $0 < \delta < 1$, is equivalent to the parameter regimes of § 3 and 4 combined; hence, we have covered all possible unbounded non-negative values of x, μ and τ, with one or both of the parameters being large. Explicit error bounds are available for all our approximations, via the various general rigorous asymptotic results that we shall use, but we do not include them in this paper.

For economy of notation, we shall use various symbols in different contexts. For example, we shall use ξ and ζ as certain transformed independent variables, and these will vary from section to section: see, for example, (2.6) and (8.3). Consistency is maintained within any one given section.

We remark that significant numerical algorithms have recently been developed for the computation of some of our approximants, namely the modified Bessel functions of imaginary order and the modified parabolic cylinder functions (see [11]).

Regarding earlier results in the literature, the most powerful asymptotic approximations previously derived for conical functions are given in [4]. Therein μ is real, the degree is $-\frac{1}{2}+\mathrm{i}\tau$, $\tau \geqslant 0$, the argument z is real or complex and the cases of one or both τ and μ being large are considered. As $\tau \to \infty$, expansions are furnished that involve Bessel functions of order μ, and are valid for $0 \leqslant \mu \leqslant A\tau$ (A an arbitrary positive constant). These are uniformly valid for $\mathrm{Re}(z) \geqslant 0$ in the complex argument case, and z non-negative in the real argument case. The case $\mu \to \infty$ was also considered, and expansions were furnished that are valid for $0 \leqslant \tau \leqslant B\mu$ (B an arbitrary positive constant) uniformly for $\mathrm{Re}(z) \geqslant 0$; in the cases where z is complex, as well as real with $z \in (1, \infty)$, the given expansions involve Bessel functions of purely imaginary order $\mathrm{i}\tau$, and in the real-variable case where $z \in [0, 1)$ the expansions involve elementary functions.

In [8] uniform asymptotic approximations are derived for the conical functions of purely imaginary order $\mathrm{P}^{\mathrm{i}\mu}_{-1/2+\mathrm{i}\tau}(x)$ and $\mathrm{Q}^{\mathrm{i}\mu}_{-1/2+\mathrm{i}\tau}(x)$, where x is real and $\tau \to \infty$. These approximations involve parabolic cylinder functions, and are uniformly valid for $-1 < x < 1$ and $1 - \delta \leqslant \mu/\tau \leqslant 1 + \delta$, $0 < \delta < 1$.

We also note that, in [2], asymptotic expansions were derived for associated Legendre functions of large real degree n, purely imaginary order $\mathrm{i}\mu$ and real or complex argument. These expansions are uniformly valid for unbounded argument, with $0 \leqslant \mu/n \leqslant A < \infty$.

For a summary of other asymptotic results for associated Legendre functions, see [11].

2. Large τ, $0 \leqslant \mu/\tau \leqslant 1 - \delta$, $\delta > 0$, and $-\infty < x < \infty$

As usual for the asymptotic study of linear second-order differential equations, we first remove the first derivative in (1.11), which in this case is achieved by the following change of dependent variable:

$$y(x) = (1 + x^2)^{-1/2} w(x). \tag{2.1}$$

We thus obtain

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \left\{ -\frac{\tau^2}{1+x^2} + \frac{\mu^2}{(1+x^2)^2} + \frac{3-x^2}{4(1+x^2)^2} \right\} w. \tag{2.2}$$

Here we consider $\tau \to \infty$, and to do so we introduce

$$\mu = \beta\tau, \tag{2.3}$$

yielding

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \left\{ -\frac{\tau^2(1-\beta^2+x^2)}{(1+x^2)^2} + \frac{3-x^2}{4(1+x^2)^2} \right\} w. \tag{2.4}$$

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For large τ this equation has turning points (zeros of the dominate term) at $x = \pm\sqrt{\beta^2 - 1}$, and these can be real or imaginary in our parameter range. Let us consider the situation where they are purely imaginary, so that

$$0 \leqslant \beta \leqslant 1 - \delta, \quad \delta > 0. \tag{2.5}$$

Consequently, there is no turning point in (or close) to the interval $-\infty < x < \infty$. This allows us to apply the Liouville–Green expansions given by [10, ch. 10]. We next make the Liouville transformation of independent variable (see [10, ch. 10, (2.02)])

$$\begin{aligned}\xi &= \int_0^x \frac{(1-\beta^2+p^2)^{1/2}}{1+p^2}\,\mathrm{d}p \\ &= \operatorname{arctanh}\left\{\frac{x}{(1-\beta^2+x^2)^{1/2}}\right\} - \beta \operatorname{arctanh}\left\{\frac{\beta x}{(1-\beta^2+x^2)^{1/2}}\right\}.\end{aligned} \tag{2.6}$$

The lower limit of integration in the integral was chosen to ensure that ξ is an odd function of x.

As $x \to 0$,

$$\xi = (1-\beta^2)^{1/2} x + O(x^3), \tag{2.7}$$

and, as $x \to \infty$,

$$\xi = \ln(2x) - \beta \operatorname{arctanh}(\beta) + O(x^{-2}). \tag{2.8}$$

Then, with the new dependent variable defined by

$$W = \frac{(1-\beta^2+x^2)^{1/4}}{(1+x^2)^{1/2}} w, \tag{2.9}$$

we obtain

$$\frac{\mathrm{d}^2 W}{\mathrm{d}\xi^2} = \{-\tau^2 + \psi(\xi)\} W, \tag{2.10}$$

where

$$\psi(\xi) = \frac{(1+x^2)(1-\beta^4+x^2-4\beta^2x^2)}{4(1-\beta^2+x^2)^3}. \tag{2.11}$$

We apply [10, ch. 10, theorem 3.1], with $u = \mathrm{i}\tau$ in the solution (3.02) of that theorem, and, defining

$$W_{2n+1,1}(\tau,\xi) = \tfrac{1}{2}\operatorname{Re}\{W_{n,1}(\mathrm{i}\tau,\xi) + W_{n,1}(\mathrm{i}\tau,-\xi)\} \tag{2.12}$$

and

$$W_{2n+1,2}(\tau,\xi) = \tfrac{1}{2}\operatorname{Im}\{W_{n,1}(\mathrm{i}\tau,\xi) - W_{n,1}(\mathrm{i}\tau,-\xi)\}, \tag{2.13}$$

we obtain the following even asymptotic solution of (2.10):

$$W_{2n+1,1}(\tau,\xi) = \cos(\tau\xi)\sum_{s=0}^{n}(-1)^s\frac{A_{2s}(\xi)}{\tau^{2s}} + \sin(\tau\xi)\sum_{s=0}^{n-1}(-1)^s\frac{A_{2s+1}(\xi)}{\tau^{2s+1}} + \varepsilon_{2n+1,1}(\tau,\xi) \tag{2.14}$$

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and the odd asymptotic solution

$$W_{2n+1,2}(\tau,\xi) = \sin(\tau\xi)\sum_{s=0}^{n}(-1)^s\frac{A_{2s}(\xi)}{\tau^{2s}} - \cos(\tau\xi)\sum_{s=0}^{n-1}(-1)^s\frac{A_{2s+1}(\xi)}{\tau^{2s+1}} + \varepsilon_{2n+1,2}(\tau,\xi), \tag{2.15}$$

where $A_0(\xi) = 1$, and

$$A_{s+1}(\xi) = \tfrac{1}{2}\{A_s'(0) - A_s'(\xi)\} + \frac{1}{2}\int_0^\xi \psi(\eta)A_s(\eta)\,\mathrm{d}\eta, \quad s = 0, 1, 2. \tag{2.16}$$

Note that we have chosen the integration constants in (2.16) so that $A_s(0) = 0$ for $s = 1, 2, 3, \ldots$, and, consequently, since $\psi(\xi)$ is an even function of ξ, it can be shown by induction that

$$A_s(-\xi) = (-1)^s A_s(\xi), \quad s = 0, 1, 2, \ldots. \tag{2.17}$$

In (2.14) and (2.15) the error terms $\varepsilon_{2n+1,j}(\tau,\xi)$ have explicit bounds, and are $O(\tau^{-2n-1})$ uniformly for $-\infty < \xi < \infty$ (equivalently, $-\infty < x < \infty$). Moreover, they, and their first derivatives, vanish at $\xi = 0$ (equivalently, $x = 0$) if we take $\alpha_1 = 0$ in the $j = 1$ bound of [10, ch. 10, theorem 3.1]. As a result, we find by uniqueness of even and odd solutions, along with (2.1), (2.7) and (2.9), that

$$e^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x) = \left(\frac{1-\beta^2}{1-\beta^2+x^2}\right)^{1/4} W_{2n+1,1}(\tau,\xi) \tag{2.18}$$

and

$$o^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x) = \left\{\tau - \sum_{s=0}^{n-1}(-1)^s\frac{A'_{2s+1}(0)}{\tau^{2s+1}}\right\}^{-1} \times \{(1-\beta^2)(1-\beta^2+x^2)\}^{-1/4} W_{2n+1,2}(\tau,\xi), \tag{2.19}$$

for $0 \leqslant \mu/\tau \leqslant 1-\delta$, $\delta > 0$, uniformly for $-\infty < x < \infty$. The corresponding expansions for $R^\mu_\tau(\theta, \pm x)$ are derivable from (1.28).

Returning to the original variables, and taking $n = 0$, we have

$$e^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x) = \left(\frac{\tau^2-\mu^2}{\tau^2x^2+\tau^2-\mu^2}\right)^{1/4}\left\{\cos(\tau\xi) + O\left(\frac{1}{\tau}\right)\right\} \tag{2.20}$$

and

$$o^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x) = \{(\tau^2-\mu^2)(\tau^2x^2+\tau^2-\mu^2)\}^{-1/4}\left\{\sin(\tau\xi) + O\left(\frac{1}{\tau}\right)\right\}, \tag{2.21}$$

where

$$\xi = \ln\left\{\frac{\tau x + (\tau^2x^2+\tau^2-\mu^2)^{1/2}}{(\tau^2-\mu^2)^{1/2}}\right\} - \frac{\mu}{\tau}\operatorname{arctanh}\left\{\frac{\mu x}{(\tau^2x^2+\tau^2-\mu^2)^{1/2}}\right\}. \tag{2.22}$$

3. Large τ, $1-\delta \leqslant \mu/\tau \leqslant 1+\delta$, $\delta > 0$, and $-\infty < x < \infty$

Again we consider the equation in the form (2.4), but now assume that $1-\delta \leqslant \beta \leqslant 1+\delta$, $\delta > 0$. Let us break this into two subcases, $\beta \leqslant 1$ and $\beta \geqslant 1$. We first consider the former, namely

$$1-\delta \leqslant \beta \leqslant 1. \tag{3.1}$$

Thus, the turning points of (3.1) are purely imaginary, located at $x = \pm\mathrm{i}\sqrt{1-\beta^2}$, and coalesce at $x = 0$ when $\beta \to 1$. The appropriate theory is given by [9]. To apply it, we make the Liouville transformation (see [7, § 2])

$$\int_0^x \frac{(p^2+1-\beta^2)^{1/2}}{p^2+1}\,\mathrm{d}p = \int_0^\zeta (\eta^2+\hat{\beta}^2)^{1/2}\,\mathrm{d}\eta, \tag{3.2}$$

where $\hat{\beta}$ is defined by

$$\int_{-\mathrm{i}\sqrt{1-\beta^2}}^{\mathrm{i}\sqrt{1-\beta^2}} \frac{(x^2+1-\beta^2)^{1/2}}{x^2+1}\,\mathrm{d}x = \int_{-\mathrm{i}\hat{\beta}}^{\mathrm{i}\hat{\beta}} (\zeta^2+\hat{\beta}^2)^{1/2}\,\mathrm{d}\zeta. \tag{3.3}$$

Upon explicit integration we find from (3.3) that

$$\hat{\beta} = |2(1-\beta)|^{1/2}, \tag{3.4}$$

and, from (3.2),

$$\begin{aligned} \ln\{x+(x^2+1-\beta^2)^{1/2}\} + \beta\ln\{(x^2+1-\beta^2)^{1/2} - \beta x\} \\ - \tfrac{1}{2}\beta\ln(x^2+1) - \tfrac{1}{2}(\beta+1)\ln(1-\beta^2) \\ = \tfrac{1}{2}\zeta(\zeta^2+\hat{\beta}^2)^{1/2} + \tfrac{1}{2}\hat{\beta}^2 \operatorname{arcsinh}\left(\frac{\zeta}{\hat{\beta}}\right). \end{aligned} \tag{3.5}$$

We use the absolute sign in (3.4) to take into account that later we will consider the case $\beta \geqslant 1$.

The x interval $(-\infty,\infty)$ is mapped one-to-one to the ζ interval $(-\infty,\infty)$, with the points $x = 0, \pm\mathrm{i}\sqrt{1-\beta^2}, \pm\infty$ mapped to $\zeta = 0, \pm\mathrm{i}\hat{\beta}, \pm\infty$, respectively. We observe that ζ is an odd function of x, and $\zeta \to \pm\infty$ as $x \to \pm\infty$, such that

$$x = \pm(\tfrac{1}{2}(\beta+1))^{(\beta+1)/2}|\zeta|^{1-\beta}\exp\{\tfrac{1}{2}(\zeta^2+1-\beta)\}\{1+O(\zeta^{-2})\}. \tag{3.6}$$

Next we introduce the dependent variable

$$W = \frac{(x^2+1-\beta^2)^{1/4}}{(x^2+1)^{1/2}(\zeta^2+\hat{\beta}^2)^{1/4}}w, \tag{3.7}$$

which, in conjunction with (3.2), recasts (2.4) in the form

$$\frac{\mathrm{d}^2W}{\mathrm{d}\zeta^2} = \{-\tau^2(\zeta^2+\hat{\beta}^2) + \psi(\hat{\beta},\zeta)\}W, \tag{3.8}$$

where

$$\psi(\hat{\beta},\zeta) = \frac{3\zeta^2-2\hat{\beta}^2}{4(\zeta^2+\hat{\beta}^2)^2} + \frac{(\zeta^2+\hat{\beta}^2)(x^2+1)(x^2+1-4\beta^2x^2-\beta^4)}{4(x^2+1-\beta^2)^3}. \tag{3.9}$$

Note from (3.6) that

$$\psi(\hat{\beta},\zeta) = O(\zeta^{-2}) \quad \text{as } \zeta \to \pm\infty. \tag{3.10}$$

Equation (3.8) has turning points at $\zeta = \pm\mathrm{i}\hat{\beta}$, which coalesce at $\zeta = 0$ as $\beta \to 1$ ($\hat{\beta} \to 0$). The dominant term therefore is of a similar form to (2.4), but is simpler.

We apply the theorem of [9] to (3.8) to obtain the asymptotic solutions

$$w(\tau,\hat{\beta},\pm\zeta) = W(\mu-\tau, \pm\zeta\sqrt{2\tau}) + \varepsilon(\tau,\hat{\beta},\pm\zeta), \tag{3.11}$$

where $W(b,x)$ is a (real-valued) modified parabolic cylinder function defined by

$$W(b,x) = \{\tfrac{1}{2}k(b)\}^{1/2}\mathrm{e}^{\pi b/4}\{\mathrm{e}^{\mathrm{i}\phi_1}U(\mathrm{i}b, x\mathrm{e}^{-\pi\mathrm{i}/4}) + \mathrm{e}^{-\mathrm{i}\phi_1}U(-\mathrm{i}b, x\mathrm{e}^{\pi\mathrm{i}/4})\}, \tag{3.12}$$

with

$$k(b) = \sqrt{1+\mathrm{e}^{2\pi b}} - \mathrm{e}^{\pi b}, \tag{3.13}$$

$$\phi_1(b) = \tfrac{1}{8}\pi + \tfrac{1}{2}\phi_2(b), \tag{3.14}$$

$$\phi_2(b) = \arg\Gamma(\tfrac{1}{2}+\mathrm{i}b), \tag{3.15}$$

the latter being defined such that $\phi_2(0) = 0$, and to be continuous for all real-valued b. Here $b = \mu - \tau \leqslant 0$.

We remark that $W(\mu-\tau, \pm\zeta\sqrt{2\tau})$ are solutions of the comparison equation to (3.8), namely

$$\frac{\mathrm{d}^2W}{\mathrm{d}\zeta^2} = -\tau^2(\zeta^2+\hat{\beta}^2)W. \tag{3.16}$$

From [7] we note that, as $x \to \infty$,

$$W(b,x) = \left\{\frac{2k(b)}{x}\right\}^{1/2}\cos\{\tfrac{1}{4}x^2 - b\ln(x) + \tfrac{1}{2}\phi_2(b) + \tfrac{1}{4}\pi\} + O\left(\frac{1}{x^2}\right) \tag{3.17}$$

and

$$W(b,-x) = \left\{\frac{2}{k(b)x}\right\}^{1/2}\sin\{\tfrac{1}{4}x^2 - b\ln(x) + \tfrac{1}{2}\phi_2(b) + \tfrac{1}{4}\pi\} + O\left(\frac{1}{x^2}\right). \tag{3.18}$$

The error term in (3.11) satisfies

$$\varepsilon(\tau,\hat{\beta},\zeta) = \operatorname{env} W(\mu-\tau, \zeta\sqrt{2\tau})O(\tau^{-1}\ln(\tau)), \tag{3.19}$$

as $\tau \to \infty$, uniformly for $-\infty < \zeta < \infty$, where

$$\operatorname{env} W(b,x) = \begin{cases} \{W^2(b,x) + k^{-2}(b)W^2(b,-x)\}^{1/2}, & -\infty < x \leqslant -\sigma(b), \\ \sqrt{2}W(b,x), & -\sigma(b) \leqslant x \leqslant \sigma(b), \\ \{W^2(b,x) + k^2(b)W^2(b,-x)\}^{1/2}, & \sigma(b) \leqslant x < \infty, \end{cases} \tag{3.20}$$

in which $\sigma(b)$ denotes the smallest positive root of the equation

$$W(b,x) = k(b)W(b,-x).$$

Furthermore,

$$\varepsilon(\tau,\hat{\beta},\zeta) = O(\zeta^{-5/2}), \quad \zeta \to \infty. \tag{3.21}$$

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We now can match standard solutions with the asymptotic ones, starting with solutions whose behaviour is characterized at infinity. From (1.22), (2.1), (3.6), (3.13), (3.17) and (3.21) we deduce that, with the choice $\theta = \gamma_1$, where

$$\gamma_1 = \tau\ln(2\tau) + \tfrac{1}{2}(\mu-\tau) - \tfrac{1}{2}(\tau+\mu)\ln(\tau+\mu) - \tfrac{1}{2}\arg\Gamma(\tfrac{1}{2}+\mathrm{i}(\tau-\mu)) + \tfrac{1}{4}\pi, \tag{3.22}$$

that

$$R_\tau^\mu(\gamma_1, \pm x) = K\frac{(\zeta^2+2-2\beta)^{1/4}}{(x^2+1-\beta^2)^{1/4}}w(\tau,\hat{\beta},\pm\zeta), \tag{3.23}$$

where

$$K = (\tfrac{1}{2}\tau)^{1/4}\{\sqrt{1+\mathrm{e}^{2\pi(\mu-\tau)}} + \mathrm{e}^{\pi(\mu-\tau)}\}^{1/2}. \tag{3.24}$$

The identification of the even and odd functions is also straightforward. By uniqueness of these functions, we have

$$e^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x) = C^e\left(\frac{\zeta^2+2-2\beta}{x^2+1-\beta^2}\right)^{1/4}\{w(\tau,\hat{\beta},\zeta) + w(\tau,\hat{\beta},-\zeta)\} \tag{3.25}$$

and

$$o^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x) = C^o\left(\frac{\zeta^2+2-2\beta}{x^2+1-\beta^2}\right)^{1/4}\{w(\tau,\hat{\beta},\zeta) - w(\tau,\hat{\beta},-\zeta)\}, \tag{3.26}$$

where the proportionality constant C^e can be found by setting $x = \zeta = 0$ in (3.25), and likewise C^o can be determined in the ζ differentiated form of (3.26). Thus,

$$C^e = \frac{1}{2w(\tau,\hat{\beta},0)}\left(\frac{\beta+1}{2}\right)^{1/4}, \tag{3.27}$$

and, on referring to (3.2),

$$C^o = \frac{1}{2\{2(\beta+1)\}^{1/4}\tau^{1/2}w'(\tau,\hat{\beta},0)}. \tag{3.28}$$

Remark 3.1. $W(b,0)$ and $W'(b,0)$ are non-vanishing for all real b (see [7, (8.3)]), and hence the same is true of $w(\tau,\hat{\beta},0)$ and $w'(\tau,\hat{\beta},0)$ for sufficiently large τ.

We now turn our attention to the second subcase,

$$1 \leqslant \beta \leqslant 1+\delta. \tag{3.29}$$

This time the turning points of (2.4) are real, located at $x = \pm\sqrt{\beta^2-1}$, and again have the property that they coalesce at $x = 0$ when $\beta \to 1$.

In place of (3.2) we prescribe

$$\int_0^x \frac{(\beta^2-1-p^2)^{1/2}}{p^2+1}\,\mathrm{d}p = \int_0^{\hat{\zeta}}(\hat{\beta}^2-\eta^2)^{1/2}\,\mathrm{d}\eta, \quad 0 \leqslant x \leqslant \sqrt{\beta^2-1}, \tag{3.30}$$

and

$$\int_{\sqrt{\beta^2-1}}^x \frac{(1+p^2-\beta^2)^{1/2}}{p^2+1}\,\mathrm{d}p = \int_{\hat{\beta}}^{\hat{\zeta}}(\eta^2-\hat{\beta}^2)^{1/2}\,\mathrm{d}\eta, \quad \sqrt{\beta^2-1} \leqslant x < \infty, \tag{3.31}$$

with $\hat{\beta}$ again given by (3.4). Explicit integration yields

$$\beta \arctan\left\{\frac{\beta x}{(\beta^2-1-x^2)^{1/2}}\right\} - \arctan\left\{\frac{x}{(\beta^2-1-x^2)^{1/2}}\right\} = \tfrac{1}{2}\hat{\zeta}(\hat{\beta}^2-\hat{\zeta}^2)^{1/2} - \tfrac{1}{2}\hat{\beta}^2 \arccos\left(\frac{\hat{\zeta}}{\hat{\beta}}\right) + \tfrac{1}{4}\pi\hat{\beta}^2, \quad (3.32)$$

for $0 \leqslant x \leqslant \sqrt{\beta^2-1}$, and

$$\ln\{x+(x^2+1-\beta^2)^{1/2}\} + \beta\ln\{\beta x-(x^2+1-\beta^2)^{1/2}\} - \tfrac{1}{2}\beta\ln(x^2+1) - \tfrac{1}{2}(\beta+1)\ln(\beta^2-1) = \tfrac{1}{2}\hat{\zeta}(\hat{\zeta}^2-\hat{\beta}^2)^{1/2} - \tfrac{1}{2}\hat{\beta}^2\operatorname{arccosh}\left(\frac{\hat{\zeta}}{\hat{\beta}}\right), \quad (3.33)$$

for $\sqrt{\beta^2-1} \leqslant x < \infty$. Note that in (3.30)–(3.33), and below, $\hat{\beta}^2 = 2\beta - 2$.

Now, with

$$\hat{W} = \frac{(x^2+1-\beta^2)^{1/4}}{(x^2+1)^{1/2}(\hat{\zeta}^2-\hat{\beta}^2)^{1/4}} w, \quad (3.34)$$

we obtain

$$\frac{\mathrm{d}^2\hat{W}}{\mathrm{d}\hat{\zeta}^2} = \{\tau^2(\hat{\beta}^2-\hat{\zeta}^2) + \hat{\psi}(\hat{\beta},\hat{\zeta})\}\hat{W}, \quad (3.35)$$

where

$$\hat{\psi}(\hat{\beta},\hat{\zeta}) = \frac{3\hat{\zeta}^2+2\hat{\beta}^2}{4(\hat{\zeta}^2-\hat{\beta}^2)^2} + \frac{(\hat{\zeta}^2-\hat{\beta}^2)(x^2+1)(x^2+1-4\beta^2x^2-\beta^4)}{4(x^2+1-\beta^2)^3}. \quad (3.36)$$

We again apply the theorem of [9] to obtain the solutions

$$\hat{w}(\tau,\hat{\beta},\pm\hat{\zeta}) = W(\mu-\tau, \pm\hat{\zeta}\sqrt{2\tau}) + \hat{\varepsilon}(\tau,\hat{\beta},\pm\hat{\zeta}). \quad (3.37)$$

These are of the same form as (3.11), but since $\mu - \tau \geqslant 0$ the error term now satisfies

$$\hat{\varepsilon}(\tau,\hat{\beta},\hat{\zeta}) = \operatorname{env} W(\mu-\tau,\hat{\zeta}\sqrt{2\tau})O(\tau^{-2/3}\ln(\tau)), \quad (3.38)$$

uniformly for $-\infty < \zeta < \infty$.

From (3.33) we find that (3.6) still applies when $x \to \infty$ with ζ replaced by $\hat{\zeta}$. The identification of asymptotic solutions is therefore the same, and again (3.22)–(3.28) hold, with ζ replaced by $\hat{\zeta}$, and $w(\tau,\hat{\beta},\pm\zeta)$ replaced by $\hat{w}(\tau,\hat{\beta},\pm\hat{\zeta})$.

4. Large τ, $1+\delta \leqslant \mu/\tau \leqslant B < \infty$ and $0 \leqslant x < \infty$

Here we consider $\tau \to \infty$, with β, again defined by (2.3), satisfying

$$1 + \delta \leqslant \beta \leqslant B < \infty. \quad (4.1)$$

The turning points of (2.4) are located at $x = \pm\sqrt{\beta^2-1}$ and are bounded, and also bounded away from one another. The appropriate asymptotic theory is that of a fixed simple turning point [10, ch. 11], which furnishes asymptotic expansions in terms of Airy functions.

The appropriate Liouville transformation is as follows. Firstly, we use the Liouville transformation defined by [10, ch. 11, (3.02), (3.03)], except we replace Olver's ζ by $-\zeta$. Thus, we have, on referring to (2.4), the new independent variable given by

$$\tfrac{2}{3}(-\zeta)^{3/2} = \int_{\sqrt{\beta^2-1}}^{x} \frac{(p^2-\beta^2+1)^{1/2}}{p^2+1}\,\mathrm{d}p, \quad x \geqslant \sqrt{\beta^2-1}, \tag{4.2}$$

and

$$\tfrac{2}{3}\zeta^{3/2} = \int_{x}^{\sqrt{\beta^2-1}} \frac{(\beta^2-1-p^2)^{1/2}}{p^2+1}\,\mathrm{d}p, \quad x \leqslant \sqrt{\beta^2-1}. \tag{4.3}$$

On explicit integration, for $x \geqslant \sqrt{\beta^2-1}$ we have the relationship

$$\begin{aligned}\tfrac{2}{3}(-\zeta)^{3/2} = \ln\{x+(x^2+1-\beta^2)^{1/2}\} + \beta\ln\{\beta x-(x^2+1-\beta^2)^{1/2}\}\\ -\tfrac{1}{2}\beta\ln(x^2+1) - \tfrac{1}{2}(\beta+1)\ln(\beta^2-1),\end{aligned} \tag{4.4}$$

and, for $x \leqslant \sqrt{\beta^2-1}$,

$$\tfrac{2}{3}\zeta^{3/2} = \beta\arctan\left\{\frac{(\beta^2-1-x^2)^{1/2}}{\beta x}\right\} - \arctan\left\{\frac{(\beta^2-1-x^2)^{1/2}}{x}\right\}. \tag{4.5}$$

From (4.4) we find that $\zeta \to -\infty$ as $x \to \infty$, such that

$$\tfrac{2}{3}|\zeta|^{3/2} = \ln(2x) + \tfrac{1}{2}(\beta-1)\ln(\beta-1) - \tfrac{1}{2}(\beta+1)\ln(\beta+1) + O(x^{-2}). \tag{4.6}$$

Note that the turning point $x = \sqrt{\beta^2-1}$ is mapped to $\zeta = 0$, and $x = 0$ corresponds to $\zeta = \zeta_0$, where

$$\zeta_0 = \{\tfrac{3}{4}\pi(\beta-1)\}^{2/3}. \tag{4.7}$$

Next, with

$$W = (x^2+1)^{1/2}\left(\frac{\zeta}{\beta^2-1-x^2}\right)^{1/4} w, \tag{4.8}$$

we obtain the desired equation

$$\frac{\mathrm{d}^2W}{\mathrm{d}\zeta^2} = \{\tau^2\zeta + \psi(\zeta)\}W, \tag{4.9}$$

where

$$\psi(\zeta) = \frac{5}{16\zeta^2} + \frac{(x^2+1)(4\beta^2x^2-x^2+\beta^4-1)\zeta}{4(x^2+1-\beta^2)^3}. \tag{4.10}$$

From (4.6) it is evident that

$$\psi(\zeta) = O(\zeta^{-2}), \quad \zeta \to -\infty. \tag{4.11}$$

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Applying [10, ch. 11, theorem 7.1] to the transformed equation (4.9), we obtain the asymptotic solutions

$$W_{2n+1,1}(\tau,\zeta) = \mathrm{Bi}(\tau^{2/3}\zeta)\sum_{s=0}^{n}\frac{A_s(\zeta)}{\tau^{2s}} + \frac{\mathrm{Bi}'(\tau^{2/3}\zeta)}{\tau^{4/3}}\sum_{s=0}^{n-1}\frac{B_s(\zeta)}{\tau^{2s}} + \varepsilon_{2n+1,1}(\tau,\zeta) \tag{4.12}$$

and

$$W_{2n+1,2}(\tau,\zeta) = \mathrm{Ai}(\tau^{2/3}\zeta)\sum_{s=0}^{n}\frac{A_s(\zeta)}{\tau^{2s}} + \frac{\mathrm{Ai}'(\tau^{2/3}\zeta)}{\tau^{4/3}}\sum_{s=0}^{n-1}\frac{B_s(\zeta)}{\tau^{2s}} + \varepsilon_{2n+1,2}(\tau,\zeta), \tag{4.13}$$

where $A_0(\zeta) = 1$ and, for $s = 0, 1, 2, 3, \dots$,

$$B_s(\zeta) = \tfrac{1}{2}\zeta^{-1/2}\int_0^{\zeta}\eta^{-1/2}\{\psi(\eta)A_s(\eta) - A_s''(\eta)\}\,\mathrm{d}\eta, \qquad \zeta > 0, \tag{4.14}$$

$$B_s(\zeta) = \tfrac{1}{2}|\zeta|^{-1/2}\int_{\zeta}^{0}|\eta|^{-1/2}\{\psi(\eta)A_s(\eta) - A_s''(\eta)\}\,\mathrm{d}\eta, \quad \zeta < 0 \tag{4.15}$$

and

$$A_{s+1}(\zeta) = -\tfrac{1}{2}B_s'(\zeta) + \frac{1}{2}\int \psi(\zeta)B_s(\zeta)\,\mathrm{d}\zeta. \tag{4.16}$$

The error terms satisfy explicit error bounds [10, ch. 11, theorem 7.1], and from these we obtain

$$\varepsilon_{2n+1,1}(\tau,\zeta) = \mathrm{env}\,\mathrm{Bi}(\tau^{2/3}\zeta)O(\tau^{-2n-1}) \tag{4.17}$$

and

$$\varepsilon_{2n+1,2}(\tau,\zeta) = \mathrm{env}\,\mathrm{Ai}(\tau^{2/3}\zeta)O(\tau^{-2n-1}) \tag{4.18}$$

as $\tau \to \infty$, uniformly for $0 \leqslant x < \infty$, $-\infty < \zeta \leqslant \zeta_0$. Here

$$\mathrm{env}\,f(x) = \begin{cases} \{\mathrm{Ai}^2(x) + \mathrm{Bi}^2(x)\}^{1/2}, & -\infty < x \leqslant c, \\ \sqrt{2}f(x), & c \leqslant x < \infty, \end{cases} \tag{4.19}$$

where $x = c = -0.36605\dots$ is the largest negative root of the equation $\mathrm{Ai}(x) = \mathrm{Bi}(x)$.

Moreover, with $\alpha = -\infty$ in [10, ch. 11, (7.12)], we have, by virtue of (4.11),

$$\varepsilon_{2n+1,1}(\tau,\zeta) = O(|\zeta|^{-3/2}) \tag{4.20}$$

as $\zeta \to -\infty$. Also, by choosing $\beta = \zeta_0$ in [10, ch. 11, (7.13)] we have

$$\varepsilon_{2n+1,2}(\tau,\zeta_0) = 0, \tag{4.21}$$

and similarly for the derivatives of these error terms.

We now identify the asymptotic solution given by (4.12) with the solutions defined in § 1 whose characteristic behaviour is given at infinity. Specifically, taking into account (2.9) and (4.8), we seek constants γ_2, $\mathcal{A}_n(\tau)$ and $\mathcal{B}_n(\tau)$ such that

$$W_{2n+1,1}(\tau,\zeta) = \left(\frac{\beta^2-1-x^2}{\zeta}\right)^{1/4}[\mathcal{A}_n(\tau)R^{\mu}_{\tau}(\gamma_2+\tfrac{1}{4}\pi,x)+\mathcal{B}_n(\tau)R^{\mu}_{\tau}(\gamma_2-\tfrac{1}{4}\pi,x)]. \tag{4.22}$$

To this end, it can be shown from (4.11), (4.14)–(4.16), and by induction, that, as $\zeta \to -\infty$,

$$A_s(\zeta) = k_s + O(|\zeta|^{-3/2}) \tag{4.23}$$

and

$$B_s(\zeta) = l_s|\zeta|^{-1/2} + O(|\zeta|^{-2}), \tag{4.24}$$

for some constants k_s and l_s, $s = 0, 1, 2, \dots$. Thus, from the well-known behaviour of Airy functions of large negative argument [10, ch. 11, § 1] we find from (4.12) that

$$W_{2n+1,1}(\tau,\zeta) \sim \frac{1}{\pi^{1/2}\tau^{1/6}|\zeta|^{1/4}} \times\left[-\sin(\tfrac{2}{3}\tau|\zeta|^{3/2}-\tfrac{1}{4}\pi)\sum_{s=0}^{n}\frac{k_s}{\tau^{2s}}+\cos(\tfrac{2}{3}\tau|\zeta|^{3/2}-\tfrac{1}{4}\pi)\sum_{s=0}^{n-1}\frac{l_s}{\tau^{2s+1}}\right] \tag{4.25}$$

as $\zeta \to -\infty$.

Now, in comparison, from (1.22), we observe that

$$R^{\mu}_{\tau}(\gamma_2-\tfrac{1}{4}\pi,x) \sim x^{-1/2}\cos(\tau\ln(x)+\gamma_2-\tfrac{1}{4}\pi) \tag{4.26}$$

and

$$R^{\mu}_{\tau}(\gamma_2+\tfrac{1}{4}\pi,x) \sim -x^{-1/2}\sin(\tau\ln(x)+\gamma_2-\tfrac{1}{4}\pi) \tag{4.27}$$

as $x \to \infty$. Thus, if we choose

$$\gamma_2 = \tfrac{1}{2}(\mu-\tau)\ln(\mu-\tau)-\tfrac{1}{2}(\mu+\tau)\ln(\mu+\tau)+\tau\ln(2\tau), \tag{4.28}$$

we have, from (4.6),

$$\tau\ln(x)+\gamma_2-\tfrac{1}{4}\pi = \tfrac{2}{3}\tau|\zeta|^{3/2}-\tfrac{1}{4}\pi+O(\exp\{-\tfrac{4}{3}(-\zeta)^{3/2}\}). \tag{4.29}$$

It follows from (4.22)–(4.29) that

$$\mathcal{A}_n(\tau) = \frac{1}{\pi^{1/2}\tau^{1/6}}\sum_{s=0}^{n}\frac{k_s}{\tau^{2s}} \tag{4.30}$$

and

$$\mathcal{B}_n(\tau) = \frac{1}{\pi^{1/2}\tau^{1/6}}\sum_{s=0}^{n-1}\frac{l_s}{\tau^{2s+1}}, \tag{4.31}$$

thereby completing the identification (4.22).

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Next, we identify the asymptotic solutions with the standard even and odd functions, by seeking the coefficients $C^e_{2n+1,j}(\tau)$ and $C^o_{2n+1,j}(\tau)$ in the relations

$$W_{2n+1,j}(\tau,\zeta) = \left(\frac{\beta^2-1-x^2}{\zeta}\right)^{1/4}[C^e_{2n+1,j}(\tau)e^{i\mu}_{-(1/2)+i\tau}(x) + C^o_{2n+1,j}(\tau)o^{i\mu}_{-(1/2)+i\tau}(x)] \quad (4.32)$$

for $j=1$ and $j=2$.

Firstly, setting $x=0$, $\zeta=\zeta_0$, in these and referring to (1.14) yields

$$C^e_{2n+1,j}(\tau) = \left(\frac{\zeta_0}{\beta^2-1}\right)^{1/4} W_{2n+1,j}(\tau,\zeta_0), \quad j=1,2. \quad (4.33)$$

Similarly, differentiating (4.32) with respect to ζ and referring to (1.15) and (4.3) yields

$$C^o_{2n+1,j}(\tau) = -\left(\frac{\beta^2-1}{\zeta_0}\right)^{1/4}\left\{W'_{2n+1,j}(\tau,\zeta_0) + \frac{W_{2n+1,j}(\tau,\zeta_0)}{4\zeta_0}\right\}, \quad j=1,2. \quad (4.34)$$

We observe from (4.3), (4.13) and (4.21) that

$$W_{2n+1,2}(\tau,\zeta_0) = \mathrm{Ai}(\tau^{2/3}\zeta_0)\sum_{s=0}^{n}\frac{A_s(\zeta_0)}{\tau^{2s}} + \frac{\mathrm{Ai}'(\tau^{2/3}\zeta_0)}{\tau^{4/3}}\sum_{s=0}^{n-1}\frac{B_s(\zeta_0)}{\tau^{2s}} \quad (4.35)$$

and

$$W'_{2n+1,2}(\tau,\zeta_0) = \mathrm{Ai}(\tau^{2/3}\zeta_0)\sum_{s=0}^{n}\frac{A'_s(\zeta_0)+\zeta_0 B_s(\zeta_0)}{\tau^{2s}} + \tau^{2/3}\,\mathrm{Ai}'(\tau^{2/3}\zeta_0)\sum_{s=0}^{n}\frac{A_s(\zeta_0)+B'_{s-1}(\zeta_0)}{\tau^{2s}}. \quad (4.36)$$

Similar expressions hold for $W_{2n+1,1}(\tau,\zeta_0)$ and $W'_{2n+1,1}(\tau,\zeta_0)$, but we note that the error terms for this function and its derivative do not vanish at $\zeta=\zeta_0$.

Conversely, from (4.32)–(4.34), we arrive at

$$e^{i\mu}_{-(1/2)+i\tau}(x) = \left(\frac{\zeta}{\beta^2-1-x^2}\right)^{1/4}\frac{C^o_{2n+1,1}W_{2n+1,2}(\tau,\zeta) - C^o_{2n+1,2}W_{2n+1,1}(\tau,\zeta)}{\mathcal{W}\{W_{2n+1,1},W_{2n+1,2}\}(\zeta_0)} \quad (4.37)$$

and

$$o^{i\mu}_{-(1/2)+i\tau}(x) = \left(\frac{\zeta}{\beta^2-1-x^2}\right)^{1/4}\frac{C^e_{2n+1,2}W_{2n+1,1}(\tau,\zeta) - C^e_{2n+1,1}W_{2n+1,2}(\tau,\zeta)}{\mathcal{W}\{W_{2n+1,1},W_{2n+1,2}\}(\zeta_0)}. \quad (4.38)$$

Note that, from using the well-known Wronskian

$$\mathcal{W}\{\mathrm{Ai}(x),\mathrm{Bi}(x)\} = \frac{1}{\pi}, \quad (4.39)$$

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we find from (4.12), (4.13), (4.17) and (4.21) that

$$\begin{aligned}\mathcal{W}\{W_{2n+1,1},W_{2n+1,2}\}(\zeta_0)&\\ =-\frac{\tau^{2/3}}{\pi}\biggl[\sum_{s=0}^{n}\frac{A_s(\zeta_0)}{\tau^{2s}}\sum_{s=0}^{n}\frac{A_s(\zeta_0)+B'_{s-1}(\zeta_0)}{\tau^{2s}}&\\ -\frac{1}{\tau^2}\sum_{s=0}^{n-1}\frac{B_s(\zeta_0)}{\tau^{2s}}\sum_{s=0}^{n}\frac{A'_s(\zeta_0)+\zeta_0B_s(\zeta_0)}{\tau^{2s}}+O\biggl(\frac{1}{\tau^{2n+1}}\biggr)\biggr],&\quad (4.40)\end{aligned}$$

in which $B'_{-1}(\zeta_0)$ is understood to be zero.

5. Large τ, $0\leqslant\mu/\tau\leqslant B$, $B<\infty$, and $\mathrm{Re}\sqrt{B^2-1}+\delta\leqslant x<\infty$

We can extend the results of § 2 to the case

$$0\leqslant\beta\leqslant B,\quad B<\infty, \tag{5.1}$$

provided we restrict x to lie in the interval $\mathrm{Re}\sqrt{B^2-1}+\delta\leqslant x<\infty$; in this case the turning points $x=\pm\sqrt{\beta^2-1}$ are bounded, can be real or imaginary and indeed can coalesce at the origin. However, the stated x interval is such that these critical points are avoided, which allows us to apply the simpler Liouville–Green theory of [10, ch. 10].

Proceeding as in § 2, (2.6)–(2.15) still apply, but in place of (2.16) we choose

$$A_{s+1}(\xi)=\tfrac{1}{2}\{A'_s(\infty)-A'_s(\xi)\}-\frac{1}{2}\int_{\xi}^{\infty}\psi(\eta)A_s(\eta)\,\mathrm{d}\eta,\quad s=0,1,2,\ldots, \tag{5.2}$$

so that

$$A_{s+1}(\infty)=0,\quad s=1,2,3,\ldots. \tag{5.3}$$

The error terms in (2.14) and (2.15) can also be chosen to vanish at $\xi=\infty$, $x=\infty$. Thus, from (2.14),

$$W_{2n+1,1}(\tau,\xi)\sim\cos(\tau\xi),\quad \xi\to\infty \tag{5.4}$$

and, from (2.15),

$$W_{2n+1,2}(\tau,\xi)\sim\sin(\tau\xi),\quad \xi\to\infty. \tag{5.5}$$

REMARK 5.1. Although of a similar form, the asymptotic solutions $W_{2n+1,j}(\tau,\xi)$ here differ from the corresponding ones in § 2, and in particular there is no longer a reason to suppose that they are either even ($j=1$) or odd ($j=2$).

Now, from (2.8), we have

$$\tau\xi=\tau\ln(x)+\gamma_3+O(x^{-2}),\quad x\to\infty, \tag{5.6}$$

where

$$\gamma_3=\tau\ln(2)-\mu\,\mathrm{arctanh}\left(\frac{\mu}{\tau}\right). \tag{5.7}$$

Thus, from (1.22), (2.1), (2.9) and (5.4) we arrive at

$$R^{\mu}_{\tau}(\gamma_3,x)=(1-\beta^2+x^2)^{-1/4}W_{2n+1,1}(\tau,\xi). \tag{5.8}$$

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Likewise, from (5.5), we have

$$R^{\mu}_{\tau}(\gamma_3 - \tfrac{1}{2}\pi, x) = (1 - \beta^2 + x^2)^{-1/4} W_{2n+1,2}(\tau, \xi). \tag{5.9}$$

The corresponding asymptotic results for $e^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x)$ and $o^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x)$ are now readily derivable from (1.24)–(1.27) (with $\theta = \gamma_3$ in the latter two equations).

6. Large μ, bounded τ and $0 \leqslant x < \infty$

We now consider the case for μ large. The form (2.2) of the differential equation is not appropriate to obtain asymptotic solutions which are uniformly valid for unbounded x. Therefore, it is necessary to redefine the independent variable. There are several ways of doing this, and we choose the following. Let

$$s = 1 - \frac{x}{(x^2 + 1)^{1/2}}, \tag{6.1}$$

so that $0 \leqslant x < \infty$ is mapped one-to-one to $0 < s \leqslant 1$. If we further define

$$V(s) = (x^2 + 1)^{-3/4} w(x), \tag{6.2}$$

then (2.2) is transformed to

$$\frac{\mathrm{d}^2 V}{\mathrm{d}s^2} = \left\{ \frac{\mu^2}{s(2 - s)} + \frac{2s - s^2 - 4\tau^2 - 4}{4s^2(2 - s)^2} \right\} V. \tag{6.3}$$

In the interval $(0, 1]$ this equation is characterized by having a regular singularity at $s = 0$, and for large μ the dominant term on the right-hand side has a simple pole. There are no turning points in this case, and the other end point $s = 1$ (corresponding to $x = 0$) is an ordinary point of the differential equation.

The exponent of the pole at $s = 0$ is complex, and as such solutions are oscillatory in its neighbourhood, and so the theory in [3, § 7] is applicable. From this reference we make the following transformation of independent variables

$$\zeta^{1/2} = \int_0^s \frac{1}{p^{1/2}(2 - p)^{1/2}}\, \mathrm{d}p = \frac{\pi}{2} - \arcsin(1 - s). \tag{6.4}$$

Thus, $s = 0$ ($x = \infty$) corresponds to $\zeta = 0$, with

$$\zeta = 2s + \tfrac{1}{3}s^2 + O(s^3), \qquad s \to 0, \tag{6.5}$$

i.e.

$$\zeta = x^{-2} - \tfrac{2}{3}x^{-4} + O(x^{-6}), \quad x \to \infty. \tag{6.6}$$

We also note that $x = 0$ ($s = 1$) corresponds to $\zeta = \zeta_0$, where

$$\zeta_0 = \tfrac{1}{4}\pi^2. \tag{6.7}$$

With the change of dependent variable

$$W = \left(\frac{\zeta}{s(2 - s)} \right)^{1/4} V, \tag{6.8}$$

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we obtain the following equation:

$$\frac{\mathrm{d}^2W}{\mathrm{d}\zeta^2} = \left\{\frac{\mu^2}{4\zeta} - \frac{\tau^2+1}{4\zeta^2} + \frac{\psi(\zeta)}{\zeta}\right\}W, \tag{6.9}$$

where

$$\psi(\zeta) = \frac{(4\tau^2+1)(2s-s^2-\zeta)}{16\zeta s(2-s)}. \tag{6.10}$$

From (6.5) and (6.10) we see that if we define $\psi(0) = \lim_{\zeta\to 0}\psi(\zeta)$, then $\psi(\zeta)$ is analytic at $\zeta = 0$, and, in particular,

$$\psi(\zeta) = -\frac{4\tau^2+1}{48} + O(\zeta), \quad \zeta \to 0. \tag{6.11}$$

We now apply [3, theorem 1] to obtain the asymptotic solutions

$$W_{2n+1,1}(\mu,\zeta) = \zeta^{1/2}K_{\mathrm{i}\tau}(\mu\zeta^{1/2})\sum_{s=0}^{n}\frac{C_s(\zeta)}{\mu^{2s}} + \frac{\zeta}{\mu}K'_{\mathrm{i}\tau}(\mu\zeta^{1/2})\sum_{s=0}^{n-1}\frac{D_s(\zeta)}{\mu^{2s}} + \varepsilon_{2n+1,1}(\mu,\zeta) \tag{6.12}$$

and

$$W_{2n+1,2}(\mu,\zeta) = \zeta^{1/2}L_{\mathrm{i}\tau}(\mu\zeta^{1/2})\sum_{s=0}^{n}\frac{C_s(\zeta)}{\mu^{2s}} + \frac{\zeta}{\mu}L'_{\mathrm{i}\tau}(\mu\zeta^{1/2})\sum_{s=0}^{n-1}\frac{D_s(\zeta)}{\mu^{2s}} + \varepsilon_{2n+1,2}(\mu,\zeta), \tag{6.13}$$

where

$$L_{\mathrm{i}\tau}(x) = \frac{\pi}{2\sinh(\tau\pi)}\{I_{\mathrm{i}\tau}(x) + I_{-\mathrm{i}\tau}(x)\}. \tag{6.14}$$

In (6.12) and (6.13) $C_0(\zeta) = 1$, with the other coefficients satisfying the recursion formulae

$$D_s(\zeta) = -C'_s(\zeta) + \zeta^{-1/2}\int_0^{\zeta}\eta^{-1/2}\{\psi(\eta)C_s(\eta) - \tfrac{1}{2}C'_s(\eta) + \tau^2 D'_{s-1}(\eta)\}\,\mathrm{d}\eta \tag{6.15}$$

and

$$C_{s+1}(\zeta) = -\zeta D'_s(\zeta) + \int \psi(\zeta)D_s(\zeta)\,\mathrm{d}\zeta \tag{6.16}$$

for $s = 0, 1, 2, \dots$.

The error terms are explicitly bounded, and satisfy

$$\varepsilon_{2n+1,1}(\mu,\zeta) = \zeta^{1/2}\,\mathrm{env}\,K_{\mathrm{i}\tau}(\mu\zeta^{1/2})O(\mu^{-2n-1}) \tag{6.17}$$

and

$$\varepsilon_{2n+1,2}(\mu,\zeta) = \zeta^{1/2}\,\mathrm{env}\,L_{\mathrm{i}\tau}(\mu\zeta^{1/2})O(\mu^{-2n-1}) \tag{6.18}$$

as $\mu\to\infty$, uniformly for $0 \leqslant x < \infty$, $0 < \zeta \leqslant \zeta_0$. Here

$$\mathrm{env}\,f(x) = \begin{cases}\{K^2_{\mathrm{i}\tau}(x) + L^2_{\mathrm{i}\tau}(x)\}^{1/2}, & 0 \leqslant x \leqslant \chi_\tau,\\ \sqrt{2}f(x), & \chi_\tau \leqslant x < \infty,\end{cases} \tag{6.19}$$

where $x = \chi_\tau$ is the largest positive root of the equation $K_{\mathrm{i}\tau}(x) = L_{\mathrm{i}\tau}(x)$.

Moreover,

$$\varepsilon_{2n+1,1}(\mu,\zeta) = \zeta^{1/2} K_{\mathrm{i}\tau}(\mu\zeta^{1/2}) O(\zeta-\zeta_0), \quad \zeta \to \zeta_0, \tag{6.20}$$

and

$$\varepsilon_{2n+1,2}(\mu,\zeta) = L_{\mathrm{i}\tau}(\mu\zeta^{1/2}) O(\zeta), \qquad \zeta \to 0^+. \tag{6.21}$$

Similarly to (4.22), we seek constants γ_4, $\mathcal{C}_n(\mu)$ and $\mathcal{D}_n(\mu)$ such that

$$W_{2n+1,2}(\mu,\zeta) = \zeta^{1/4}[\mathcal{C}_n(\mu) R^{\mu}_{\tau}(\gamma_4 - \tfrac{1}{4}\pi, x) + \mathcal{D}_n(\mu) R^{\mu}_{\tau}(\gamma_4 + \tfrac{1}{4}\pi, x)]. \tag{6.22}$$

To do so we note from [3] that

$$L_{\mathrm{i}\tau}(x) = \left\{\frac{\pi}{\tau\sinh(\tau\pi)}\right\}^{1/2}\left[\cos\left\{\tau\ln\left(\frac{2}{x}\right) + \phi_{\tau,0}\right\} + O(x^2)\right] \tag{6.23}$$

and

$$L'_{\mathrm{i}\tau}(x) = \left\{\frac{\tau\pi}{\sinh(\tau\pi)}\right\}^{1/2}\left[\frac{1}{x}\sin\left\{\tau\ln\left(\frac{2}{x}\right) + \phi_{\tau,0}\right\} + O(x)\right] \tag{6.24}$$

as $x \to 0^+$, where

$$\phi_{\tau,0} = \arg\Gamma(1+\mathrm{i}\tau). \tag{6.25}$$

Thus, from (6.13) and (6.21),

$$\begin{aligned} W_{2n+1,2}(\mu,\zeta) \sim \zeta^{1/2}&\left\{\frac{\pi}{\tau\sinh(\tau\pi)}\right\}^{1/2} \\ &\times\left[\cos\left\{\tau\ln\left(\frac{2}{\mu\zeta^{1/2}}\right) + \phi_{\tau,0}\right\}\sum_{s=0}^{n}\frac{C_s(0)}{\mu^{2s}}\right. \\ &\left.\qquad + \tau\sin\left\{\tau\ln\left(\frac{2}{\mu\zeta^{1/2}}\right) + \phi_{\tau,0}\right\}\sum_{s=0}^{n-1}\frac{D_s(0)}{\mu^{2s+2}}\right] \end{aligned} \tag{6.26}$$

as $\zeta \to 0^+$.

Now from (1.22) and (6.6)

$$R^{\mu}_{\tau}(\gamma_4 - \tfrac{1}{4}\pi, x) \sim \zeta^{1/4}\cos\left\{\tau\ln\left(\frac{2}{\mu\zeta^{1/2}}\right) + \phi_{\tau,0}\right\} \tag{6.27}$$

and

$$R^{\mu}_{\tau}(\gamma_4 + \tfrac{1}{4}\pi, x) \sim -\zeta^{1/4}\sin\left\{\tau\ln\left(\frac{2}{\mu\zeta^{1/2}}\right) + \phi_{\tau,0}\right\} \tag{6.28}$$

as $x \to \infty$ and $\zeta \to 0^+$, provided that we choose

$$\gamma_4 = \tfrac{1}{4}\pi - \tau\ln(\tfrac{1}{2}\mu) + \phi_{\tau,0}. \tag{6.29}$$

Consequently, with this choice of γ_4, on comparing (6.22) and (6.6) with (6.27) and (6.28), we arrive at

$$\mathcal{C}_n(\mu) = \left\{\frac{\pi}{\tau\sinh(\tau\pi)}\right\}^{1/2}\sum_{s=0}^{n}\frac{C_s(0)}{\mu^{2s}} \tag{6.30}$$

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and

$$\mathcal{D}_n(\mu) = -\left\{\frac{\tau\pi}{\sinh(\tau\pi)}\right\}^{1/2} \sum_{s=0}^{n-1} \frac{D_s(0)}{\mu^{2s+2}}, \tag{6.31}$$

as desired.

Results complementary to (4.32)–(4.36) are derived similarly to these (we therefore omit details), and read as follows:

$$W_{2n+1,j}(\tau,\zeta) = \zeta^{1/4}[C^e_{2n+1,j}(\mu)e^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x) + C^o_{2n+1,j}(\mu)o^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x)], \tag{6.32}$$

where

$$C^e_{2n+1,j}(\mu) = (2/\pi)^{1/2} W_{2n+1,j}(\tau, \tfrac{1}{4}\pi^2) \tag{6.33}$$

and

$$C^o_{2n+1,j}(\mu) = (2/\pi^3)^{1/2} W_{2n+1,j}(\tau, \tfrac{1}{4}\pi^2) - (2\pi)^{1/2} W'_{2n+1,j}(\tau, \tfrac{1}{4}\pi^2). \tag{6.34}$$

In (6.34), for the case $j = 1$, we find from (6.12) that

$$\begin{aligned} &W'_{2n+1,1}(\mu, \tfrac{1}{4}\pi^2) \\ &\quad = \frac{K_{\mathrm{i}\tau}(\frac{1}{2}\mu\pi)}{4\pi} \sum_{s=0}^{n} \frac{4C_s(\frac{1}{4}\pi^2) + 2\pi^2 C'_s(\frac{1}{4}\pi^2) + \pi^2 D_s(\frac{1}{4}\pi^2) - 4\tau^2 D_{s-1}(\frac{1}{4}\pi^2)}{\mu^{2s}} \\ &\qquad + \frac{\mu K'_{\mathrm{i}\tau}(\frac{1}{2}\mu\pi)}{4} \sum_{s=0}^{n-1} \frac{2C_s(\frac{1}{4}\pi^2) + 2D_{s-1}(\frac{1}{4}\pi^2) + \pi^2 D'_{s-1}(\frac{1}{4}\pi^2)}{\mu^{2s}}. \end{aligned} \tag{6.35}$$

7. Large μ, $0 \leqslant \tau/\mu \leqslant 1-\delta$, $\delta > 0$, and $0 \leqslant x < \infty$

We extend the results of the previous section to the case where τ is no longer restricted to be bounded. We thus define

$$\tau = \alpha\mu, \tag{7.1}$$

to recast (6.3) in the form

$$\frac{\mathrm{d}^2 V}{\mathrm{d}s^2} = \left\{\frac{\mu^2(2s - s^2 - \alpha^2)}{s^2(2-s)^2} + \frac{2s - s^2 - 4}{4s^2(2-s)^2}\right\} V. \tag{7.2}$$

This equation has two simple turning points located at $s = 1 \pm \sqrt{1-\alpha^2}$. We assume that

$$0 \leqslant \alpha \leqslant 1 - \delta, \quad \delta > 0, \tag{7.3}$$

so that they cannot coalesce (which happens when $\alpha = 1$), but one of them ($s = 1 \pm \sqrt{1-\alpha^2}$) can coalesce with the double pole at $s = 0$ (when $\alpha \to 0$). Again we consider s lying in the interval $(0, 1]$ (which corresponds to $0 \leqslant x < \infty$).

Equation (7.2) is characterized by having a coalescing turning point and double pole with complex exponent, and the appropriate theory is given by [2]. Thus, from (7.2) and [2, (2.2a,b)], we introduce a new independent variable by

$$\int_{1-\sqrt{1-\alpha^2}}^{s} \frac{(2p - p^2 - \alpha^2)^{1/2}}{p(2-p)}\,\mathrm{d}p = \int_{\alpha^2}^{\zeta} \frac{(\eta - \alpha^2)^{1/2}}{2\eta}\,\mathrm{d}\eta \tag{7.4}$$

for $1-\sqrt{1-\alpha^2}\leqslant s\leqslant 1$, and

$$\int_s^{1-\sqrt{1-\alpha^2}} \frac{(\alpha^2-2p+p^2)^{1/2}}{p(2-p)}\,\mathrm{d}p = \int_\zeta^{\alpha^2} \frac{(\alpha^2-\eta)^{1/2}}{2\eta}\,\mathrm{d}\eta \tag{7.5}$$

for $0<s\leqslant 1-\sqrt{1-\alpha^2}$.

Integration of these yields

$$\arccos\left\{\frac{1-s}{(1-\alpha^2)^{1/2}}\right\} - \tfrac{1}{2}\alpha\arccos\left\{\frac{2\alpha^2}{(1-\alpha^2)s(2-s)} - \frac{1+\alpha^2}{1-\alpha^2}\right\} = (\zeta-\alpha^2)^{1/2} - \alpha\arctan\left\{\frac{(\zeta-\alpha^2)^{1/2}}{\alpha}\right\},\quad 1-\sqrt{1-\alpha^2}\leqslant s\leqslant 1, \tag{7.6}$$

and

$$\ln\{1-s-(\alpha^2-2s+s^2)^{1/2}\} - \tfrac{1}{2}\ln(1-\alpha^2) - \tfrac{1}{2}\alpha\ln\left\{\frac{2\alpha\{\alpha-(1-s)(\alpha^2-2s+s^2)^{1/2}\}}{(1-\alpha^2)s(2-s)} - \frac{1+\alpha^2}{1-\alpha^2}\right\} = \tfrac{1}{2}\alpha\ln\left\{\frac{\alpha+(\alpha^2-\zeta)^{1/2}}{\alpha-(\alpha^2-\zeta)^{1/2}}\right\} - (\alpha^2-\zeta)^{1/2},\quad 0<s\leqslant 1-\sqrt{1-\alpha^2}. \tag{7.7}$$

Note that $x=0$ $(s=1)$ corresponds to $\zeta=\zeta_0$, where

$$(\zeta_0-\alpha^2)^{1/2} - \alpha\arctan\left\{\frac{1}{\alpha}(\zeta_0-\alpha^2)^{1/2}\right\} = \tfrac{1}{2}\pi(1-\alpha). \tag{7.8}$$

We find from (7.7) and (7.8) that

$$s = \frac{e^2}{2(1-\alpha^2)}\left(\frac{1-\alpha}{1+\alpha}\right)^{1/\alpha}\zeta + O(\zeta^2),\qquad \zeta\to 0, \tag{7.9}$$

$$s = 1-(1-\alpha^2)^{1/2} + \frac{\zeta-\alpha^2}{2(1-\alpha^2)^{1/6}} + O\{(\zeta-\alpha^2)^2\},\quad \zeta\to\alpha^2, \tag{7.10}$$

and

$$\zeta_0 = \tfrac{1}{4}\pi^2 - \alpha^2 - \frac{4\alpha^4}{3\pi^2} + O(\alpha^6),\quad \alpha\to 0. \tag{7.11}$$

Furthermore, from (6.1) and (7.9) we see that $x\to\infty$ as $\zeta\to 0^+$, such that

$$x = \frac{(1-\alpha^2)^{1/2}}{e}\left(\frac{1+\alpha}{1-\alpha}\right)^{1/(2\alpha)}\zeta^{-1/2} + O(\zeta^{1/2}). \tag{7.12}$$

Analogous to (6.8) we next define (see [2, (2.1a,b)])

$$W = \left(\frac{\zeta}{s(2-s)}\right)^{1/2}\left(\frac{2s-s^2-\alpha^2}{\zeta-\alpha^2}\right)^{1/4} V, \tag{7.13}$$

to obtain

$$\frac{\mathrm{d}^2W}{\mathrm{d}\zeta^2} = \left\{\mu^2\frac{\zeta-\alpha^2}{4\zeta^2} - \frac{1}{4\zeta^2} + \frac{\psi(\alpha,\zeta)}{\zeta}\right\}W, \tag{7.14}$$

where

$$\psi(\alpha,\zeta)=\frac{\zeta+4\alpha^2}{16(\zeta-\alpha^2)^2}-\frac{(\zeta-\alpha^2)s(2-s)(2s-s^2+4\alpha^2(1-s)^2-\alpha^4)}{16\zeta(2s-s^2-\alpha^2)^3}. \tag{7.15}$$

Applying [2, theorem 1], we obtain the solutions

$$W_{2n+1,1}(\mu,\alpha,\zeta)=\zeta^{1/2}\tilde{I}_{\mathrm{i}\tau}(\mu\zeta^{1/2})\sum_{s=0}^{n}\frac{A_s(\alpha,\zeta)}{\mu^{2s}} + \frac{\zeta}{\mu}\tilde{I}'_{\mathrm{i}\tau}(\mu\zeta^{1/2})\sum_{s=0}^{n-1}\frac{B_s(\alpha,\zeta)}{\mu^{2s}}+\varepsilon_{2n+1,1}(\mu,\alpha,\zeta) \tag{7.16}$$

and

$$W_{2n+1,2}(\mu,\alpha,\zeta)=\zeta^{1/2}K_{\mathrm{i}\tau}(\mu\zeta^{1/2})\sum_{s=0}^{n}\frac{A_s(\alpha,\zeta)}{\mu^{2s}} + \frac{\zeta}{\mu}K_{\mathrm{i}\tau}(\mu\zeta^{1/2})\sum_{s=0}^{n-1}\frac{B_s(\alpha,\zeta)}{\mu^{2s}}+\varepsilon_{2n+1,2}(\mu,\alpha,\zeta), \tag{7.17}$$

where

$$\tilde{I}_{\mathrm{i}\tau}(x)=\pi\mathrm{e}^{-\tau\pi}\{I_{\mathrm{i}\tau}(x)+I_{-\mathrm{i}\tau}(x)\} \tag{7.18}$$

(cf. (6.14)).

In (7.16) and (7.17) $A_0(\alpha,\zeta)=1$ and, for $s=0,1,2,\dots,$

$$B_s(\alpha,\zeta)=(\zeta-\alpha^2)^{-1/2}\int_{\alpha^2}^{\zeta}(\eta-\alpha^2)^{-1/2}\{\psi(\alpha,\eta)A_s(\alpha,\eta)-\eta A''_s(\alpha,\eta)-A'_s(\alpha,\eta)\}\,\mathrm{d}\eta \tag{7.19}$$

when $\zeta>\alpha^2$,

$$B_s(\alpha,\zeta)=(\alpha^2-\zeta)^{-1/2}\int_{\zeta}^{\alpha^2}(\alpha^2-\zeta)^{-1/2}\{\psi(\alpha,\eta)A_s(\alpha,\eta)-\eta A''_s(\alpha,\eta)-A'_s(\alpha,\eta)\}\,\mathrm{d}\eta \tag{7.20}$$

when $\zeta<\alpha^2$ and

$$A_{s+1}(\alpha,\zeta)=-\zeta B'_s(\alpha,\zeta)+\int_{\alpha^2}^{\zeta}\psi(\alpha,\eta)B_s(\alpha,\eta)\,\mathrm{d}\eta+\lambda_{s+1}, \tag{7.21}$$

where λ_{s+1} are arbitrary constants.

The error terms satisfy

$$\varepsilon_{2n+1,1}(\mu,\zeta)=\zeta^{1/2}\tilde{I}_{\mathrm{i}\tau}(\mu\zeta^{1/2})O(\mu^{-2n-1}) \tag{7.22}$$

$$\varepsilon_{2n+1,2}(\mu,\zeta)=\zeta^{1/2}K_{\mathrm{i}\tau}(\mu\zeta^{1/2})O(\mu^{-2n-1}) \tag{7.23}$$

as $\mu\to\infty$, uniformly for $0\leqslant\alpha\leqslant1-\delta$, $\delta>0$, and $0\leqslant x<\infty$, $0<\zeta\leqslant\zeta_0$, except near the zeros of each Bessel function. Moreover,

$$\varepsilon_{2n+1,2}(\mu,\zeta)=\tilde{I}_{\mathrm{i}\tau}(\mu\zeta^{1/2})O(\zeta),\qquad \zeta\to0^+, \tag{7.24}$$

$$\varepsilon_{2n+1,2}(\mu,\zeta)=\zeta^{1/2}K_{\mathrm{i}\tau}(\mu\zeta^{1/2})O(\zeta-\zeta_0),\quad \zeta\to\zeta_0, \tag{7.25}$$

uniformly for $0\leqslant\alpha\leqslant1-\delta$, $\delta>0$.

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An identification similar to (6.22) now follows. Using (1.22), (2.1), (6.2), (6.14), (6.23), (6.24), (7.12), (7.13), (7.16), (7.18) and (7.24), we arrive at

$$W_{2n+1,1}(\mu,\alpha,\zeta)=\zeta^{1/4}[\mathcal{A}_n(\mu,\alpha)R^{\mu}_{\tau}(\gamma_5-\tfrac{1}{4}\pi,x)+\mathcal{B}_n(\mu,\alpha)R^{\mu}_{\tau}(\gamma_5+\tfrac{1}{4}\pi,x)],\quad(7.26)$$

where

$$\gamma_5=\tau-\tfrac{1}{2}\mu\ln\left(\frac{\mu+\tau}{\mu-\tau}\right)-\tfrac{1}{2}\tau\ln\{\tfrac{1}{4}(\mu^2-\tau^2)\}+\phi_{\tau,0}+\tfrac{1}{4}\pi,\quad(7.27)$$

$$\mathcal{A}_n(\mu,\alpha)=2\mathrm{e}^{-\tau\pi}\left\{\frac{\pi\sinh(\tau\pi)}{\mathrm{e}\mu\tau}\right\}^{1/2}(\mu^2-\tau^2)^{1/4}\left(\frac{\mu+\tau}{\mu-\tau}\right)^{\mu/(4\tau)}\sum_{s=0}^{n}\frac{A_s(\alpha,0)}{\mu^{2s}}\quad(7.28)$$

and

$$\mathcal{B}_n(\mu,\alpha)=-2\mathrm{e}^{-\tau\pi}\left\{\frac{\tau\pi\sinh(\tau\pi)}{\mathrm{e}\mu}\right\}^{1/2}(\mu^2-\tau^2)^{1/4}\left(\frac{\mu+\tau}{\mu-\tau}\right)^{\mu/(4\tau)}\sum_{s=0}^{n-1}\frac{B_s(\alpha,0)}{\mu^{2s+1}}.\quad(7.29)$$

Similarly to (6.32), we likewise obtain

$$W_{2n+1,j}(\mu,\zeta)=\left\{\frac{\zeta^2(1-\alpha^2-\alpha^2x^2)}{\zeta-\alpha^2}\right\}^{1/4}\times[C^{e}_{2n+1,j}(\mu)e^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x)+C^{o}_{2n+1,j}(\mu)o^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x)],\quad(7.30)$$

where

$$C^{e}_{2n+1,j}(\mu)=\left\{\frac{\zeta_0-\alpha^2}{\zeta_0^2(1-\alpha^2)}\right\}^{1/4}W_{2n+1,j}(\mu,\zeta_0)\quad(7.31)$$

and

$$C^{o}_{2n+1,j}(\mu)=\frac{\frac{1}{2}(\zeta_0-2\alpha^2)W_{2n+1,j}(\mu,\zeta_0)-2\zeta_0(\zeta_0-\alpha^2)W'_{2n+1,j}(\mu,\zeta_0)}{\zeta_0^{1/2}(\zeta_0-\alpha^2)^{5/4}(1-\alpha^2)^{3/4}}.\quad(7.32)$$

8. Large μ, $0\leqslant\tau/\mu\leqslant A$, $0<A<1$, and $-A^{-1}\sqrt{1-A^2}+\delta\leqslant x\leqslant A^{-1}\sqrt{1-A^2}-\delta$, $\delta>0$

We can simplify the results of the previous two sections if we restrict x to be bounded. In this case we can forgo the preliminary transformation (6.1) and consider (2.4) in the form

$$\frac{\mathrm{d}^2w}{\mathrm{d}x^2}=\left\{\frac{\mu^2(1-\alpha^2x^2-\alpha^2)}{(1+x^2)^2}+\frac{3-x^2}{4(1+x^2)^2}\right\}w,\quad(8.1)$$

where α is again given by (7.1).

The turning points of (8.1) are located at $x=\pm\alpha^{-1}\sqrt{1-\alpha^2}$. In this section we assume that

$$0\leqslant\alpha\leqslant A,\quad 0<A<1,\quad(8.2)$$

so that the turning points are real, with the one lying in $(0,\infty)$ being bounded away from the origin. Moreover, we assume that $-A^{-1}\sqrt{1-A^2}+\delta\leqslant x\leqslant A^{-1}\sqrt{1-A^2}-$

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δ, so that this variable lies in an interval that excludes any turning points, and as such must be bounded.

Similarly to § 2 and 5 the Liouville–Green expansions of [10, ch. 10] are applicable. This time the transformation of the independent variable is given by

$$\begin{aligned}\xi &= \int_0^x \frac{(1-\alpha^2-\alpha^2p^2)^{1/2}}{p^2+1}\,\mathrm{d}p \\ &= \arctan\left\{\frac{x}{(1-\alpha^2-\alpha^2x^2)^{1/2}}\right\} - \alpha\arctan\left\{\frac{\alpha x}{(1-\alpha^2-\alpha^2x^2)^{1/2}}\right\}. \end{aligned}\tag{8.3}$$

It is important to note that ξ is an odd function of x. We also observe from (8.3) that

$$\xi = (1-\alpha^2)^{1/2}x + O(x^3) \tag{8.4}$$

as $x \to 0$.

Next, setting

$$w = \frac{(1+x^2)^{1/2}}{(1-\alpha^2-\alpha^2x^2)^{1/4}}W, \tag{8.5}$$

we obtain

$$\frac{\mathrm{d}^2W}{\mathrm{d}\xi^2} = \{\mu^2 + \psi(\xi)\}W, \tag{8.6}$$

where

$$\psi(\xi) = -\frac{(1+x^2)(1-\alpha^4+4\alpha^2x^2-\alpha^4x^2)}{4(1-\alpha^2-\alpha^2x^2)^3}. \tag{8.7}$$

Applying Olver's Liouville–Green theory [10, ch. 10, theorem 3.1], and defining

$$W_{2n+1,1}(\mu,\xi) = \tfrac{1}{2}\{W_{n,1}(u,\xi) + W_{n,1}(u,-\xi)\} \tag{8.8}$$

and

$$W_{2n+1,2}(\tau,\xi) = \tfrac{1}{2}\{W_{n,1}(u,\xi) - W_{n,1}(u,-\xi)\}, \tag{8.9}$$

we obtain two asymptotic solutions of (8.6) of the form

$$W_{2n+1,1}(\mu,\xi) = \cosh(\mu\xi)\sum_{s=0}^{n}\frac{A_{2s}(\xi)}{\mu^{2s}} + \sinh(\mu\xi)\sum_{s=0}^{n-1}\frac{A_{2s+1}(\xi)}{\mu^{2s+1}} + \varepsilon_{2n+1,1}(\mu,\xi) \tag{8.10}$$

and

$$W_{2n+1,2}(\mu,\xi) = \sinh(\mu\xi)\sum_{s=0}^{n}\frac{A_{2s}(\xi)}{\mu^{2s}} + \cosh(\mu\xi)\sum_{s=0}^{n-1}\frac{A_{2s+1}(\xi)}{\mu^{2s+1}} + \varepsilon_{2n+1,2}(\mu,\xi), \tag{8.11}$$

where $A_0(\xi) = 1$, and

$$A_{s+1}(\xi) = \tfrac{1}{2}\{A_s'(0) - A_s'(\xi)\} + \frac{1}{2}\int_0^\xi \psi(\eta)A_s(\eta)\,\mathrm{d}\eta, \quad s = 0,1,2,\dots. \tag{8.12}$$

Note that, for $s = 1, 2, 3, \dots$,

$$A_s(0) = 0, \qquad A_s(-\xi) = (-1)^s A_s(\xi). \tag{8.13}$$

In (8.10) and (8.11) the error terms $\varepsilon_{2n+1,j}(\tau,\xi)$ have explicit bounds, with the properties

$$\varepsilon_{2n+1,1}(\mu,\xi) = \cosh(\mu\xi) O(\mu^{-2n-1}) \tag{8.14}$$

and

$$\varepsilon_{2n+1,2}(\mu,\xi) = \sinh(\mu\xi) O(\mu^{-2n-1}) \tag{8.15}$$

as $\mu \to \infty$, uniformly for $-A^{-1}\sqrt{1-A^2} + \delta \leqslant x \leqslant A^{-1}\sqrt{1-A^2} - \delta$, $\delta > 0$.

Now, from (8.8) and (8.9) it is evident that, when regarded as functions of x, $W_{2n+1,1}(\mu,\xi)$ is even and $W_{2n+1,2}(\mu,\xi)$ is odd. It follows by uniqueness of solutions having these properties, along with (2.1), (8.4), (8.5), (8.10), (8.11) and (8.13), that

$$e^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x) = \frac{1}{1+\varepsilon_{2n+1,1}(\mu,0)} \left(\frac{1-\alpha^2}{1-\alpha^2-\alpha^2x^2}\right)^{1/4} W_{2n+1,1}(\mu,\xi) \tag{8.16}$$

and

$$\begin{aligned} o^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x) = {} & \left\{\mu + \sum_{s=0}^{n-1} \frac{A'_{2s+1}(0)}{\mu^{2s+1}} + \varepsilon'_{2n+1,2}(\mu,0)\right\}^{-1} \\ & \times \{(1-\alpha^2)(1-\alpha^2-\alpha^2x^2)\}^{-1/4} W_{2n+1,2}(\mu,\xi), \end{aligned} \tag{8.17}$$

for $0 \leqslant \tau/\mu \leqslant A < 1$, uniformly for $-A^{-1}\sqrt{1-A^2} + \delta \leqslant x \leqslant A^{-1}\sqrt{1-A^2} - \delta$, $\delta > 0$.

Returning to the original variables, and taking $n = 0$, we have

$$e^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x) = \left(\frac{\mu^2-\tau^2}{\mu^2-\tau^2-\tau^2x^2}\right)^{1/4} \cosh(\mu\xi) \left\{1 + O\left(\frac{1}{\mu}\right)\right\} \tag{8.18}$$

and

$$o^{\mathrm{i}\mu}_{-(1/2)+\mathrm{i}\tau}(x) = \{(\mu^2-\tau^2)(\mu^2-\tau^2-\tau^2x^2)\}^{-1/4} \sinh(\mu\xi)\{1 + O(\mu^{-1})\}, \tag{8.19}$$

where

$$\xi = \arctan\left\{\frac{\mu x}{(\mu^2-\tau^2-\tau^2x^2)^{1/2}}\right\} - \frac{\tau}{\mu} \arctan\left\{\frac{\tau x}{(\mu^2-\tau^2-\tau^2x^2)^{1/2}}\right\}. \tag{8.20}$$

Acknowledgements

I am grateful to Junehyuk Jung for bringing to my attention the applications in [5, 12, 13]. I also thank the referee for a number of helpful suggestions.

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(*Issued 4 October 2013*)