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One-Level Density of Low-lying Zeros of Quadratic and Quartic Hecke L-functions

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Abstract. In this paper we prove some one-level density results for the low-lying zeros of families of quadratic and quartic Hecke L-functions of the Gaussian field. As corollaries, we deduce that at least 94.27% and 5%, respectively, of the members of the quadratic family and the quartic family do not vanish at the central point.

1 Introduction

Given a natural family of L-functions, the density conjecture of N. Katz and P. Sarnak [25, 26] states that the distribution of zeros near the central point of a family of L-functions is the same as that of eigenvalues near 1 of a corresponding classical compact group. An important example is the family of quadratic Dirichlet characters. Let χ be a primitive Dirichlet character and denote the non-trivial zeroes of the Dirichlet L-function $L(s,\chi)$ by $\frac{1}{2}+i\gamma_{\chi,j}$. Without assuming the generalized Riemann hypothesis (GRH), we order them as

$$\cdots \le \Re\gamma_{\chi,-2} \le \Re\gamma_{\chi,-1} < 0 \le \Re\gamma_{\chi,1} \le \Re\gamma_{\chi,2} \le \cdots. \tag{1.1}$$

For any primitive Dirichlet character χ with conductor q of size X, we set

$$\widetilde{\gamma}_{\chi,j} = \frac{\gamma_{\chi,j}}{2\pi}\log X \tag{1.2}$$

and define, for an even Schwartz class function ϕ,

$$S(\chi,\phi) = \sum_j \phi(\widetilde{\gamma}_{\chi,j}). \tag{1.3}$$

For positive, odd, and square-free integers d, the Kronecker symbol $\chi_{8d} = (\frac{8d}{\cdot})$ is primitive. Let $D(X)$ denote the set of such d satisfying $X \le d \le 2X$. A. E. Özluk and C. Snyder studied the family of quadratic Dirichlet L-functions [36]. It follows from their work that, assuming GRH for this family, we have

$$\lim_{X\to+\infty} \frac{1}{\#D(X)} \sum_{d\in D(X)} S(\chi_{8d},\phi) = \int_{\mathbb{R}} \phi(x) W_{USp}(x)\,\mathrm{d}x, \tag{1.4}$$

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where $W_{USp}(x) = 1 - \frac{\sin(2\pi x)}{2\pi x}$, provided that the support of $\widehat{\phi}$, the Fourier transform of ϕ, is contained in the interval $(-2, 2)$. The expression on the left-hand side of (1.4) is known as the one-level density of the low-lying zeros for this family of L-functions under consideration.

The kernel of the integral W_{USp} in (1.4) is the same function that occurs on the random-matrix theory side when studying the eigenvalues of unitary symplectic matrices. This shows that the family of quadratic Dirichlet L-functions is a symplectic family. M. O. Rubinstein extended the work of A. E. Özluk and C. Snyder to all n-level densities (roughly speaking, investigating n-tuples of zeros) without assuming GRH [40]. He showed that the n-level analogues of the limit in (1.4) converge to the symplectic densities for test functions $\phi(x_1, \dots, x_n)$ whose Fourier transforms $\widehat{\phi}(u_1, \dots, u_n)$ is supported in $\sum_{i=1}^n |u_i| < 1$.

Assuming the truth of GRH, Gao [11] computed the n-level densities for low-lying zeros of the family for quadratic Dirichlet L-functions when $\widehat{\phi}(u_1, \dots, u_n)$ is supported in $\sum_{i=1}^n |u_i| < 2$. That this result agrees with random matrix theory was shown by J. Levinson and S. J. Miller [28] for $n \le 7$ and by A. Entin, E. Roditty-Gershon, and Z. Rudnick [9] for all n. A. Mason and N. C. Snaith developed a new formula for the n-level densities of eigenvalues of unitary symplectic matrices [30]; they showed [31] that their result leads to a relatively straightforward way to match the result in [11] with random matrix theory.

Besides the family of quadratic Dirichlet L-functions, the density conjecture has been confirmed for many other families of L-functions, such as different types of Dirichlet L-functions [11, 21, 33, 36, 40], L-functions with characters of the ideal class group of the imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$ [10], automorphic L-functions [7, 20, 24, 38, 39], elliptic curve L-functions [1, 3, 18, 32, 42], symmetric powers of $\mathrm{GL}(2)$ L-functions [8, 16], and a family of $\mathrm{GL}(4)$ and $\mathrm{GL}(6)$ L-functions [8].

Among the many results concerning the n-level densities of low-lying zeroes for various families of L-functions, A. M. Güloğlu studied the one-level density of the low-lying zeros of a family of Hecke L-functions of $\mathbb{Q}(\omega)$ ($\omega = \exp(2\pi i/3)$) associated with cubic symbols $\chi_c = (\frac{\cdot}{c})_3$ with c square-free and congruent to 1 modulo 9, regarded as primitive ray class characters of the ray class group $h_{(c)}$ [17]. We recall here that for any c, the ray class group $h_{(c)}$ is defined to be $I_{(c)}/P_{(c)}$, where $I_{(c)} = \{\mathcal{A} \in I, (\mathcal{A}, (c)) = 1\}$ and $P_{(c)} = \{(a) \in P, a \equiv 1 \pmod{c}\}$ with I and P denoting the group of fractional ideals in $\mathbb{Q}(\omega)$ and the subgroup of principal ideals, respectively. Assuming GRH, Güloğlu [17] obtained the one-level density for this family when the Fourier transform of the test function is supported in $(\frac{-31}{30}, \frac{31}{30})$. An unconditional result was obtained for the same family with a more restricted range for the support of the Fourier transform of the test function [12].

A general approach towards establishing the n-level densities involves converting the sum over zeros of the L-functions under consideration into sums over primes using the relevant versions of the explicit formula (see Section 2.3). This leads to the estimation of certain character sums. It is this estimation that mainly affects the breadth of the support of the Fourier transform of the test function in the resulting expression. A common key ingredient is the Poisson summation essentially to convert the corresponding character sums to other character sums over dual lattices [11, 17].

In this process, the length of the character sum is shortened, which allows one to get a better estimation resulting in the enlargement of the support of the Fourier transform of the test function in the expression for n-level densities. Following a method of K. Soundararajan [41], Gao used the Poisson summation over $\mathbb{Z}$ [11]. Güloğlu [17] used a two-dimensional Poisson summation, which is similar to the Poisson summation over $\mathbb{Z}(\omega)$ developed by D. R. Heath-Brown [19, Lemma 10].

Motivated by the results in [11,17], it is our goal to further explore the application of Poisson summation in the study of one level density results for the low-lying zeros. We focus our attention on the family of quadratic and quartic Hecke L-functions in the Gaussian field $K = \mathbb{Q}(i)$.

Let χ be a primitive Hecke character; the Hecke L-function associated with χ is defined for $\Re(s) > 1$ by

$$L(s,\chi) = \sum_{0\neq \mathcal{A}\subset \mathcal{O}_K} \chi(\mathcal{A})(N(\mathcal{A}))^{-s},$$

where $\mathcal{A}$ runs over all non-zero integral ideals in K and $N(\mathcal{A})$ is the norm of $\mathcal{A}$. As shown by E. Hecke, $L(s,\chi)$ admits analytic continuation to an entire function and satisfies a functional equation. We refer the reader to [14,15,17,29] for more detailed discussion of these Hecke characters and L-functions. We denote non-trivial zeroes of $L(s,\chi)$ by $\frac{1}{2} + i\gamma_{\chi,j}$ and order them in a fashion similar to (1.1).

Set

$$C(X) = \left\{c \in \mathbb{Z}[i] : (c, 1+i) = 1, c \text{ square-free}, X \le N(c) \le 2X\right\}.$$

We shall define (Section 2.1) the primitive quadratic Kronecker symbol $\chi_{i(1+i)^5c}$ and the primitive quartic Kronecker symbol $\chi_{(1+i)^7c}$. For $\chi = \chi_{i(1+i)^5c}$ or $\chi_{(1+i)^7c}$, we set $\widetilde{\gamma}_{\chi,j}$ as in (1.2) and $S(\chi,\phi)$ as in (1.3) for an even Schwartz class function ϕ. Furthermore, let $\Phi_X(t)$ be a non-negative smooth function supported on $(1,2)$, satisfying $\Phi_X(t) = 1$ for $t \in (1+\frac{1}{U}, 2-\frac{1}{U})$ with $U = \log\log X$ and such that $\Phi_X^{(j)}(t) \ll_j U^j$ for all integers $j \ge 0$. Our results are as follows.

Theorem 1.1 *Suppose that GRH is true. Let $\phi(x)$ be an even Schwartz function whose Fourier transform $\widehat{\phi}(u)$ has compact support in $(-2,2)$. Then*

$$\lim_{X\to+\infty} \frac{1}{\#C(X)} \sideset{}{^*}\sum_{(c,1+i)=1} S(\chi_{i(1+i)^5c},\phi)\Phi_X\Big(\frac{N(c)}{X}\Big) = \int_{\mathbb{R}} \phi(x)\Big(1-\frac{\sin(2\pi x)}{2\pi x}\Big)\,dx. \tag{1.5}$$

Here the "$*$" on the sum over c means that the sum is restricted to square-free elements c of $\mathbb{Z}[i]$.

S. Chowla conjectured that $L(\frac{1}{2},\chi) \neq 0$ for any primitive Dirichlet character χ [5]. One expects that the same statement should hold for Hecke characters as well. Using Theorem 1.1, we can deduce the following non-vanishing result.

Corollary 1.2 *Suppose that the GRH is true and that $1/2$ is a zero of $L(s,\chi_{i(1+i)^5c})$ of order $m_c \ge 0$. As $X \to \infty$,*

$$\sideset{}{^*}\sum_{(c,1+i)=1} m_c \Phi_X\Big(\frac{N(c)}{X}\Big) \le \Big(\frac{\cot\frac{1}{4}-3}{8} + o(1)\Big)\#C(X).$$

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Moreover, as $X \to \infty$

$$\#\{c \in C(X) : L(1/2, \chi_{i(1+i)^5c}) \neq 0\} \geq \Big(\frac{19 - \cot\frac{1}{4}}{16} + o(1)\Big)\#C(X).$$

Proof The proof goes along the same line as that of [4, Corollary 2.1]. We thus omit the details here. ∎

For a family of quartic characters, we have the following.

Theorem 1.3 *Suppose that GRH is true. Let $\phi(x)$ be an even Schwartz function whose Fourier transform $\widehat{\phi}(u)$ has compact support in $(\frac{-20}{19}, \frac{20}{19})$. Then*

$$\lim_{X\to+\infty}\frac{1}{\#C(X)}\sideset{}{^*}\sum_{(c,1+i)=1} S(\chi_{(1+i)^7c}, \phi)\Phi_X\Big(\frac{N(c)}{X}\Big) = \int_{\mathbb{R}}\phi(x)\,\mathrm{d}x. \tag{1.6}$$

Here the "$*$" on the sum over c means that the sum is restricted to square-free elements c of $\mathbb{Z}[i]$.

Similar to Corollary 1.2, we have a non-vanishing result for the family of quartic Hecke L-functions under our consideration.

Corollary 1.4 *Suppose that the GRH is true and that $1/2$ is a zero of $L(s, \chi_{(1+i)^7c})$ of order $n_c \geq 0$. As $X \to \infty$,*

$$\sideset{}{^*}\sum_{(c,1+i)=1} n_c\Phi_X\Big(\frac{N(c)}{X}\Big) \leq \Big(\frac{19}{20} + o(1)\Big)\#C(X). \tag{1.7}$$

Moreover, as $X \to \infty$

$$\#\{c \in C(X) : L(1/2, \chi_{(1+i)^7c}) \neq 0\} \geq \Big(\frac{1}{20} + o(1)\Big)\#C(X). \tag{1.8}$$

Proof Consider $\phi_0(x) = \big(\frac{\sin \pi x}{\pi x}\big)^2$. It is well known that

$$\int_{\mathbb{R}}\phi_0(x)\,\mathrm{d}x = 1 \quad \text{and} \quad \widehat{\phi_0}(x) = \max\{|x|, 0\}.$$

Take $0 \leq \theta < \frac{20}{19}$; then $\phi(x) = \phi_0(\theta x)$ satisfies the requirements of Theorem 1.3. The truth of GRH implies that all the non-trivial zeros of $L(s, \chi_{(1+i)^7c})$ are of the form $\frac{1}{2} + i\gamma$ with $\gamma \in \mathbb{R}$. So $n_c \leq S(\chi_{(1+i)^7c}, \phi)$ for every c. Now using (1.6), we get that as $X \to \infty$

$$\begin{aligned}\frac{1}{\#C(X)}\sideset{}{^*}\sum_{(c,1+i)=1} n_c\Phi_X\Big(\frac{N(c)}{X}\Big) &\leq \frac{1}{\#C(X)}\sideset{}{^*}\sum_{(c,1+i)=1} S(\chi_{(1+i)^7c}, \phi)\Phi_X\Big(\frac{N(c)}{X}\Big)\\ &= \int_{\mathbb{R}}\phi(x)\,\mathrm{d}x + o(1).\end{aligned}$$

By taking θ arbitrarily close to $\frac{20}{19}$, the last integral above is $\frac{19}{20} + o(1)$ and (1.7) follows from this.

To prove (1.8), we start with

$$\#C(X) = \#\{c \in C(X) : L(1/2, \chi_{(1+i)^7c}) \neq 0\} + \#\{c \in C(X) : L(1/2, \chi_{(1+i)^7c}) = 0\}.$$

As $X \to \infty$,

$$\#\{c \in C(X) : L(1/2, \chi_{(1+i)^7 c}) = 0\} \le \sum_{(c,1+i)=1}^{*} \Big(n_c \Phi_X\Big(\frac{N(c)}{X}\Big) + o(1)\Big) \le \Big(\frac{19}{20} + o(1)\Big) \#C(X),$$

using (1.7). Now (1.8) follows easily from the above. ∎

Note that it follows from (1.4) that Theorem 1.1 shows that the family of quadratic Hecke L-functions is a symplectic family. We also note that, as

$$\int_{\mathbb{R}} f(x)\,dx = \int_{\mathbb{R}} f(x) W_U(x)\,dx,$$

with $W_U(x) = 1$, Theorem 1.3 gives that the family of quartic Hecke L-functions is a unitary family. This was already observed in [12], where an unconditional result was obtained for the family of quartic Hecke L-functions with smaller support of the Fourier transform of the test function.

1.1 Notations

The following notations and conventions are used throughout the paper.

- $\Phi(t)$ for $\Phi_X(t)$.
- $e(z) = \exp(2\pi i z) = e^{2\pi i z}$.
- $f = O(g)$ or $f \ll g$ means $|f| \le cg$ for some unspecified positive constant c.
- $f = o(g)$ means $\lim_{x\to\infty} f(x)/g(x) = 0$.
- $\mu_{[i]}$ denotes the Möbius function on $\mathbb{Z}[i]$.
- The letter ϖ is reserved for primes in $\mathbb{Z}[i]$.
- $\zeta_{\mathbb{Q}(i)}(s)$ denotes the Dedekind zeta function of $\mathbb{Q}(i)$.
- $\chi_{[-1,1]}$ denotes the characteristic function of $[-1,1]$.

2 Preliminaries

2.1 Quadratic and Quartic Characters and Kronecker Symbols

The symbol $(\frac{\cdot}{n})_4$ is the quartic residue symbol in the ring $\mathbb{Z}[i]$. For a prime $\varpi \in \mathbb{Z}[i]$ with $N(\varpi) \ne 2$, the quartic character is defined for $a \in \mathbb{Z}[i]$, $(a,\varpi) = 1$ by $(\frac{a}{\varpi})_4 \equiv a^{(N(\varpi)-1)/4} \pmod{\varpi}$, with $(\frac{a}{\varpi})_4 \in \{\pm 1, \pm i\}$. When $\varpi | a$, we define $(\frac{a}{\varpi})_4 = 0$. Then the quartic character can be extended to any composite n with $(N(n), 2) = 1$ multiplicatively. We extend the definition of $(\frac{\cdot}{n})_4$ to $n = 1$ by setting $(\frac{\cdot}{1})_4 = 1$. We further define $(\frac{\cdot}{n}) = (\frac{\cdot}{n})_4^2$ to be the quadratic residue symbol for these n.

Note that in $\mathbb{Z}[i]$, every ideal co-prime to 2 has a unique generator congruent to 1 modulo $(1+i)^3$. Such a generator is called primary. Recall [27, Theorem 6.9] that the quartic reciprocity law states that for two primary integers $m, n \in \mathbb{Z}[i]$,

$$\Big(\frac{m}{n}\Big)_4 = \Big(\frac{n}{m}\Big)_4 (-1)^{((N(n)-1)/4)((N(m)-1)/4)}.$$

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As a consequence, the following quadratic reciprocity law holds for two primary integers $m, n \in \mathbb{Z}[i]$:

$$\left(\frac{m}{n}\right) = \left(\frac{n}{m}\right). \tag{2.1}$$

Observe that a non-unit $n = a+bi$ in $\mathbb{Z}[i]$ is congruent to 1 mod $(1+i)^3$ if and only if $a \equiv 1 \pmod 4, b \equiv 0 \pmod 4$ or $a \equiv 3 \pmod 4, b \equiv 2 \pmod 4$ [23, Lemma 6, p. 121].

From the supplement theorem to the quartic reciprocity law (see [2, Lemma 8.2.1, Theorem 8.2.4]), we have, for $n = a + bi$ being primary,

$$\left(\frac{i}{n}\right)_4 = i^{(1-a)/2} \quad \text{and} \quad \left(\frac{1+i}{n}\right)_4 = i^{(a-b-1-b^2)/4}. \tag{2.2}$$

We now define a character of order 2 modulo $(1+i)^5$. In fact, for any element $c \in \mathbb{Z}[i]$, $(c, 1+i) = 1$, we can define a Dirichlet character $\chi_{i(1+i)^5c} \pmod{(1+i)^5c}$ by noting that the ring $(\mathbb{Z}[i]/(1+i)^5c\mathbb{Z}[i])^*$ is isomorphic to the direct product of the group of units $U = \langle i \rangle$ and the group $N_{(1+i)^5c}$ formed by elements in $(\mathbb{Z}[i]/(1+i)^5 c\mathbb{Z}[i])^*$ congruent to 1 $\pmod{(1+i)^3}$, *i.e.*, primary. Under this isomorphism, any element $n \in (\mathbb{Z}[i]/(1+i)^5c\mathbb{Z}[i])^*$ can be written uniquely as $n = u_n \cdot n_0$ with $u_n \in U$, $n_0 \in N_{(1+i)^5c}$. We can now define $\chi_{i(1+i)^5c} \pmod{(1+i)^5c}$ such that for any $n \in (\mathbb{Z}[i]/(1+i)^5c\mathbb{Z}[i])^*$,

$$\chi_{i(1+i)^5c}(n) = \left(\frac{i(1+i)^5c}{n_0}\right).$$

One deduces from (2.2) and the quadratic reciprocity that $\chi_{i(1+i)^5c}(n) = 1$ when $n_0 \equiv 1 \pmod{(1+i)^5c}$. It follows from this that $\chi_{i(1+i)^5c}(n)$ is well defined, *i.e.*, $\chi_{i(1+i)^5c}(n) = \chi_{i(1+i)^5c}(n')$ when $n \equiv n' \pmod{(1+i)^5c}$. As $\chi_{i(1+i)^5c}(n)$ is clearly multiplicative, of order 2, and trivial on units, it can be regarded as a primitive Hecke character $(\text{mod}(1+i)^5c)$ of trivial infinite type. Let $\chi_{i(1+i)^5c}$ stand for this Hecke character as well; we call it the Kronecker symbol. Furthermore, when c is square-free, $\chi_{i(1+i)^5c}$ is non-principal and primitive. To see this, we write $c = u_c \cdot \varpi_1 \cdots \varpi_k$ with $u_c \in U$ and ϖ_j being primary primes. Suppose $\chi_{i(1+i)^5c}$ is induced by some χ modulo c' with $\varpi_j \nmid c'$, then by the Chinese Remainder Theorem, there exists an n such that $n \equiv 1 \pmod{(1+i)^5c/\varpi_j}$ and $\left(\frac{n}{\varpi_j}\right) \neq 1$. It follows that $\chi(n) = 1$ but $\chi_{i(1+i)^5c}(n) \neq 1$, a contradiction. Thus, $\chi_{i(1+i)^5c}$ can possibly be induced only by some χ modulo $(1+i)^4c$. By the Chinese Remainder Theorem, there exists an n such that $n \equiv 1 \pmod c$ and $n \equiv 5 \pmod{(1+i)^5}$. As this n satisfies $n \equiv 1 \pmod{(1+i)^4}$, it follows that $n \equiv 1 \pmod{(1+i)^4c}$, hence $\chi(n) = 1$, but $\chi_{i(1+i)^5c}(n) = -1 \neq 1$ (note that $\left(\frac{i}{n}\right) = 1$ when $n \equiv 5 \pmod{(1+i)^5}$, so we may assume that c is primary) and this implies that $\chi_{i(1+i)^5c}$ is primitive. This also shows that $\chi_{i(1+i)^5c}$ is non-principal.

Similarly, for any element $c \in \mathbb{Z}[i]$, $(c, 1+i) = 1$, we define the Kronecker symbol $\chi_{(1+i)^7c}$ as a character of order 4 modulo $(1+i)^7c$ such that for any $n \in (\mathbb{Z}[i]/(1+i)^7 c\mathbb{Z}[i])^*$, with $n = u_n \cdot n_0$, $u_n \in U$ and n_0 being primary,

$$\chi_{(1+i)^7c}(n) = \left(\frac{(1+i)^7c}{n_0}\right)_4.$$

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2.2 The Gauss Sums

For any $n \in \mathbb{Z}[i]$, $n \equiv 1 \pmod{(1+i)^3}$, the quadratic and quartic Gauss sums $g_2(n)$, $g_4(n)$ are defined by

$$g_2(n) = \sum_{x \bmod n} \left(\frac{x}{n}\right) \widetilde{e}\left(\frac{x}{n}\right) \quad \text{and} \quad g_4(n) = \sum_{x \bmod n} \left(\frac{x}{n}\right)_4 \widetilde{e}\left(\frac{x}{n}\right),$$

where $\widetilde{e}(z) = \exp(2\pi i(\frac{z}{2i} - \frac{\bar{z}}{2i}))$. Note that $g_2(1) = g_4(1) = 1$ by definition.

More generally, for any $n \in \mathbb{Z}[i]$, $n \equiv 1 \pmod{(1+i)^3}$, we set

$$g_2(r,n) = \sum_{x \bmod n} \left(\frac{x}{n}\right) \widetilde{e}\left(\frac{rx}{n}\right) \quad \text{and} \quad g_4(r,n) = \sum_{x \bmod n} \left(\frac{x}{n}\right)_4 \widetilde{e}\left(\frac{rx}{n}\right).$$

The following properties of $g_4(r,n)$ can be found in [6].

Lemma 2.1 *We have*

(2.3) $$g_4(rs,n) = \overline{\left(\frac{s}{n}\right)_4} g_4(r,n), \quad (s,n) = 1,$$

(2.4) $$g_4(r, n_1 n_2) = \left(\frac{n_2}{n_1}\right)_4 \left(\frac{n_1}{n_2}\right)_4 g_4(r,n_1) g_4(r,n_2), \quad (n_1,n_2) = 1,$$

$$g_4(\varpi^k, \varpi^l) = \begin{cases} N(\varpi)^k g_4(\varpi) & \text{if } l = k+1, k \equiv 0 \pmod 4, \\ N(\varpi)^k g_2(\varpi) & \text{if } l = k+1, k \equiv 1 \pmod 4, \\ N(\varpi)^k (\frac{-1}{\varpi})_4 \overline{g_4}(\varpi) & \text{if } l = k+1, k \equiv 2 \pmod 4, \\ -N(\varpi)^k, & \text{if } l = k+1, k \equiv 3 \pmod 4, \\ \varphi(\varpi^l) = \#(\mathbb{Z}[i]/(\varpi^l))^* & \text{if } k \geq l, l \equiv 0 \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, the next lemma allows us to evaluate $g_2(r,n)$ for $n \equiv 1 \pmod{(1+i)^3}$ explicitly.

Lemma 2.2 (i) *We have*

(2.5) $$g_2(rs,n) = \overline{\left(\frac{s}{n}\right)} g_2(r,n), \quad (s,n) = 1,$$

$$g_2(k, mn) = g_2(k,m) g_2(k,n), \quad m, n \text{ primary and } (m,n) = 1.$$

(ii) *Let ϖ be a primary prime in $\mathbb{Z}[i]$. Suppose ϖ^h is the largest power of ϖ dividing k. (If $k = 0$, then set $h = \infty$.) Then for $l \geq 1$,*

$$g_2(k, \varpi^l) = \begin{cases} 0 & \text{if } l \leq h \text{ is odd,} \\ \varphi(\varpi^l) = \#(\mathbb{Z}[i]/(\varpi^l))^* & \text{if } l \leq h \text{ is even,} \\ -N(\varpi)^{l-1} & \text{if } l = h+1 \text{ is even,} \\ \left(\frac{ik\varpi^{-h}}{\varpi}\right) N(\varpi)^{l-1/2} & \text{if } l = h+1 \text{ is odd,} \\ 0 & \text{if } l \geq h+2. \end{cases}$$

Proof (i) Using the quadratic reciprocity (2.1), the proof is similar to those of (2.3) and (2.4).

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(ii) The case $l \le h$ is easily verified. If $l > h$, then

$$\sum_{a \bmod \varpi^l} \Big(\frac{a}{\varpi^l}\Big)\widetilde{e}\Big(\frac{ka}{\varpi^l}\Big) = \sum_{b \bmod \varpi} \Big(\frac{b}{\varpi^l}\Big) \sum_{c \bmod \varpi^{l-1}} \widetilde{e}\Big(\frac{k(c\varpi + b)}{\varpi^l}\Big). \tag{2.6}$$

We write the inner sum above as

$$\widetilde{e}\Big(\frac{kb}{\varpi^l}\Big) \sum_{c \bmod \varpi^{l-1}} \widetilde{e}\Big(\frac{k\varpi^{-h}c}{\varpi^{l-h-1}}\Big) = \widetilde{e}\Big(\frac{kb}{\varpi^l}\Big) \sum_{c \bmod \varpi^{l-1}} \widetilde{e}\Big(\frac{c}{\varpi^{l-h-1}}\Big).$$

When $l \ge h+2$, we write $c = c_1\varpi^{l-h-1} + c_2$, where c_1 varies over a set of representatives in $\mathbb{Z}[i]$ (mod ϖ^h), and c_2 varies over a set of representatives in $\mathbb{Z}[i]$ (mod ϖ^{l-h-1}) to see that

$$\sum_{c \bmod \varpi^{l-1}} \widetilde{e}\Big(\frac{c}{\varpi^{l-h-1}}\Big) = N(\varpi^h) \sum_{c_2 \bmod \varpi^{l-h-1}} \widetilde{e}\Big(\frac{c_2}{\varpi^{l-h-1}}\Big).$$

Now we can find a c_3 such that $\widetilde{e}(c_3/\varpi^{l-h-1}) \ne 1$ (for example, take $c_3 = 1$, when ϖ is not rational and $c_3 = i$, when ϖ is rational) to deduce that

$$\begin{aligned}\widetilde{e}\Big(\frac{c_3}{\varpi^{l-h-1}}\Big) \sum_{c_2 \bmod \varpi^{l-h-1}} \widetilde{e}\Big(\frac{c_2}{\varpi^{l-h-1}}\Big) &= \sum_{c_2 \bmod \varpi^{l-h-1}} \widetilde{e}\Big(\frac{c_2+c_3}{\varpi^{l-h-1}}\Big)\\ &= \sum_{c_2 \bmod \varpi^{l-h-1}} \widetilde{e}\Big(\frac{c_2}{\varpi^{l-h-1}}\Big).\end{aligned}$$

This implies that

$$\sum_{c_2 \bmod \varpi^{l-h-1}} \widetilde{e}\Big(\frac{c_2}{\varpi^{l-h-1}}\Big) = 0. \tag{2.7}$$

This proves the last case when $l \ge h+2$.

When $l = h+1$, the right-hand side of (2.6) is

$$N(\varpi)^{l-1} \sum_{b \bmod \varpi} \Big(\frac{b}{\varpi^l}\Big)\widetilde{e}\Big(\frac{kb}{\varpi^l}\Big).$$

If l is even then the last sum above is -1 (using (2.7)), and if l is odd the last sum above is

$$\sum_{b \bmod \varpi} \Big(\frac{b}{\varpi}\Big)\widetilde{e}\Big(\frac{b(k\varpi^{-h})}{\varpi}\Big) = \Big(\frac{k\varpi^{-h}}{\varpi}\Big) g_2(\varpi) = \Big(\frac{ik\varpi^{-h}}{\varpi}\Big) N(\varpi)^{1/2},$$

where the expression of $g_2(\varpi)$ follows from [35, Proposition 2.2]. (Note that in [35] the definition of the Gauss sum is different from the one here). This completes the proof of the lemma. ∎

2.3 The Explicit Formula

Our approach in this paper relies on the following explicit formula, which essentially converts a sum over zeros of an L-function to a sum over primes.

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Lemma 2.3 *Let $\phi(x)$ be an even Schwartz function whose Fourier transform $\widehat{\phi}(u)$ has compact support. Let $c \in \mathbb{Z}[i]$ be square-free satisfying $(c, 1+i) = 1$, $X \le N(c) \le 2X$ and let $\chi = \chi_{i(1+i)^5c}$ or $\chi_{(1+i)^7c}$. We have*

$$S(\chi, \phi) = \int_{-\infty}^{\infty} \phi(t)\,\mathrm{d}t - \sum_{j=1}^{2}(S_j(\chi, X; \widehat{\phi}) + S_j(\overline{\chi}, X; \widehat{\phi})) + O\Big(\frac{1}{\log X}\Big),$$

where

$$S_j(\chi, X; \widehat{\phi}) = \frac{1}{\log X} \sum_{\varpi \equiv 1 \bmod (1+i)^3} \frac{\log N(\varpi)}{\sqrt{N(\varpi^j)}} \chi(\varpi^j) \widehat{\phi}\Big(\frac{\log N(\varpi^j)}{\log X}\Big)$$

with the sum over ϖ running over primes in $\mathbb{Z}[i]$.

Proof The proof is rather standard and goes along the same line as [17, Lemma 4.1]. ∎

We write $S(\chi, X; \widehat{\phi})$ for $S_1(\chi, X; \widehat{\phi})$ in the rest of the paper and we deduce the following from Lemma 2.3.

Lemma 2.4 *Let $\phi(x)$ be an even Schwartz function whose Fourier transform $\widehat{\phi}(u)$ is compactly supported. For any square-free $c \in \mathbb{Z}[i]$, $(c, 1+i) = 1$, $X \le N(c) \le 2X$, we have*

$$\begin{aligned} S(\chi_{i(1+i)^5c}, \phi) = &\int_{-\infty}^{\infty} \phi(t)\,\mathrm{d}t - \frac{1}{2}\int_{-\infty}^{\infty} \widehat{\phi}(u)\,\mathrm{d}u \\ &- 2S(\chi_{i(1+i)^5c}, X; \widehat{\phi}) + O\Big(\frac{\log\log 3X}{\log X}\Big), \end{aligned}$$

with the implicit constant depending on ϕ.

Proof Note first that

$$\sum_{\varpi | i(1+i)^5c} \frac{\log N(\varpi)}{N(\varpi)} \ll \log\log 3X.$$

It follows that

$$\begin{aligned} &-S_2(\chi_{i(1+i)^5c}, X; \widehat{\phi}) - S_2(\overline{\chi}_{i(1+i)^5c}, X; \widehat{\phi}) \\ &\qquad = -\frac{2}{\log X} \sum_{\varpi \equiv 1 \bmod (1+i)^3} \frac{\log N(\varpi)}{N(\varpi)} \widehat{\phi}\Big(\frac{2\log N(\varpi)}{\log X}\Big) + O\Big(\frac{\log\log 3X}{\log X}\Big). \end{aligned}$$

The prime ideal theorem [34, Theorem 8.9], together with partial summation, gives that for $x \ge 1$,

(2.8) $$\sum_{\substack{N(\varpi) \le x \\ \varpi \equiv 1 \bmod (1+i)^3}} \frac{\log N(\varpi)}{N(\varpi)} = \log x + O(\log\log 3x).$$

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From this and partial summation, we see that

$$\begin{aligned}
-S_2(\chi_{i(1+i)^5c}, X; \widehat{\phi}) - S_2(\overline{\chi}_{i(1+i)^5c}, X; \widehat{\phi}) \\
&= -\frac{2}{\log X}\int_1^\infty \widehat{\phi}\Big(\frac{2\log t}{\log X}\Big)\frac{dt}{t} + O\Big(\frac{\log\log 3X}{\log X}\Big) \\
&= -\frac{1}{2}\int_{-\infty}^{\infty} \widehat{\phi}(t)\,dt + O\Big(\frac{\log\log 3X}{\log X}\Big).
\end{aligned}$$

The assertion of the lemma follows from this and Lemma 2.3. ∎

To estimate the terms $S_2(\chi, X; \widehat{\phi})$ and $S_2(\overline{\chi}, X; \widehat{\phi})$ in Lemma 2.3 for $\chi = \chi_{(1+i)^7c}$, we need the following lemma.

Lemma 2.5 *Suppose that GRH is true. For any non-principal Hecke character χ of trivial infinite type with modulus n, we have, for $x \geq 1$,*

$$(2.9)\qquad S(x,\chi) = \sum_{\substack{N(\varpi)\leq x \\ \varpi\equiv 1 \bmod (1+i)^3}} \chi(\varpi)\log N(\varpi) \ll \min\left\{x, \sqrt{x}\log^3 x \log N(n)\right\}.$$

Proof The proof of this lemma is rather standard and we only give a sketch here. The details can be found in the proof of [17, Lemma 2.19]. First, it follows from the prime ideal theorem [34, Theorem 8.9] and partial summation that $(x,\chi) \ll x$. Now using Perron's formula with $a = 1 + 1/\log x$, we have

$$(2.10)\qquad -\frac{1}{2\pi i}\int_{a-i\sqrt{x}}^{a+i\sqrt{x}} \frac{L'(s,\chi)}{L(s,\chi)}\frac{x^s}{s}\,ds - S(x,\chi) \ll \sqrt{x}\log x + \sqrt{x}\sum_{\mathfrak{a}} \frac{\Lambda(\mathfrak{a})}{N(\mathfrak{a})^a|\log(x/N(\mathfrak{a}))|},$$

where $\Lambda(\mathfrak{a})$ is the analogue of the von Mangoldt function in $\mathbb{Z}[i]$. Breaking up the sum over $\mathfrak{a}$ into the ranges

$$N(\mathfrak{a}) \leq x/2,\quad x/2 < N(\mathfrak{a}) \leq x,\quad x < N(\mathfrak{a}) \leq 3x/2,\quad 3x/2 < N(\mathfrak{a}),$$

and using different lower bounds for $|\log(x/N(\mathfrak{a}))|$ for each of these ranges, we get that

$$\sum_{\mathfrak{a}} \frac{\Lambda(\mathfrak{a})}{N(\mathfrak{a})^a|\log(x/N(\mathfrak{a}))|} \ll \log^2 x.$$

Now, assuming the truth of GRH, we can move the line of integration in (2.10) to $1/2 + 1/\log x$ without picking up any residue. Applying standard bounds for $L'(s,\chi)/L(s,\chi)$ (see [17, Lemma 2.18]) implies that the integral in (2.10) is

$$\ll \sqrt{x}\log^3 x \log N(n),$$

which gives the second term inside the minimum in (2.9) and completes the proof. ∎

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Applying Lemma 2.5, we see that term corresponding to the second sum in the expression of $S(\chi,\phi)$ in Lemma 2.3 for $\chi=\chi_{(1+i)^7c}$ (note that in this case χ is non-principal) contributes

$$\begin{aligned}&\frac{1}{\log X}\int_1^\infty \frac{1}{t}\phi\Big(\frac{2\log t}{\log X}\Big)\,\mathrm{d}S(t,\chi)\\&\quad\ll \frac{1}{\log X}\int_1^\infty S(t,\chi)\frac{1}{t^2}\phi\Big(\frac{2\log t}{\log X}\Big)\,\mathrm{d}t+\frac{1}{\log^2 X}\int_1^\infty S(t,\chi)\frac{1}{t^2}\phi'\Big(\frac{2\log t}{\log X}\Big)\,\mathrm{d}t\\&\quad\ll \frac{1}{\log X}\int_1^{\log^3 N(c)}\frac{1}{t}\,\mathrm{d}t+\frac{\log N(c)}{\log X}\int_{\log^3 N(c)}^\infty t^{-3/2+\varepsilon}\,\mathrm{d}t\ll\frac{\log\log N(c)}{\log X}.\end{aligned}$$

We then arrive at the following.

Lemma 2.6 *Let $\phi(x)$ be an even Schwartz function whose Fourier transform $\widehat{\phi}(u)$ has compact support. Let $\chi=\chi_{(1+i)^7c}$ for any square-free $c\in\mathbb{Z}[i]$, $(c,1+i)=1$, $X\le N(c)\le 2X$. We have*

$$S(\chi,\phi)=\int_{-\infty}^\infty \phi(t)\,\mathrm{d}t-S(\chi_{i(1+i)^7c},X;\widehat{\phi})-S(\overline{\chi}_{(1+i)^7c},X;\widehat{\phi})+O\Big(\frac{\log\log 3X}{\log X}\Big),$$

with the implicit constant depending on ϕ.

2.4 Poisson Summation

The proofs of Theorems 1.1 and 1.3 require the following Poisson summation formula.

Lemma 2.7 *Let $n\in\mathbb{Z}[i]$, $n\equiv 1 \pmod{(1+i)^3}$, and let χ be a quadratic or quartic character $(\bmod\, n)$. For any Schwartz class function W, we have, for all $a>0$,*

$$\sum_{m\in\mathbb{Z}[i]}\chi(m)W\Big(\frac{aN(m)}{X}\Big)=\frac{X}{aN(n)}\sum_{k\in\mathbb{Z}[i]}g(k,n)\widetilde{W}\Bigg(\sqrt{\frac{N(k)X}{aN(n)}}\Bigg),$$

where

$$\widetilde{W}(t)=\int_{-\infty}^\infty\int_{-\infty}^\infty W(N(x+yi))\widetilde{e}(-t(x+yi))\,\mathrm{d}x\,\mathrm{d}y,\qquad t\ge 0$$

and

$$g(k,n)=\sum_{r \bmod n}\chi(r)\widetilde{e}\Big(\frac{kr}{n}\Big).$$

Proof We first recall the following Poisson summation formula for $\mathbb{Z}[i]$ (see the proof of [13, Lemma 4.1]), which is itself an easy consequence of the classical Poisson summation formula in two dimensions:

$$\sum_{j\in\mathbb{Z}[i]}f(j)=\sum_{k\in\mathbb{Z}[i]}\int_{-\infty}^\infty\int_{-\infty}^\infty f(x+yi)\widetilde{e}(-k(x+yi))\,\mathrm{d}x\,\mathrm{d}y.$$

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We get

$$\sum_{m\in\mathbb{Z}[i]}\chi(m)W\Big(\frac{aN(m)}{X}\Big)=\sum_{r \bmod n}\chi(r)\sum_{j\in\mathbb{Z}[i]}W\Big(\frac{aN(r+jn)}{X}\Big)$$
$$=\sum_{r \bmod n}\chi(r)\sum_{k\in\mathbb{Z}[i]}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W\Big(\frac{aN(r+(x+yi)n)}{X}\Big)\widetilde{e}(-k(x+yi))\,\mathrm{d}x\,\mathrm{d}y.$$

We make a change of variables in the integral, writing

$$\sqrt{N\Big(\frac{n}{k}\Big)}\frac{k}{n}\frac{(r+(x+yi)n)}{\sqrt{X/a}}=u+vi,$$

with $u,v\in\mathbb{R}$. (If $k=0$, we omit the factors involving k/n.) With the Jacobian of this transformation being $aN(n)/X$, we find that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W\Big(\frac{aN(r+(x+yi)n)}{X}\Big)\widetilde{e}(-k(x+yi))\,\mathrm{d}x\,\mathrm{d}y$$
$$=\frac{X}{aN(n)}\widetilde{e}\Big(\frac{kr}{n}\Big)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W(N(u+vi))\widetilde{e}\Big(-(u+vi)\sqrt{N\Big(\frac{k}{n}\Big)\frac{X}{a}}\Big)\,\mathrm{d}u\,\mathrm{d}v,$$

whence

$$\sum_{m\in\mathbb{Z}[i]}W\Big(\frac{aN(m)}{X}\Big)\chi(m)=\frac{X}{aN(n)}\sum_{k\in\mathbb{Z}[i]}\widetilde{W}\Big(\sqrt{\frac{N(k)X}{aN(n)}}\Big)\sum_{r \bmod n}\chi(r)\widetilde{e}\Big(\frac{kr}{n}\Big).$$

As the inner sum of the last expression above is $g(k,n)$ by definition, this completes the proof of the lemma. ∎

From Lemma 2.7, we readily deduce the following.

Corollary 2.8 *Let $n\in\mathbb{Z}[i]$, $n\equiv 1 \pmod{(1+i)^3}$ and χ be a quadratic or quartic character $\pmod{n}$. For any Schwartz class function W, we have*

$$\sum_{\substack{m\in\mathbb{Z}[i]\\(m,1+i)=1}}\chi(m)W\Big(\frac{N(m)}{X}\Big)=\frac{X}{2N(n)}\chi(1+i)\sum_{k\in\mathbb{Z}[i]}(-1)^{N(k)}g(k,n)\widetilde{W}\Big(\sqrt{\frac{N(k)X}{2N(n)}}\Big).$$

Proof It follows from Lemma 2.7 that

$$\sum_{\substack{m\in\mathbb{Z}[i]\\(m,1+i)=1}}\chi(m)W\Big(\frac{N(m)}{X}\Big)\tag{2.11}$$
$$=\sum_{m}\chi(m)W\Big(\frac{N(m)}{X}\Big)-\chi(1+i)\sum_{m}\chi(m)W\Big(\frac{2N(m)}{X}\Big)$$
$$=\frac{X}{N(n)}\sum_{k\in\mathbb{Z}[i]}g(k,n)\widetilde{W}\Big(\sqrt{\frac{N(k)X}{N(n)}}\Big)$$
$$-\chi(1+i)\frac{X}{2N(n)}\sum_{k\in\mathbb{Z}[i]}g(k,n)\widetilde{W}\Big(\sqrt{\frac{N(k)X}{2N(n)}}\Big).$$

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Using the relation (see (2.3) and (2.5))

$$g((1+i)k,n) = \overline{\chi}(1+i)g(k,n),$$

we can rewrite the first sum in the last expression of (2.11) as

$$\begin{aligned}\sum_{k\in\mathbb{Z}[i]} g(k,n)\widetilde{W}\Big(\sqrt{\frac{N((1+i)k)X}{2N(n)}}\Big) &= \chi(1+i)\sum_{k\in\mathbb{Z}[i]} g((1+i)k,n)\widetilde{W}\Big(\sqrt{\frac{N((1+i)k)X}{2N(n)}}\Big)\\ &= \chi(1+i)\sum_{\substack{k\in\mathbb{Z}[i]\\ 1+i|k}} g(k,n)\widetilde{W}\Big(\sqrt{\frac{N(k)X}{2N(n)}}\Big).\end{aligned}$$

Substituting this back to the last expression in (2.11), we get the desired result. ■

Suppose that $W(t)$ is a non-negative smooth function supported on $(1,2)$, satisfying $W(t) = 1$ for $t \in (1+1/U, 2-1/U)$ and $W^{(j)}(t) \ll_j U^j$ for all integers $j \geq 0$. Using integration by parts and our assumptions on W, one shows that

$$\widetilde{W}^{(\mu)}(t) \ll_j U^{j-1}|t|^{-j}, \tag{2.12}$$

for all integers $\mu \geq 0$, $j \geq 1$, and all real t.

On the other hand, evaluating $\widetilde{W}(t)$ with polar coordinate gives

$$\begin{aligned}\widetilde{W}(t) &= \int_{\mathbb{R}^2} \cos(2\pi t y) W(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_0^\infty \int_0^{2\pi} \cos(2\pi t r\sin\theta) W(r^2) r\,\mathrm{d}r\,\mathrm{d}\theta\\ &= \int_1^{\sqrt{2}} \int_0^{2\pi} \cos(2\pi t r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta + O\Big(\frac{1}{U}\Big).\end{aligned}$$

In particular, we have

$$\widetilde{W}(0) = \pi + O\big(\tfrac{1}{U}\big). \tag{2.13}$$

Similarly, for any $j \geq 0$, we have

$$\widetilde{W}^{(j)}(t) \ll 1. \tag{2.14}$$

As $\Phi(t)$ satisfies the assumptions on $W(t)$, the estimations (2.12)–(2.14) are also valid for $\widetilde{\Phi}(t)$. So in the sequel, we shall use these estimations for $\widetilde{\Phi}(t)$ without further justification.

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3 Proof of Theorem 1.1

3.1 Evaluation of $C(X)$

We have

$$\sum^*_{\substack{N(c)\le X\\(c,1+i)=1}} 1=\sum_{\substack{N(c)\le X\\(c,1+i)=1}}\mu^2_{[i]}(c)=\sum_{\substack{N(c)\le X\\(c,1+i)=1}}\sum_{\substack{d^2|c\\d\equiv 1 \bmod (1+i)^3}}\mu_{[i]}(d)$$
$$=\sum_{\substack{N(d)\le\sqrt{X}\\d\equiv 1 \bmod (1+i)^3}}\mu_{[i]}(d)\sum_{\substack{N(c)\le X/N(d^2)\\(c,1+i)=1}}1, \tag{3.1}$$

where the "$*$" on the sum over c means that the sum is restricted to square-free elements c of $\mathbb{Z}[i]$.

The best-known result [22] for the Gauss circle problem gives that

$$\sum_{N(a)\le x}1=\pi x+O(x^\theta)$$

with $\theta=131/146$. This implies that

$$\sum_{\substack{N(c)\le X/N(d^2)\\(c,1+i)=1}}1=\frac{\pi X}{2N(d^2)}+O\Big(\Big(\frac{X}{N(d^2)}\Big)^\theta\Big).$$

Inserting the above into (3.1), we get

$$\sum^*_{\substack{N(c)\le X\\(c,1+i)=1}}1=\pi X\sum_{\substack{N(d)\le\sqrt{X}\\d\equiv 1 \bmod (1+i)^3}}\frac{\mu_{[i]}(d)}{2N(d^2)}+O(X^\theta)=\frac{2\pi X}{3\zeta_{\mathbb{Q}(i)}(2)}+O(X^\theta).$$

We conclude from this that as $X\to\infty$,

$$\sum^*_{(c,1+i)=1}\Phi\Big(\frac{N(c)}{X}\Big)\sim\#C(X)\sim\frac{2\pi X}{3\zeta_{\mathbb{Q}(i)}(2)}.$$

It follows from this and Lemma 2.4 that the left-hand side of (1.5) equals

$$\int_{-\infty}^{\infty}\phi(t)\,\mathrm{d}t-\frac{1}{2}\int_{-\infty}^{\infty}\widehat{\phi}(u)\,\mathrm{d}u-2\lim_{X\to\infty}\frac{S(X,Y;\widehat{\phi},\Phi)}{\#C(X)\log X}, \tag{3.2}$$

where

$$S(X,Y;\widehat{\phi},\Phi)=\sum^*_{(c,1+i)=1}\sum_{\substack{N(\varpi)\le Y\\\varpi\equiv 1 \bmod (1+i)^3}}\frac{\chi_{i(1+i)^5c}(\varpi)\log N(\varpi)}{\sqrt{N(\varpi)}}\widehat{\phi}\Big(\frac{\log N(\varpi)}{\log X}\Big)\Phi\Big(\frac{N(c)}{X}\Big).$$

Here $\widehat{\phi}(u)$ is smooth and has its support contained in the interval $(-2+\varepsilon,2-\varepsilon)$ for some $0<\varepsilon<1$. To emphasize this condition, we shall set $Y=X^{2-\varepsilon}$ and write the condition $N(\varpi)\le Y$ explicitly throughout this section.

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On the other hand, observe that when $\widehat{\phi}$ is supported in $(-2,2)$, we have

$$\int_{-\infty}^{\infty}\phi(t)\,\mathrm{d}t-\frac{1}{2}\int_{-\infty}^{\infty}\widehat{\phi}(u)\,\mathrm{d}u+\frac{1}{2}\int_{-\infty}^{\infty}(1-\chi_{[-1,1]}(u))\widehat{\phi}(u)\,\mathrm{d}u = \int_{-\infty}^{\infty}\phi(t)\Big(1-\frac{\sin(2\pi t)}{2\pi t}\Big)\,\mathrm{d}t. \tag{3.3}$$

Comparing (3.2) with (3.3), we see that, in order to establish Theorem 1.1, it suffices to show that, for any Schwartz function ϕ with $\widehat{\phi}$ supported in $(-2+\varepsilon,2-\varepsilon)$, for any $0<\varepsilon<1$,

$$\lim_{X\to\infty}\frac{S(X,Y;\widehat{\phi},\Phi)}{X\log X}=-\frac{\pi}{6\zeta_{\mathbb{Q}(i)}(2)}\int_{-\infty}^{\infty}(1-\chi_{[-1,1]}(t))\widehat{\phi}(t)\,\mathrm{d}t. \tag{3.4}$$

3.2 Expressions $S_M(X,Y;\widehat{\phi},\Phi)$ and $S_R(X,Y;\widehat{\phi},\Phi)$

Let $Z>1$ be a real parameter to be chosen later and write $\mu^2_{[i]}(c)=M_Z(c)+R_Z(c)$, where

$$M_Z(c)=\sum_{\substack{l^2|c\\ N(l)\le Z}}\mu_{[i]}(l)\quad\text{and}\quad R_Z(c)=\sum_{\substack{l^2|c\\ N(l)>Z}}\mu_{[i]}(l).$$

Define

$$S_M(X,Y;\widehat{\phi},\Phi) = \sum_{(c,1+i)=1}M_Z(c)\sum_{\substack{\varpi\equiv1 \bmod (1+i)^3\\ N(\varpi)\le Y}}\frac{\log N(\varpi)}{\sqrt{N(\varpi)}}\Big(\frac{i(1+i)c}{\varpi}\Big)\widehat{\phi}\Big(\frac{\log N(\varpi)}{\log X}\Big)\Phi\Big(\frac{N(c)}{X}\Big),$$

and

$$S_R(X,Y;\widehat{\phi},\Phi) = \sum_{(c,1+i)=1}R_Z(c)\sum_{\substack{\varpi\equiv1\,(\bmod\,(1+i)^3)\\ N(\varpi)\le Y}}\frac{\log N(\varpi)}{\sqrt{N(\varpi)}}\Big(\frac{i(1+i)c}{\varpi}\Big)\widehat{\phi}\Big(\frac{\log N(\varpi)}{\log X}\Big)\Phi\Big(\frac{N(c)}{X}\Big),$$

so that $S(X,Y;\widehat{\phi},\Phi)=S_M(X,Y;\widehat{\phi},\Phi)+S_R(X,Y;\widehat{\phi},\Phi)$.

Using standard techniques (see (3.7) below), we can show that, by choosing Z appropriately, $S_R(X,Y;\widehat{\phi},\Phi)$ is small. We now give another expression for $S_M(X,Y;\widehat{\phi},\Phi)$ using Poisson summation. We write it as

$$S_M(X,Y;\widehat{\phi},\Phi)=\sum_{\varpi\equiv1 \bmod (1+i)^3}\frac{\log N(\varpi)}{\sqrt{N(\varpi)}}\Big(\frac{i(1+i)}{\varpi}\Big)\widehat{\phi}\Big(\frac{\log N(\varpi)}{\log X}\Big) \times\sum_{\substack{N(l)\le Z\\ l\equiv1 \bmod (1+i)^3}}\mu_{[i]}(l)\Big(\frac{l^2}{\varpi}\Big)\sum_{\substack{c\in\mathbb{Z}[i]\\ (c,1+i)=1}}\Big(\frac{c}{\varpi}\Big)\Phi\Big(\frac{N(cl^2)}{X}\Big).$$

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Applying Corollary 2.8, we obtain that

$$\begin{aligned}
&\sum_{\substack{c\in\mathbb{Z}[i]\\(c,1+i)=1}} \Big(\frac{c}{\varpi}\Big)\Phi\Big(\frac{N(cl^2)}{X}\Big)\\
&\quad= \frac{X}{2N(l^2\varpi)}\Big(\frac{1+i}{\varpi}\Big)\sum_{k\in\mathbb{Z}[i]}(-1)^{N(k)}g_2(k,\varpi)\widetilde{\Phi}\Bigg(\sqrt{\frac{N(k)X}{2N(l^2\varpi)}}\Bigg)\\
&\quad= \frac{X}{2N(l^2)\sqrt{N(\varpi)}}\Big(\frac{1+i}{\varpi}\Big)\sum_{k\in\mathbb{Z}[i]}(-1)^{N(k)}\Big(\frac{ik}{\varpi}\Big)\widetilde{\Phi}\Bigg(\sqrt{\frac{N(k)X}{2N(l^2\varpi)}}\Bigg),
\end{aligned}$$

where the last equality above follows from Lemma 2.2 by noting that

$$g_2(k,\varpi) = \Big(\frac{ik}{\varpi}\Big)N(\varpi)^{1/2}.$$

We can now recast $S_M(X,Y;\widehat{\phi},\Phi)$ as

$$\begin{aligned}
&S_M(X,Y;\widehat{\phi},\Phi)\\
&=\frac{X}{2}\sum_{\substack{N(l)\le Z\\ l\equiv 1 \bmod (1+i)^3}}\frac{\mu_{[i]}(l)}{N(l^2)}\sum_{k\in\mathbb{Z}[i]}(-1)^{N(k)}\\
&\qquad\times\sum_{\varpi\equiv 1 \bmod (1+i)^3}\frac{\log N(\varpi)}{N(\varpi)}\Big(\frac{kl^2}{\varpi}\Big)\widehat{\phi}\Big(\frac{\log N(\varpi)}{\log X}\Big)\widetilde{\Phi}\Bigg(\sqrt{\frac{N(k)X}{2N(l^2\varpi)}}\Bigg).
\end{aligned}\tag{3.5}$$

The remainder of this section is devoted to the evaluations of $S_R(X,Y;\widehat{\phi},\Phi)$ and $S_M(X,Y;\widehat{\phi},\Phi)$.

3.3 Estimation of $S_R(X,Y;\widehat{\phi},\Phi)$

We first seek a bound for

$$E(Y;\chi_{i(1+i)^5cl^2},\widehat{\phi}) = \sum_{\substack{N(\varpi)\le Y\\ \varpi\equiv 1 \bmod (1+i)^3}}\frac{\chi_{i(1+i)^5cl^2}(\varpi)\log N(\varpi)}{\sqrt{N(\varpi)}}\widehat{\phi}\Big(\frac{\log N(\varpi)}{\log X}\Big),$$

with $Y\le X^{2-2\varepsilon}$, $(cl,1+i)=1$, $X\le N(cl^2)\le 2X$.

Note that $\chi_{i(1+i)^5cl^2}$ is non-principal. It follows from Lemma 2.5 and partial summation that

$$\begin{aligned}
E(Y;\chi_{i(1+i)^5cl^2},\widehat{\phi}) &= \int_1^Y \frac{1}{\sqrt{u}}\widehat{\phi}\Big(\frac{\log u}{\log X}\Big)\,dO(u^{1/2}\log^3(u)\log N(cl^2))\\
&\ll \log^5(X).
\end{aligned}\tag{3.6}$$

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We then deduce that

$$S_R(X,Y;\widehat{\phi},\Phi)=\sum_{\substack{N(l)>Z\\ l\equiv 1 \bmod (1+i)^3}}\mu_{[i]}(l)\sum_{\substack{c\in\mathbb{Z}[i]\\ (c,1+i)=1}}E(Y;\chi_{i(1+i)^5cl^2},\widehat{\phi})\Phi\Big(\frac{N(cl^2)}{X}\Big)\tag{3.7}$$

$$\ll\sum_{N(l)>Z}\sum_{X/N(l)^2\leq N(c)\leq 2X/N(l)^2}\log^5(X)\ll\frac{X\log^5 X}{Z}.$$

3.4 Estimation of $S_M(X,Y;\widehat{\phi},\Phi)$, the Second Main Term

It follows from (3.5) that the sum in $S_M(X,Y;\widehat{\phi},\Phi)$ corresponding to $k=0$ is zero. We show in what follows that terms $k=\square$ (k is a square), $k\neq 0$ in $S_M(X,Y;\widehat{\phi},\Phi)$ contribute a second main term. Before we proceed, we need the following result.

Lemma 3.1 *For $y>0$,*

$$\sum_{\substack{k\in\mathbb{Z}[i]\\ k\neq 0}}(-1)^{N(k)}\widetilde{\Phi}\Big(\frac{N(k)}{y}\Big)=-\widetilde{\Phi}(0)+O\Big(\frac{U^2}{y^{1/2}}\Big).$$

Proof Note that

$$\sum_{\substack{k\in\mathbb{Z}[i]\\ k\neq 0}}(-1)^{N(k)}\widetilde{\Phi}\Big(\frac{N(k)}{y}\Big)=\sum_{k\in\mathbb{Z}[i]}(-1)^{N(k)}\widetilde{\Phi}\Big(\frac{N(k)}{y}\Big)-\widetilde{\Phi}(0)$$

and

$$\sum_{k\in\mathbb{Z}[i]}(-1)^{N(k)}\widetilde{\Phi}\Big(\frac{N(k)}{y}\Big)=2\sum_{k\in\mathbb{Z}[i]}\widetilde{\Phi}\Big(\frac{2N(k)}{y}\Big)-\sum_{k\in\mathbb{Z}[i]}\widetilde{\Phi}\Big(\frac{N(k)}{y}\Big).$$

By taking $n=1$ in Lemma 2.7, we immediately obtain that

$$2\sum_{k\in\mathbb{Z}[i]}\widetilde{\Phi}\Big(\frac{2N(k)}{y}\Big)=y\sum_{j\in\mathbb{Z}[i]}\check{\Phi}\Big(\sqrt{\frac{N(j)y}{2}}\Big)$$
$$\sum_{k\in\mathbb{Z}[i]}\widetilde{\Phi}\Big(\frac{N(k)}{y}\Big)=y\sum_{j\in\mathbb{Z}[i]}\check{\Phi}\big(\sqrt{N(j)y}\big),$$

where

$$\check{\Phi}(t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\widetilde{\Phi}(N(u+vi))\widetilde{e}(-t(u+vi))\,\mathrm{d}u\,\mathrm{d}v,\qquad t\geq 0.$$

It follows that

$$\sum_{k\in\mathbb{Z}[i]}(-1)^{N(k)}\widetilde{\Phi}\Big(\frac{N(k)}{y}\Big)=y\sum_{\substack{j\in\mathbb{Z}[i]\\ (j,1+i)=1}}\check{\Phi}\Big(\sqrt{\frac{N(j)y}{2}}\Big).\tag{3.8}$$

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We have, when $t>0$, via integration by parts (and noting (2.12))

$$\begin{aligned}\check{\Phi}(t) &= \int_{\mathbb{R}^2}\cos(2\pi t y)\widetilde{\Phi}(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y = -\frac{1}{\pi t}\int_{\mathbb{R}^2}\sin(2\pi t y)\widetilde{\Phi}'(x^2+y^2)y\,\mathrm{d}x\,\mathrm{d}y\\ &= -\frac{1}{2(\pi t)^2}\int_{\mathbb{R}^2}\cos(2\pi t y)\big(\widetilde{\Phi}'(x^2+y^2)+\widetilde{\Phi}''(x^2+y^2)2y^2\big)\,\mathrm{d}x\,\mathrm{d}y\\ &= \frac{1}{4(\pi t)^3}\int_{\mathbb{R}^2}\sin(2\pi t y)\big(\widetilde{\Phi}''(x^2+y^2)6y+\widetilde{\Phi}'''(x^2+y^2)4y^3\big)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

We can evaluate the last integral above with polar coordinates. Using (2.14) for $\widetilde{\Phi}''$, $\widetilde{\Phi}'''$ if $r\le 1$ and using (2.12) with $j=3$ for $\widetilde{\Phi}''$, $\widetilde{\Phi}'''$ if $r>1$, we get

$$\check{\Phi}(t)\ll \frac{U^2}{t^3}.$$

The lemma follows immediately from this bound and (3.8). ∎

By a change of variables $k\mapsto k^2$ and noting that $k^2=k_1^2$ if and only if $k=\pm k_1$, we see that the terms $k=\square$, $k\ne 0$ in $S_M(X,Y;\widehat{\phi},\Phi)$ contribute

$$\begin{aligned}S_{M,\square}(X,Y;\widehat{\phi},\Phi) &= \frac{X}{4}\sum_{\substack{N(l)\le Z\\ l\equiv 1 \bmod (1+i)^3}}\frac{\mu_{[i]}(l)}{N(l^2)}\sum_{\substack{(\varpi,l)=1\\ \varpi\equiv 1 \bmod (1+i)^3}}\frac{\log N(\varpi)}{N(\varpi)}\widehat{\phi}\Big(\frac{\log N(\varpi)}{\log X}\Big)\\ &\qquad\times\sum_{\substack{k\in\mathbb{Z}[i],k\ne 0\\ (k,\varpi)=1}}(-1)^{N(k)}\widetilde{\Phi}\left(N(k)\sqrt{\frac{X}{2N(l^2\varpi)}}\right)\\ &= S_\square - S'_\square,\end{aligned}$$

where

$$\begin{aligned}S_\square &= \frac{X}{4}\sum_{\substack{N(l)\le Z\\ l\equiv 1 \bmod (1+i)^3}}\frac{\mu_{[i]}(l)}{N(l^2)}\sum_{\substack{(\varpi,l)=1\\ \varpi\equiv 1 \bmod (1+i)^3}}\frac{\log N(\varpi)}{N(\varpi)}\widehat{\phi}\Big(\frac{\log N(\varpi)}{\log X}\Big)\\ &\qquad\times\sum_{\substack{k\in\mathbb{Z}[i]\\ k\ne 0}}(-1)^{N(k)}\widetilde{\Phi}\left(N(k)\sqrt{\frac{X}{2N(l^2\varpi)}}\right)\end{aligned}$$

and

$$\begin{aligned}S'_\square &= \frac{X}{4}\sum_{\substack{N(l)\le Z\\ l\equiv 1 \bmod (1+i)^3}}\frac{\mu_{[i]}(l)}{N(l^2)}\sum_{\substack{(\varpi,l)=1\\ \varpi\equiv 1 \bmod (1+i)^3}}\frac{\log N(\varpi)}{N(\varpi)}\widehat{\phi}\Big(\frac{\log N(\varpi)}{\log X}\Big)\\ &\qquad\times\sum_{\substack{k\in\mathbb{Z}[i]\\ k\ne 0}}(-1)^{N(k)}\widetilde{\Phi}\left(N(k)\sqrt{\frac{XN(\varpi)}{2N(l^2)}}\right).\end{aligned}$$

To estimate $S'_\square$, (2.12) with $j=2$ gives

$$\sum_{\substack{k\in\mathbb{Z}[i]\\ k\ne 0}}(-1)^{N(k)}\widetilde{\Phi}\left(N(k)\sqrt{\frac{XN(\varpi)}{2N(l^2)}}\right)\ll\sum_{\substack{k\in\mathbb{Z}[i]\\ N(k)\ge 1}}\frac{UN(l^2)}{N^2(k)XN(\varpi)}\ll\frac{UN(l^2)}{XN(\varpi)}.$$

From this we deduce that $S'_{\square} \ll UZ$.

Now we further rewrite $S_{\square} = S_{\square,1} + S_{\square,2}$, where

$$S_{\square,1} = \frac{X}{4} \sum_{\substack{N(l)\le Z \\ l\equiv 1 \bmod (1+i)^3}} \frac{\mu_{[i]}(l)}{N(l^2)} \sum_{\substack{\varpi\equiv 1 \bmod (1+i)^3 \\ (\varpi,l)=1 \\ N(\varpi)\ge X/N(l^2)}} \frac{\log N(\varpi)}{N(\varpi)} \widehat{\phi}\Big(\frac{\log N(\varpi)}{\log X}\Big) \times \sum_{\substack{k\in\mathbb{Z}[i] \\ k\ne 0}} (-1)^{N(k)} \widetilde{\Phi}\left(N(k)\sqrt{\frac{X}{2N(l^2\varpi)}}\right),$$

and

$$S_{\square,2} = \frac{X}{4} \sum_{\substack{N(l)\le Z \\ l\equiv 1 \bmod (1+i)^3}} \frac{\mu_{[i]}(l)}{N(l^2)} \sum_{\substack{\varpi\equiv 1 \bmod (1+i)^3 \\ (\varpi,l)=1 \\ N(\varpi)< X/N(l^2)}} \frac{\log N(\varpi)}{N(\varpi)} \widehat{\phi}\Big(\frac{\log N(\varpi)}{\log X}\Big) \times \sum_{\substack{k\in\mathbb{Z}[i] \\ k\ne 0}} (-1)^{N(k)} \widetilde{\Phi}\left(N(k)\sqrt{\frac{X}{2N(l^2\varpi)}}\right).$$

To estimate $S_{\square,2}$, we use (2.12) with $j = 2$ to get that

$$\sum_{\substack{k\in\mathbb{Z}[i] \\ k\ne 0}} (-1)^{N(k)} \widetilde{\Phi}\left(N(k)\sqrt{\frac{X}{2N(l^2\varpi)}}\right) \ll \sum_{\substack{k\in\mathbb{Z}[i] \\ k\ne 0}} U\frac{N(l^2\varpi)}{N^2(k)X} \ll U\frac{N(l^2\varpi)}{X}.$$

From this and (2.8), we obtain $S_{\square,2} \ll XU \log\log X$.

We now apply Lemma 3.1 to see that $S_{\square,1} = S_{\square,M} + S_{\square,R}$, where

$$S_{\square,M} = -\widetilde{\Phi}(0)\frac{X}{4} \sum_{\substack{N(l)\le Z \\ l\equiv 1 \bmod (1+i)^3}} \frac{\mu_{[i]}(l)}{N(l^2)} \sum_{\substack{\varpi\equiv 1 \bmod (1+i)^3 \\ (\varpi,l)=1 \\ N(\varpi)\ge X/N(l^2)}} \frac{\log N(\varpi)}{N(\varpi)} \widehat{\phi}\Big(\frac{\log N(\varpi)}{\log X}\Big),$$

and

$$S_{\square,R} \ll X^{1+1/4}U^2 \sum_{\substack{N(l)\le Z \\ l\equiv 1 \bmod (1+i)^3}} \frac{1}{N(l^2)^{1+1/4}} \sum_{\substack{\varpi\equiv 1 \bmod (1+i)^3 \\ N(\varpi)\ge X/N(l^2)}} \frac{\log N(\varpi)}{N(\varpi)^{1+1/4}}.$$

Using (2.8) again, we arrive at $S_{\square,R} \ll XU^2 \log\log X$.

Let $\omega(l)$ denote the number of distinct primes in $\mathbb{Z}[i]$ dividing l. It is well known that for $N(l) \ge 3$,

$$\omega(l) \ll \frac{\log N(l)}{\log\log N(l)}.$$

It follows that

$$S_{\square,M} = -\widetilde{\Phi}(0)\frac{X}{4} \sum_{\substack{N(l)\le Z \\ l\equiv 1 \bmod (1+i)^3}} \frac{\mu_{[i]}(l)}{N(l^2)} \sum_{\substack{\varpi\equiv 1 \bmod (1+i)^3 \\ N(\varpi)\ge X/N(l^2)}} \frac{\log N(\varpi)}{N(\varpi)} \widehat{\phi}\Big(\frac{\log N(\varpi)}{\log X}\Big) + O(X).$$

Utilizing (2.8) again, we get that

$$\begin{aligned}
S_{\square,M} &= -\widetilde{\Phi}(0)\frac{X}{4}\sum_{\substack{N(l)\le Z\\ l\equiv 1 \bmod (1+i)^3}}\frac{\mu_{[i]}(l)}{N(l^2)}\int_{X/N(l^2)}^{\infty}\widehat{\phi}\Big(\frac{\log u}{\log X}\Big)\,\mathrm{d}\log u\\
&\quad + O(X\log\log X)\\
&= -\widetilde{\Phi}(0)\frac{X\log X}{4}\sum_{\substack{N(l)\le Z\\ l\equiv 1 \bmod (1+i)^3}}\frac{\mu_{[i]}(l)}{N(l^2)}\int_{1-\log N(l^2)/\log X}^{\infty}\widehat{\phi}(t)\,\mathrm{d}t\\
&\quad + O(X\log\log X)\\
&= -\widetilde{\Phi}(0)\frac{X\log X}{4}\sum_{\substack{N(l)\le Z\\ l\equiv 1 \bmod (1+i)^3}}\frac{\mu_{[i]}(l)}{N(l^2)}\int_{1}^{\infty}\widehat{\phi}(t)\,\mathrm{d}t + O(X\log\log X)\\
&= -\widetilde{\Phi}(0)\frac{X\log X}{3\zeta_{\mathbb{Q}(i)}(2)}\int_{1}^{\infty}\widehat{\phi}(t)\,\mathrm{d}t + O\Big(\frac{X\log X}{Z} + X\log\log X\Big)\\
&= -\widetilde{\Phi}(0)\frac{X\log X}{6\zeta_{\mathbb{Q}(i)}(2)}\int_{-\infty}^{\infty}(1-\chi_{[-1,1]}(t))\widehat{\phi}(t)\,\mathrm{d}t\\
&\quad + O\Big(\frac{X\log X}{Z} + X\log\log X\Big)\\
&= -\frac{\pi X\log X}{6\zeta_{\mathbb{Q}(i)}(2)}\int_{-\infty}^{\infty}(1-\chi_{[-1,1]}(t))\widehat{\phi}(t)\,\mathrm{d}t\\
&\quad + O\Big(\frac{X\log X}{Z} + \frac{X\log X}{U} + X\log\log X\Big),
\end{aligned}$$

where the last equality follows from (2.13).

Gathering the estimations for $S'_{\square}$, $S_{\square,2}$, $S_{\square,M}$, and $S_{\square,R}$, we obtain that

$$\begin{aligned}
S_{M,\square}(X,Y;\widehat{\phi},\Phi) &= -\frac{\pi X\log X}{6\zeta_{\mathbb{Q}(i)}(2)}\int_{-\infty}^{\infty}\big(1-\chi_{[-1,1]}(t)\big)\,\widehat{\phi}(t)\,\mathrm{d}t\\
&\quad + O\Big(\frac{X\log X}{Z} + \frac{X\log X}{U} + XU^2\log\log X + UZ\Big).
\end{aligned}$$

3.5 Estimation of $S_M(X,Y;\widehat{\phi},\Phi)$, the Remainder

Now the sums in $S_M(X,Y;\widehat{\phi},\Phi)$ corresponding to the contribution of $k\neq 0,\square$ can be written as $XR/2$, where R is

$$\sum_{\substack{N(l)\le Z\\ l\equiv 1 \bmod (1+i)^3}}\frac{\mu_{[i]}(l)}{N(l^2)}\sum_{\substack{k\in\mathbb{Z}[i]\\ k\neq 0,\square}}(-1)^{N(k)}\sum_{\varpi\equiv 1 \bmod (1+i)^3}\times\frac{\log N(\varpi)}{N(\varpi)}\Big(\frac{kl^2}{\varpi}\Big)\widehat{\phi}\Big(\frac{\log N(\varpi)}{\log X}\Big)\widetilde{\Phi}\Bigg(\sqrt{\frac{N(k)X}{2N(l^2\varpi)}}\Bigg).$$

We define χ_{kl^2} to be $\left(\frac{kl^2}{\cdot}\right)$. Similar to our discussions in Section 2.1, when k is not a square, χ_{kl^2} can be regarded as a non-principle Hecke character modulo $(1+i)^5 kl^2$ of trivial infinite type. Analogous to (3.6), one has the bound $E(Y;\chi_{kl^2},\widehat{\phi}) \ll \log^4(X(N(kl^2)+2))$.

We then deduce, by partial summation, that

$$\begin{aligned}
\sum_{\varpi\equiv 1 \bmod (1+i)^3} & \frac{\log N(\varpi)}{N(\varpi)}\left(\frac{kl^2}{\varpi}\right)\widehat{\phi}\left(\frac{\log N(\varpi)}{\log X}\right)\widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2\varpi)}}\right) \\
&= \int_1^Y \frac{1}{\sqrt{V}}\widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2)V}}\right)\mathrm{d}E(V;\chi_{kl^2},\widehat{\phi}) \\
&\ll \log^4(X(N(kl^2)+2))\Bigg(\frac{1}{\sqrt{Y}}\left|\widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2)Y}}\right)\right| \\
&\qquad + \int_1^Y \frac{1}{V^{3/2}}\left|\widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2)V}}\right)\right|\mathrm{d}V 1V^2\left|\widetilde{\Phi}'\left(\sqrt{\frac{N(k)X}{2N(l^2)V}}\right)\right|\mathrm{d}V\Bigg).
\end{aligned}$$

This gives rise to

$$R \ll \sum_{\substack{N(l)\le Z \\ l\equiv 1 \bmod (1+i)^3}} \frac{1}{N(l^2)}(R_1+R_2+R_3),$$

where

$$\begin{aligned}
R_1 &= \frac{1}{\sqrt{Y}}\sum_{\substack{k\in\mathbb{Z}[i] \\ k\neq 0}} \log^4(X(N(kl^2)+2))\left|\widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2)Y}}\right)\right|, \\
R_2 &= \int_1^Y \frac{1}{V^{3/2}}\sum_{\substack{k\in\mathbb{Z}[i] \\ k\neq 0}} \log^4(X(N(kl^2)+2))\left|\widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2)V}}\right)\right|\mathrm{d}V, \\
R_3 &= \int_1^Y \sqrt{\frac{X}{N(l^2)}}\frac{1}{V^2}\sum_{\substack{k\in\mathbb{Z}[i] \\ k\neq 0}} \log^4(X(N(kl^2)+2))\sqrt{N(k)}\left|\widetilde{\Phi}'\left(\sqrt{\frac{N(k)X}{2N(l^2)V}}\right)\right|\mathrm{d}V.
\end{aligned}$$

Similar to the estimations done in [12, §3.3], we use (2.14) when $VN(l^2)/X \ge 1$, $N(k) \le VN(l^2)/X$ and (2.12) when $VN(l^2)/X \le 1$ or $VN(l^2)/X \ge 1$, $N(k) \ge YN(l^2)/X$ with $j = 3$ or 4. These estimates give

$$R \ll \frac{\log^4 XZ\sqrt{Y}U^3}{X}.$$

Thus we conclude that the contribution of $k \neq 0, \square$ is

$$\ll \log^4 XZ\sqrt{Y}U^3. \tag{3.9}$$

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3.6 Conclusion

We now combine the bounds (3.7) and (3.9), and take $Y = X^{2-2\varepsilon}$, $Z = \log^5 X$ (recall that $U = \log\log X$) with any fixed $\varepsilon > 0$ to obtain

$$\begin{aligned} S(X,Y;\widehat{\phi},\Phi) &= -\frac{\pi X\log X}{6\zeta_{\mathbb{Q}(i)}(2)}\int_{-\infty}^{\infty}\big(1-\chi_{[-1,1]}(t)\big)\,\widehat{\phi}(t)\,\mathrm{d}t \\ &\quad + O\Big(\frac{X\log X}{Z}+\frac{X\log X}{U}+XU^2\log\log X \\ &\quad + UZ+\frac{X\log^5 X}{Z}+\log^4 XZ\sqrt{Y}U^3\Big) \\ &= -\frac{\pi X\log X}{6\zeta_{\mathbb{Q}(i)}(2)}\int_{-\infty}^{\infty}(1-\chi_{[-1,1]}(t))\widehat{\phi}(t)\,\mathrm{d}t + o\,(X\log X)\,, \end{aligned}$$

which implies (3.4); this completes the proof of Theorem 1.1.

4 Proof of Theorem 1.3

The proof of Theorem 1.3 is similar to that of Theorem 1.1. We define

$$S(X,Y;\widehat{\phi},\Phi),\quad S_M(X,Y;\widehat{\phi},\Phi),\quad S_R(X,Y;\widehat{\phi},\Phi)$$

in this section similar to those defined in Section 3, with the necessary modifications on characters, as we replace $\chi_{i(1+i)^5c}$ by $\chi_{(1+i)^7c}$ in the definition of $S(X,Y;\widehat{\phi},\Phi)$ and we replace $\big(\frac{i(1+i)c}{\varpi}\big)$ by $\big(\frac{(1+i)^3c}{\varpi}\big)_4$ in the definitions of $S_M(X,Y;\widehat{\phi},\Phi)$ and $S_R(X,Y;\widehat{\phi},\Phi)$ here. We assume that $\widehat{\phi}(u)$ is smooth and has its support contained in the interval $(-20/19+\varepsilon, 20/19-\varepsilon)$ for some $0<\varepsilon<1$. We also set $Y = X^{20/19-\varepsilon}$ throughout this section. Similar to the proof of Theorem 1.1, we see that in order to establish Theorem 1.3, it suffices to show that

$$\lim_{X\to\infty}\frac{S(X,Y;\widehat{\phi},\Phi)}{X\log X} = 0.$$

The relation $S(X,Y;\widehat{\phi},\Phi) = S_M(X,Y;\widehat{\phi},\Phi)+S_R(X,Y;\widehat{\phi},\Phi)$ and the estimation (3.7) for $S_R(X,Y;\widehat{\phi},\Phi)$ is still valid. To estimate $S_M(X,Y;\widehat{\phi},\Phi)$, we recast it as

$$\begin{aligned} S_M(X,Y;\widehat{\phi},\Phi) &= \sum_{\varpi\equiv 1 \bmod (1+i)^3}\frac{\log N(\varpi)}{\sqrt{N(\varpi)}}\Big(\frac{(1+i)^3}{\varpi}\Big)_4\widehat{\phi}\Big(\frac{\log N(\varpi)}{\log X}\Big) \\ &\quad\times\sum_{\substack{N(l)\le Z\\ l\equiv 1 \bmod (1+i)^3}}\mu_{[i]}(l)\Big(\frac{l^2}{\varpi}\Big)_4\sum_{\substack{c\in\mathbb{Z}[i]\\ (c,1+i)=1}}\Big(\frac{c}{\varpi}\Big)_4\Phi\Big(\frac{N(cl^2)}{X}\Big). \end{aligned}$$

Applying Corollary 2.8 , we obtain that

$$\sum_{\substack{c\in\mathbb{Z}[i]\\(c,1+i)=1}} \Big(\frac{c}{\varpi}\Big)_4 \Phi\Big(\frac{N(cl^2)}{X}\Big)$$
$$= \frac{X}{2N(l^2\varpi)}\Big(\frac{1+i}{\varpi}\Big)_4 \sum_{k\in\mathbb{Z}[i]} (-1)^{N(k)} g_4(k,\varpi)\widetilde{W}\Big(\sqrt{\frac{N(k)X}{2N(l^2\varpi)}}\Big)$$
$$= \frac{X}{2N(l^2\varpi)}\Big(\frac{1+i}{\varpi}\Big)_4 \sum_{k\in\mathbb{Z}[i]} (-1)^{N(k)} \overline{\Big(\frac{k}{\varpi}\Big)_4} g_4(\varpi)\widetilde{W}\Big(\sqrt{\frac{N(k)X}{2N(l^2\varpi)}}\Big),$$

as one checks easily from Lemma 2.1 that

$$g_4(k,\varpi) = \overline{\Big(\frac{k}{\varpi}\Big)_4} g_4(\varpi).$$

We can now rewrite $S_M(X,Y;\widehat{\phi},\Phi)$ as

(4.1) $S_M(X,Y;\widehat{\phi},\Phi)$
$$= \frac{X}{2}\sum_{\substack{N(l)\le Z\\ l\equiv 1 \bmod (1+i)^3}} \frac{\mu_{[i]}(l)}{N(l^2)} \sum_{\substack{k\in\mathbb{Z}[i]\\k\neq 0}} (-1)^{N(k)}$$
$$\times \sum_{\varpi\equiv 1 \bmod (1+i)^3} \frac{\log N(\varpi)}{N(\varpi)^{3/2}} \overline{\Big(\frac{kl^2}{\varpi}\Big)_4} g_4(\varpi)\widehat{\phi}\Big(\frac{\log N(\varpi)}{\log X}\Big)\widetilde{W}\Big(\sqrt{\frac{N(k)X}{2N(l^2\varpi)}}\Big).$$

4.1 Average of Quartic Gauss Sums at Prime Arguments

For any ray class character $\chi \pmod{16}$, we let

$$h(r,s;\chi) = \sum_{\substack{(n,r)=1\\ n\equiv 1 \bmod (1+i)^3}} \frac{\chi(n)g_4(r,n)}{N(n)^s}.$$

The following lemma, a consequence of [37, Lemma, p. 200], gives the analytic behavior of $h(r,s;\chi)$ for $\Re(s)>1$.

Lemma 4.1 ([14, Lemma 2.5]) *The function $h(r,s;\chi)$ has meromorphic continuation to the entire complex plane. It is holomorphic in the region $\sigma=\Re(s)>1$ except, possibly, for a pole at $s=5/4$. Let $\sigma_1=3/2+\varepsilon$, for any $\varepsilon>0$. Then for $\sigma_1\ge\sigma\ge\sigma_1-1/2$ and $|s-5/4|>1/8$, we have*

$$h(r,s;\chi) \ll N(r)^{\frac{1}{2}(\sigma_1-\sigma+\varepsilon)}(1+t^2)^{\frac{3}{2}(\sigma_1-\sigma+\varepsilon)},$$

where $t=\Im(s)$. Moreover, the residue satisfies

$$\mathrm{Res}_{s=5/4}\, h(r,s;\chi) \ll N(r)^{1/8+\varepsilon}.$$

We derive the following from Lemma 4.1.

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Lemma 4.2 *Let* $(b, 1+i) = 1$. *For any* $d \in \mathbb{Z}[i]$, *we have*

$$\sum_{\substack{N(c)\le x \\ c\equiv 1 \bmod (1+i)^3 \\ c\equiv 0 \bmod b}} \overline{\left(\frac{d}{c}\right)_4} g_4(c) N(c)^{-1/2} \ll \tag{4.2}$$
$$N(d)^{1/10} N(b)^{-3/5} x^{4/5+\varepsilon} + N(d)^{1/8+\varepsilon} N(b)^{-1/2+\varepsilon} x^{3/4+\varepsilon}.$$

Proof This follows essentially from the proof of [37, Proposition 1, p. 198] with a few modifications. Using the notations in [37], in the inclusion-exclusion type estimation of ψ (the first expression below [37, Lemma, p. 200]), one needs to replace the estimation for a corresponding $h((b/\delta)^{n-2}, s; \chi)$ by $h(d(b/\delta)^{n-2}, s; \chi)$ (using (2.3)). Lemma 4.1 then yields (with $n = 4$ in our case)

$$\begin{aligned} \psi &\ll N(d)^{(n-2)(3/2-\Re(s)+\varepsilon)/4} N(b)^{3n/4-1-n\Re(s)/2+2\varepsilon} \\ &\quad \times (1+|s|^2)^{\mathrm{Card}\,\Sigma_\infty(k)\cdot(n/2-1/2)(3/2-\Re(s)+\varepsilon)}, \\ \mathrm{Res}_{s=1+1/n}\psi &\ll N(d)^{(n/4-1/2)(1/2-1/n+\varepsilon)} N(b)^{n/4-3/2+\varepsilon}. \end{aligned}$$

Using this and Perron's formula, we see that the "horizontal" integrals can be estimated by observing that the estimates for the integrand (with $\sigma = \Re(s)$)

$$N(d)^{(n/4-1/2)(1-\sigma+\varepsilon)} N(b)^{n/2-1-n\sigma/2+\varepsilon} T^{(n-1)\,\mathrm{Card}\,\Sigma_\infty(k)(1-\sigma+\varepsilon)-1} X^{\sigma}$$

are monotonic in σ and can therefore be estimated as the sum of the values at $\sigma = c - 1/2$ and $\sigma = c$, where $c = 1 + \varepsilon$. Thus, we obtain that the "horizontal" integrals are

$$\ll N(d)^{(n-2)/8} N(b)^{n/4-1} T^{(n/2-1/2)\,\mathrm{Card}\,\Sigma_\infty(k)-1} X^{1/2+\varepsilon} + N(b)^{-1} T^{-1} X^{1+\varepsilon}.$$

The "vertical" integral can be estimated by

$$\begin{aligned} &\ll N(d)^{(n/8-1/4)} N(b)^{n/4-1} X^{1/2+\varepsilon} \int_{-T}^{T} (1+t^2)^{(n/4-1/4)\,\mathrm{Card}\,\Sigma_\infty(k)-1/2}\,\mathrm{d}t \\ &\ll N(d)^{(n/8-1/4)} N(b)^{n/4-1} T^{(n/2-1/2)\,\mathrm{Card}\,\Sigma_\infty(k)} X^{1/2+\varepsilon}. \end{aligned}$$

Together, these estimates yield that the left-hand side of (4.2) is

$$\begin{aligned} &\ll N(d)^{(n/8-1/4)} N(b)^{n/4-1} T^{(n/2-1/2)\,\mathrm{Card}\,\Sigma_\infty(k)} X^{1/2+\varepsilon} + N(b)^{-1} T^{-1} X^{1+\varepsilon} \\ &\qquad + N(d)^{(n-2)(1/2-1/n+\varepsilon)/4} N(b)^{n/4-3/2+\varepsilon} X^{1/2+1/n+\varepsilon}. \end{aligned}$$

With $R = 2 + (n-1)\,\mathrm{Card}\,\Sigma_\infty(k)$, we now take

$$T = \left(\frac{X}{N(b)^{n/2} N(d)^{(n-2)/4}}\right)^{1/R}$$

to get the desired result. ∎

We now use Lemma 4.2 instead of [37, Proposition 1, p. 198] in the sieve identity in [37, §4] (note that in our case [37, Proposition 2, p. 206] is still valid) to get that

$$
\begin{aligned}
(4.3)\quad & \sum_{\substack{n\equiv 1 (\bmod (1+i)^3)\\ N(n)\le x}} \overline{\Big(\frac{kl^2}{n}\Big)_4}\frac{g_4(n)\Lambda(n)}{\sqrt{N(n)}}\\
&\ll x^{4\varepsilon}\bigg(N(kl^2)^{(n-2)/(4R)}\Big(\frac{x}{u_3}\Big)^{n/(2R)}x^{1-1/R}\\
&\qquad + N(kl^2)^{(n/4-1/2)(1/2-1/n+\varepsilon)}\Big(\frac{x}{u_3}\Big)^{n/4-1/2}x^{1/2+1/n}+x^{1-1/20}+xu_1^{-1/5}\bigg)\\
&\ll x^{4\varepsilon}\big(N(kl^2)^{1/10}x^{1-1/10}+N(kl^2)^{1/8+\varepsilon}x^{1-1/8}+x^{1-1/20}\big),
\end{aligned}
$$

where we take $u_3 = x/u_1$, $u_1 = x^{1/4}/8$ as in [37] and we note that $n = 4$, $R = 5$ in our case.

Since $g_4(n) = 0$ if n is not square-free, the proper prime powers contribute nothing to the left-hand side of (4.3). Thus we have

$$
\begin{aligned}
E(x;k,l) := & \sum_{\substack{N(\varpi)\le x\\ \varpi\equiv 1 \bmod (1+i)^3}} \overline{\Big(\frac{kl^2}{\varpi}\Big)_4}\frac{g_4(\varpi)\Lambda(\varpi)}{\sqrt{N(\varpi)}}\\
&\ll x^{\varepsilon}\big(N(kl^2)^{1/10}x^{1-1/10}+N(kl^2)^{1/8+\varepsilon}x^{1-1/8}+x^{1-1/20}\big).
\end{aligned}
$$

4.2 Estimation of $S_M(X,Y;\widehat{\phi},\Phi)$

It follows from partial summation that

$$
\begin{aligned}
&\sum_{\varpi\equiv 1 \bmod (1+i)^3}\frac{\log N(\varpi)}{N(\varpi)^{3/2}}\overline{\Big(\frac{kl^2}{\varpi}\Big)_4}g_4(\varpi)\widehat{\phi}\Big(\frac{\log N(\varpi)}{\log X}\Big)\widetilde{W}\bigg(\sqrt{\frac{N(k)X}{2N(l^2\varpi)}}\bigg)\\
&= \int_1^Y \frac{1}{V}\widehat{\phi}\Big(\frac{\log V}{\log X}\Big)\widetilde{W}\bigg(\sqrt{\frac{N(k)X}{2N(l)^2V}}\bigg)\,\mathrm{d}E(V;k,l)\\
&\ll \frac{E(Y;k,l)}{Y}\widehat{\phi}\Big(\frac{\log Y}{\log X}\Big)\widetilde{W}\bigg(\sqrt{\frac{N(k)X}{2N(l)^2Y}}\bigg)\\
&\quad + \int_1^Y \frac{E(V;k,l)}{V^2}\bigg|\widehat{\phi}\Big(\frac{\log V}{\log X}\Big)\widetilde{W}\bigg(\sqrt{\frac{N(k)X}{2N(l)^2V}}\bigg)\bigg|\,\mathrm{d}V\\
&\quad + \frac{1}{\log X}\int_1^Y \frac{E(V;k,l)}{V^2}\bigg|\widehat{\phi}'\Big(\frac{\log V}{\log X}\Big)\widetilde{W}\bigg(\sqrt{\frac{N(k)X}{2N(l)^2V}}\bigg)\bigg|\,\mathrm{d}V\\
&\quad + \sqrt{\frac{N(k)X}{N(l)^2}}\int_1^Y \frac{E(V;k,l)}{V^{5/2}}\bigg|\widehat{\phi}\Big(\frac{\log V}{\log X}\Big)\widetilde{W}'\bigg(\sqrt{\frac{N(k)X}{2N(l)^2V}}\bigg)\bigg|\,\mathrm{d}V.
\end{aligned}
$$

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It follows that

$$\sum_{\substack{k\in\mathbb{Z}[i]\\k\neq 0}}(-1)^{N(k)}\sum_{\varpi\equiv 1 \bmod (1+i)^3}\frac{\log N(\varpi)}{N(\varpi)^{3/2}}\overline{\Big(\frac{kl^2}{\varpi}\Big)_4}g(\varpi)\widehat{\phi}\Big(\frac{\log N(\varpi)}{\log X}\Big)\widetilde{W}\Big(\sqrt{\frac{N(k)X}{2N(l^2\varpi)}}\Big) \ll R_1+R_2+R_3+R_4,$$

where

$$R_1=\sum_{\substack{k\in\mathbb{Z}[i]\\k\neq 0}}\frac{M(Y)}{Y}\widehat{\phi}\Big(\frac{\log Y}{\log X}\Big)\widetilde{W}\Big(\sqrt{\frac{N(k)X}{2N(l)^2Y}}\Big),$$

$$R_2=\int_1^Y\sum_{\substack{k\in\mathbb{Z}[i]\\k\neq 0}}\frac{M(V)}{V^2}\Big|\widehat{\phi}\Big(\frac{\log V}{\log X}\Big)\widetilde{W}\Big(\sqrt{\frac{N(k)X}{2N(l)^2V}}\Big)\Big|\,\mathrm{d}V,$$

$$R_3=\frac{1}{\log X}\int_1^Y\sum_{\substack{k\in\mathbb{Z}[i]\\k\neq 0}}\frac{M(V)}{V^2}\Big|\widehat{\phi}'\Big(\frac{\log V}{\log X}\Big)\widetilde{W}\Big(\sqrt{\frac{N(k)X}{2N(l)^2V}}\Big)\Big|\,\mathrm{d}V,$$

$$R_4=\int_1^Y\sum_{\substack{k\in\mathbb{Z}[i]\\k\neq 0}}\sqrt{\frac{N(k)X}{N(l)^2}}\frac{M(V)}{V^{5/2}}\Big|\widehat{\phi}\Big(\frac{\log V}{\log X}\Big)\widetilde{W}'\Big(\sqrt{\frac{N(k)X}{2N(l)^2V}}\Big)\Big|\,\mathrm{d}V,$$

with

$$M(W)=N(kl^2)^{1/10}W^{1-1/10+\varepsilon}+N(kl^2)W^{1-1/8+\varepsilon}+W^{1-1/20+\varepsilon}.$$

We use (2.14) when $VN(l^2)/X\geq 1$, $N(k)\leq VN(l^2)/X$ and (2.12) when $VN(l^2)/X\leq 1$ or $VN(l^2)/X\geq 1$, and $N(k)\geq YN(l^2)/X$ with $j=4$ to arrive at

$$\begin{aligned}R_4 &\ll \int_1^{X/N(l^2)}\sum_{\substack{k\in\mathbb{Z}[i]\\N(k)\geq 1}}\frac{M(V)}{V^{5/2}}\sqrt{\frac{N(k)X}{N(l)^2}}\Big(\frac{N(l)^2V}{N(k)X}\Big)^2U^3\,\mathrm{d}V\\
&\quad+\int_{X/N(l^2)}^Y\sum_{\substack{k\in\mathbb{Z}[i]\\0<N(k)\leq VN(l^2)/X}}\sqrt{\frac{N(k)X}{N(l)^2}}\frac{M(V)}{V^{5/2}}\,\mathrm{d}V\\
&\quad+\int_{X/N(l^2)}^Y\sum_{\substack{k\in\mathbb{Z}[i]\\N(k)\geq VN(l^2)/X}}\frac{M(V)}{V^{5/2}}\sqrt{\frac{N(k)X}{N(l)^2}}\Big(\frac{N(l)^2V}{N(k)X}\Big)^2U^3\,\mathrm{d}V\\
&\ll\frac{N(l^2)^{6/5}Y^{1+\varepsilon}U^3}{X^{11/10}}+\frac{N(l^2)^{5/4+\varepsilon}Y^{1+\varepsilon}U^3}{X^{9/8+\varepsilon}}+\frac{N(l^2)Y^{19/20+\varepsilon}U^3}{X}.\end{aligned}$$

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The estimations for R_1, R_2, and R_3 are similar. We then conclude from (3.7) and (4.1) that

$$S(X, Y; \widehat{\phi}, \Phi) \ll X\Big(\frac{Z^{6/5}Y^{1+\varepsilon}U^3}{X^{11/10}} + \frac{Z^{5/4+\varepsilon}Y^{1+\varepsilon}U^3}{X^{9/8+\varepsilon}} + \frac{ZY^{19/20+\varepsilon}U^3}{X} + \frac{\log^5 X}{Z}\Big)$$
$$= o(X\log X),$$

when $U = \log\log X$, $Z = \log^5 X$. This completes the proof of Theorem 1.3.

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