SIMULTANEOUS ASYMPTOTIC DIOPHANTINE APPROXIMATIONS TO A BASIS OF A REAL NUMBER FIELD

WILLIAM W. ADAMS

1. Introduction

The purpose of this paper is to prove the following result.

THEOREM 1. *Let K be a real algebraic number field of degree $m = n + 1$. Let $1, \beta_1, \cdots, \beta_n$ be a basis of K. For a given constant $C > 0$ set $\lambda_B = \lambda_B(\beta_1, \cdots, \beta_n, C)$ equal to the number of solutions in integers $q, p_1, \cdots, p_n$ of the inequalities*

$$* \qquad\qquad 0 < q\beta_i - p_i < C/q^{\frac{1}{n}} \qquad\qquad (1 \leq i \leq n)$$

$$** \qquad\qquad 1 \leq q \leq B.$$

Then $\lambda_B = 0(1)$ or there is a $C' > 0$ such that $\lambda_B \sim C' \log B \ (B \to \infty)$.

There is a dual theorem.

THEOREM 2. *With $\beta_1, \cdots, \beta_n, C$ as in Theorem 1 set Λ_B equal to the number of solutions in integers $q_1, \cdots, q_n, p$ of the inequalities*

$$0 < q_1\beta_1 + \cdots + q_n\beta_n - p < C/q^n$$

$$1 \leq q_1, \cdots, q_n \leq B$$

where $q = \max(q_1, \cdots, q_n)$. Then either $\Lambda_B = 0(1)$ or there is a $C'' > 0$ such that $\Lambda_B \sim C'' \log B \ (B \to \infty)$.

These results generalize the results of [2, 3]. However the work of [2, 3] had the advantage that the constant C' was more precisely defined. For related work see [1, 4, 5].

The author would like to thank W.M. Schmidt and S. Kuroda for not believing that the results of [2] could not be generalized to higher dimensions.

The proof of the present result closely parallels the proof in [2]. Hence this paper will sometimes be sketchy.

2. The reduction to counting units

Denote by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{C}$ the integers, rationals, reals and complexes. Let $\tau_0 =$ identity, $\tau_1, \cdots, \tau_n$ denote the distinct embeddings of K into $\boldsymbol{C}$. Assume $\tau_0, \cdots, \tau_r$ are the real embeddings. For $\alpha \in K$, set $\tau_i \alpha = \alpha^{(i)}$ and assume as usual $n = r + 2s$ and $\alpha^{(r+i)} = \bar{\alpha}^{(r+s+i)}$ $(1 \le i \le s)$.

LEMMA 1. *The dual basis* $\alpha_0, \alpha_1, \cdots, \alpha_n$ *of* K *(with respect to the trace) has the following properties*

$$\alpha_0^{(i)} + \alpha_1^{(i)} \beta_1 + \cdots + \alpha_n^{(i)} \beta_n = 0 \quad (1 \le i \le n) \tag{1}$$

$$\alpha_0 + \alpha_1 \beta_1 + \cdots + \alpha_n \beta_n = 1 \tag{2}$$

$$A = (\alpha_j^{(i)}) \quad (1 \le i, j \le n) \text{ is nonsingular.} \tag{3}$$

For the proof see [2, 5].

Let M be the free $\boldsymbol{Z}$-module generated by $\alpha_0, \cdots, \alpha_n$. Let $\mathscr{O}$ be the associated order. Let U be the group of units of $\mathscr{O}$. Then by the Dirichlet Unit Theorem there are units $\zeta_1, \cdots, \zeta_{r+s} > 1$ in U such that $U = \{\pm \zeta_1^{\nu_1} \cdots \zeta_{r+s}^{\nu_{r+s}}\}$. For $\nu = (\nu_1, \cdots, \nu_{r+s}) \in \boldsymbol{Z}^{r+s}$ set $\zeta^\nu = \zeta_1^{\nu_1} \cdots \zeta_{r+s}^{\nu_{r+s}}$ so $U = \{\pm \zeta^\nu\}$ $(\nu \in \boldsymbol{Z}^{r+s})$.

For $\xi_1, \xi_2 \in M$ write $\xi_1 \sim \xi_2$ if there is a $\zeta \in U$ such that $\xi_1 = \zeta \xi_2$ (this is an equivalence relation). If $\Omega \subseteq M$ is an equivalence class, then for all $\xi \in M$, $|N\xi|$ is the same and we denote this by $N\Omega$ (N denotes the norm of $K/\boldsymbol{Q}$).

Now for $\zeta \in M$ we have unique $q, p_1, \cdots, p_n \in \boldsymbol{Z}$ such that $\xi = q\alpha_0 + p_1\alpha_1 + \cdots + p_n\alpha_n$. In this way we view $\xi \in M$ as being in 1–1 correspondence with possible solutions to * and **. Moreover by (1) we see

$$-\xi^{(i)} = \alpha_1^{(i)}(q\beta_1 - p_1) + \cdots + \alpha_n^{(i)}(q\beta_n - p_n) \quad (1 \le i \le n). \tag{4}$$

LEMMA 2. *There are only finitely many classes* $\Omega \subseteq M$ *which yield solutions to* *.

Proof. Combine * and (4) with the known fact that there are only finitely many Ω with $N\Omega$ below a given value.

It suffices to show the following: *if for a fixed class* $\Omega \subseteq M$ *there are infinitely many solutions to* * *then there is a* $C_1 > 0$ *such that the number of* $\xi \in \Omega$ *satisfying* * *and* ** *is asymptotic to* $C_1 \log B$.

We now need some notation. Set $\Omega = \{\pm \zeta^\nu \xi_0\}$ $(\nu \in \boldsymbol{Z}^{r+s})$ for any $\xi_0 > 0$ in Ω. Let Ω^+ (Ω^-) denote the positive (negative) elements for Ω. Let

$\xi_\nu = \zeta^\nu \xi_0 \ (\nu \in \mathbf{Z}^{r+s})$ denote a typical element of Ω^+. Set

$$\xi_\nu = q_\nu \alpha_0 + p_{1\nu} \alpha_1 + \cdots + p_{n\nu} \alpha_n.$$

Write

$$\gamma_{i\nu} = q_\nu \beta_i - p_{i\nu} \quad (1 \leq i \leq n) \tag{5}$$

and so (4) is

$$(\xi_\nu^{(1)}, \cdots, \xi_\nu^{(n)})^t = -A(\gamma_{1\nu}, \cdots, \gamma_{n\nu})^t. \tag{6}$$

For $q_\nu > 0$ let $\delta_{i\nu} = q_\nu^{\frac{1}{n}} \gamma_{i\nu}$. Then we wish to count $\lambda_B(\Omega^+)$, the number of $\xi_\nu \in \Omega^+$ such that

$$0 < \delta_{1\nu}, \cdots, \delta_{n\nu} < C \tag{7}$$

$$1 \leq q_\nu \leq B \tag{8}$$

(the corresponding $\lambda_B(\Omega^-)$ will be shown to be bounded). For $q_\nu \neq 0$ set

$$\kappa_\nu' = \alpha_0 + \frac{p_{1\nu}}{q_\nu} \alpha_1 + \cdots + \frac{p_{n\nu}}{q_\nu} \alpha_n$$

so that

$$\xi_\nu = q_\nu \kappa_\nu' \tag{9}$$

$$\kappa_\nu' = 1 + \varepsilon_\nu/q_\nu \tag{10}$$

where

$$\varepsilon_\nu = \alpha_1 \gamma_{1\nu} + \cdots + \alpha_n \gamma_{n\nu}. \tag{11}$$

For $q_\nu > 0$

$$\delta_{i\nu} = (\xi_0 \kappa_\nu'^{-1})^{\frac{1}{n}} \zeta^{\frac{\nu}{n}} \gamma_{i\nu}. \tag{12}$$

Set

$$\eta_{i\nu} = \zeta^{\frac{\nu}{n}} \gamma_{i\nu} \tag{13}$$

$$\kappa_\nu = (\xi_0 \kappa_\nu'^{-1})^{\frac{1}{n}} \tag{14}$$

so that

$$\delta_{i\nu} = \kappa_\nu \eta_{i\nu}. \tag{15}$$

LEMMA 3. *Let $\lambda_B'(\Omega^+)$ be the number of solutions, v, of*

i) ξ_ν *is sufficiently large*

ii) $0 < max\,(\eta_{1\nu}, \cdots, \eta_{n\nu}) < 10\xi_0^{-\frac{1}{n}}C = C_2$

iii) $0 < \eta_{1\nu}, \cdots, \eta_{n\nu} < C\kappa_\nu^{-1}$

iv) $1 \le \xi_\nu \le 2B$.

Then $\lambda_B(\Omega^+) + 0(1) \le \lambda_B'(\Omega^+) \le \lambda_{4B}(\Omega^+) + 0(1)$.

Proof. By i) and ii) we see $q_\nu \ne 0$ since $q_\nu = 0$ implies for $\eta_{i\nu} > 0$

$$C_2 > \eta_{i\nu} = \zeta^{\frac{\nu}{n}}(-p_{i\nu}) \ge \zeta^{\frac{\nu}{n}} = \hat{\xi}_0^{-\frac{1}{n}}\xi_\nu^{\frac{1}{n}}$$

which violates i). Now by i) ζ^ν is large so by ii) and (13) $r_{i\nu}$ is small so by (11) ε_ν is small and so from (10) we may assume $\frac{1}{2} \le \kappa_\nu' \le \frac{3}{2}$. From this and (9) we have q_ν is large and positive. In this situation (7) and iii) are equivalent (see (15)). Finally by iv) $q_\nu = \xi_\nu\kappa_\nu'^{-1} \le 4B$. Thus the right hand inequality is true. Conversely assume (7) and (8). We may assume q_ν is large. So from (7), (10), (11), (12) we see that κ_ν' is close to 1 and so by (9) ξ_ν is large. Again (7) and iii) are equivalent. Here iii) implies ii). Finally by (8), $\xi_\nu = q_\nu\kappa_\nu' \le 2B$ and so we are done.

It follows from the above argument that solutions of * are such that ξ_ν and q_ν have the same mangitude; in particular $\xi_\nu > 0$ and so $\lambda_B(\Omega^-) = 0(1)$.

So now we know it suffices to show that if there are an infinite number of solutions, $\lambda_B'(\Omega^+) \sim C_1 \log B$ (with $C_1 > 0$).

3. Counting the relevant units

We now essentially prove the theorem except that instead of counting $1 \le q_\nu \le B$ we count $1 \le \nu_1 \le N$. We then put B back.

For $1 \le i \le r + s$ set

$$X_i = (\log|\zeta_i^{\frac{1}{n}}\zeta_i^{(2)}|, \cdots, \log|\zeta_i^{\frac{1}{n}}\zeta_i^{(r+s)}|, \arg\zeta_i^{(r+1)}, \cdots, \arg\zeta_i^{(r+s)})^t$$

where we assume for $Z \in C$ that the argument of Z satisfies $0 \le \arg Z < 2\pi$. So $X_i \in R^{n-1}$. Further for $r + s + 1 \le i \le n$ set $X_i = (0, \cdots, 0, 2\pi, 0, \cdots, 0)^t$ with 2π in the ith spot.

Now $\det(X_1, \cdots, \hat{X}_i, \cdots, X_n) = 0$ for $i = 1, 2, \cdots, r + s$ violates the known rank for the regulator matrix. So without loss of generality (rela-

beling $\zeta_1, \cdots \zeta_{r+s}$) we may assume

$$\det (X_2, \cdots, X_n) \neq 0. \tag{16}$$

LEMMA 4. *Let $\sharp_N'$ be the number of solutions of the inequalities*

$$0 < \eta_{1\nu}, \cdots, \eta_{n\nu} < C_3 \tag{17}$$

$$1 \leq \nu_1 \leq N \quad (\text{or } 1 \leq -\nu_1 \leq N).$$

Then

$$\sharp_N' \sim C_4 N \quad (N \to \infty)$$

where $C_4 > 0$ is some constant.

Proof. Set

$$A_1 = - \begin{pmatrix} \xi_0^{(1)-1} & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & \cdot \; \xi_0^{(n)-1} \end{pmatrix} A$$

with A as in (3). Also set $\rho_{i\nu} = \zeta^{\frac{\nu}{2}} \zeta^{(i)\nu} \, (1 \leq i \leq n)$. Then by (6) and (13)

$$A_1(\eta_{1\nu}, \cdots, \eta_{n\nu})^t = (\rho_{1\nu}, \cdots, \rho_{n\nu})^t.$$

Now $\rho_{r+s+i,\nu} = \bar{\rho}_{r+i,\nu} \, (1 \leq i \leq s)$ so we omit the last s coordinates in the vector of ρ's.

That is, define the linear transformation A_2 by

$$\boldsymbol{R}^n \xrightarrow{A_1} \boldsymbol{R}^r \times \boldsymbol{C}^{2s} \xrightarrow{\text{proj.}} \boldsymbol{R}^r \times \boldsymbol{C}^s = \boldsymbol{R}^n \text{ so}$$

$$A_2(\eta_{1\nu}, \cdots, \eta_{n\nu})^t = (\rho_{1\nu}, \cdots, \rho_{r+s,\nu})^t.$$

We show A_2 is non singular. We know from (3) of Lemma 1 that A is non singular and so also A_1 is non sigular (i.e., $\det A_1 \neq 0$). Now if the row vectors of A_1 are $A^{(1)}, \cdots, A^{(n)}$ we see $A^{(r+i)} = \bar{A}^{(r+s+i)} \, (1 \leq i \leq s)$ and that the row vectors of A_2 are $A^{(1)}, \cdots, A^{(r)}, Re \, (A^{(r+1)}), \cdots, Re \, (A^{(r+s)})$, $Im(A^{(r+1)}), \cdots, Im(A^{(r+s)})$. So we see that $\det A_2 = 2^{-s} \det A_1 \neq 0$, as desired.

Now

$$\rho_{1\nu}\rho_{2\nu} \cdots \rho_{n\nu} = \rho_{1\nu} \cdots \rho_{r\nu} |\rho_{r+1,\nu}|^2 \cdots |\rho_{r+s,\nu}|^2$$

$$= (N\zeta_1)^{\nu_1} \cdots (N\zeta_{r+s})^{\nu_{r+s}}$$

$$= \pm 1$$

So the images of the points $(\eta_{1\nu}, \cdots, \eta_{n\nu})$ under A_2 lie on the surface $\mathscr{S}$:

$u_1 u_2 \cdots u_r |u_{r+1}|^2 \cdots |u_{r+s}|^2 = \pm 1$ in $\boldsymbol{R}^n = \boldsymbol{R}^r \times \boldsymbol{C}^s$ (so $(u_1, \cdots, u_r) \in \boldsymbol{R}^r$, $(u_{r+1}, \cdots, u_{r+s}) \in \boldsymbol{C}^s$). Define $\phi \colon \mathscr{S} \to \boldsymbol{R}^{n-1}$ by

$$\phi(u_1, \cdots, u_{r+s}) = (\log|u_2|, \cdots, \log|u_{r+s}|, \arg u_{r+1}, \cdots, \arg u_{r+s})^t.$$

Then on any part of $\mathscr{S}$ where the signs of $u_1, \cdots, u_r$ are fixed we see ϕ is 1–1 and has as image

$$\mathscr{D} = \{(x_1, \cdots, x_{n-1}) \mid 0 \le x_i < 2\pi \text{ for } r+s \le i < n\}.$$

Now our problem is to count the number of $(\eta_{1\nu}, \cdots, \eta_{n\nu})$ lying in an open box in $\boldsymbol{R}^n$. This then is equivalent to counting the number of $(\rho_{1\nu}, \cdots, \rho_{r+s,\nu})$ lying in some open parallelepiped in $\boldsymbol{R}^n = \boldsymbol{R}^r \times \boldsymbol{C}^s$ which meets $\mathscr{S}$ in a set open in $\mathscr{S}$. Then again restricting ourselves to a portion of $\mathscr{S}$ with the signs of $u_1, \cdots, u_r$ fixed we see the image $\mathscr{R}$ under ϕ of this subset of $\mathscr{S}$ is a bounded open set together with some of its boundary. Moreover it is clear that the volume of the boundaries of all the sets involved is zero. We wish to count the number of $\phi(\rho_{1\nu}, \cdots, \rho_{r+s,\nu})$ lying in $\mathscr{R}$.

We see readily

$$\phi(\rho_{1\nu}, \cdots, \rho_{r+s,\nu}) \equiv \nu_1 X_1 + \cdots + \nu_{r+s} X_{r+s} \pmod{2\pi}$$

where the congruence is read only in the last s coordinates. So when the last s coordinates are reduced mod 2π we want to count which $\nu_1 X_1 + \cdots + \nu_{r+s} X_{r+s}$ lies in $\mathscr{R}$. This is clearly the same as counting the number of $\nu_1 X_1 + \cdots + \nu_n X_n \in \mathscr{R}$ (see the definition of $X_{r+s+1}, \cdots, X_n$).

Let $\varLambda$ be the lattice in $\boldsymbol{R}^{n-1}$ spanned by $X_2, \cdots, X_n$ (a lattice by (16)). Thus we now see that we wish to count the number of $\nu_1 X_1$ ($1 \le \nu_1 \le N$) lying in $\mathscr{R}$ mod $\varLambda$. It is a well known theorem in uniform distribution theory that if $\mathscr{R}$ is a set whose boundary has zero volume, then there is a $C_4 \ge 0$ such that the number of $\nu_1 X_1$ in $\mathscr{R}$ mod $\varLambda$, $1 \le \nu_1 \le N$ ($1 \le -\nu_1 \le N$) is asymptotic to $C_4 N$.

It should be recalled that the region $\mathscr{R}$ depended on the sign of the coordinates $\rho_{1\nu}, \cdots, \rho_{r\nu}$. The sign of $\rho_{i\nu}$ depends only on the sign of $\zeta_1^{(i)\nu_1}, \cdots, \zeta_{r+s}^{(i)\nu_{r+s}}$ ($1 \le i \le r$) and so for a fixed parity of $\nu_1, \cdots, \nu_{r+s}$ the region $\mathscr{R}$ does not change. Then the argument should be repeated as above for these 2^{r+s} cases.

This concludes the proof of Lemma 4.

We now solve for ξ_ν in terms of ν_1.

LEMMA 5. *There is a constant $C_7 \neq 0$ such that for all ν satisfying* i), ii), iii) *of Lemma* 3 *we have*

$$\log \xi_\nu = \nu_1 C_7 + 0(1) \qquad (\nu_1 \to \pm \infty). \tag{18}$$

Proof. As in the proof of Lemma 3 we see i) and ii) imply κ_ν' is close to 1 and so from (10) and (14)

$$C\kappa_\nu^{-1} = C\xi_0^{-\frac{1}{n}}(1 + \varepsilon_\nu/q_\nu)^{\frac{1}{n}} = C_6 + 0(q_\nu^{-1}).$$

So we are interested in the ν such that

$$0 < \eta_{1\nu}, \cdots, \eta_{n\nu} < C_6 + 0(q_\nu^{-1}). \tag{19}$$

The C_7 will depend only on $\zeta_1, \cdots, \zeta_{r+s}$ so it suffices to show (18) for all ν such that

$$0 < \eta_{1\nu}, \cdots, \eta_{n\nu} < C_8.$$

Then as in Lemma 4 there is a bounded subset $\mathscr{R} \subseteq \mathbf{R}^{n-1}$ such that we want all $\nu_1 X_1 + \cdots + \nu_n X_n \in \mathscr{R}$. This says that for $2 \leq i \leq r+s$

$$\nu_2 \log|\zeta_2^{\frac{1}{n}}\zeta_2^{(i)}| + \cdots + \nu_{r+s}\log|\zeta_{r+s}^{\frac{1}{n}}\zeta_{r+s}^{(i)}| = -\nu_1\log|\zeta_1^{\frac{1}{n}}\zeta_1^{(i)}| + 0(1).$$

Set $X_i' = (\log|\zeta_i^{\frac{1}{n}}\zeta_i^{(2)}|, \cdots, \log|\zeta_i^{\frac{1}{n}}\zeta_i^{(r+s)}|)$, $(1 \leq i \leq r+s)$. Then $\det(X_2, \cdots, X_n) \neq 0$, (16), implies $\det(X_2', \cdots, X_{r+s}') \neq 0$, Solving for ν_j $(2 \leq j \leq r+s)$ yields with $Y = -\nu_1 X_1' + 0(1)$

$$\nu_j = \frac{\det(X_2', \cdots, X_{j-1}', Y, X_{j+1}', \cdots, X_{r+s}')}{\det(X_2', \cdots, X_{r+s}')}$$

$$= (-1)^{j-1}\nu_1\frac{\det(X_1', \cdots, \hat{X}_j', \cdots, X_{r+s}')}{\det(X_2', \cdots, X_{r+s}')} + 0(1).$$

Thus

$$\log \xi_\nu = \log \xi_0 + \nu_1 \log \zeta_1 + \cdots + \nu_{r+s}\log \zeta_{r+s}$$

$$= \nu_1 \sum_{j=1}^{r+s}(-1)^{j-1}\frac{\det(X_1', \cdots, \hat{X}_j', \cdots, X_{r+s}')}{\det(X_2', \cdots, X_{r+s}')}\log \zeta_j + 0(1)$$

$$= \nu_1 C_7 + 0(1).$$

Now $C_7 = 0$ implies

$$\sum_{j=1}^{r+s}(-1)^{j-1}\det(X_1', \cdots, \hat{X}_j', \cdots, X_{r+s}')\log \zeta_j = 0$$

and this is the regulator of $\mathcal{O}$. So $C_7 \neq 0$.

LEMMA 6. *Let $\#_N$ be the number of solutions of* i), ii), iii) *of Lemma* 3 *such that* $1 \leq \nu_1 \leq N$ $(1 \leq -\nu_1 \leq N)$. *Then for some* $C_9 > 0$, $\#_N \sim C_9 N$.

Proof. We just observed that we must count the number of ν such that ξ_ν and q_ν are large, (19) holds and $1 \leq \nu_1 \leq N$ $(1 \leq -\nu_1 \leq N)$. Then examining the proof of Lemma 4 we see that we wish to count the number of ν_1, $1 \leq \nu_1 \leq N$ $(1 \leq -\nu_1 \leq N)$ such that

$$\nu_1 X_1 + \cdots + \nu_n X_n \in \mathcal{R} + 0(q_\nu^{-1}).$$

Let $F_N(\varepsilon)$ be the number lying in $\mathcal{R}_\varepsilon$ (all $X \in \boldsymbol{R}^{n-1}$ no further from $\mathcal{R}$ than ε). Then $F_N(\varepsilon) \sim C(\varepsilon)N$ $(N \to \infty)$. Moreover from the uniform distribution theory $\lim C(\varepsilon) = C(0)$ $(\varepsilon \to 0)$. Then given ε there is an $N_0(\varepsilon)$ such that $|\nu_1| > N_0(\varepsilon)$ implies $\mathcal{R} + 0(q_\nu^{-1}) \subseteq \mathcal{R}_\varepsilon$. Now clearly the q_ν yielding solutions to * are such that $q_\nu \to \infty$ so also $\xi_\nu \to \infty$ and so by Lemma 5, $\nu_1 \to \infty$. Hence there is a constant C_{10} independent of N and ε such that

$$\#_N \leq F_N(\varepsilon) + C_{10} N_0(\varepsilon).$$

Thus as $N \to \infty$

$$\limsup \#_N / N \leq \limsup (F_N(\varepsilon) + C_{10} N_0(\varepsilon))/N = C(\varepsilon).$$

Since this is true for all $\varepsilon > 0$ we see

$$\limsup \#_N / N \leq C(0).$$

Similarly for $\liminf$ and so $\lim \#_N / N = C(0)$.

We now prove the theorem. It follows from Lemma 5 that ξ_ν is large if and only if ν_1 is large and has the same sign as C_7; we now restrict ν_1 to these values. Set $N = |C_7|^{-1} \log B$. Then $1 \leq \xi_\nu \leq 2B$ implies by Lemma 5

$$0 \leq \log \xi_\nu = \nu_1 C_7 + 0(1) \leq \log B + 0(1)$$

or

$$0(1) \leq |\nu_1| \leq N + 0(1).$$

Thus

$$\lambda'_B(\Omega^+) \leq \#_{N+0(1)}.$$

The lower bound is similar and so there are constants C_{11} and C_{12} such that

$$\frac{\#_{N-C_{11}}}{N} \le \frac{\lambda'_B(\Omega^+)}{|C_7|^{-1}\log B} \le \frac{\#_{N+C_{12}}}{N}$$

and so letting N (hence B) tend to ∞ we have the desired result.

REFERENCES

[1] W.W. Adams. Simultaneous Asymptotic Diophantine Approximations. Mathematika 14 (1967) 173–180.

[2] W.W. Adams. Simultaneous Asymptotic Diophantine Approximations to a Basis of a Real Cubic Number Field. J. of Number Theory 1 (1969) 179–194.

[3] S. Lang. Asymptotic Approximation to Quadratic Irrationals. Am. J. of Mathematics 87 (1965) 481–487.

[4] L.G. Peck. Simultaneous Rational Approximations to Algebraic Numbers. Bull. AMS 67 (1961) 197–201.

[5] W.M. Schmidt. Simultaneous Approximations to a Basis of a Real Algebraic Number Field. Am. J. of Mathematics 88 (1966) 517–527.

Department of Mathematics
University of Maryland