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SOME $\mathbb{Z}_{n-1}$ TERRACES FROM $\mathbb{Z}_n$ POWER-SEQUENCES, n BEING AN ODD PRIME POWER

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Abstract A terrace for $\mathbb{Z}_m$ is a particular type of sequence formed from the m elements of $\mathbb{Z}_m$. For m odd, many procedures are available for constructing power-sequence terraces for $\mathbb{Z}_m$; each terrace of this sort may be partitioned into segments, of which one contains merely the zero element of $\mathbb{Z}_m$, whereas every other segment is either a sequence of successive powers of an element of $\mathbb{Z}_m$ or such a sequence multiplied throughout by a constant. We now refine this idea to show that, for $m = n - 1$, where n is an odd prime power, there are many ways in which power-sequences in $\mathbb{Z}_n$ can be used to arrange the elements of $\mathbb{Z}_n \setminus \{0\}$ in a sequence of distinct entries i, $1 \leqslant i \leqslant m$, usually in two or more segments, which becomes a terrace for $\mathbb{Z}_m$ when interpreted modulo m instead of modulo n. Our constructions provide terraces for $\mathbb{Z}_{n-1}$ for all prime powers n satisfying $0 < n < 300$ except for $n = 125$, 127 and 257.

Keywords: 2-sequencings; number theory; power-sequence terraces; primitive roots

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1. Basic definitions and notation

Let $\boldsymbol{a} = (a_1, a_2, \ldots, a_m)$ be an arrangement of the elements of $\mathbb{Z}_m$, and let $\boldsymbol{b} = (b_1, b_2, \ldots, b_{m-1})$ be the ordered sequence $b_i = a_{i+1} - a_i$ for $i = 1, 2, \ldots, m-1$. For m odd, the arrangement $\boldsymbol{a}$ is a *terrace* for $\mathbb{Z}_m$, with $\boldsymbol{b}$ as the corresponding 2-*sequencing* or *quasi-sequencing* for $\mathbb{Z}_m$, if, for each element x from $\mathbb{Z}_m \setminus \{0\}$, the sequence $\boldsymbol{b}$ contains exactly two occurrences of x but none of $-x$, or exactly two occurrences of $-x$ but none of x, or exactly one occurrence of each of x and $-x$. For m even, the definitions of a terrace $\boldsymbol{a}$ and 2-sequencing $\boldsymbol{b}$ for $\mathbb{Z}_m$ are as just given, save that the element $\frac{1}{2}m$ (the *involution*) from $\mathbb{Z}_m \setminus \{0\}$ occurs exactly once in $\boldsymbol{b}$.

Some expositions include the zero element of $\mathbb{Z}_m$ in $\boldsymbol{b}$, as an extra element at the start, but we find this practice inconvenient and we follow various precedents by not adopting it. For convenience we often write '$\mathbb{Z}_m$ terrace' in place of 'terrace for $\mathbb{Z}_m$'.

Terraces for $\mathbb{Z}_m$ have been used in the construction of solutions to the Lucas round-dance problem [**7**] and the generalized Oberwolfach problem [**9**], and of combinatorial designs used in statistical applications involving carry-over effects [**1**, **6**] and neighbour effects. However, the present paper provides new constructions for terraces, not for designs.

Terraces were originally defined by Bailey [**6**] for a general finite group G, but the general case does not concern us here. A detailed review of related results is provided in [**8**].

Generalizing our previous definition [**2**, **4**] to cover both odd and even values of m, we say that a terrace $\boldsymbol{a}$ for $\mathbb{Z}_m$ is *narcissistic* if the corresponding 2-sequencing $\boldsymbol{b}$ has $b_i = b_{m-i}$ for all i satisfying $1 \leqslant i \leqslant m-1$.

For many series of odd values m, Anderson and Preece [**2**–**5**] gave general constructions for 'power-sequence' terraces for $\mathbb{Z}_m$. Each of these terraces can be partitioned into segments, one of which contains merely the zero element of $\mathbb{Z}_m$, whereas every other segment is either a sequence of successive powers of an element of $\mathbb{Z}_m$, or such a sequence multiplied throughout by a constant. Many of the sequences $x^0, x^1, \dots, x^{s-1}$ of distinct elements are 'full-cycle' sequences such that $x^s = x^0$, but partial cycles are used too.

The techniques used in [**2**–**5**] are not adaptable to producing terraces from power-sequences in $\mathbb{Z}_m$, where m is even. Nevertheless, we now show that, with $m = n-1$, where n an odd prime power, there are many ways in which power-sequences in $\mathbb{Z}_n$ can be used to arrange the elements of $\mathbb{Z}_n \setminus \{0\}$ in a sequence of distinct elements, usually in two or more segments, which becomes a terrace for $\mathbb{Z}_m$, i.e. for $\mathbb{Z}_{n-1}$, when interpreted modulo $m = n-1$. We restrict our constructions to those where each segment is a full-cycle sequence modulo n, but we draw on some general theory that also covers certain half-cycle sequences.

We use notation taken from our previous papers, but our current exposition needs further terminology and notation. Throughout the rest of this paper n is always an odd prime power, $n > 1$. We write $\mathcal{S}_k$ for the set of integers $\{1, 2, \dots, k\}$. When we evaluate the entries in a sequence $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_s)$, $1 < s < n$, of distinct elements of $\mathbb{Z}_n \setminus \{0\}$, these entries are always to be written so that $0 < \alpha_i < n$ for all i; in particular, α_i so defined is never to be replaced by $\alpha_i - n$, even though these two values are congruent modulo n.

Take such a sequence $\boldsymbol{\alpha}$. Using subtraction modulo n, write

$$d_i = \alpha_{i+1} - \alpha_i, \quad 0 < d_i < n, \qquad e_i = \alpha_i - \alpha_{i+1}, \quad 0 < e_i < n,$$

for $i = 1, 2, \dots, s-1$. Likewise, using subtraction modulo $n-1$, write

$$d_i^* = \alpha_{i+1} - \alpha_i, \quad 0 < d_i^* < n-1, \qquad e_i^* = \alpha_i - \alpha_{i+1}, \quad 0 < e_i^* < n-1,$$

for $i = 1, 2, \dots, s-1$. Write $\mu_i = \min(d_i, e_i)$ and $\mu_i^* = \min(d_i^*, e_i^*)$ for $i = 1, 2, \dots, s-1$. We call the values μ_i the *μ-differences* for $\boldsymbol{\alpha}$ (from μ = mu = minimum unsigned), and we call the values μ_i^* the corresponding μ^*-*differences* for $\boldsymbol{\alpha}$. For any particular value of i we have either $\mu_i^* = \mu_i$ or $\mu_i^* = \mu_i - 1$. If $\mu_i^* = \mu_i - 1$, we call the μ-difference μ_i a *reducing difference*; the corresponding μ^*-difference μ_i^* is then a *reduced difference*.

The definition of a 2-sequencing implies that, when $s = n - 1$, the sequence $\boldsymbol{\alpha}$, interpreted modulo $n - 1$, is a terrace for $\mathbb{Z}_{n-1}$ if its μ^*-differences comprise exactly one occurrence of the involution $\frac{1}{2}(n-1)$ of $\mathbb{Z}_{n-1}$, and exactly two occurrences of each member of $\mathcal{S}_{(n-3)/2}$. If $\boldsymbol{\alpha}$ is indeed a terrace for $\mathbb{Z}_{n-1}$, its μ^*-differences may or may not include reduced differences. Of the $\mathbb{Z}_{n-1}$ terraces constructed in this paper, many have a reduced difference at a join between two segments, and some have reduced differences within a segment. In most but not all of our constructions, the 'successive powers' in each segment are successive positive or negative powers of 2.

When we present a terrace, we print it as a display, with the commas between successive entries replaced by spaces, and with vertical bars (*fences*) in the joins between segments. For terraces with many segments, we use the notation

$$| \quad c \quad \xrightarrow{2} \quad |$$

for a segment $| \quad 2^0c \quad 2^1c \quad 2^2c \quad \cdots \quad |$, and the notation

$$| \quad c \quad \xleftarrow{2} \quad |$$

for $| \quad 2^0c \quad 2^{-1}c \quad 2^{-2}c \quad \cdots \quad |$, each arrow indicating the direction of successive multiplications by 2. More generally, but much less commonly, we use $| \quad c \quad \xrightarrow{x} \quad |$ for a segment $| \quad x^0c \quad x^1c \quad x^2c \quad \cdots \quad |$, and $| \quad c \quad \xleftarrow{x} \quad |$ for $| \quad x^0c \quad x^{-1}c \quad x^{-2}c \quad \cdots \quad |$.

2. Some preliminary number theory

Many of our constructions draw on ideas in [**2**]. However, we also need a few non-standard number theoretic results, as follows, that are relevant to reduced differences for $\mathbb{Z}_{n-1}$ terraces.

Lemma 2.1. *If the sequence $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_s)$, $1 < s < n$, of distinct elements of $\mathbb{Z}_n \setminus \{0\}$ has $\alpha_i = 2\alpha_{i-1}$ or $\alpha_i = 2^{-1}\alpha_{i-1}$ for all i satisfying $2 \leqslant i \leqslant s$, then the μ-differences and μ^*-differences for the sequence satisfy $\mu_i^* = \mu_i$ for all $i = 1, 2, \dots, s-1$, so that $\boldsymbol{\alpha}$ has no reducing differences.*

Lemma 2.2. *Let p be a prime, $p \equiv 23 \pmod{24}$ such that q, given by $q = \frac{1}{2}(p-1)$, is prime. Write $a = \frac{1}{2}(p+3)$. Then $\operatorname{ord}_p(2) = \operatorname{ord}_p(a) = q$.*

Proof. As $p \equiv 7 \pmod 8$ and $p \equiv 11 \pmod{12}$, both 2 and 3, and hence both 2 and a, are squares in $\mathbb{Z}_p$ and hence are not primitive roots of p. Their orders must divide $p - 1 = 2q$ and hence must both be q. □

Remark 2.3. If p is a prime satisfying $p \equiv 23 \pmod{24}$ and $\operatorname{ord}_p(2) = \operatorname{ord}_p(a) = q$, where a and q are defined as above, then q is not necessarily prime, the two counter-examples with $p < 300$ being provided by $p = 191$ and $p = 239$. Also, if p is prime with

$p \equiv 23 \pmod{24}$ and $\mathrm{ord}_p(2) = q$, then $\mathrm{ord}_p(a)$ is not necessarily equal to q. The two smallest counter-examples are $p = 71$, with $\mathrm{ord}_p(a) = \mathrm{ord}_{71}(37) = 7$, and $p = 431$, with $\mathrm{ord}_p(a) = \mathrm{ord}_{431}(217) = 43$.

Lemma 2.4. *Let $a = \frac{1}{2}(p+3)$, where p is an odd prime. For each $x \in \mathbb{Z}_p \setminus \{0\}$ consider ax to be reduced modulo p so as to lie in $\mathcal{S}_{p-1}$. Let $\mu_x(a)$ and $\mu^*_x(a)$ respectively denote the μ-difference and the μ^*-difference between x and ax. Then*

(i) *if x is odd and $x < \frac{1}{3}p$, then $\mu_x(a) = \frac{1}{2}(p - x)$ and $\mu^*_x(a) = \mu_x(a) - 1$,*

(ii) *if x is odd and $x > \frac{1}{3}p$, then $\mu_x(a) = \mu^*_x(a) = \frac{1}{2}(p - x)$,*

(iii) *if x is even and $x < \frac{2}{3}p$, then $\mu_x(a) = \mu^*_x(a) = \frac{1}{2}x$,*

(iv) *if x is even and $x > \frac{2}{3}p$, then $\mu_x(a) = \frac{1}{2}x$ and $\mu^*_x(a) = \mu_x(a) - 1$.*

Thus, if $p \equiv \delta \pmod 3$, where $\delta = 1$ or 2, then, as x varies over all elements of $\mathbb{Z}_p \setminus \{0\}$, the μ^*-difference $\mu^*_x(a)$ takes the value $\frac{1}{3}(p - \delta)$ four times and all other values from $\mathcal{S}_{(p-3)/2}$ twice. The reducing μ-differences are precisely the μ-differences greater than $\frac{1}{3}p$.

Proof. Consider each of the cases (i)–(iv) separately. For example, case (i) has $ax = \frac{1}{2}(3x + p)$ and $p > ax - x = \frac{1}{2}(p + x) > \frac{1}{2}p$, so that $\mu_x(a) = p - \frac{1}{2}(p + x) = \frac{1}{2}(p - x)$ and $\mu^*_x(a) = \mu_x(a) - 1$.

Straightforward checking shows that $\mu^*_x(a) = \frac{1}{3}(p - \delta)$ for four values of x, namely the odd numbers on each side of $\frac{1}{3}p$ and the even numbers on each side of $\frac{2}{3}p$. Every other value of $\mu^*_x(a)$ occurs twice: $\mu^*_x(a) = \mu^*_{p-x}(a)$. □

Theorem 2.5. *Let p be a prime, $p > 5$, $p \not\equiv 1$ or $5 \pmod{24}$, such that $\mathrm{ord}_p(a) = p-1$ or q, where $a = \frac{1}{2}(p + 3)$ and $q = \frac{1}{2}(p - 1)$. Consider the sequence*

$$\boldsymbol{\alpha} = (1, a, a^2, \dots, a^{q-1}) \pmod p,$$

where successive elements $\alpha_i = a^{i-1}$, $i = 1, 2, \dots, q$, are written so as to satisfy $0 < \alpha_i < p$. Then the μ^*-differences comprise exactly one occurrence of each member of $\mathcal{S}_{q-1}$.

Proof. When $\mathrm{ord}_p(a) = q$, the value a is a square, modulo p, so that $p = 1$, 5, 19 or 23 (mod 24). We have to avoid $p \equiv 1 \pmod 4$ as q is then even and $a^{q/2} = -1$, so that the differences in the second half of $\boldsymbol{\alpha}$ would be the same as those in the first half. For $p \equiv 19$ or $23 \pmod{24}$, the element x is a square, modulo p, if and only if $p - x$ is a non-square, so any x is in $\boldsymbol{\alpha}$ precisely when $p - x$ is not. This property of $\boldsymbol{\alpha}$ also holds when $\mathrm{ord}_p(a) = p - 1$.

As x and $p - x$ lead to the same μ^*-differences in Lemma 2.4, the lemma shows that the μ^*-differences for those x in $\boldsymbol{\alpha}$ will comprise each element of $\mathcal{S}_{q-1}$ once, with an extra occurrence of $\frac{1}{3}(p - \delta)$. But the missing μ^*-difference between the first and last elements of $\boldsymbol{\alpha}$ is $\frac{1}{3}(p - \delta)$, so each element of $\mathcal{S}_{q-1}$ will occur once. □

Remark 2.6. The values of p satisfying the conditions of Theorem 2.5 are $p = 7, 11, 17, 23, 31, 37, 41, 43, 47, 59, 67, 73, 83, 89, 103, 113, \dots$. In the range $3 < p < 300$, primes p satisfying $p \not\equiv 1$ or $5 \pmod{24}$ but not satisfying $\operatorname{ord}_p(a) = p-1$ or q are as follows:

p	13	19	61	71	79	181	199	211	281
$(p-1)/\operatorname{ord}_p(a)$	3	6	5	10	3	5	3	42	7

Remark 2.7. We use Theorem 2.5 for our construction in Theorem 4.9, below, which requires $\operatorname{ord}_p(2) = \operatorname{ord}_p(a) = q$, thus restricting us to $p \equiv 23 \pmod{24}$. (Values p with $p \equiv 19$ are excluded, as the condition $\operatorname{ord}_p(2) = q$ requires 2 to be a square in $\mathbb{Z}_p$.)

Remark 2.8. For some values of p that do not satisfy the conditions of Theorem 2.5, good alternatives to $\boldsymbol{\alpha}$ nevertheless exist. Suppose that $(p-1)/\operatorname{ord}_p(a) = 2\nu$ for some integer ν with $\nu > 1$. Suppose further that all or half of the elements in $\mathbb{Z}_n \setminus \{0\}$ belong to $\langle a, 3\rangle$. There may then be an element y in $\mathbb{Z}_n \setminus \{0\}$ such that the μ^*-differences for the sequence

$$3^0 y \quad \xrightarrow{a} \quad | \; 3^{-1} y \quad \xrightarrow{a} \quad | \quad \cdots \quad | \; 3^{-(\nu-1)} y \quad \xrightarrow{a} \quad \pmod{p}$$

comprise exactly one occurrence of each member of $\mathcal{S}_{q-1}$. Examples include

$$\begin{aligned}(p, a, \nu, y) = {} & (19, 11, 3, 1), (29, 16, 2, 23), (53, 28, 2, 47), (71, 37, 5, 20), \\ & (101, 52, 2, 95), (197, 100, 2, 191), (269, 136, 2, 263), (293, 148, 2, 287).\end{aligned}$$

As we note below, the first of these readily provides a terrace for $\mathbb{Z}_{18}$. No such example exists for $p = 211$.

3. The 'powers of k and $2k-1$' method

To aid understanding of some of the constructions later in this paper, we now informally outline an approach used in creating certain power-sequence terraces for $\mathbb{Z}_n$, where n is odd. If $k = 2$, this approach readily carries over to terraces for $\mathbb{Z}_{n-1}$.

Let p be an odd prime. Suppose that k is an element of $\mathbb{Z}_p \setminus \{0\}$, $k \neq \frac{1}{2}(p+1)$, such that $\operatorname{ord}_p(k) = \omega$, where $1 < k < p-1$ and $1 < \omega < p-1$. Write $c = (2k-1)^{-1}$. Suppose further that either

(i) every element of $\mathbb{Z}_p \setminus \{0\}$ belongs to $\langle k, c\rangle$, or

(ii) if $\omega < \frac{1}{2}(p-1)$, exactly half of the elements of $\mathbb{Z}_p \setminus \{0\}$ belong to $\langle k, c\rangle$.

Then, if we write $\boldsymbol{s} = s_1, s_2, \ldots, s_{2\omega}$ for either of the sequences

$$k^0 \quad k^1 \quad \cdots \quad k^{\omega-1} \mid ck^0 \quad ck^1 \quad \cdots \quad ck^{\omega-1} \tag{3.1}$$

and

$$k^0 \quad k^{\omega-1} \quad \cdots \quad k^1 \mid ck^2 \quad ck^3 \quad \cdots \quad ck^{\omega-1} \quad ck^0 \quad ck^1 \tag{3.2}$$

(mod p, with $0 < s_i < p$ for $i = 1, 2, \ldots, 2\omega$), we have $s_{2\omega} - s_{\omega+1} = s_{\omega+1} - s_\omega$. The differences between successive entries in the second half of each sequence are the quantities $ck^i - ck^{i-1}$, $i = 1, 2, \ldots, \omega$, except that the difference for one particular value of i is missing, namely $s_{\omega+1} - s_{2\omega}$, whose absence is compensated for in the sequence by the difference $s_\omega - s_{\omega+1}$.

If we now append further terms $s_{2\omega+1}, s_{2\omega+2}, \ldots, s_{3\omega}$ to either sequence, with $s_{2\omega+i} = cs_{\omega+i}$, $i = 1, 2, \ldots, \omega$, there will again be a difference 'missing' from the appended terms, but it will be compensated for by the difference at the point where the appended terms abut the previous one, and so on.

This is readily illustrated for $p = 13$ by taking $k = 3$, so that $\text{ord}_p(k) = 3$ and condition (i) is satisfied with $c = 5^{-1} = 8$. When prolonged to four segments as just described, sequence (3.2) becomes

$$1 \quad 9 \quad 3 \mid 7 \quad 8 \quad 11 \mid 4 \quad 12 \quad 10 \mid 6 \quad 5 \quad 2$$

where, for example, $s_{12} - s_{10} = s_{10} - s_9$. The only 'missing' difference not compensated for at a fence is $s_3 - s_1 = 2$, so the 12-term sequence becomes a terrace for $\mathbb{Z}_{13}$ when the missing element 0 is put at the end. Standard number theory ensures correct frequencies of occurrence for differences not involved in the compensations.

Condition (i) applies also for $p = 17$ if we take $k = 4$. This gives $\text{ord}_p(k) = 4$ and $c = 7^{-1} = 5$. Sequence (3.2) is now

$$1 \quad 13 \quad 16 \quad 4 \mid 12 \quad 14 \quad 5 \quad 3$$

where $s_8 - s_5 = -9 \equiv +8 = s_5 - s_4 \pmod{17}$. The difference 'missing' from segment 1 is $s_4 - s_1 = 3$, which can be compensated for by appending 0 after (3.2). For this particular case, a terrace for $\mathbb{Z}_{17}$ can now be completed by multiplying (3.2) throughout by 2 and placing the reverse of this eight-term sequence after the zero, to give

$$1 \quad 13 \quad 16 \quad 4 \mid 12 \quad 14 \quad 5 \quad 3 \mid 0 \mid 6 \quad 10 \quad 11 \quad 7 \mid 8 \quad 15 \quad 9 \quad 2.$$

Condition (ii) applies for $p = 17$, $k = 13$ and $c = 15$, for which (3.2) becomes

$$1 \quad 4 \quad 16 \quad 13 \mid 2 \quad 9 \quad 15 \quad 8;$$

multiplying this by 7 we have

$$7 \quad 11 \quad 10 \quad 6 \mid 14 \quad 12 \quad 3 \quad 5,$$

whence, in this particular case, we are able to write down the further $\mathbb{Z}_{17}$ terrace

$$1 \quad 4 \quad 16 \quad 13 \mid 2 \quad 9 \quad 15 \quad 8 \mid 7 \quad 11 \quad 10 \quad 6 \mid 14 \quad 12 \quad 3 \quad 5 \mid 0.$$

In general, compensation for a difference 'missing' from a segment will not be achievable by appending 0 at the start or end of an otherwise promising sequence. However, if $k = 2$, so that we have the 'powers of 2 and 3' (P2&3) method, putting 0 at the start of (3.2) will always compensate for the difference 1 that is 'missing' from

$$\mid \quad k^0 \quad k^{\omega-1} \quad k^{\omega-2} \quad \cdots \quad k^1 \quad \mid.$$

Thus, for $p = 17$, which satisfies condition (i) with $k = 2$ and $c = 3^{-1} = 6$, appending 0 at the start of (3.2) gives the $\mathbb{Z}_{17}$ terrace

$$0 \mid 1 \quad 9 \quad 13 \quad 15 \quad 16 \quad 8 \quad 4 \quad 2 \mid 7 \quad 14 \quad 11 \quad 5 \quad 10 \quad 3 \quad 6 \quad 12. \tag{3.3}$$

Alternatively, the third segment here can be moved to the front to give the $\mathbb{Z}_{17}$ terrace

$$7 \quad 14 \quad 11 \quad 5 \quad 10 \quad 3 \quad 6 \quad 12 \mid 0 \mid 1 \quad 9 \quad 13 \quad 15 \quad 16 \quad 8 \quad 4 \quad 2, \tag{3.4}$$

where the first segment is merely 12 times the reverse of the third. If we now replace the last two segments of (3.4) by the first segment of (3.1), we obtain

$$7 \quad 14 \quad 11 \quad 5 \quad 10 \quad 3 \quad 6 \quad 12 \mid 1 \quad 2 \quad 4 \quad 8 \quad 16 \quad 15 \quad 13 \quad 9. \tag{3.5}$$

This sequence of distinct elements of $\mathbb{Z}_{17} \setminus \{0\}$ has identical μ-differences and μ^*-differences except at the fence, where the μ^*-difference 5 compensates for the fact that the μ^*-difference 5 is not duplicated in the first segment; thus, reinterpreted modulo 16, the 16-element sequence is a $\mathbb{Z}_{16}$ terrace, as the difference 'missing' from the second segment is the involution and so does not have to be compensated for. The key to this construction is recognizing that the difference, modulo p, across the fence in $\cdots\ 12 \mid 0\ \cdots$, as in (3.4), is the same as the difference, modulo $p-1$, for $\cdots\ 12 \mid 1\ \cdots$, as in (3.5). More generally, the sequence (3.5) remains a terrace for $\mathbb{Z}_{16}$ when multiplied throughout, modulo 17, by any element of $\mathbb{Z}_{17} \setminus \{0\}$ such that the difference at the fence remains a reducing/reduced difference.

The value $p = 71$ satisfies condition (ii) with $k = 2 \cdot 3^{-1}$ and $c = 3$. Then, multiplying by 25 the sequence obtained by prolonging (3.1) to five segments, we have

$$25 \quad 64 \quad \cdots \quad 2 \mid 4 \quad 50 \quad \cdots \quad 6 \mid 12 \quad 8 \quad \cdots \quad 18 \mid 36 \quad 24 \quad \cdots \quad 54 \mid 37 \quad 1 \quad \cdots \quad 20.$$

For this sequence of distinct elements from $\mathbb{Z}_{71} \setminus \{0\}$, the μ-differences comprise one occurrence of each member of $\mathcal{S}_{35} \setminus \{23\}$, where 23 is the difference 'missing' from the first segment, but the μ^*-differences comprise exactly one occurrence of each member of $\mathcal{S}_{34}$.

As $\operatorname{ord}_p(k) = \operatorname{ord}_p(k^{-1})$, the parameters k and c in (3.1) and (3.2) can sometimes, for $k > 2$, be replaced by $k^* = k^{-1}$ and $c^* = (2k^{-1} - 1)^{-1}$, respectively. The same is true, of course, of prolonged versions of (3.1) and (3.2). If we take $k = a$, where $a = \frac{1}{2}(p+3)$ as in § 2, we have $k^* = 2 \cdot 3^{-1}$ and $c^* = 3$, in agreement with the result given for $p = 71$ in the previous paragraph.

4. $\mathbb{Z}_{n-1}$ terraces for prime n

Theorem 4.1. *Let n be an odd prime having 2 as a primitive root. When reinterpreted modulo $n-1$, the sequence*

$$2^0 \quad 2^1 \quad \cdots \quad 2^{n-2}$$

of elements from $\mathbb{Z}_n \setminus \{0\}$ is a terrace for $\mathbb{Z}_{n-1}$.

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Proof. The μ-differences for the sequence consist of exactly one occurrence of $\frac{1}{2}(n-1)$ (exactly in the middle of the terrace) and exactly two occurrences of each member of $\mathcal{S}_{(n-3)/2}$. Clearly, $\mu_i^* = \mu_i$ for all i satisfying $1 \leqslant i \leqslant n-2$, as in Lemma 2.1. □

Example 4.2. Taking $n = 11$ gives the single-segment $\mathbb{Z}_{10}$ terrace

$$1 \quad 2 \quad 4 \quad 8 \quad 5 \quad 10 \quad 9 \quad 7 \quad 3 \quad 6.$$

Remark. In the range $2 < n < 300$, Theorem 4.1 provides $\mathbb{Z}_{n-1}$ terraces for

$$n = 3, 5, 11, 13, 19, 29, 37, 53, 59, 61, 67, 83,$$
$$101, 107, 131, 139, 149, 163, 173, 179, 181, 197, 211, 227, 269, 293.$$

Theorem 4.3. *Let n be any prime satisfying $n \equiv 1$ or $7 \pmod 8$ and $\mathrm{ord}_n(2) = \frac{1}{2}(n-1)$. Let x be any non-square element of $\mathbb{Z}_n$ that satisfies $\frac{1}{2}(n+1) < x < n$. When reinterpreted modulo $n-1$, the sequence*

$$2^1x \quad 2^2x \quad \cdots \quad 2^{(n-3)/2}x \quad 2^0x \mid 2^0 \quad 2^1 \quad \cdots \quad 2^{(n-3)/2}$$

of elements from $\mathbb{Z}_n \setminus \{0\}$ is a terrace for $\mathbb{Z}_{n-1}$.

Proof. Because of Lemma 2.1 and other standard results, the only differences that need attention are those between the two ends of a segment, lest they be under-represented in the proposed terrace, and the difference across the fence, lest it be over-represented. The difference across the fence is $2^0x - 2^0 = x - 1$, which gives a μ-difference of $n-(x-1)$ and therefore a μ^*-difference of $n - (x-1) - 1 = n - x = 2^0x - 2^1x$, which is the difference between the last and first elements of the first segment. The difference between the last and first elements of the second segment is $2^{(n-3)/2} - 1 = \frac{1}{2}(n-1)$, the very μ^*-difference that must appear once, not twice, throughout the terrace. □

Case 1 (special case of Theorem 4.3). Let n be any prime satisfying $n \equiv 7 \pmod 8$ and $\mathrm{ord}_n(2) = \frac{1}{2}(n-1)$. When reinterpreted modulo $n-1$, the sequence

$$-2^1 \quad -2^2 \quad \cdots \quad -2^{(n-3)/2} \quad -2^0 \mid 2^0 \quad 2^1 \quad \cdots \quad 2^{(n-3)/2}$$

of elements from $\mathbb{Z}_n \setminus \{0\}$ is a terrace for $\mathbb{Z}_{n-1}$.

Example 4.4. Taking $n = 7$ in the special case gives the $\mathbb{Z}_6$ terrace

$$5 \quad 3 \quad 6 \mid 1 \quad 2 \quad 4.$$

Example 4.5. Taking $n = 17$ and $x = 10$ in Theorem 4.3 gives the $\mathbb{Z}_{16}$ terrace

$$3 \quad 6 \quad 12 \quad 7 \quad 14 \quad 11 \quad 5 \quad 10 \mid 1 \quad 2 \quad 4 \quad 8 \quad 16 \quad 15 \quad 13 \quad 9.$$

Remark. In the range $2 < n < 300$, Theorem 4.3 provides $\mathbb{Z}_{n-1}$ terraces for $n = 7, 17, 23, 41, 47, 71, 79, 97, 103, 137, 167, 191, 193, 199, 239, 263, 271$.

Theorem 4.6. *Let n be a prime satisfying $n \equiv 7$ or $17 \pmod{24}$ and $\mathrm{ord}_n(2) = \frac{1}{2}(n-1)$. When reinterpreted modulo $n-1$, the sequence*

$$2^0 \quad 2^1 \quad \cdots \quad 2^{(n-3)/2} \mid 3^{-1}\cdot 2^0 \quad 3^{-1}\cdot 2^1 \quad \cdots \quad 3^{-1}\cdot 2^{(n-3)/2}$$

of elements from $\mathbb{Z}_n \setminus \{0\}$ is a terrace for $\mathbb{Z}_{n-1}$.

Proof. The conditions on n ensure that 3 is not a square in $\mathbb{Z}_n$, and thus that 3^{-1} is not a power of 2 in $\mathbb{Z}_n$. The rest of the proof becomes straightforward when we note that the P2&3 method of construction is used for the terrace. The last element of the first segment, the first element of the second segment, and the last element of the second segment are, respectively, $\frac{1}{2}(n+1)$, $\frac{1}{3}(2n+1)$ and $\frac{1}{6}(5n+1)$ if $n \equiv 7 \pmod{24}$, or $\frac{1}{2}(n+1)$, $\frac{1}{3}(n+1)$ and $\frac{1}{6}(n+1)$ if $n \equiv 17 \pmod{24}$. The terrace has no reduced difference. □

Example 4.7. Taking $n = 7$ gives the $\mathbb{Z}_6$ terrace

$$1 \quad 2 \quad 4 \mid 5 \quad 3 \quad 6.$$

Example 4.8. Taking $n = 17$ gives the $\mathbb{Z}_{16}$ terrace

$$1 \quad 2 \quad 4 \quad 8 \quad 16 \quad 15 \quad 13 \quad 9 \mid 6 \quad 12 \quad 7 \quad 14 \quad 11 \quad 5 \quad 10 \quad 3.$$

Remark. In the range $2 < n < 300$, Theorem 4.6 provides $\mathbb{Z}_{n-1}$ terraces for $n = 7, 17, 41, 79, 103, 137, 199, 271$.

Theorem 4.9. *Let n be a prime, $n \equiv 23 \pmod{24}$, such that $\mathrm{ord}_n(2) = \mathrm{ord}_n(a) = \frac{1}{2}(n-1)$, where $a = \frac{1}{2}(n+3)$. Let i be an integer such that the element 2^i from $\mathbb{Z}_n$ satisfies $0 < 2^i < \frac{1}{2}(n-1)$. When reinterpreted modulo $n-1$, the sequence*

$$-a^{(n-3)/2} \quad -a^{(n-5)/2} \quad \cdots \quad -a^0 \mid 2^i \quad 2^{i-1} \quad \cdots \quad 2^{i-(n-3)/2}$$

of elements from $\mathbb{Z}_n \setminus \{0\}$ is a terrace for $\mathbb{Z}_{n-1}$. The integer i can always be given the value $i = \frac{1}{2}(n-5)$, to place the involution at the extreme right-hand end of the 2-sequencing for the terrace.

Proof. With $n = p$, the first segment of the proposed terrace is the reverse of the negative of the sequence $\boldsymbol{\alpha}$ in Theorem 2.5. Thus, the μ^*-differences for the first segment comprise exactly one occurrence of each member of $\mathcal{S}_{(n-1)/2}$. The μ^*-difference at the fence is a reduced difference identical to the first entry in the second segment and so is equal to the difference $2^{i-(p-3)/2} - 2^i = 2^{i+1} - 2^i = 2^i$ between the last and first entries of the second segment. Thus, the complete set of μ^*-differences for the proposed terrace is correct.

If we take $i = \frac{1}{2}(p-5)$, then the first element of the second segment is $\frac{1}{4}(n+1)$, which is less than $\frac{1}{2}(n-1)$, as required. The last two elements of the second segment are 1 and $\frac{1}{2}(n+1)$, which provide the μ^*-difference $\frac{1}{2}(n-1)$; this puts the involution at the very end of the 2-sequencing. □

Example 4.10. Taking $n = 23$ and $i = 0$ gives the $\mathbb{Z}_{22}$ terrace

$$7 \quad 20 \quad 21 \quad 14 \quad 17 \quad 19 \quad 5 \quad 11 \quad 15 \quad 10 \quad 22 \mid 1 \quad 12 \quad 6 \quad 3 \quad 13 \quad 18 \quad 9 \quad 16 \quad 8 \quad 4 \quad 2.$$

Remark. In the range $2 < n < 300$, Theorem 4.9 provides $\mathbb{Z}_{n-1}$ terraces for $n = 23, 47, 167, 191, 239, 263$, but not (see the Remark 2.3) for $n = 71$. Despite the result in Remark 2.8, we have found no modification of Theorem 4.9 that covers $n = 71$.

We now move on to values of n satisfying $\operatorname{ord}_n(2) = \frac{1}{3}(n-1)$. Here 2 cannot be a square modulo n, so $n \equiv 3$ or $5 \pmod 8$. But $n \equiv 1 \pmod 3$, so $n \equiv 13$ or $19 \pmod{24}$.

Lemma 4.11. *Let p be a prime, $p \equiv 1 \pmod 6$, such that $\operatorname{ord}_p(2) = \frac{1}{3}(p-1)$ and $3 \notin \langle 2 \rangle$. Then $\mathbb{Z}_p \setminus \{0\} = \langle 2 \rangle \cup 3\langle 2 \rangle \cup 3^{-1}\langle 2 \rangle$.*

Proof. Let $p = 6k+1$. We have to show that $3^2 \notin \langle 2 \rangle$. Let θ be any primitive root of p and suppose that $2 = \theta^v$. As $\operatorname{ord}_p(2) = \frac{1}{3}(p-1)$, we have $\gcd(v, 6k) = 3$, so $v = 6u + 3$ for some u and $2^k \equiv \theta^{6ku+3k} \equiv \theta^{3k} \equiv -1 \pmod p$.

Suppose that $3^2 \equiv 2^i$ for some i. As 2 is a non-square, we have $i = 2j$ for some j, so $3^2 \equiv 2^{2j}$. Thus, $3 \equiv 2^j$ or -2^j, which is to say that $3 \equiv 2^j$ or 2^{j+k}. In either case $3 \in \langle 2 \rangle$, which gives us a contradiction. □

Theorem 4.12. *Let n be a prime, $n \equiv 13$ or $19 \pmod{24}$, such that $\operatorname{ord}_n(2) = \frac{1}{3}(n-1)$, with $3 \notin \langle 2 \rangle$ in $\mathbb{Z}_n$. Let x be an element of $\mathbb{Z}_n$ that satisfies $\frac{1}{2}(n+1) < x < n$ and $x \in 3\langle 2 \rangle$. When reinterpreted modulo $n-1$, the sequence*

$$2x \quad \xrightarrow{2} \quad \mid 1 \quad \xrightarrow{2} \quad \mid 3^{-1} \quad \xrightarrow{2}$$

of elements from $\mathbb{Z}_n \setminus \{0\}$ is a terrace for $\mathbb{Z}_{n-1}$; the involution falls exactly in the middle of the 2-sequencing for the terrace, and the only reduced difference occurs at the first fence.

Proof. Lemma 4.11 applies. The differences are readily checked if it is noted that the first and last entries in the third segment are $\frac{1}{3}(2n+1)$ and $\frac{1}{6}(5n+1)$, respectively. □

Example 4.13. Taking $n = 43$ and $x = 33$ in Theorem 4.12 gives the $\mathbb{Z}_{42}$ terrace

$$23 \quad 3 \quad \cdots \quad 38 \quad 33 \mid 1 \quad 2 \quad \cdots \quad 11 \quad 22 \mid 29 \quad 15 \quad \cdots \quad 18 \quad 36.$$

Remark. In the range $2 < n < 300$, Theorem 4.12 provides $\mathbb{Z}_{n-1}$ terraces for $n = 43, 109, 157, 229, 277, 283$. These are indeed the only primes less than 300 that have $\operatorname{ord}_n(2) = \frac{1}{3}(n-1)$. However, the prime value $n = 307$, despite satisfying $n \equiv 19 \pmod{24}$ and $\operatorname{ord}_n(2) = \frac{1}{3}(n-1)$, is not covered by the theorem, as it has $3 \equiv 2^{93} \pmod n$. Also, for example, the prime value $n = 997$, despite satisfying $n \equiv 13 \pmod{24}$ and $\operatorname{ord}_n(2) = \frac{1}{3}(n-1)$, has $3 \equiv 2^{114} \pmod n$.

Theorem 4.14. *Let n be a prime, $n \equiv 13$ or $19 \pmod{24}$, such that $\operatorname{ord}_n(2) = \frac{1}{3}(n-1)$ with $3 \notin \langle 2 \rangle$ in $\mathbb{Z}_n$. Let x be an element of $\mathbb{Z}_n$ that satisfies $\frac{1}{2}(n+1) < x < n$, with $x \in 3^{-1}\langle 2 \rangle$ and $x \not\equiv 0 \pmod 3$. When reinterpreted modulo $n-1$, the sequence*

$$3^{-1} \cdot 2x \quad \xleftarrow{2} \quad \mid 2x \quad \xrightarrow{2} \quad \mid 1 \quad \xrightarrow{2}$$

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is a terrace for $\mathbb{Z}_{n-1}$; the involution in the 2-sequencing occurs exactly in the middle of the second segment, and the sole reduced difference occurs at the second fence.

Proof. If $x \equiv 1 \pmod 3$, the entries $3^{-1} \cdot 2x$, $3^{-1} \cdot 4x$ and $2x$ (at the start and end of the first segment and at the start of the second segment) are $\frac{1}{3}(2x+n)$, $\frac{1}{3}(4x-n)$ and $2x-n$, respectively; these are in decreasing order of magnitude, with common difference $\frac{2}{3}(n-x) < \frac{1}{2}(n-1)$. If $x \equiv 2 \pmod 3$, the three entries are $\frac{1}{3}(2x-n)$, $\frac{1}{3}(4x-2n)$ and $2x-n$; these are in increasing order of magnitude, with common difference $\frac{1}{3}(2x-n) < \frac{1}{2}(n-1)$. But if we were to take a value x satisfying $x = 3v$, the three entries would be $2v$, then $4v$ or $4v-n$, and then $6v-n$, and in either case there would be a reducing difference at the first fence. $\square$

Case 2 (special case of Theorem 4.14). We can always take $x = 3^2 \cdot 2^{-1}$, so that the terrace becomes

$$3 \overset{2}{\leftarrow} \mid 3^2 \overset{2}{\rightarrow} \mid 1 \overset{2}{\rightarrow} .$$

Example 4.15. For $n = 43$, the special case has $x = 26$, which gives the $\mathbb{Z}_{42}$ terrace

$$3 \quad 23 \quad \cdots \quad 6 \mid 9 \quad 18 \quad \cdots \quad 26 \mid 1 \quad 2 \quad \cdots \quad 22,$$

whereas $x = 34$ gives

$$37 \quad 40 \quad \cdots \quad 31 \mid 25 \quad 7 \quad \cdots \quad 34 \mid 1 \quad 2 \quad \cdots \quad 22.$$

This latter terrace should be compared with the second terrace in Example 4.18, below.

Example 4.16. For $n = 109$ and $x = 91$, Theorem 4.14 gives the $\mathbb{Z}_{108}$ terrace

$$97 \quad 103 \quad \cdots \quad 85 \mid 73 \quad 37 \quad \cdots \quad 91 \mid 1 \quad 2 \quad \cdots \quad 55,$$

which should be compared with the terrace in Example 4.19, below.

Remark. Theorem 4.14 covers the same values of n as Theorem 4.12.

Theorem 4.17. *Let n be a prime, $n \equiv 13$ or $19 \pmod{24}$, such that $\mathrm{ord}_n(2) = \frac{1}{3}(n-1)$. Suppose that y and z are both odd integers from $\mathcal{S} \setminus \{\frac{1}{2}(n-1)\}$, with $\frac{1}{3}(n-1) < y < \frac{2}{3}(n-1)$, such that $\mathbb{Z}_n \setminus \{0\} = \langle 2\rangle \cup y\langle 2\rangle \cup z\langle 2\rangle$. Suppose further that $z \equiv 3y+1 \pmod n$ if $0 < y < \frac{1}{2}(n-1)$, and $z \equiv 3y-1 \pmod n$ if $\frac{1}{2}(n-1) < y < n-1$. Reinterpreted modulo $n-1$, the sequence*

$$y \overset{2}{\leftarrow} \mid z \overset{2}{\rightarrow} \mid 1 \overset{2}{\rightarrow}$$

of elements from $\mathbb{Z}_n \setminus \{0\}$ is a terrace for $\mathbb{Z}_{n-1}$; the μ^*-differences at the two fences are both reduced differences.

Proof. If $y < \frac{1}{2}(n-1)$, the sequence is

$$y \ \cdots \ 2y \mid 3y+1-n \ \cdots \ \tfrac{1}{2}(3y+1) \mid 1 \ \cdots \ \tfrac{1}{2}(n+1)$$

and so the 'missing' differences are y, $\frac{1}{2}(3y+1)-n$ and $\frac{1}{2}(n-1)$, while the reduced μ^*-differences at the fences are y and $\frac{1}{2}(3y+1)-n$. If $y > \frac{1}{2}(n-1)$, the sequence is

$$y \ \cdots \ 2y-n \mid 3y-1-n \ \cdots \ \tfrac{1}{2}(3y-1) \mid 1 \ \cdots \ \tfrac{1}{2}(n+1),$$

and again the μ^*-differences at the two fences are reduced. □

Case 3 (special case of Theorem 4.17). If $3 \notin \langle 2 \rangle$ in $\mathbb{Z}_n$, in Theorem 4.17 we can take $y = \frac{1}{3}(n+2)$ in conjunction with $z = 3$.

Example 4.18. For $n = 43$, the possible parameter sets are $(y, z) = (15, 3)$, $(19, 15)$, $(25, 31)$, $(23, 25)$. The fact that the y-value for the first (third) parameter set is equal to the z-value for the second (fourth) does not reflect any known general result. The first parameter set yields the special case $\mathbb{Z}_{42}$ terrace

$$15 \quad 29 \quad \cdots \quad 30 \mid 3 \quad 6 \quad \cdots \quad 23 \mid 1 \quad 2 \quad \cdots \quad 22,$$

whereas the fourth yields

$$23 \quad 33 \quad \cdots \quad 3 \mid 25 \quad 7 \quad \cdots \quad 34 \mid 1 \quad 2 \quad \cdots \quad 22.$$

The fact that the second segment of the first of these terraces is the reverse of the first segment of the second terrace again does not reflect a known general result. In the second of these terraces, the second and third segments are exactly as in the second terrace from Example 4.15.

Example 4.19. For $n = 109$, the parameter set $(y, z) = (61, 73)$ yields the $\mathbb{Z}_{108}$ terrace

$$61 \quad 85 \quad \cdots \quad 13 \mid 73 \quad 37 \quad \cdots \quad 91 \mid 1 \quad 2 \quad \cdots \quad 55.$$

Here the second and third segments are exactly as in Example 4.16.

Remark. Theorem 4.17 provides several terraces for the previously excluded value $n = 307$, e.g. the $\mathbb{Z}_{306}$ terraces

$$159 \quad 233 \quad \cdots \quad 11 \mid 169 \quad 31 \quad \cdots \quad 238 \mid 1 \quad 2 \quad \cdots \quad 154$$

and

$$165 \quad 236 \quad \cdots \quad 23 \mid 187 \quad 67 \quad \cdots \quad 247 \mid 1 \quad 2 \quad \cdots \quad 154.$$

We have no proof that a pair (y, z) can be found for *any* value n satisfying the conditions of Theorem 4.17 and such that $3 \in \langle 2 \rangle$ in $\mathbb{Z}_n$. However, the next such n-value after 307 is 499, for which we can take $(y, z) = (241, 225)$.

We now move on to values of n such that $\mathrm{ord}_n(2) = \frac{1}{4}(n-1)$. Now 2 is a square (and indeed a fourth power) in $\mathbb{Z}_n$. If we also require 3 to be a non-square, we must have $n \equiv 17 \pmod{24}$, i.e. $n \equiv 17$ or 41 or 65 $(\mathrm{mod}\, 72)$.

Theorem 4.20. *Let n be a prime, $n \equiv 17$ or $65 \pmod{72}$, such that $\mathrm{ord}_n(2) = \frac{1}{4}(n-1)$. Let x be an element of $\mathbb{Z}_n$ that satisfies $\frac{1}{2}(n+1) < x < n$ and $x \in 3\langle 2\rangle$. When reinterpreted modulo $n-1$, the sequence*

$$2x \quad \xrightarrow{2} \quad \mid 1 \quad \xrightarrow{2} \quad \mid 3^{-1} \quad \xrightarrow{2} \quad \mid 3^{-2} \quad \xrightarrow{2}$$

of elements from $\mathbb{Z}_n \setminus \{0\}$ is a terrace for $\mathbb{Z}_{n-1}$, with the 2-sequencing's involution arising exactly in the middle of the second segment of the terrace.

Proof. This is similar to that of Theorem 4.12. The requirement $3^2 \notin \langle 2\rangle$ is automatically satisfied. Again, the only reduced difference occurs at the first fence. The differences are readily checked on noting the following values for elements at the ends of segments:

$$\begin{aligned} 2^{(n-5)/4} &= \tfrac{1}{2}(n+1), \\ 3^{-1} &= \tfrac{1}{3}(n+1), \\ 3^{-1} \cdot 2^{(n-5)/4} &= \tfrac{1}{6}(n+1), \\ 3^{-2} &= \begin{cases} \tfrac{1}{9}(n+1) & \text{if } n \equiv 17 \pmod{72}, \\ \tfrac{1}{9}(4n+1) & \text{if } n \equiv 65 \pmod{72}, \end{cases} \end{aligned}$$

and

$$3^{-2} \cdot 2^{(n-5)/4} = \begin{cases} \tfrac{1}{18}(n+1) & \text{if } n \equiv 17 \pmod{72}, \\ \tfrac{1}{18}(13n+1) & \text{if } n \equiv 65 \pmod{72}. \end{cases}$$

If we were to take $n \equiv 41 \pmod{72}$, there would be an unwanted reducing μ-difference at the final fence. $\square$

Example 4.21. Taking $n = 281$ and $x = 142$ in Theorem 4.20 gives the $\mathbb{Z}_{280}$ terrace

$$3 \quad 6 \quad \cdots \quad 142 \mid 1 \quad 2 \quad \cdots \quad 141 \mid 94 \quad 188 \quad \cdots \quad 47 \mid 125 \quad 250 \quad \cdots \quad 203.$$

Remark 4.22. In the range $2 < n < 300$, the only n-value covered by Theorem 4.20 is 281, for which $n \equiv 65 \pmod{72}$. For $n \equiv 17 \pmod{72}$, the smallest n-value covered by the theorem is 593.

Remark 4.23. The value $n = 113$ satisfies $\mathrm{ord}_n(2) = \frac{1}{4}(n-1)$ and is such that 3^2 is not a power of 2 in $\mathbb{Z}_n$. However, the construction in Theorem 4.20 fails if, as here, $n \equiv 41 \pmod{72}$. We therefore proceed to the P2&3 construction in the next theorem.

Theorem 4.24. *Let n be a prime, $n \equiv 17 \pmod{24}$, such that $\mathrm{ord}_n(2) = \frac{1}{4}(n-1)$. Let x be an element of $\mathbb{Z}_n$ that satisfies $\frac{1}{2}(n+1) < x < n$, $x \not\equiv 0 \pmod 3$ and $x \in 3^2\langle 2\rangle$ in $\mathbb{Z}_n$. When reinterpreted modulo $n-1$, the sequence*

$$3^{-1} \cdot 2x \quad \xleftarrow{2} \quad \mid 2x \quad \xrightarrow{2} \quad \mid 1 \quad \xrightarrow{2} \quad \mid 3^{-1} \quad \xrightarrow{2}$$

of elements from $\mathbb{Z}_n \setminus \{0\}$ is a terrace for $\mathbb{Z}_{n-1}$, with the 2-sequencing's involution falling in the middle of the third segment of the terrace.

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Proof. This is similar to that of Theorem 4.20. Again, the only reduced difference is at the fence followed by the element 1. If we try $x \equiv 0 \pmod 3$, we obtain an unwanted reducing μ-difference at the first fence. □

Example 4.25. Taking $n = 113$ and $x = 95$ in Theorem 4.24 gives the $\mathbb{Z}_{112}$ terrace

$$101 \quad 107 \quad \cdots \quad 89 \,|\, 77 \quad 41 \quad \cdots \quad 95 \,|\, 1 \quad 2 \quad \cdots \quad 57 \,|\, 38 \quad 76 \quad \cdots \quad 19.$$

Example 4.26. Taking $n = 281$ and $x = 250$ in Theorem 4.24 gives the $\mathbb{Z}_{280}$ terrace

$$73 \quad 118 \quad \cdots \quad 146 \,|\, 219 \quad 157 \quad \cdots \quad 250 \,|\, 1 \quad 2 \quad \cdots \quad 141 \,|\, 94 \quad 188 \quad \cdots \quad 47.$$

Remark 4.27. In the range $2 < n < 300$, the only n-values covered by Theorem 4.24 are $n = 113$ and $n = 281$, these being the range's only primes n with $\operatorname{ord}_n(2) = \frac{1}{4}(n-1)$.

Remark 4.28. Although Theorems 4.20 and 4.24 produce terraces with a reduced difference at just one of the three fences, some similar terraces exist, $\operatorname{ord}_n(2) = \frac{1}{4}(n-1)$, with reduced differences at exactly two of the three fences or at all three. We content ourselves with giving the following examples for $n = 281$; first, with a reduced difference at each of the first two fences but not at the third:

$$135 \quad \overset{2}{\leftarrow} \quad |\, 125 \quad \overset{2}{\rightarrow} \quad |\, 1 \quad \overset{2}{\rightarrow} \quad |\, 94 \quad \overset{2}{\rightarrow}\,;$$

second, with a reduced difference at each fence:

$$135 \quad \overset{2}{\leftarrow} \quad |\, 125 \quad \overset{2}{\leftarrow} \quad |\, 95 \quad \overset{2}{\rightarrow} \quad |\, 1 \quad \overset{2}{\rightarrow}\,.$$

Remark 4.29. The principle underlying the construction in Theorem 4.24 can be extended to other values of n, even though a fully general theorem would be notationally unmanageable. We now proceed to a P2&3 theorem that sweeps up special cases within the range $2 < n < 300$.

Theorem 4.30. *Let n be a prime satisfying* $\operatorname{ord}_n(2) = (n-1)/(f+g+2)$ *for some non-negative integers f and g such that either*

(i) $\mathbb{Z}_n \setminus \{0\} = \langle 2, 3\rangle$, *or*

(ii) $f = g$ *and $\langle 2, 3\rangle$ comprises exactly half of the members of $\mathbb{Z}_n \setminus \{0\}$.*

Suppose that there exists a value y from $\mathbb{Z}_n$ with $y \in 3^{-(f+1)}\langle 2\rangle$ in case (i) and $y \notin \langle 2, 3\rangle$ in case (ii), such that $3^0y, 3^{-1}y, \dots, 3^{-g}y$ are all odd, and such that $3^0 2^{-1}, 3^{-1}2^{-1}, \dots, 3^{-f}2^{-1}$ are either all even or all odd when they are evaluated, modulo n, as elements of $\mathcal{S}_{n-1}$. Then, when reinterpreted modulo $n-1$, the sequence

$$3^{-g}y \ \overset{2}{\leftarrow} \,|\, 3^{-(g-1)}y \ \overset{2}{\leftarrow} \,|\, \cdots \,|\, 3^{-1}y \ \overset{2}{\leftarrow} \,|\, y \ \overset{2}{\rightarrow} \,|\, 3^0 \ \overset{2}{\rightarrow} \,|\, 3^{-1} \ \overset{2}{\rightarrow} \,|\, \cdots \,|\, 3^{-f} \ \overset{2}{\rightarrow}\,.$$

of elements from $\mathbb{Z}_n \setminus \{0\}$ is a terrace for $\mathbb{Z}_{n-1}$. (The possible values of f and g for $n < 300$ are listed in Table 1.)

Table 1. *The special cases covered by Theorem 4.30*

n	type	f	g	y
31	(i)	1	3	19
73	(ii)	3	3	21, 43, 63
89	(i)	0	6	35, 51
		1	5	17, 71
		2	4	33, 65, 83
		3	3	11, 81, 87
113	(i)	0	2	5, 27, 33, 35, 59
		1	1	9, 11, 41, 51, 63, 69, 77, 87, 95
127	(i)	—	—	—
151	(i)	0	8	129
		1	7	43
		2	6	115
		3	5	139
223	(i)	0	4	75, 79, 117, 187
		1	3	25, 39, 81, 121, 133, 175, 211
		2	2	13, 27, 103, 111, 157, 189, 193, 207, 219
233	(i)	0	6	99
		1	5	33
		2	4	11, 119
241	(ii)	4	4	31, 39, 43, 63, 85, 93, 117, 129, 189
251	(i)	0	3	17, 81, 83, 87, 89, 105, 137, 159
		1	2	27, 29, 35, 53, 99, 135, 143, 173, 195, 197, 213
		2	1	9, 33, 45, 65, 71, 93, 107, 119, 147, 161, 165, 177, 179, 185, 215, 225, 233
		3	0	3, 11, 15, 31, 49, 55, 59, 75, 101, 127, 131, 133, 141, 153, 155, 163, 189, 191, 203, 207, 221, 227, 229, 239, 245
257	(i)	—	—	—
281	(i)	0	2	15, 27, 41, 47, 51, 65, 77, 83, 95, 99, 113, 117, 159, 171, 173, 207, 221, 227, 239
		1	1	5, 9, 17, 33, 39, 53, 57, 69, 125, 143, 149, 167, 201, 203, 209, 213, 215, 219, 225, 245, 261, 263, 267
		2	0	3, 11, 13, 19, 23, 67, 71, 73, 75, 87, 89, 97, 103, 105, 107, 129, 131, 135, 139, 147, 177, 185, 189, 193, 205, 229, 233, 235, 237, 243, 255, 257, 259, 269, 275

Proof. To the left of the segment starting with y, each part of the sequence is of the form $x\ \cdots\ 2x \mid 3x\ \cdots$ with x odd. If $x < \frac{1}{3}n$, then the missing difference and fence difference are both x and non-reducing. If $\frac{1}{3}n < x < \frac{2}{3}n$, then $3x$ is in fact $3x-n$, which is even; therefore, this possibility does not arise here. If $x > \frac{2}{3}n$, then $3x$ is $3x-2n$, which is

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odd; the missing difference is $n-x$ and the fence difference is $(2x-n)-(3x-2n) = n-x$, which is also non-reducing.

The segment starting with y yields a reducing difference of $\frac{1}{2}(n-y)$ at the fence following, as y is odd, and this equals the μ-difference. The segment starting with 1 has missing difference $\frac{1}{2}(n-1)$, as required.

On the right, each part of the sequence is of the form $\cdots\ 3z \mid 2z\ \cdots\ z$ and straightforward checking shows that the fence differences cancel out the missing differences, as on the left. □

Example 4.31 ($n = 31$). The $\mathbb{Z}_{30}$ terrace:

$$3\ \ 17\ \ 24\ \ 12\ \ 6 \mid 9\ \ 20\ \ 10\ \ 5\ \ 18 \mid 27\ \ 29\ \ 30\ \ 15\ \ 23 \mid$$
$$19\ \ 7\ \ 14\ \ 28\ \ 25 \mid 1\ \ 2\ \ 4\ \ 8\ \ 16 \mid 21\ \ 11\ \ 22\ \ 13\ \ 26.$$

Example 4.32 ($n = 73$). The $\mathbb{Z}_{72}$ terrace for $y = 43$:

$$7\ \ 40\ \cdots\ 14 \mid 21\ \ 47\ \cdots\ 42 \mid 63\ \ 68\ \cdots\ 53 \mid 43\ \ 13\ \cdots\ 58 \mid$$
$$1\ \ 2\ \cdots\ 37 \mid 49\ \ 25\ \cdots\ 61 \mid 65\ \ 57\ \cdots\ 69 \mid 46\ \ 19\ \cdots\ 23.$$

Example 4.33 ($n = 89$). The $\mathbb{Z}_{88}$ terrace for $f = 3$, $y = 81$:

$$3\ \ 46\ \cdots\ 6 \mid 9\ \ 49\ \cdots\ 18 \mid 27\ \ 58\ \cdots\ 54 \mid 81\ \ 73\ \cdots\ 85 \mid$$
$$1\ \ 2\ \cdots\ 45 \mid 30\ \ 60\ \cdots\ 15 \mid 10\ \ 20\ \cdots\ 5 \mid 33\ \ 66\ \cdots\ 61.$$

Example 4.34 ($n = 151$). The $\mathbb{Z}_{150}$ terrace for $f = 3$:

$$13\ \ 82\ \cdots\ 26 \mid 39\ \ 95\ \cdots\ 78 \mid 117\ \ 134\ \cdots\ 83 \mid$$
$$49\ \ 100\ \cdots\ 98 \mid 147\ \ 149\ \cdots\ 143 \mid 139\ \ 127\ \cdots\ 145 \mid 1\ \ 2\ \cdots\ 76 \mid$$
$$101\ \ 51\ \cdots\ 126 \mid 84\ \ 17\ \cdots\ 42 \mid 28\ \ 56\ \cdots\ 14.$$

Remark 4.35. As can be seen from the entries that Table 1 contains for $n = 281$, the $\mathbb{Z}_{280}$ terraces include three unusually elegant specimens, namely

$$3^1 \overset{2}{\leftarrow} \mid 3^2 \overset{2}{\leftarrow} \mid 3^3 \overset{2}{\rightarrow} \mid 3^0 \overset{2}{\rightarrow},$$
$$3^1 \overset{2}{\leftarrow} \mid 3^2 \overset{2}{\rightarrow} \mid 3^0 \overset{2}{\rightarrow} \mid 3^{-1} \overset{2}{\rightarrow},$$
$$3^1 \overset{2}{\rightarrow} \mid 3^0 \overset{2}{\rightarrow} \mid 3^{-1} \overset{2}{\rightarrow} \mid 3^{-2} \overset{2}{\rightarrow}.$$

A further elegant possibility, not covered by Theorem 4.30, is the $\mathbb{Z}_{280}$ terrace

$$3^0 \overset{2}{\rightarrow} \mid 3^{-1} \overset{2}{\rightarrow} \mid 3^{-2} \overset{2}{\rightarrow} \mid 3^{-3} \overset{2}{\rightarrow}.$$

Remark 4.36. The absence from Table 1 of details of any $\mathbb{Z}_{n-1}$ terrace with $n = 257$ reflects an extraordinary phenomenon. Taking $f = 1$, $g = 13$ and $y = 207$ for $n = 257$

gives a 16-segment sequence which, when interpreted modulo 256, fails to be a terrace merely because the single element $3^{-9}y$, at the start of the fifth segment, is even, namely 112, instead of odd. Likewise if we take $f = 0$, $g = 14$ and $y = 107$, we again have the value 112, now equal to $3^{-10}y$, at the start of the fifth segment, and this value is again the sole cause of failure. There is no way of overcoming this perversity of the elements of $\mathbb{Z}_{257}$. No such 'near miss' is available for $n = 127$, which can only be regarded as a 'hopeless' case.

Theorem 4.37. *Suppose that a prime n satisfies $n = 2^{m+1} - 3$, that 2 is a primitive root of n and that $r = \mathrm{ord}_n(a) = (n-1)/m$, where $a = \frac{1}{2}(n+3)$. When reinterpreted modulo $n-1$, the sequence*

$$2^0a^0 \quad 2^0a^1 \quad \cdots \quad 2^0a^{r-1} \mid 2^1a^0 \quad 2^1a^1 \quad \cdots \quad 2^1a^{r-1} \mid \quad \cdots \quad \mid \\ 2^{m-1}a^0 \quad 2^{m-1}a^1 \quad \cdots \quad 2^{m-1}a^{r-1}$$

of elements from $\mathbb{Z}_n \setminus \{0\}$ is a terrace for $\mathbb{Z}_{n-1}$. The involution in the 2-sequencing for the terrace occurs at the final fence.

Proof. We use Lemma 2.4. If $n \equiv 1 \pmod 3$, the first two missing μ^*-differences correspond to $x = \frac{1}{3}(n+2)$ and $x = \frac{2}{3}(n+2)$ and so give $\frac{1}{3}(n-1)$ twice. If $n \equiv 2 \pmod 3$, they correspond to $x = \frac{1}{3}(2n+2)$ and $x = \frac{2}{3}(2n+2) - n = \frac{1}{3}(n+4)$, and so give $\frac{1}{3}(n-2)$ twice. So two of the four copies of $\frac{1}{3}(n-\delta)$ remain. Straightforward checking shows that the missing difference in the ith segment ($i \geqslant 3$) equals the difference at the $(i-2)$th fence. As $2^{m-2}a^{r-1} = \frac{1}{3}(2^{m-1}+2n)$, the final fence difference is $q = \frac{1}{2}(n-1)$. So it follows from Lemma 2.4 that the reduced differences in the sequence comprise each element from $\mathcal{S}_{q-1}$ twice, and q once. □

Remark 4.38. The first four parameter sets covered by Theorem 4.37 are $(n,m) = (13,3), (29,4), (61,5), (4093,11)$.

Example 4.39. For $(n,m) = (13,3)$, Theorem 4.37 yields the $\mathbb{Z}_{12}$ terrace

$$1 \quad 8 \quad 12 \quad 5 \mid 2 \quad 3 \quad 11 \quad 10 \mid 4 \quad 6 \quad 9 \quad 7.$$

5. $\mathbb{Z}_{n-1}$ terraces for $n = p^r$, where $r > 1$

Theorem 5.1. *Let $n = p^2$, where p is an odd prime having 2 as a primitive root. Let c be any integer that satisfies $\frac{1}{2}p < c < p$. When reinterpreted modulo $n-1$, the sequence*

$$2^1cp \quad 2^2cp \quad \cdots \quad 2^{p-2}cp \quad cp \mid 1 \quad 2 \quad 4 \quad \cdots \quad 2^{n-p-1}$$

of elements from $\mathbb{Z}_n \setminus \{0\}$ is a terrace for $\mathbb{Z}_{n-1}$; the only reduced difference is at the fence, and the involution in the 2-sequencing falls in the middle of the second segment of the terrace.

Proof. This is straightforward. □

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Example 5.2. Taking $p = 3$ gives the $\mathbb{Z}_8$ terrace

$$3 \quad 6 \mid 1 \quad 2 \quad 4 \quad 8 \quad 7 \quad 5.$$

Example 5.3. Taking $p = 5$ gives the $\mathbb{Z}_{24}$ terraces

$$5 \quad 10 \quad 20 \quad 15 \mid 1 \quad 2 \quad 4 \quad 8 \quad \cdots \quad 19 \quad 13$$

and

$$15 \quad 5 \quad 10 \quad 20 \mid 1 \quad 2 \quad 4 \quad 8 \quad \cdots \quad 19 \quad 13.$$

Remark 5.4. In the range $2 < n < 300$, Theorem 5.1 provides $\mathbb{Z}_{n-1}$ terraces for $n = 9, 25, 121, 169$.

Theorem 5.5. *Let $n = p^2$, where p is a prime satisfying $p \equiv 7$ or $17 \pmod{24}$ and such that $\mathrm{ord}_p(2) = \frac{1}{2}(p-1)$ and $\mathrm{ord}_n(2) = \frac{1}{2}p(p-1)$. Let c be any odd number satisfying $1 \leqslant c < \frac{1}{3}p$ or $\frac{2}{3}p < c < p$. Then, when reinterpreted modulo $n-1$, the sequence*

$$cp \quad \overset{2}{\leftarrow} \quad \mid 3cp \quad \overset{2}{\rightarrow} \quad \mid 1 \quad \overset{2}{\rightarrow} \quad \mid 3^{-1} \quad \overset{2}{\rightarrow}$$

of elements from $\mathbb{Z}_n \setminus \{0\}$ is a terrace for $\mathbb{Z}_{n-1}$, with the multiples of p occurring in the first two segments. The only reduced difference is at the second fence. The involution in the 2-sequencing falls in the middle of the third segment of the terrace if $p \equiv 17 \pmod{24}$; if $p \equiv 7 \pmod{24}$ it occurs in the final segment.

Proof. As $p \equiv 7$ or $17 \pmod{24}$, the element 2 is a square in $\mathbb{Z}_p$ and 3 is a non-square. Thus, precisely one of $2c$ and $3c$ is a square in $\mathbb{Z}_p$, and hence the first two segments of the sequence include all the multiples of p in $\mathbb{Z}_p \setminus \{0\}$.

As $3 \notin \langle 2 \rangle$ in $\mathbb{Z}_p$, we have $3 \notin \langle 2 \rangle$ in $\mathbb{Z}_n$. Thus, the sequence does indeed contain every element of $\mathbb{Z}_{n-1}$ exactly once.

The following are easily checked for $\mathbb{Z}_n$:

$$\begin{aligned} 2^{-1} \cdot 3cp &= \tfrac{1}{2}(3cp + p^2), \\ 2^{-1} &= \tfrac{1}{2}(p^2 + 1), \\ 3^{-1} &= \tfrac{1}{3}(2p^2 + 1), \\ 2^{-1} \cdot 3^{-1} &= \tfrac{1}{6}(5p^2 + 1). \end{aligned}$$

If $c < p/3$, the terrace becomes

$$cp \ \cdots \ 2cp \mid 3cp \ \cdots \ \tfrac{1}{2}(3cp + p^2) \mid 1 \ \cdots \ \tfrac{1}{2}(p^2 + 1) \mid \tfrac{1}{3}(2p^2 + 1) \ \cdots \ \tfrac{1}{6}(5p^2 + 1).$$

Thus, the ‘missing’ differences in the four segments are, respectively:

(i) cp, which is the difference at the first fence,

(ii) $\frac{1}{2}(p^2 - 3cp) = p^2 - \frac{1}{2}(3cp + p^2)$, the second fence's reduced difference,

(iii) $\frac{1}{2}(p^2 - 1)$, the involution in the 2-sequencing, and

(iv) $\frac{1}{6}(p^2 - 1) = 3^{-1} - 2^{-1}$, the difference at the third fence.

If $c > \frac{2}{3}p$, the terrace becomes

$$cp\ \cdots\ 2cp - p^2 \mid 3cp - 2p^2\ \cdots\ \tfrac{1}{2}(3cp - p^2) \mid 1\ \cdots$$

and so the 'missing' differences in the first two segments are, respectively, $p^2 - cp$, which is the difference at the first fence, and $\frac{3}{2}(p^2 - cp)$, the second fences's reduced difference. □

Example 5.6. For $n = 49$, $p = 7$, we must take $c = 1$ or 5. For $c = 1$ we have the $\mathbb{Z}_{48}$ terrace

$$7\quad 28\quad 14 \mid 21\quad 42\quad 35 \mid 1\quad 2\quad \cdots\quad 25 \mid 33\quad 17\quad \cdots\quad 41.$$

Example 5.7. For $n = 289$, $p = 17$, we must take $c = 1, 3, 5, 13$ or 15. Taking $c = 1$ gives the $\mathbb{Z}_{288}$ terrace

$$17\quad 153\quad \cdots\quad 34 \mid 51\quad 102\quad \cdots\quad 170 \mid 1\quad 2\quad \cdots\quad 145 \mid 193\quad 97\quad \cdots\quad 241,$$

while $c = 3$ gives the $\mathbb{Z}_{288}$ terrace

$$51\quad 170\quad \cdots\quad 102 \mid 153\quad 17\quad \cdots\quad 221 \mid 1\quad 2\quad \cdots\quad 145 \mid 193\quad 97\quad \cdots\quad 241.$$

These examples illustrate the fact that, in general, either of the two cosets of multiples of p may appear in the first segment.

Remark 5.8.

(a) If $p \equiv 7 \pmod{24}$, say $p = 24k + 7$, we can choose $c = \frac{1}{3}(p - 4) = 8k + 1$. The second segment then starts with $3cp = -4p = -2^2p$.

(b) If $p \equiv 17 \pmod{24}$, say $p = 24k + 17$, we can choose $c = \frac{1}{3}(p - 2) = 8k + 5$, and the second segment then starts with $-2p$.

(c) In the range $2 < n < 300$, Theorem 5.5 provides $\mathbb{Z}_{n-1}$ terraces for $n = 49$ and 289.

Theorem 5.9. *Let $n = 3^r$, where $r > 2$. Write $s = r - 2$ and, for each $i = 1, 2, \ldots, s$, let $t_i = 2\cdot 3^i$ so that t_i is the order of 2 modulo 3^{i+1}. Define a sequence $\{c_i\}$ with $c_i \in 3^i\langle 2\rangle$ by*

$$c_s = 3^s \text{ or } 3^s + 2 \cdot 3^{r-1},$$
$$c_{i-1} = \tfrac{1}{3}a_i \text{ or } \tfrac{1}{3}a_i + 2 \cdot 3^{r-1}, \quad i = 2, 3, \ldots, s.$$

Then the sequence

$$2^{t_s}c_1\quad 2^{t_s-1}c_1\quad \cdots\quad 2c_1 \mid 2^{t_{s-1}}c_2\quad 2^{t_{s-1}-1}c_2\quad \cdots\quad 2c_2 \mid\ \cdots\ \mid$$
$$2^{t_1}c_s\quad 2^{t_1-1}c_s\quad \cdots\quad 2c_s \mid 3^{r-1}\quad 2\cdot 3^{r-1} \mid 2^0\quad 2^1\quad \cdots\quad 2^{-1}$$

of the elements of $\mathbb{Z}_n \setminus \{0\}$ is, when reinterpreted modulo $n - 1$, a terrace for $\mathbb{Z}_{n-1}$; the involution in the 2-sequencing falls in the middle of the final segment of the terrace. This construction yields 2^s different terraces for $\mathbb{Z}_{n-1}$.

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Table 2. *Summary of constructions for $\mathbb{Z}_{n-1}$ terraces,* $2 < n < 300$
(For non-prime values of n, the prime p is given by $n = p^i$.)

$n-1$	$\text{ord}_n(2)$	Theorem(s)
2	$n-1$	4.1
4	$n-1$	4.1
6	$\frac{1}{2}(n-1)$	4.1, 4.6
8	$p(p-1)$	5.1
10	$n-1$	4.1
12	$n-1$	4.1, 4.37
16	$\frac{1}{2}(n-1)$	4.3, 4.6
18	$n-1$	4.1, § 6
22	$\frac{1}{2}(n-1)$	4.3, 4.9
24	$p(p-1)$	5.1
26	$p^2(p-1)$	5.9
28	$n-1$	4.1, 4.37
30	$\frac{1}{6}(n-1)$	4.30
36	$n-1$	4.1
40	$\frac{1}{2}(n-1)$	4.3, 4.6
42	$\frac{1}{3}(n-1)$	4.12, 4.14, 4.17
46	$\frac{1}{2}(n-1)$	4.3, 4.9
48	$\frac{1}{2}p(p-1)$	5.5
52	$n-1$	4.1
58	$n-1$	4.1
60	$n-1$	4.1, 4.37
66	$n-1$	4.1
70	$\frac{1}{2}(n-1)$	4.3
72	$\frac{1}{8}(n-1)$	4.30
78	$\frac{1}{2}(n-1)$	4.3, 4.6
80	$p^3(p-1)$	5.9
82	$n-1$	4.1
88	$\frac{1}{8}(n-1)$	4.30
96	$\frac{1}{2}(n-1)$	4.3
100	$n-1$	4.1
102	$\frac{1}{2}(n-1)$	4.3, 4.6
106	$n-1$	4.1
108	$\frac{1}{3}(n-1)$	4.12, 4.14, 4.17
112	$\frac{1}{4}(n-1)$	4.24, 4.30
120	$p(p-1)$	5.1
124	$p^2(p-1)$	—
126	$\frac{1}{18}(n-1)$	—
130	$n-1$	4.1
136	$\frac{1}{2}(n-1)$	4.3, 4.6

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Table 2. (Cont.) *Summary of constructions for $\mathbb{Z}_{n-1}$ terraces,* $2 < n < 300$

$n-1$	$\mathrm{ord}_n(2)$	Theorem(s)
138	$n-1$	4.1
148	$n-1$	4.1
150	$\frac{1}{10}(n-1)$	4.30
156	$\frac{1}{3}(n-1)$	4.12, 4.14, 4.17
162	$n-1$	4.1
166	$\frac{1}{2}(n-1)$	4.3, 4.9
168	$p(p-1)$	5.1
172	$n-1$	4.1
178	$n-1$	4.1
180	$n-1$	4.1
190	$\frac{1}{2}(n-1)$	4.3, 4.9
192	$\frac{1}{2}(n-1)$	4.3
196	$n-1$	4.1
198	$\frac{1}{2}(n-1)$	4.3, 4.6
210	$n-1$	4.1
222	$\frac{1}{6}(n-1)$	4.30
226	$n-1$	4.1
228	$\frac{1}{3}(n-1)$	4.12, 4.14, 4.17
232	$\frac{1}{8}(n-1)$	4.30
238	$(n-1)2$	4.3, 4.9
240	$\frac{1}{10}(n-1)$	4.30
242	$p^4(p-1)$	5.9
250	$\frac{1}{5}(n-1)$	4.30
256	$\frac{1}{16}(n-1)$	—
262	$\frac{1}{2}(n-1)$	4.3, 4.9
268	$n-1$	4.1
270	$\frac{1}{2}(n-1)$	4.3, 4.6
276	$\frac{1}{3}(n-1)$	4.12, 4.14, 4.17
280	$\frac{1}{4}(n-1)$	4.20, 4.24, 4.30
282	$\frac{1}{3}(n-1)$	4.12, 4.14, 4.17
288	$p(p-1)$	5.5
292	$n-1$	4.1

Proof. The last segment of the terrace has missing difference $\frac{1}{2}(n-1)$ (even though this difference occurs in the middle of the segment). The missing difference in the penultimate segment is 3^{r-1}, which is the reduced difference at the last fence. To ensure that the ith missing difference equals the ith fence difference, $i < s$, we need c_i, $2c_i$ and c_{i+1} to be in arithmetic progression. This is trivially true when $c_i = \frac{1}{3}c_{i+1}$. If $c_i = \frac{1}{3}c_{i+1} + 2 \cdot 3^{r-1}$, we have

$$2c_i \equiv \tfrac{2}{3}c_{i+1} + 3^{r-1} \pmod{3^r}$$

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and

$$c_i - 2c_i = \tfrac{1}{3}c_{i+1} + 2 \cdot 3^{r-1} - \tfrac{2}{3}c_{i+1} - 3^{r-1} = 3^{r-1} - \tfrac{1}{3}c_{i+1} = 2c_i - c_{i+1}.$$

□

Remark 5.10. The special case where $c_{i+1} = \frac{1}{3}c_i$ for each i and $c_1 = 3^s$ yields a particularly elegant terrace:

$$3 \quad 2^{-1} \cdot 3 \quad \cdots \quad 2 \cdot 3 \mid 3^2 \quad 2^{-1} \cdot 3^2 \quad \cdots \quad 2 \cdot 3^2 \mid \quad \cdots \quad \mid$$
$$3^s \quad 2^{-1} \cdot 3^s \quad \cdots \quad 2 \cdot 3^s \mid 3^{r-1} \quad 2 \cdot 3^{r-1} \mid 2^0 \quad 2^1 \quad \cdots \quad 2^{-1}.$$

Example 5.11. For $n = 27$ we have $r = 3$ and can take $c_1 = 3$ or 21. These respectively give the $\mathbb{Z}_{26}$ terraces

$$3 \quad 15 \quad 21 \quad 24 \quad 12 \quad 6 \mid 9 \quad 18 \mid 1 \quad 2 \quad \cdots \quad 14$$

and

$$21 \quad 24 \quad 12 \quad 6 \quad 3 \quad 15 \mid 9 \quad 18 \mid 1 \quad 2 \quad \cdots \quad 14.$$

Example 5.12. For $n = 81$ we have $r = 4$ and can take $c_2 = 9$ or 63. With $c_2 = 9$ we can take $c_1 = 3$ or 57; with $c_2 = 63$ we can take $c_1 = 21$ or 75. The choice $c_1 = 75$ and $c_2 = 63$ yields the $\mathbb{Z}_{80}$ terrace

$$75 \quad 78 \quad \cdots \quad 69 \mid 63 \quad 72 \quad \cdots \quad 45 \mid 27 \quad 54 \mid 1 \quad 2 \quad \cdots \quad 41.$$

6. A narcissistic terrace for $\mathbb{Z}_{18}$

Previous sections of this paper give only general constructions and ignore examples for which no generalization is apparent. However, as a special isolated example, we now give the following terrace for $\mathbb{Z}_{18}$:

$$1 \quad 11 \quad 7 \mid 13 \quad 10 \quad 15 \mid 17 \quad 16 \quad 5 \mid 14 \quad 3 \quad 2 \mid 4 \quad 9 \quad 6 \mid 12 \quad 8 \quad 18.$$

Here, the first three segments are obtained from

$$3^0 \quad \xrightarrow{a} \quad \mid 3^{-1} \quad \xrightarrow{a} \quad \mid 3^{-2} \quad \xrightarrow{a} \quad (\text{mod}\, 19),$$

as described in Remark 2.8. As the second half of the terrace is the reverse of the negative, modulo 19, of the first half, the terrace is narcissistic. Generally, however, the methodology of the present paper does not lend itself to the construction of narcissistic terraces.

7. Overview

In § 4 we have provided $\mathbb{Z}_{n-1}$ terraces for all prime n in the range $2 < n < 300$ except for the two values $n = 127$ and 257, for which our methodology fails. Within the same range, in § 5 we have provided $\mathbb{Z}_{n-1}$ terraces for all prime-power values $n = p^r$, where p is prime and $r > 1$, except for $n = 5^3 = 125$; we have failed to find a construction that covers $n = 5^r$ for any $r > 2$.

As a quick-reference guide to our constructions, Table 2 lists the relevant theorem or theorems for each odd prime power n in the range $2 < n < 300$.

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References

1. I. Anderson and D. A. Preece, Locally balanced change-over designs, *Util. Math.* **62** (2002), 33–59.
2. I. Anderson and D. A. Preece, Power-sequence terraces for $\mathbb{Z}_n$ where n is an odd prime power, *Discr. Math.* **261** (2003), 31–58.
3. I. Anderson and D. A. Preece, Some narcissistic half-and-half power-sequence $\mathbb{Z}_n$ terraces with segments of different lengths, *Congr. Numer.* **163** (2003), 5–26.
4. I. Anderson and D. A. Preece, Narcissistic half-and-half power-sequence terraces for $\mathbb{Z}_n$ with $n = pq^t$, *Discr. Math.* **279** (2004), 33–60.
5. I. Anderson and D. A. Preece, Some power-sequence terraces for $\mathbb{Z}_{pq}$ with as few segments as possible, *Discr. Math.* **293** (2005), 29–59.
6. R. A. Bailey, Quasi-complete Latin squares: construction and randomisation, *J. R. Stat. Soc.* B **46** (1984), 323–334.
7. R. A. Bailey, M. A. Ollis and D. A. Preece, Round-dance neighbour designs from terraces, *Discr. Math.* **266** (2003), 69–86.
8. M. A. Ollis, Sequenceable groups and related topics, *Electron. J. Combinat.*, paper no. DS10 (available at www.combinatorics.org/surveys/).
9. M. A. Ollis and D. A. Preece, Sectionable terraces and the (generalised) Oberwolfach problem, *Discr. Math.* **266** (2003), 399–416.