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Higher horospherical limit sets for G-modules over $CAT(0)$-spaces

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Abstract

The Σ-invariants of Bieri–Neumann–Strebel and Bieri–Renz involve an action of a discrete group G on a geometrically suitable space M. In the early versions, M was always a finite-dimensional Euclidean space on which G acted by translations. A substantial literature exists on this, connecting the invariants to group theory and to tropical geometry (which, actually, Σ-theory anticipated). More recently, we have generalized these invariants to the case where M is a proper $CAT(0)$ space on which G acts by isometries. The "zeroth stage" of this was developed in our paper [**BG16**]. The present paper provides a higher-dimensional extension of the theory to the "nth stage" for any n.

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1. *Introduction*

1·1. *Background*

The Bieri–Neumann–Strebel invariant $\Sigma(G, \mathbb{Z})$ of a finitely generated group G is a certain subset of the sphere at infinity of $\mathbb{R}^d$, where d is the rank of the abelianisation of G. Over the years this elusive subset has been computed in many cases, and has been related to a variety of issues in group theory and tropical geometry. The first major generalisation was the higher-dimensional Bieri–Renz invariant $\Sigma^n(G; A)$, where A is a G-module of type FP_n. (The case $n = 1$ with A the trivial G-module $\mathbb{Z}$ gives back the original $\Sigma(G, \mathbb{Z})$).

The invariant $\Sigma^n(G; A)$ is intrinsically associated with the natural action of G on $\mathbb{R}^d$ by translations. This led us to a generalisation of the fundamental idea, in which, given G and a G-module A of type FP_n, the translation action of G on $\mathbb{R}^d$ is replaced by an isometric action of G on an arbitrary proper $CAT(0)$ space M, leading us to a subset of the boundary ∂M playing the role previously played by the sphere at infinity. By clear analogy, we call this subset $\Sigma^n(M; A)$.

Even the case $\Sigma^0(M; A)$ has turned out to be remarkably interesting. Although computation is still in its infancy, it has already been deeply linked to buildings associated with certain arithmetic groups, where M in that case is a symmetric space. The basic theory of $\Sigma^0(G; A)$ is set out in our paper [**BG16**].

In this paper we set out the corresponding theory of $\Sigma^n(G; A)$, thus exhibiting the natural place of $\Sigma^0(M; A)$ within a richer theory. We prove appropriate analogs of theorems already known in the "classical" (i.e. Euclidean) case. This is far from routine, and new methods have to be developed. Besides the basic theory, we find a product formula for Σ invariants, and an interpretation of the whole theory in terms of Novikov homology.

1·2. *Quick definition of* $\Sigma^n(G; A)$

We prefer to give this definition in context later in the paper, but for readers who wish to know now, we include a short version here.

The given G-module A has type FP_n, so there is a free resolution $\mathbf{F} \twoheadrightarrow A$ with chosen finitely generated n-skeleton. Here, the lowest possible value of n is 0 so A is assumed to be finitely generated throughout the paper. We choose a base point $b \in M$, and we define a G-equivariant "control map" $h : \mathbf{F} \to fM$ where fM denotes the G-set of finite subsets of M.

The definition of this map h proceeds in stages:

(1) first we define $h : \mathbb{Z}G \to fM$ taking $\lambda \in \mathbb{Z}G$ to the finite subset $h(\lambda) := \mathrm{supp}(\lambda) \cdot b \subseteq M$;
(2) we extend this to a finitely generated free G-module F with chosen basis by defining $h : F \to fM$ separately as above on each $\mathbb{Z}G$-summand and then taking the union;
(3) this is then applied to the n-skeleton of the resolution $\mathbf{F}$.

Roughly, $\Sigma^n(M; A)$ consists of those points $e \in \partial M$ whose horoballs at e, $H_e \subseteq \partial M$, have the property that every cycle z in the $(n-1)$-skeleton $\mathbf{F}^{(n-1)}$ with $h(z) \subseteq H_e$ bounds a chain in $\mathbf{F}^{(n)}$ with $h(c) \subseteq H'_e$ where H'_e depends only on (and is only slightly larger than) H_e.

1·3. *The dynamical invariant*

Given a base point $b \in M$ we use the Busemann function $\beta_e : M \to \mathbb{R}$, normalised by $\beta_e(b) = 0$, to measure the effect of the G-action and chain endomorphisms on the images $h(c)$ of the elements $c \in \mathbf{F}$ of dimension $\leq n$. Specifically, we consider chain endomorphisms $\varphi : \mathbf{F} \to \mathbf{F}$ with the properties:

(1) φ lifts the identity map of A;
(2) the difference $\beta_e(h(\varphi(c))) - \beta_e(h(c))$ has a positive lower bound as c runs through the n-skeleton $\mathbf{F}^{(n)}$ of $\mathbf{F}$.

We call φ a "push" of the n-skeleton towards e. In the classical case, [**BR88**, theorem 4·1] shows that the existence of such a G-equivariant push is equivalent to $e \in \Sigma^n(G; A)$ – a key fact often referred to as the "Σ^n criterion". Here in the $CAT(0)$ situation we find that the set of boundary points $e \in \Sigma^n(M; A)$ for which a G-equivariant push $\varphi : \mathbf{F}^{(n)} \to \mathbf{F}^{(n)}$ towards e exists is in general a proper subset – potentially interesting but not sufficiently closely related to $\Sigma^n(M; A)$ since it vanishes in some of the most interesting examples. In [**BG16**] (which was about $\Sigma^0(M; A)$ in the $CAT(0)$ case) we introduced the notion of G-finitary homomorphisms between G-modules. These are more general than G-homomorphisms but still share their coarse metric properties. We define the subset ${}^\circ\Sigma^n(M; A) \subseteq \Sigma^n(M; A)$ to be

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the set of those points $e \in \partial M$ such that there is a G-finitary push of the n-skeleton towards e, and we call it the "dynamical invariant". By using G-finitary chain homotopies in higher dimensional homological algebra arguments, we prove:

THEOREM 1·1 (Σ^n Criterion).

$$^\circ\Sigma^n(M;\, A) = \{e \in \partial M \mid \mathrm{cl}(Ge) \subseteq \Sigma^n(M;\, A)\}.$$

(Here, closure is taken in the cone topology on ∂M.)

In the classical theory G acts trivially on ∂M, hence Theorem 1·1 is a true generalisation of the classical Σ^n criterion, and, just as back then, it is again the fundamental tool for all further results.

1·4. *The main results*

THEOREM 1·2. *Assume that the isometric action ρ of G on M is cocompact and that its orbits are discrete subsets of M. Then $\Sigma^n(M;\, A) = \partial M$ if and only if A has type FP_n as a G_b-module, where b is any point of M and G_b denotes its stabiliser.*

We have openness theorems as follows:

THEOREM 1·3. (i) *With $\rho \in \mathrm{Hom}(G, \mathrm{Isom}(M))$ an isometric action of G on M as above, if $\Sigma^n({}_\rho M;\, A) = \partial M$ then there is a neighbourhood N of ρ in this space (with the compact-open topology) such that $\Sigma^n({}_{\rho'} M;\, A) = \partial M$ for all $\rho' \in N$.*

(ii) *$^\circ\Sigma^n({}_\rho M;\, A)$ is open in the Tits metric topology on ∂M.*

$\Sigma^n_\rho(M;\, A)$ is not in general open in ∂M with the cone topology.

The next two theorems concerning the dynamical invariant $^\circ\Sigma^n(M;\, A)$ – a homological description and a product formula – are new, even in the 0-dimensional case [**BG16**].

Given a base point $b \in M$ and a boundary point $e \in \partial M$ we write $\widehat{\mathbb{Z}G}^e$ for the set all infinite sums $\sigma = \Sigma_{g\in G} a_g g$ with the property that each horoball HB_e at e contains all but a finite subset of $\mathrm{supp}(\sigma)b \subseteq M$. This $\widehat{\mathbb{Z}G}^e$ is a right G-module which we call the *Novikov module at e*.

THEOREM 1·4. *If A is a $\mathbb{Z}G$-module of type FP_n then*[1]

$$^\circ\Sigma^n(M;\, A) = \{e \in \partial M \mid \ \mathrm{Tor}_k(\widehat{\mathbb{Z}G}^{e'}, A) = 0 \textit{ for all } e' \in \mathrm{cl}Ge \textit{ and all } k \leq n\}.$$

If (M', A') is second pair consisting of a proper $CAT(0)$ space M' and an abelian group A', both acted on by a group H, then we have a corresponding pair $(M \times M', A \otimes A')$ with the obvious $G \times H$ action. Assuming A and A' are of type FP_n as G- [resp. H-] modules, we can take advantage of the identification $\partial(M \times M') = \partial M \star \partial M'$ to ask for a formula expressing $\Sigma^*(M \times M';\, A \otimes A')$ in terms of $\Sigma^*(M;\, A)$ and $\Sigma^*(M';\, A')$. While there is no intrinsic relationship between the subsets $\Sigma^*(M;\, A)$ and $\Sigma^*(M';\, A')$ of $\partial(M \times M')$, the tensor product $A \otimes A'$ with ground ring $\mathbb{Z}$ could be zero. Hence there is no hope for a simple formula without restrictions on the modules. For this reason we replace the ground ring $\mathbb{Z}$

[1]The special case of Theorem 1·4 when $M = G_{ab} \otimes \mathbb{R}$ is Euclidean is proved in Pascal Schweitzer's appendix to [**Bie07**].

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by a field K in our product formula, and we interpret the Σ-invariants correspondingly. As usual, formulas for these invariants are best expressed in terms of their complements $\Sigma^c = \partial M - \Sigma$.

THEOREM 1·5. *Let K be a field and let A, A' be KG- (resp. KH-)modules of type F_n, with the additional assumption that ${}^\circ\Sigma^0(M; A) = \partial M$ and ${}^\circ\Sigma^0(M'; A') = \partial M'$. Then*

$$^\circ\Sigma^n(M \times M'; A \otimes_K A')^c = \bigcup_{p=0}^{n} {}^\circ\Sigma^p(M; A)^c * {}^\circ\Sigma^{n-p}(M'; A')^c.$$

Remarks 1·6.

(1) Theorem 1·5 extends our product formula for $\Sigma^n(G \times H; K)$ in [**BG10**].

(2) In the discrete case the assumption that $\Sigma^0(M; A) = \partial M$ is equivalent to saying that the G-action on M is cocompact and that A is finitely generated over any point stabilizer; see [**BG16**]. For more details see the remarks in Section 12.

(3) An early (sometimes forgotten) product formula for the original Bieri–Strebel invariant, defined for modules over finitely generated abelian groups, is not covered by Theorem 1·5. It asserts that for such groups G and H and arbitrary KG- and KH-modules A and A',

$$\Sigma^0(G \times H; A \otimes_K A')^c = \Sigma^0(G; A)^c * \Sigma^0(H; A')^c. \qquad (*)$$

It is a fact that when G is finitely generated and abelian then $\Sigma^n(G; A) = \Sigma^0(G; A)$ for all $n \geq 0$. So, in this "classical case" Theorem 1·5 holds without any restriction on Σ^0. The formula $(*)$ appeared in [**BGr82**] as the key to proving that every metabelian group of type FP_∞ is virtually of type FP. It would be highly interesting to have a general formula for $\Sigma^n(G \times H; A \otimes_K A')$ that explains the role of $\Sigma^0(G; A)$ and $\Sigma^0(H; A')$.

2. *Finitary homological algebra*

We use the symbol fS to denote the set of all finite subsets of a given set S.

An additive homomorphism $\varphi : A \to B$ between G-modules[2] is *G-finitary* (or just *finitary*) if it is captured by a G-map $\Phi : A \to fB$, in the sense that $\varphi(a) \in \Phi(a)$ for every $a \in A$. We call the G-map Φ a *G-volley* (or just a *volley*) φ, and we say that φ is a *selection* from the volley Φ.

Two volleys $\Phi : A \to fB$ and $\Psi : B \to fC$ can be "composed" to give the volley $\Psi\Phi : A \to fC$ defined by $\Psi\Phi(s) := \bigcup_{t \in \Phi(s)} \Psi(t)$. A G-map $\varphi : A \to B$ may be regarded as the G-volley which assigns to every element $s \in S$ the singleton set $\{\varphi(s)\}$. Hence G-volleys and G-homomorphisms can be composed in the above sense.

Every G-homomorphism is, of course, G-finitary, but G-finitary homomorphisms are much more general. Unlike a G-homomorphism, a G-finitary map $\varphi : A \to B$ is not uniquely determined by its values on a $\mathbb{Z}G$-generating set X of A; however, the possible values on $a = gx$ (where $g \in G$ and $x \in X$) are restricted to be in the finite set $\Phi(a) = g\Phi(x) \subseteq g\Phi(X)$.

[2] Until Section 11 the ground ring in this paper will be $\mathbb{Z}$.

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LEMMA 2·1. *If $\varphi : A \to B$ and $\psi : B \to C$ are G-finitary, so is the composition $\psi\varphi : A \to C$.*

Proof. If $\varphi : A \to B$ and $\psi : B \to C$ are selections from the volleys $\Phi : A \to fB$, $\Psi : B \to fC$, respectively, then $\psi\varphi : A \to C$ is a selection from the composed volley $\Psi\Phi : A \to fC$.

Thus there is a G-finitary category of G-modules.

By a *based* free $\mathbb{Z}G$-module we mean a free (left) G-module F with a specified basis. We write $F = F_X$ when we wish to emphasise this basis X. The letter Y will always stand for the induced $\mathbb{Z}$-basis $Y = GX$.

Example 2·2. In this paper a G-volley will usually be given on a based free G-module F_X. Indeed, if B is an arbitrary G-module, every map $\Phi : X \to fB$ extends to a *canonical volley* $\Phi : F_X \to fB$ as follows: On elements $y = gx$ of the $\mathbb{Z}$-basis $Y = GX$, Φ is uniquely determined by G-equivariance: $\Phi(y) := g\Phi(x)$; and for arbitrary elements $c = \sum_{y \in Y} n_y y \in F_X$, in the unique expansion, we put

$$\Phi(c) := \sum_{y\in Y} n_y \Phi(y) := \left\{ \sum_{y\in Y} n_y b_y \mid b_y \in \Phi(y), \text{ for all } y \in Y \right\}.$$

It is straighforward to check that $\Phi(gc) = g\Phi(c)$. We call $\Phi : F_X \to fB$ the *canonical G-volley induced by* $\Phi : X \to fB$.

With respect to the canonical G-volley, finitary homomorphisms are easy to construct on F_X: a finitary homomorphism $\varphi : F_X \to B$ can be given by first choosing $\Phi(x) \in fB$ for each $x \in X$, and then picking $\varphi(gx) \in g\Phi(x)$ for all $(g, x) \in G \times X$.

Examples 2·3. (1) If $H \leq G$ is a subgroup of finite index, and A, B are G-modules then every H-homomorphism $\varphi : A \to B$ is G-finitary.

(2) If $N \leq G$ is a finite normal subgroup, and A is a G-module then the additive endomorphism of A given by multiplication by $\lambda \in \mathbb{Z}N$ is G-finitary.

For more details, see [**BG16**].

2·1. *Graded volleys and finitary chain maps*

In order to extend the results of [**BG16**] to higher dimensions we need to know that a Comparison Theorem (Theorem 2·4) for projective resolutions is available for G-finitary homomorphisms.

Here are our standing notations and conventions:

Until Section 11, $\mathbf{F} \twoheadrightarrow A$ denotes a free $\mathbb{Z}G$-resolution of the G-module A by free G-modules F_k, i.e. $\mathbf{F}$ is graded as $\bigoplus_{k\geq 0} F_k$ with boundary morphism $\partial : \mathbf{F} \to \mathbf{F}$; in the final sections we allow more general ground rings than just $\mathbb{Z}$. Usually, the free G-modules F_k will be finitely generated in dimensions $\leq n$. The truncation of $\mathbf{F}$ obtained by setting $F_k = 0$ when $k > n$ is the *n-skeleton* of $\mathbf{F}$ and is denoted by $\mathbf{F}^{(n)}$. Our free resolutions are *based*, meaning that each F_k comes with a specified free $\mathbb{Z}G$-basis X_k. We write Y_k for the induced $\mathbb{Z}$-basis

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GX_k. We write X and Y for the unions of the X_k and of the Y_k respectively. Motivated by topology, we often refer to members of $\mathbf{F}$ as *chains* and to members of Y as *cells*.

The augmentation morphism is $\epsilon : F_0 \twoheadrightarrow A$. The corresponding *augmented resolution* is the acyclic chain complex $\mathbf{F} \twoheadrightarrow A$.

The based free resolution $\mathbf{F} \twoheadrightarrow A$ is *admissible* if its basis X has the feature that $\partial x \neq 0$ for every $x \in X$, and $\epsilon(x) \neq 0$ for every $x \in X_0$. It is easy to replace an arbitrary basis X by a basis which makes $\mathbf{F}$ admissible – either by deleting basis elements x with $\partial x = 0$, or by replacing them by $x + x'$ if there is some $x' \in X$ with $\partial x' \neq 0$. *We will always assume that our based free resolutions are admissible.*

THEOREM 2·4 (Finitary Comparison Theorem). *Let $\mathbf{F} \twoheadrightarrow A$ and $\mathbf{F}' \twoheadrightarrow A'$ be admissible free resolutions of G-modules A and A'. Then every G-finitary homomorphism $f : A \to A'$ can be lifted to a G-finitary chain homomorphism $\varphi : \mathbf{F} \to \mathbf{F}'$ and any two such lifts are chain homotopic by a G-finitary chain homotopy.*

In view of the straightforward construction of G-finitary maps on a based free G-module (see Example 2·2) the first part of the theorem — the existence of a lift — is obvious. The rest of this section is concerned with the second assertion.

To get control of G-finitary chain maps and chain homotopies on free resolutions $\mathbf{F} \to \mathbf{F}'$ we need volleys $\Phi : \mathbf{F} \to f\mathbf{F}'$ of chain complexes. We assume such volleys to be graded of some degree k, i.e., $\Phi(F_n) \subseteq F'_{n+k}$, for all $n \geq 0$; but we do not require compatibility with the differentials. However, degree 0 volleys $\Phi : \mathbf{F} \to f\mathbf{F}'$ will only be used in connection with chain maps, and hence *in degree 0 a selection will always be understood to be a chain-map-selection from* Φ. And *graded volleys will always be understood to be degree 0 unless some other degree is specified*. We say that the volley $\Phi : \mathbf{F} \to f\mathbf{F}'$ *induces* the G-homomorphism $f : A \to A'$, if, for each $c \in F_0$, $\epsilon'\Phi(c)$ is (the singleton) $f\epsilon(c)$. This implies that all chain-map-selections from Φ induce f.

PROPOSITION 2·5. *Let $\mathbf{F} \twoheadrightarrow A$ and $\mathbf{F}' \twoheadrightarrow A'$ be admissible free resolutions of G-modules A and A', and $\Phi : \mathbf{F} \to f\mathbf{F}'$ a degree 0 G-volley, inducing the zero map $A \to A'$, i.e., with $\epsilon'\Phi(F_0) = 0$. Then there is a degree 1 G-volley, $\Sigma : \mathbf{F} \to f\mathbf{F}'$, with the property that every chain map φ which is a selection from Φ is homotopic to 0 by a homotopy which is a selection from Σ.*

Proof. We construct the volleys $\Sigma : \mathbf{F}^{(n)} \to f\mathbf{F}'^{(n+1)}$ by induction on n, starting with $n = 0$. As im $\Phi_0 \subseteq \ker \epsilon' = \operatorname{im} \partial'_1$, we can find, for each element x of the G-basis X_0 of F_0, a finite subset $\Sigma_0(x) \subseteq F'{}_1$ with $\partial'\Sigma_0(x) = \Phi_0(x)$. This defines a canonical G-volley $\Sigma_0 : F_0 \to fF'_1$, and by G-equivariance we have $\partial'\Sigma_0(y) = \Phi_0(y)$ for all $y \in Y_0 = GX_0$. Selections are determined by their restrictions to the $\mathbb{Z}$-basis $Y_0 = GX_0$, so for each selection φ from Φ_0 there is a selection σ from Σ_0, with $\varphi = \partial'\sigma$.

Now we take $n \geq 1$, assuming the volley $\Sigma \mid \mathbf{F}^{(n-1)} : \mathbf{F}^{(n-1)} \to f\mathbf{F}'^{(n)}$ is already constructed, with the property that every (chain-map) selection $\varphi \mid \mathbf{F}^{(n-1)} : \mathbf{F}^{(n-1)} \to f\mathbf{F}'^{(n-1)}$ is homotopic to zero by a homotopy which is a chain-homotopy selection from $\Sigma \mid \mathbf{F}^{(n-1)}$. For every chain-map-selection φ from $\Phi \mid \mathbf{F}^{(n)}$ there are (possibly several) selections σ from $\Sigma \mid \mathbf{F}^{(n-1)}$ with $\varphi \mid \mathbf{F}^{(n-1)} = \partial'\sigma + \sigma\partial$. We consider all of them and use them to define, for each $c \in F_n$,

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$$\Gamma(c) := \{\varphi(c) - \sigma\partial(c) \mid \varphi \text{ is a selection from } \Phi : \mathbf{F}^{(n)} \to f\mathbf{F}'^{(n)}, \text{ and } \sigma \text{ is a selection from } \Sigma : \mathbf{F}^{(n-1)} \longrightarrow f\mathbf{F}'^{(n)}, \text{ with } \varphi \mid \mathbf{F}^{(n-1)} = \partial'\sigma + \sigma\partial\}. \tag{2·1}$$

We claim that $\Gamma : F_n \to f F'_n$ is a G-volley. To see this, let $g \in G$, and let φ and σ be as in (2·1). Then[3] $(g\varphi) \mid F^{(n-1)} = g(\varphi \mid F^{(n-1)}) = g(\partial'\sigma + \sigma\partial) = \partial'(g\sigma) + (g\sigma)\partial$. Moreover, since φ and σ are selections from the G-volleys $\Phi \mid \mathbf{F}^{(n)}$ resp. $\Sigma \mid \mathbf{F}^{(n-1)}$, so are $g\varphi$ and $g\sigma$. Hence $(g(\varphi - \sigma\partial))(c) = g((\varphi - \sigma\partial)(g^{-1}(c)) \in \Gamma(c)$, for all c. Replacing c by gc shows $g\Gamma(c) \subseteq \Gamma(gc)$, and replacing g by g^{-1} establishes the opposite inclusion. This shows that Γ is G-equivariant. As Γ is given in terms of the $\mathbb{Z}$-homomorphisms $\varphi - \sigma\partial$, the requirements for a volley hold.

Now we claim that $\partial'\Gamma = 0$. Indeed, with φ and σ as in (2·1), we find for all $c \in F_n$,

$$\partial'(\varphi(c) - \sigma\partial(c)) = \varphi\partial(c) - \partial'\sigma\partial(c) = \varphi\partial(c) - (\varphi \mid \mathbf{F}^{(n-1)} - \sigma\partial)\partial(c) = 0.$$

For every basis element $x \in X_n$, we can now choose a finite subset $\Sigma_n(x) \subseteq F'_{n+1}$, with $\partial'\Sigma_n(x) = \Gamma(x)$. This defines a canonical G-volley $\Sigma_n : F_n \to f F'_{n+1}$, extending $\Sigma \mid \mathbf{F}^{(n-1)}$ to $\Sigma \mid \mathbf{F}^{(n)}$. By G-equivariance we have $\partial'\Sigma_n(y) = \Gamma(y)$, for all $y \in Y_n = GX_n$.

Let φ be a chain-map-selection from $\Phi \mid \mathbf{F}^{(n)}$. By induction there is a selection σ from $\Sigma \mid \mathbf{F}^{(n-1)}$, with $\varphi \mid \mathbf{F}^{(n-1)} = \partial'\sigma + \sigma\partial$. Then $\varphi(c) - \sigma\partial(c) \in \Gamma(c)$, for every $c \in F_n$, hence $\gamma = (\varphi - \sigma\partial) \mid F_n$ is a selection from $\Gamma : F_n \to f F'_n$. Since selections are determined by their restrictions to the $\mathbb{Z}$-basis $Y_n = GX_n$, there is a selection $\sigma_n : F_n \to F'_{n+1}$ from Σ_n with $\partial'\sigma_n = \gamma = (\varphi - \sigma\partial) \mid F_n$. Thus σ_n extends σ to a selection τ from $\Sigma \mid \mathbf{F}^{(n)}$, with $\varphi = \partial'\tau + \tau\partial$, as asserted.

COROLLARY 2·6. *Let $\mathbf{F} \twoheadrightarrow A$ and $\mathbf{F}' \twoheadrightarrow A'$ be admissible free resolutions of G-modules A and A', Φ, $\Psi : \mathbf{F} \to f\mathbf{F}'$ two degree 0 volleys. Then there is a degree 1 volley, $\Sigma : \mathbf{F} \to f\mathbf{F}'$, with the property that any two chain-map-selections φ from Φ and ψ from Ψ, inducing the same G-homomorphism $f : A \to A'$, are homotopic by a chain-homotopy-selection of Σ.*

Proof. Consider the map $\Gamma : \mathbf{F} \to f\mathbf{F}'$, given by

$$\Gamma(c) = \{\varphi(c) - \psi(c) \mid \varphi, \psi \text{ chain-map-selections of } \Phi, \text{ resp. } \Psi, \text{ both lifting } f\}.$$

Then Γ is a degree 0 volley inducing the zero map, and $\epsilon'\Gamma(F_0) = 0$. Hence the Corollary follows from Proposition 2·5.

The Finitary Comparison Theorem 2·4 follows from Corollary 2·6.

3. *Controlled based free resolutions*

3·1. *The control space*

Throughout the paper (M, d) is a proper non-compact $CAT(0)$ metric space. The closed ball of radius r centered at b is denoted by $B_r(b)$. We write $\rho : G \to \mathrm{Isom}(M)$ for an action of the group G on M by isometries. Unless specified, there are no further assumptions about

[3]Recall that the natural G-action on $\mathrm{Hom}_{\mathbb{Z}}(A, B)$ is defined by $(g\varphi)(a) = g\varphi(g^{-1}a)$. It follows that if φ is a selection from Φ so is $g\varphi$.

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ρ; its orbits might be indiscrete, and it might not be cocompact[4]. Except in connection with the Openness Theorem in Section 10, the action ρ is fixed throughout.

The boundary of M at infinity, denoted by ∂M, is the set of asymptoty classes of geodesic rays in M. It is assumed to carry the (compact metrisable) cone topology, unless it is clear from context that ∂M is being considered with the Tits metric topology. If γ is a geodesic ray in M determining $e \in \partial M$ we write $\gamma(\infty) = e$. (For given e there is such a γ with $\gamma(0)$ arbitrary.) We write $\beta_\gamma : M \to \mathbb{R}$ for the Busemann function[5] determined by γ and we write $HB_{(\gamma,t)}$ for the (closed) horoball about e determined by the point $\gamma(t)$. Usually we are interested in a difference of the form $\beta_\gamma(p) - \beta_\gamma(q)$ and such a difference depends on e, rather than on the particular choice of γ with $\gamma(\infty) = e$.

3·2. *Controlled based free G-modules*

The *support* $c \in F_X$, $\mathrm{supp}(c) \subseteq Y$, is the set of all $y \in Y = GX$ occurring in the unique expansion of c over $\mathbb{Z}$. By a *control map* on F we mean a G-map $h : F \to fM$ given[6] by composing the support function $\mathrm{supp} : F \to fY$ with an arbitrary G-equivariant map $fY \to fM$, where $h(0)$ is defined to be the empty set. Thus h is uniquely given by its restriction $h|X : X \to fM$. We will always assume that our control maps h are *centerless* in the sense that $h(x)$ is non-empty for all $x \in X$ (and hence $h(c) \neq \emptyset$ for all $0 \neq c \in F$). Controlled free based G-modules and resolutions are always understood to come with a specific control map h.

3·3. *Valuations on free modules*

Let the point $e \in \partial M$ be determined by the geodesic ray $\gamma : [0, \infty) \to M$. Composition of the control map $h : F \to fM$, with the Busemann function $\beta_\gamma : M \to \mathcal{R}$ assigns to each element of F a finite set of real numbers; taking minima defines the function

$$v_\gamma := \min \beta_\gamma h : F \to \mathcal{R} \cup \{\infty\}. \tag{3·1}$$

In particular $v_\gamma(c) = \infty$ if and only if $c = 0$.

Following [BR88] we call v_γ a *valuation* on F.

LEMMA 3·1. (i) $v_\gamma(-c) = v_\gamma(c)$, *for all* $c \in F$.

(ii) $v_\gamma(c + c') \geq \min\{v_\gamma(c), v_\gamma(c')\}$, *for all* $c, c' \in F$.

(iii) $v_\gamma(c) = v_{g\gamma}(gc)$, *for all* $c \in F$, $g \in G$.

(iv) *If c and c' are non-zero then $v_\gamma(c) - v_\gamma(c')$ depends only on the endpoint $\gamma(\infty) = e$, not on the ray γ, and $|v_\gamma(c) - v_\gamma(c')| \leq d_H(h(c), h(c'))$ where d_H denotes Hausdorff distance.*

Once we have picked the control map $h : \mathbf{F} \to fM$, our free resolution is equipped with a valuation $v_\gamma := \min \beta_\gamma h : \mathbf{F} \to \mathcal{R} \cup \{\infty\}$, for each geodesic ray $\gamma : [0, \infty) \to M$. On the augmented resolution we have $v_\gamma(a) = \infty$, for each $a \in A$. In our applications the finitely generated free module F will be the n-skeleton $\mathbf{F}^{(n)}$ of a free resolution $\mathbf{F} \twoheadrightarrow A$ where A is a G-module.

[4] The action is *cocompact* if there is a compact subset D such that every point lies in the translate of D under some isometry.

[5] Our convention is that $\beta_\gamma(x)$ goes to $+\infty$ as x approaches e.

[6] Recall that we write fM for the G-set of all finite subsets of M.

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Example 3·2. This example comes from topology. Take $A = \mathbb{Z}$ and $\mathbf{F} = C_*(\widetilde{K})$, the integral simplicial chains in the universal cover of a simplicial $K(G, 1)$-complex K. In this case $\mathbf{F}$ comes with a canonical $\mathbb{Z}$-basis, the simplexes of $\widetilde{K}$, and we can define a G-map $\widehat{h}: C_*(\widetilde{K}) \to fM$ on each simplex σ of K by

$$\widehat{h}(\sigma) = h(\{\text{vertices of } \sigma\}).$$

Remark. In this example we have $h(\partial c) \subseteq h(c)$ for each $c \in C_*(K)$. As a consequence we have $v_\gamma(\partial c) \geq v_\gamma(c)$, so that the chains with non-negative valuation form a subcomplex. One could mimic that in the general situation by first choosing $h(x) \neq \emptyset$ for each $x \in X_0$, and then defining $h(x)$ on the higher skeleta by $h(x) := h(\partial x)$. However, there is no need for this in general and so our control maps can ignore the boundary aspect of the resolution.

4. *Controlling homomorphisms over M*

4·1. *Controlling homomorphisms on free modules*

Let the based free modules F_X and $F'_{X'}$ be endowed with control maps h and h' mapping to M. We want to measure how far, in terms of the metric d on M, an additive homomorphism $\varphi: F \to F'$ moves the members of F. We define the *norm* of φ by

$$||\varphi|| := \inf\{r \geq 0 \mid h'(\varphi(c)) \subseteq N_r(h(c));\, c \in F\} \in \mathbb{R} \cup \{\infty\} \tag{4·1}$$

the shift function towards e, $\text{sh}_{\varphi,e}: F \to \mathbb{R} \cup \{\infty\}$, by

$$\text{sh}_{\varphi,e}(c) := v'_\gamma(\varphi(c)) - v_\gamma(c) \in \mathbb{R} \cup \{\infty\},\ \ c \in F, \tag{4·2}$$

and the *guaranteed shift towards e* by,

$$\text{gsh}_e(\varphi) := \inf\{\text{sh}_{\varphi,e}(c) \mid c \in F\}. \tag{4·3}$$

We call a $\mathbb{Z}$-submodule $L \leq F_X$ *cellular* if it is generated by $L \cap Y$. Sometimes L will be given, and we will be interested in the norm or guaranteed shift of $\varphi \mid L$. To have information for that case we include L in the next lemmas.

LEMMA 4·1. *Let $\varphi: L \to F'$ be the restriction to L of an additive homomorphism $F \to F'$.*

(i) $\text{sh}_{\varphi,e}(y) \geq -||\varphi||$, *for all $y \in L \cap Y$; hence* $\text{gsh}_e(\varphi) \geq -||\varphi||$.
(ii) $||g\varphi|| = ||\varphi||$, $\text{sh}_{g\varphi,ge} = \text{sh}_{\varphi,e}$ *and* $\text{gsh}_{ge}(g\varphi) = \text{gsh}_e(\varphi)$, *for all $g \in G$.*

LEMMA 4·2. *Let $\varphi: F \to F'$ and $\psi: F' \to F''$ be two additive endomorphisms, and let $K \leq F$ and $L \leq F'$ be cellular $\mathbb{Z}$-submodules with $\psi(K) \subseteq L$. Then*

$$\text{gsh}_e(\varphi|L \circ \psi|K) \geq \text{gsh}_e(\varphi|K) + \text{gsh}_e(\psi|L).$$

In particular,

$$\text{gsh}_e(\varphi^k) \geq k \cdot \text{gsh}_e(\varphi),\ \ \textit{for all natural numbers } k.$$

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Proof.

$$\begin{aligned}\mathrm{gsh}_e(\varphi \mid L \circ \psi \mid K) &= \inf_{c\in K}\{v''_\gamma(\varphi\psi(c)) - v_\gamma(c)\}\\ &= \inf_{c\in K}\{v''_\gamma(\varphi\psi(c)) - v'_\gamma(\psi(c)) + v'_\gamma(\psi(c)) - v_\gamma(c)\}\\ &\geq \inf_{c\in K}\{v''_\gamma(\varphi\psi(c)) - v'_\gamma(\psi(c))\} + \inf_{c\in K}\{v'_\gamma(\psi(c)) - v_\gamma(c)\}\\ &\geq \inf_{b\in L}\{v''_\gamma(\varphi(b) - v'_\gamma(b)\} + \inf_{c\in K}\{v'_\gamma(\psi(c)) - v_\gamma(c)\}\\ &= \mathrm{gsh}_e(\varphi|L) + \mathrm{gsh}_e(\psi|K).\end{aligned}$$

We say that an additive endomorphism $\varphi : F \to F$ *pushes* L *towards* $e \in \partial M$, and we call φ a *push towards* e, if the guaranteed shift of $\varphi|L$ towards e is positive; i.e., $\mathrm{gsh}_e(\varphi|L) > 0$.

4·2. *Pushing submodules towards limit points of orbits in* ∂M

When φ is G-finitary then $||\varphi||$ and $\mathrm{gsh}_e(\varphi)$ are finite. When Φ is a finite G-volley we call the number $||\Phi|| := \inf\{r \geq 0 \mid h'(\Phi(c)) \subseteq N_r(h(c))\}$ the *norm* of Φ. Then $||\varphi|| \leq ||\Phi||$ for all selections φ from Φ.

In this subsection we assume that the cellular submodule $L \leq F$ is in fact a $\mathbb{Z}G$-submodule. It will then be generated, as a $\mathbb{Z}G$-module, by $X' = L \cap X \subseteq X$. The map φ is assumed to push the G-submodule L towards e with guaranteed shift $\delta > 0$; it follows that the G-translate $g\varphi$ of φ pushes L with the same guaranteed shift δ towards ge. In the special case when $\varphi|L$ is G-finitary we can do better: given any $\hat{e} \in \mathrm{cl}(Ge)$, the closure of the G-orbit $Ge \in \partial M$, we can still construct endomorphisms pushing towards $\hat{e}$ which are "approximated" by G-translates of $\varphi \mid L$:

THEOREM 4·3. *Let $L \leq F$ be a cellular G-submodule of $F = F_X$ and let $\varphi : L \to F$ be a selection from the finite G-volley $\Phi : L \to fF$ with $\mathrm{gsh}_e(\varphi) = \delta > 0$. Then for every endpoint $\hat{e} \in \mathrm{cl}(Ge)$ there is a selection $\psi : L \to F$ of Φ with $\mathrm{gsh}_{\hat{e}}(\psi) \geq \delta/2$. In fact, this can be done so that on each finitely generated $\mathbb{Z}$-submodule $L' \subseteq L$, ψ coincides with some G-translate $g\varphi$.*

The proof can be found in [**BG16**].

5. *The dynamical invariants* $^\circ\Sigma^n(M; A)$

Let $\mathbf{F} \twoheadrightarrow A$ be a controlled based free resolution of the G-module A, with finitely generated n-skeleton $\mathbf{F}^{(n)}$. For $n \geq 0$ we define the nth *dynamical invariant* of the pair (M, A) to be

$$^\circ\Sigma^n(M;A) := \{e \in \partial M \mid \text{there is a } G\text{-finitary chain map inducing } \mathrm{id}_A \text{ pushing } \mathbf{F}^{(n)} \text{ towards } e\}.$$

PROPOSITION 5·1 (Invariance). *Let $e \in \partial M$. The existence of a G-finitary chain map $\varphi : \mathbf{F}^{(n)} \to \mathbf{F}^{(n)}$ inducing id_A and pushing $\mathbf{F}^{(n)}$ towards e is independent of the choice of the resolution $\mathbf{F} \twoheadrightarrow A$ and of the control map $h : \mathbf{F} \to fM$. In other words, $^\circ\Sigma^n(M; A)$ is well defined.*

Proof. Let $\mathbf{F}' \twoheadrightarrow A$ be a second such resolution with finitely generated n-skeleton. The identity map id_A can be lifted to G-chain homomorphisms $\alpha : \mathbf{F} \to \mathbf{F}'$, $\beta : \mathbf{F}' \to \mathbf{F}$ which are

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chain homotopy inverse to one another. Assume there exists a G-finitary push $\varphi : \mathbf{F}^{(n)} \to \mathbf{F}^{(n)}$ lifting id_A. Then $\alpha\varphi\beta : \mathbf{F}'^{(n)} \to \mathbf{F}'^{(n)}$ is a G-finitary chain endomorphism lifting id_A. By Lemmas 4·1 and 4·2, $\mathrm{gsh}_e(\alpha\varphi^k\beta) \geq -||\alpha|| + k \cdot \mathrm{gsh}_e(\varphi^k) - ||\beta||$. If we choose k large enough to ensure that $k \cdot \mathrm{gsh}_e(\varphi) > ||\alpha|| + ||\beta||$, the map $\alpha\varphi^k\beta : \mathbf{F}'^{(n)} \to \mathbf{F}'^{(n)}$ becomes a G-finitary push towards e. This shows that the existence of a finitary push towards e lifting id_A is independent of the particular free resolution. Independence of the control map is proved as a special case: take $\mathbf{F} = \mathbf{F}'$, α an automorphism, and β the inverse of α.

The set ${}^\circ\Sigma^n(M; A)$ is invariant under the topological action of G on ∂M induced by the isometric action of G on M. For inductions we define ${}^0\Sigma^{-1}(M; A) = \partial M$.

A slight adaptation of Theorem 4·3 yields a closure result:

THEOREM 5·2. *The G-set ${}^\circ\Sigma^n(M; A)$ contains the closure of each of its orbits.*

Proof. Let $\varphi : \mathbf{F}^{(n)} \to \mathbf{F}^{(n)}$ be a G-finitary chain map pushing $\mathbf{F}^{(n)}$ towards $e \in {}^\circ\Sigma^n(M; A)$. The proof of Theorem 4·3 constructs a G-finitary map $\psi : \mathbf{F}^{(n)} \to \mathbf{F}^{(n)}$ pushing towards an arbitrary point of the closure of Ge with the property that for every finitely generated $\mathbb{Z}$-submodule $L \leq \mathbf{F}^{(n)}$ there is some element $g \in G$, with $\psi \mid L = (g\varphi) \mid L$. It remains to show that ψ is a chain map. But since $g\varphi$ is a chain map, so is $\psi \mid L$, for each L. This suffices.

6. *The geometric invariants* $\Sigma^n(M; A)$

Here we define the nth geometric invariant $\Sigma^n(M; A)$. It is the strict homological analog of the "homotopical" invariant $\Sigma^n(\rho)$ described in [**BGe03**].

6·1. *Controlled acyclicity*

Recall that once we have picked the control map $h : \mathbf{F} \to fM$, our free resolution is equipped with valuation $v_\gamma := \min \beta_\gamma h : \mathbf{F} \to \mathcal{R} \cup \{\infty\}$, for each geodesic ray $\gamma : [0, \infty) \to M$. On the augmented resolution we have $v_\gamma(a) = \infty$, for each $a \in A$.

Let $n \geq 0$ and let $\gamma(\infty) = e$. We say that the augmented controlled based free resolution $\mathbf{F} \twoheadrightarrow A$ is *controlled* $(n-1)$*-acyclic over* $e \in \partial M$, in short CA^{n-1} over e, if for every real number s there is a *lag* $\lambda(s) \geq 0$, with $s - \lambda(s) \to +\infty$ as $s \to +\infty$, and such that, for $-1 \leq i \leq n-1$, every i-cycle $z \in \mathbf{F}$ with $v_\gamma(z) \geq s$ is the boundary, $z = \partial c$, of an $(i+1)$-chain $c \in \mathbf{F}$ with $v_\gamma(c) \geq s - \lambda(s)$. When $n = 0$ this is to be understood as a condition on the augmented chain complex.

6·2. *Invariance*

Let $(\mathbf{F}, \partial)$ and $(\mathbf{F}', \partial')$ be controlled based free resolutions of the G-module A, with ϵ, ϵ' the corresponding augmentation maps.

PROPOSITION 6·1 (Invariance). *Let* $\mathbf{F}$ *and* $\mathbf{F}'$ *have finitely generated n-skeleta. If* $\mathbf{F}'$ *is* CA^{n-1} *over e, so is* $\mathbf{F}$.

Proof. Consider $\mathbf{F}$ and $\mathbf{F}'$ with bases X, X' and corresponding control maps h and h'. There are G-chain-homomorphisms $\varphi : \mathbf{F} \to \mathbf{F}'$ and $\psi : \mathbf{F}' \to \mathbf{F}$, inducing the identity on A, and a chain-homotopy $\sigma : \mathbf{F} \to \mathbf{F}$, with $\psi\varphi = 1 + \partial\sigma + \sigma\partial$. For $0 \leq i < n$ let z be an i-cycle of $\mathbf{F}$ with $v_\gamma(z) \geq s$. We denote by φ, ψ, σ, and ∂ the chain maps, the homotopy, and the boundary

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as above, but restricted to the n-skeleta. By Lemma 4·1, $v'_\gamma(\varphi(z)) \geq v_\gamma(z) - ||\varphi||$. As $\mathbf{F}'$ is CA^{n-1} over e there is a chain c' in $\mathbf{F}'$, with $\partial' c' = \varphi(z)$, and

$$v'_\gamma(c') \geq v'_\gamma(\varphi(z)) - \lambda(v'_\gamma(\varphi(z))) \geq v_\gamma(z) - ||\varphi|| - \lambda(v'_\gamma(\varphi(z))), \text{ where } \lambda \text{ is independent of } c'.$$

Put $c'' = \psi(c') - \sigma(z)$. Then

$$\partial c'' = \partial\psi(c') - \partial\sigma(z) = \psi\partial'(c') - \partial\sigma(z) = \psi\varphi(z) - \partial\sigma(z) = z + \sigma\partial z = z,$$

and we have,

$$\begin{aligned} v_\gamma(c'') &\geq \min\{v_\gamma(\psi(c')), v_\gamma(\sigma(z))\} \\ &\geq \min\{v'_\gamma(c') - ||\psi||, v_\gamma(z) - ||\sigma||\} \\ &\geq \min\{v_\gamma(z) - ||\varphi|| - \lambda(v'_\gamma(\varphi(z))) - ||\psi||, v_\gamma(z) - ||\sigma||\} \end{aligned}$$

proving that $\mathbf{F}$ is CA^{n-1} over e.

6·3. *The geometric invariants*

We can now introduce the geometric (or Σ-) invariants of the pair (M, A) where the G-module A is of type FP_n. Choosing a free resolution with finitely generated n-skeleton $\mathbf{F} \twoheadrightarrow A$, we define

$$\Sigma^n(M; A) := \{e \in \partial M \mid \mathbf{F} \twoheadrightarrow A \text{ is } CA^k \text{ over } e \text{ for all } k \text{ with } -1 \leq k < n\}.$$

By Theorem 6·1 this is an invariant of $(M; A)$, i.e. $(n-1)$-acyclicity over e is independent of the choice of free resolution $\mathbf{F} \twoheadrightarrow A$ such that $\mathbf{F}^{(n)}$ is finitely generated, and of the choice of control map. In particular, this subset of ∂M is invariant under the topological action of G on ∂M induced by the isometric action ρ of G on M. For proofs using induction on n we define $\Sigma^{-1}(M; A) := \partial M$.

We will use the phrase "$e \in \Sigma^n(M; A)$ *with constant lag* $\lambda \in \mathbb{R}$" if the function $\lambda(s)$ in the definition of CA^{n-1} in Section 6·1 can be taken to be the constant λ. For trivial reasons, all members of $\Sigma^0(M; A)$ have constant lag.

7. *Characterisation of* $^\circ\Sigma^n(M; A)$ *in terms of* $\Sigma^n(M; A)$

In this section we characterize $^\circ\Sigma^n(M; A)$ as a specific subset of $\Sigma^n(M; A)\}$ (Theorem 7·1), and we give conditions under which $^\circ\Sigma^n(M; A) = \Sigma^n(M; A)$ (Theorem 7·4).

7·1. *Statement of the theorem*

THEOREM 7·1 (Characterisation Theorem). *For each G-module A of type FP_n, $n \geq 0$, we have*

$$^\circ\Sigma^n(M; A) = \{e \in \partial M \mid \mathrm{cl}(Ge) \subseteq \Sigma^n(M; A)\}.$$

This shows that $^\circ\Sigma^n(M; A)$ is determined by $\Sigma^n(M; A)\}$.

Remarks. (i) In the special case where all the endpoints $e \in \partial M$ are fixed under the induced action of G on ∂M, Theorem 7·1 implies $\Sigma^n(M; A) = {^\circ\Sigma^n}(M; A)$. Hence Theorem 7·1 is a direct generalisation of the various "Σ-Criteria" found in [**BS80**, proposition 2·1], [**BNS87**, proposition 2·1], [**BR88**, theorem C]. These were the main

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technical tools in all previous stages of Σ-theory. In all those cases, the action of G was by translations on a Euclidean space, so that all end points were fixed.

(ii) The homotopy version of Theorem 7·1 was proved in [**BGe03**].

The proof of Theorem 7·1 will be given in two steps: the inclusion $\subseteq$ follows from Proposition 7·2 together with Theorem 5·2. The other inclusion $\supseteq$ follows from the (stronger) Theorem 7·4, below.

7·2. *From pushing skeletons to constant lag*

We start by proving that ${}^{\circ}\Sigma^n(M; A) \subseteq \Sigma^n(M; A)$; while doing that we will also collect important information on the lag. More precisely, we prove:

PROPOSITION 7·2. *Let $\mathbf{F} \twoheadrightarrow A$ be a controlled based free resolution with finitely generated n-skeleton, $n \geq 0$. Let $\varphi : \mathbf{F}^{(n)} \to \mathbf{F}^{(n)}$ be a G-finitary chain endomorphism inducing id_A, and let $\sigma : \varphi \sim \mathrm{id}_F$ be a G-finitary chain homotopy. If φ pushes $\mathbf{F}^{(n)}$ towards e then the following hold*:

(i) *$e \in \Sigma^n(M; A)$, with constant lag $||\sigma||$, and*

(ii) *If $\mathbf{F}^{(n+1)}$ is finitely generated and e is in $\Sigma^{n+1}(M; A)$ then it is so with constant lag $||\sigma||$.*

Proof. (i) Let z be a j-cycle of $\mathbf{F}$, with $j \leq n-1$, and let c be a chain, with $z = \partial c$. Using $\varphi - \mathrm{id}_F = \sigma\partial + \partial\sigma$, and writing φ^k for the k-th iterate of φ, we find that $\tau := (\varphi^k + \varphi^{k-1} + \cdots \varphi + 1)\sigma$ is a G-finitary homotopy $\varphi^{k+1} \sim \mathrm{id}_F$. So

$$\begin{aligned} \varphi^{k+1}(c) &= c + \partial\tau(c) + \tau(\partial c), \quad \text{hence} \\ z &= \partial c \\ &= \partial\varphi^{k+1}(c) - \partial\tau(\partial c) \\ &= \partial c', \quad \text{where } c' = \varphi^{k+1}(c) - \tau(\partial c). \end{aligned}$$

Let $\mu > 0$ be a guaranteed shift of φ towards e. Using Lemma 4·2 we find

$$v_\gamma(\varphi^k(c)) \geq v_\gamma(c) + k \cdot \mu;$$

hence we may choose k so large that $v_\gamma(\varphi^{k+1}(c)) \geq v_\gamma(z)$. Then

$$v_\gamma(c') \geq \min\{v_\gamma(\varphi^{(k+1)}(c)), v_\gamma(\tau(z))\} \geq \min(v_\gamma(z), v_\gamma(\tau(z))).$$

But

$$\begin{aligned} v_\gamma(\tau(z)) &= v_\gamma((\varphi^k + \varphi^{k-1} + \cdots \varphi + 1)\sigma(z)) \\ &\geq \min\{v_\gamma(\varphi^p\sigma(z)) \mid 0 \leq p \leq k\} \\ &\geq \min\{v_\gamma(\sigma(z)) + p \cdot \mu \mid 0 \leq p \leq k\} \text{ by Lemma 4·2,} \\ &= v_\gamma(\sigma(z)). \end{aligned}$$

So

$$v_\gamma(c') \geq \min(v_\gamma(z), v_\gamma(\sigma(z))).$$

By the definitions in Section 4·1, $v_\gamma(\sigma(z)) \geq v_\gamma(z) - ||\sigma||$; hence $v_\gamma(c') \geq v_\gamma(z) - ||\sigma||$, showing that $e \in \Sigma^n(M; A)$, with lag $||\sigma||$. This proves statement (i).

(ii) The assumption $e \in \Sigma^{n+1}(M; A)$ asserts that $\mathbf{F}$ is CA^n over e. Thus there is a lag $\lambda(s)$ with the property that every n-cycle z with $v_\gamma(z) \geq s$ is the boundary of some $(n+1)$-chain c with $v_\gamma(c) \geq s - \lambda(s)$, and $(s - \lambda(s)) \to \infty$, as $s \to \infty$. We fix an n-cycle z and apply the lag condition to the sequence of n-cycles $\varphi^k(z)$. Put $s_k := v_\gamma(\varphi^k(z))$. As before let $\mu > 0$ be a guaranteed shift of φ towards e. By Lemma 4·2, $s_k \geq v_\gamma(z) + k \cdot \mu$, hence $s_k \to \infty$. Thus we can choose k so that $s_{k+1} - \lambda(s_{k+1}) > s_0 = v_\gamma(z)$. It follows that there is some $(n+1)$-chain c' with $\partial c' = \varphi^{k+1}(z)$ and $v_\gamma(c') \geq s_{k+1} - \lambda(s_{k+1}) > v_\gamma(z)$.

Much as in part (i), and using this new choice of k, we put $\tau := \sigma(\varphi^k + \varphi^{k-1} + \cdots \varphi + 1)$. This is a G-finitary homotopy $\varphi^{k+1} \sim \mathrm{id}_F$. Since z is a cycle we have $\varphi^{k+1}(z) = z + \partial\tau(z)$. Writing $c'' = c' - \tau(z)$ we have $\partial c'' = z$ and

$$\begin{aligned} v_\gamma(c'') &\geq \min\{v_\gamma(c'), v_\gamma(\tau(z)\} \\ &\geq \min\{v_\gamma(z), v_\gamma(\tau(z))\}. \end{aligned}$$

Now,

$$\begin{aligned} v_\gamma(\tau(z)) &= v_\gamma(\sigma(\varphi^k + \varphi^{k-1} + \cdots \varphi + 1)(z)) \\ &\geq \min\{v_\gamma(\sigma\varphi^p(z)) \mid 0 \leq p \leq k\} \\ &\geq \min\{v_\gamma(\varphi^p(z)) - ||\sigma|| \mid 0 \leq p \leq k\}, \quad \text{by Lemma 4·2} \\ &\geq \min\{v_\gamma(z) + p \cdot \mu - ||\sigma|| \mid 0 \leq p \leq k\} \\ &= v_\gamma(z) - ||\sigma||. \end{aligned}$$

Thus $v_\gamma(c'') \geq v_\gamma(z) - ||\sigma||$ and we conclude that $||\sigma||$ is a constant lag for the CA^n-property of $\mathbf{F}$ over e.

If $\Phi : \mathbf{F}^{(n)} \to f\mathbf{F}^{(n)}$ is a G-volley then, by Corollary 2·6, there is a finite degree 1 volley $\Sigma : \mathbf{F}^{(n)} \to f\mathbf{F}^{(n+1)}$ with the property that every selection φ of Φ is chain contractible by a selection σ of Σ. The norm of σ has an upper bound depending only on Σ. This yields the following uniform version of Proposition 7·2.

COROLLARY 7·3. *Let $E \subseteq \partial M$ be a set of endpoints. Let $\Phi : \mathbf{F}^{(n)} \to f\mathbf{F}^{(n)}$ a G-volley inducing id_A, with the property that each $e \in E$ admits a (chain map) selection $\varphi_e : \mathbf{F}^{(n)} \to \mathbf{F}^{(n)}$ of Φ pushing the n-skeleton towards e. Then the following hold*:

(i) *$E \subseteq \Sigma^n(M; A)$, with uniform constant lag, (i.e., the same constant lag for all $e \in E$)*;

(ii) *if $\mathbf{F}^{(n+1)}$ is finitely generated and $E \subseteq \Sigma^{n+1}(M; A)$ then it is so with uniform constant lag.*

Proof. We apply 7·2 to each φ_e; with Σ as above, there is a chain contraction $\sigma_e : \varphi_e \sim \mathrm{id}$ which is a selection from Σ. The lags are therefore independent of e.

7·3. *Closed G-invariant subsets of ∂M*

The next theorem gives conditions under which ${}^\circ\Sigma^n(M; A)$ and $\Sigma^n(M; A)$ agree. In particular, it contains Theorem 7·1.

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THEOREM 7·4. *Let* $\mathbf{F} \twoheadrightarrow A$ *be a controlled based free resolution with finitely generated* n*-skeleton. The following conditions are equivalent for a closed* G*-invariant set of endpoints* $E \subseteq \partial M$:

(i) $E \subseteq \Sigma^n(M; A)$;
(ii) $E \subseteq {}^\circ\Sigma^n(M; A)$;
(iii) *there is a uniform constant* λ *such that for all* $e \in E$, $e \in \Sigma^n(M; A)$ *with constant lag* λ;
(iv) *there is a uniform constant* $\nu > 0$ *and a* G*-volley* $\Phi : \mathbf{F}^{(n)} \to f\mathbf{F}^{(n)}$ *inducing* id_A *with the property that for each* $e \in E$ *there is a chain map selection* φ_e *from* Φ *with* $\mathrm{gsh}_e(\varphi_e) \geq \nu$.

Proof. All implications except (i) ⇒ (ii) and (i) ⇒ (iv) have been proved, and the latter is the stronger of these two, so we prove (i) ⇒ (iv). The proof is by induction on n.

When $n = 0$ it follows from [**BG16**, theorem 7·2] that there is no difference between (ii) and (iv). Assume $E \subseteq \Sigma^0(M; A)$. For each $x \in X_0$ and $e \in E$ we choose $\bar{c}(e, x) \in F_0$ such that $\epsilon(\bar{c}(e, x)) = \epsilon(x)$ and $v_\gamma(\bar{c}(e, x)) - v_\gamma(x) > 0$, where $\gamma(\infty) = e$. If this inequality holds for e and x, then it also holds for e' and x when e' lies in a suitably small neighbourhood of e. Since E is compact there is a finite subset $E_f \subseteq E$ such that for all $e \in E$ there is some $e' \in E_f$ such that $v_\gamma(\bar{c}(e', x)) - v_\gamma(x) > 0$ when $\gamma(\infty) = e$. For every $e \in E$ we choose such an e' and define $c(e, x) := \bar{c}(e', x)$. Thus $\inf_{e \in E}\{v_\gamma(c(e, x)) - v_\gamma(x)\} > 0$. Define $\Psi(x) = \{c(e, x) \mid e \in E\}$, a finite set of 0-chains. For $y = gx$, define $\Psi(y) = g\Psi(x)$. Then the induced $\Psi : F_0 \to fF_0$ is a G-volley inducing id_A.

For $e \in E$ and $y = gx$ define an additive endomorphism $\psi_e : F_0 \to F_0$ by $\psi_e(y) := gc(g^{-1}e, x)$; this makes sense because E is G-invariant. Then $\epsilon\psi_e = \epsilon$, and

$$
\begin{aligned}
v_\gamma(\psi_e(y)) - v_\gamma(y) &= v_\gamma(gc(g^{-1}e, x)) - v_\gamma(gx)) \\
&= v_{g^{-1}\gamma}(c(g^{-1}e, x)) - v_{g^{-1}\gamma}(x).
\end{aligned}
$$

Thus $\inf_{e \in E}\{\mathrm{gsh}_e(\psi_e)\} > 0$. This proves $E \subseteq {}^\circ\Sigma^0(M; A)$ and finishes the case $n = 0$.

Now we assume $n \geq 1$. We are given that $E \subseteq \Sigma^n(M; A)$, and by induction we may assume the statement of (iv) when n is replaced by $n - 1$. We also know that F_n is finitely generated. So, by Corollary 7·3(ii) we conclude that $E \subseteq \Sigma^n(M; A)$ with a uniform constant lag $\lambda \geq 0$. We have a G-volley $\Phi : \mathbf{F}^{(n-1)} \to f\mathbf{F}^{(n-1)}$ inducing id_A such that for each $e \in E$ there is a chain map selection φ_e from Φ with $\mathrm{gsh}_e(\varphi_e) \geq \nu > 0$. Hence, by Lemma 4·2, for any positive integer k we have $\mathrm{gsh}_e(\varphi_e^k) \geq k\nu$. We may choose k so that $k\nu \geq \lambda + ||\partial \mid \mathbf{F}^{(n)}|| + \delta$ where $\delta > 0$ is arbitrary. The endomorphisms φ_e^k are selections from the finite G-volley $\Phi^k : \mathbf{F}^{(n-1)} \to f\mathbf{F}^{(n-1)}$.

For $x \in X_n$ define $\Pi(x) := \{g^{-1}\varphi_e^k(g\partial x) \mid g \in G, e \in E\}$. This is a finite set of cycles, hence of boundaries. For each $(e, p) \in E \times \Pi(x)$ we choose $c(e, x, p) \in F_n$ such that $\partial(c(e, x, p)) = p$ and

$$v_\gamma(c(e, x, p)) > v_\gamma(p) - \lambda \geq v_\gamma(p) - k\nu + ||\partial \mid \mathbf{F}^{(n)}|| + \delta.$$

Just as in the case $n = 0$, above, the compactness of E allows us to make our choices $c(e, x, p)$ from a finite set $\Psi(x) \subseteq F_n$. Putting $\Psi(y) := g\Psi(x)$, when $y = gx \in Y_n$, we get a G-volley $\Psi : \mathbf{F}^{(n)} \to f\mathbf{F}^{(n)}$ extending Φ^k. Let $\psi_e : F_n \to F_n$ be the homomorphism defined by

$$\psi_e(y) := gc(g^{-1}e, x, g^{-1}\varphi_e^k \partial y).$$

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Then $\partial\psi_e = \varphi_e^k\partial$, so ψ_e is a chain map extending φ_e^k. When $\gamma(\infty) = e$ we have

$$\begin{aligned} v_\gamma(\psi_e(y)) &= v_\gamma(gc(g^{-1}e, x, g^{-1}\varphi_e^k(g\partial x))) \\ &= v_{g^{-1}\gamma}(c(g^{-1}e, x, g^{-1}\varphi_e^k(\partial y))) \\ &> v_{g^{-1}\gamma}(g^{-1}\varphi_e^k(\partial y)) - k\nu + ||\partial \mid \mathbf{F}^{(n)}|| + \delta \\ &= v_\gamma(\varphi_e^k(\partial y)) - k\nu + ||\partial \mid \mathbf{F}^{(n)}|| + \delta \\ &\geq v_\gamma(\partial y) + \mathrm{gsh}_e(\varphi_e^k) - k\nu + ||\partial \mid \mathbf{F}^{(n)}|| + \delta \\ &\geq v_\gamma(y) + \delta. \end{aligned}$$

So $\mathrm{gsh}_e(\psi_e) \geq \delta$. Thus (iv) holds for n, and the induction is complete.

8. *The meaning of* $\Sigma^n(M; A) = \partial M$

From now on we assume the module A is non-zero. In this section and the next we study the meaning of $\Sigma^n(M; A) = \partial M$. We note that, by Theorem 7·4, the statements $\Sigma^n(M; A) = \partial M$ and ${}^\circ\Sigma^n(M; A) = \partial M$ are equivalent. Our first goal is Theorem 8·10, which explains how these properties are also equivalent to what we will call "controlled $(n-1)$-acyclicity over M".

8·1. *Controlled acyclicity over points $b \in M$*

Controlled acyclicity over a point $b \in M$ is analogous to controlled acyclicity over an end-point $e \in \partial M$. The role of the valuation on the augmented controlled based free resolution $\mathbf{F} \twoheadrightarrow A$ is played by the function $D_b : \mathbf{F} \to \mathbb{R}_{\geq 0}$ defined by $D_b(c) := \max\{d(p, b) \mid p \in h(c)\}$ when $c \neq 0$ and $D_b(c) = 0$ when $c = 0$. We extend D_b to the module A by $D_b(a) = 0$ for all $a \in A$. We say $\mathbf{F} \twoheadrightarrow A$ is *controlled* $(n-1)$*-acyclic over* $b \in M$, in short CA^{n-1} over b, if there is a *lag* function $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that, for any $-1 \leq i \leq n-1$, each (augmented) i-cycle is the boundary, $z = \partial c$ of some $(i+1)$-chain c satisfying

$$D_b(c) \leq D_b(z) + \lambda(D_b(z)). \tag{8·1}$$

PROPOSITION 8·1. (i) *If $\lambda_R : [R, \infty) \to \mathbb{R}_{\geq 0}$ satisfies the inequality (8·1) when $D_b(z) \geq R$, and if λ_R is extended to $[0, \infty)$ by defining $\lambda_R(s) = \lambda_R(R) + R - s$ when $0 \leq s \leq R$, then the extended λ_R satisfies that inequality for all z, and hence is a lag function in the above sense.*

(ii) *If $d(b, b') = \delta$ and λ_R is a lag function with respect to b, then $\lambda_R + 2\delta$ is a lag function with respect to b'.*

Proposition 8·1(i) implies that if in the definition of "CA^{n-1} over b" the lag function can be chosen to be constant on some interval $[R, \infty)$ then it can be chosen to be constant over all. Proposition 8·1(ii) implies that if the action ρ is cocompact (so that there is a fundamental domain of finite diameter) and if we have CA^{n-1} over some point b, then we have CA^{n-1} over all points b with the same lag function being applicable everywhere. We refer to this as a *uniform lag*. We say that $\mathbf{F} \twoheadrightarrow A$ is CA^{n-1} *over* M if it is CA^{n-1} over each $b \in M$ with a uniform lag function.

In particular, "CA^{-1} over M" means that there is a number λ $(= \lambda(0))$ such that for every $a \in A$ and $b \in M$ there is a 0-chain c with $\epsilon(c) = a$ and $h(c) \subseteq B_\lambda(b)$. Since A is non-zero and finitely generated this implies that the action ρ is cocompact.

PROPOSITION 8·2. *Let* $\mathbf{F} \twoheadrightarrow A$ *be a controlled based free resolution with finitely generated n-skeleton. Let* $\varphi : \mathbf{F}^{(n)} \to \mathbf{F}^{(n)}$ *be a G-finitary chain endomorphism inducing* id_A*, and let* $\sigma : \varphi \sim \mathrm{id}_F$ *be a G-finitary chain homotopy. If* φ *pushes* $\mathbf{F}^{(n)}$ *towards some point of M then the following hold*:

(i) ρ *is cocompact, and* $\mathbf{F}$ *is* CA^{n-1} *over M with constant lag*;

(ii) *if* $\mathbf{F}^{(n+1)}$ *is finitely generated and* $\mathbf{F}$ *is* CA^n *over M, then it is so with constant lag.*

Proof. The proof is entirely analogous to that of Proposition 7·2, and the details are therefore omitted. Instead of pushing towards some $e \in \partial M$ we are now pushing towards b. Of course, there exists $R \geq 0$ such that those portions of our chains already over the ball of radius R about b do not make progress towards b, but Proposition 8·1 implies that this makes no difference.

Remark. In fact there is a radius R such that for any $b \in M$ the lag outside $B_R(b)$ can be $||\sigma||$.

8·2. *Bounded support and cocompactness of* ρ

We first consider the special case $n = 0$. We say the module A has *bounded support over M* if there is a bounded subset $B \subseteq M$ with the property that for each $a \in A$ there exists $c \in F_0$ with $\epsilon(c) = a$ and $h(c) \subseteq B$. By the triangle inequality, it is easy to see that this property is independent of the point $b \in M$, though the number r varies with b. It is also independent of the choice of $\mathbf{F}$ and h.

THEOREM 8·3. *Let A be a finitely generated non-zero G-module. The following are equivalent*:

(i) $\Sigma^0(M; A) = \partial M$;

(ii) $\mathbf{F} \twoheadrightarrow A$ *is* CA^{-1} *over M*;

(iii) ρ *is cocompact and A has bounded support over M.*

The example of $SL_2(\mathbb{Z})$ acting on the hyperbolic plane – here A is the trivial module $\mathbb{Z}$ – shows that "having bounded support" does not imply "cocompact".

Remarks on the proof. The equivalence of (ii) and (iii) is clear. We spelled out the meaning of "CA^{-1} over M", and (using that notation) there is a ball of radius λ inside any horoball; so (ii) implies (i). That (i) implies cocompactness follows from [**BG16**, proposition 6·6]. The remaining item, the fact that (i) implies bounded support, is given in the next subsection[7]. Since it is the $n = 0$ case of a bigger theorem, Theorem 8·10, some of the methods get used later. In the next section we sketch the rest of the proof of Theorem 8·3 referring to [**BG16**] for some details.

8·3. *Shifting towards base points*

Let $F = F_X$ be a finitely generated based free G-module, choose a base point $b \in M$ and consider the canonical control map $h : F \to M$ with respect to b and the basis X. Let $\varphi : F \to F$ be an additive endomorphism, and let $L \leq F$ be a cellular $\mathbb{Z}$-submodule of F.

[7] Theorem 8·3 appears with full proof in our paper [**BG16**]

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The *shift function of* $\varphi \mid L$ *towards* $b \in M$ measures the loss of distance from $b \in M$; it is denoted by $\mathrm{sh}_{\varphi,b} : L \cap Y \to \mathbb{R}$, and is defined by

$$\mathrm{sh}_{\varphi,b}(y) := D_b(y) - D_b(\varphi(y)) \in \mathbb{R}, \quad y \in L \cap Y. \tag{8·2}$$

The notion of guaranteed shift towards $b \in M$ is more subtle than the corresponding notion for endpoints $e \in \partial M$ because if elements are already too close to b it may not be possible to push them any closer. Therefore we have to restrict attention to elements $y \in L \cap Y$ with $h(y)$ outside some ball centered at b. When $\alpha \in \mathbb{R}$ and $R \geq 0$, the pair (α, R) *defines a guaranteed shift* for $\varphi \mid L$ if $\mathrm{sh}_{\varphi,b}(y) \geq \alpha$ whenever $y \in L \cap Y$ and $D_b(y) > R$. The *(almost) guaranteed shift of* φ *on* L is

$$\mathrm{gsh}_b(\varphi \mid L) := \sup\{\alpha \mid \text{ for some } R,\ (\alpha, R) \text{ defines a guaranteed shift for } \varphi \mid L\}.$$

We call a number R occurring in such a pair (α, R) an *event radius* for φ.

For a proof of the following lemma see [**BG16**, section 9]:

LEMMA 8·4. (i) $-||\varphi \mid L|| \leq \mathrm{gsh}_b(\varphi \mid L) \leq ||\varphi \mid L||$.

(ii) *If* $\psi : F \to F$, *and* K *is a cellular submodule with* $\psi(K) \subseteq L$ *then*

$$\mathrm{gsh}_b(\varphi \mid L \circ \psi \mid K) \geq \mathrm{gsh}_b(\varphi \mid L) + \mathrm{gsh}_b(\psi \mid K).$$

We note that when φ is G-finitary then $||\varphi \mid L|| < \infty$ and $\mathrm{gsh}_b(\varphi \mid L)$ is attained. If $\mathrm{gsh}_b(\varphi \mid L) > 0$ we say that φ *pushes* L *towards* $b \in M$.

COROLLARY 8·5. *If* φ *in Lemma 8·4* (ii) *pushes* L *towards* b, *and* $\varphi(L) \subseteq L$ *then* $\varphi^k \circ \psi$ *pushes* L *towards* b *when* $k > -\mathrm{gsh}_b(\psi \mid K)/\mathrm{gsh}_b(\varphi \mid L)$. *In fact,* $\mathrm{gsh}_b(\varphi^k \circ \psi \mid L) > \eta$ *when* $k > (\eta - \mathrm{gsh}_b(\psi \mid K))/\mathrm{gsh}_b(\varphi \mid L)$.

The $CAT(0)$ metric space M is *almost geodesically complete* if there is a number $\mu \geq 0$ such that for any b and $b' \in M$ there is a geodesic ray γ starting at b and passing within μ of b'. An example lacking this property is the half line $[0, \infty)$. The following is proved in [**GO07**]:

THEOREM 8·6. *Every non-compact cocompact proper CAT(0) space is almost geodesically complete.*

LEMMA 8·7. *Let* M *be a proper* $CAT(0)$ *space, and let* $r \geq 0$, $\delta > 0$.

(i) *If* $(b, e) \in M \times \partial M$ *is given then, by choosing* $p = p(b, e) \in M$ *sufficiently far out on the geodesic ray* γ *from* b *to* e, *we can achieve*

$$|(\beta_\gamma(q) - \beta_\gamma(p)) - (d(p, b) - d(q, b))| < \delta \text{ whenever } d(p, q) \leq r. \tag{8·3}$$

(ii) *Assume* M *is almost geodesically complete and let* μ *be as in that definition. There is a number* $R = R(r, \delta)$ *such that (8·3) holds whenever* $d(p, b) \geq R$, γ *is a geodesic ray starting at* b *and passing within* μ *of* p, *and* $d(p, q) \leq r$.

Proof. (i) is immediate from the definition of horoballs and Busemann functions. The proof of (ii) is contained in the proof of theorem 15·3 of [**BGe03**].

The next proposition is proved in [**BG16**, section 9]:

PROPOSITION 8·8. *Let M be an almost geodesically complete proper $CAT(0)$ space. The following are equivalent for a G-volley $\Phi : \mathbf{F} \to f\mathbf{F}$:*

(i) $\forall e \in \partial M$ *Φ admits a selection pushing* $\mathbf{F}$ *towards e which induces* id_A;
(ii) $\forall b \in M$ *Φ admits a selection pushing* $\mathbf{F}$ *towards b which induces* id_A.

Completion of proof of Theorem 8·3. We assume (i) and we know that this implies cocompactness since A is finitely generated. Hence, by Theorem 8·6 M is almost geodesically complete. So for any $b \in M$ Theorem 7·4 and Proposition 8·8 give us a G-volley φ having a selection φ with $\mathrm{gsh}_b(\varphi) > 0$. Let (α, R) define a guaranteed shift for φ, where $\alpha > 0$. For any $a \in A$ there is a 0-chain c mapped by ϵ to a such that

$$D_b(\varphi(c)) \leq \max\{R + ||\varphi||, D_b(c) - \alpha\}.$$

By iterating φ we can thus move c over M to a new c' such that $\epsilon(c') = a$ and $h(c')$ lies over the ball centered at b with radius $R + ||\varphi||$, a number independent of a.

8·4. *The higher dimensional case*

PROPOSITION 8·9. *Let $\mathbf{F} \twoheadrightarrow A$ be a based free resolution over M with finitely generated n-skeleton, where $n \geq 1$, and let $h : \mathbf{F} \to fM$ be a canonical control function at $b \in M$ with respect to the basis X. Let $\varphi : \mathbf{F}^{(n-1)} \to \mathbf{F}^{(n-1)}$ be a G-finitary chain map inducing the identity on A, with φ pushing $\mathbf{F}^{(n-1)}$ towards b. If $\Sigma^n(M; A) = \partial M$ then some iterate φ^k of φ admits a G-finitary chain map extension $\psi : \mathbf{F}^{(n)} \to \mathbf{F}^{(n)}$ pushing $\mathbf{F}^{(n)}$ towards $b \in M$. The guaranteed shift, the event radius of ψ, and the number k depend on φ and the uniform lag, but are independent of $b \in M$.*

Proof. Let $\Phi : \mathbf{F}^{(n-1)} \to f\mathbf{F}^{(n-1)}$ be a G-volley from which φ is a selection. By Theorem 7·4 we know that $\mathbf{F}^{(n)}$ is CA^{n-1} with respect to every endpoint $e \in \partial M$, with uniform constant lag λ. Now, Corollary 8·5 asserts that for suitable k, $\varphi^k\partial : \mathbf{F}^{(n)} \to \mathbf{F}^{(n)}$ pushes all of $\mathbf{F}^{(n)}$ towards $b \in M$; in fact, by choosing k sufficiently large, we can achieve

$$\mathrm{gsh}_b(\varphi^k\partial) \geq \lambda + 3\delta, \text{ where } \delta > 0 \text{ is arbitrary.} \tag{8·4}$$

We aim to extend the kth iterate Φ^k to a G-volley on the n-skeleton by our usual compactness argument. It suffices to define this extension on the finite G-basis X_n. Let $x \in X_n$. We observe that the set of chains

$$\Pi(x) := \{g^{-1}\varphi^k(g\partial x) \mid g \in G\}$$

lies in $g^{-1}\Phi^k(\partial gx)$ and hence is finite, with $\partial\Pi(x) = 0$. For each pair $(e', p) \in \partial M \times \Pi(x)$ we can choose an n-chain $c(e', p) \in F_n$, with $\partial c(e', p) = p$, and

$$v_{\gamma'}(c(e', p)) > v_{\gamma'}(p) - \lambda, \tag{8·5}$$

where $\gamma'(\infty) = e'$.

Fixing p for a moment, we observe that if (8·5) holds for some $e' \in \partial M$ then there is a neighborhood $N(e')$ of e in ∂M such that

$$v_{\gamma''}(c(e', p)) > v_{\gamma''}(p) - \lambda \text{ for all } e'' = \gamma''(\infty) \in N(e').$$

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Since ∂M is compact, finitely many of these neighbourhoods cover ∂M. This shows that we can improve on the choice of the n-chains $c(e', p)$ as follows: we can find a *finite* set of n-chains, which we denote $\Psi(x) \subseteq F_n$, with the property that for each $(e', p) \in \partial M \times \Pi(x)$ there is some $c(e', p) \in \Psi(x)$, with $\partial c(e', p) = p$ which satisfies the inequality (8·5). Putting $\Psi(gx) := g\Psi(x)$ defines a G-volley $\Psi : F_n \to f F_n$.

Since $\Sigma^n(M; A) = \partial M$, Theorems 8·3 and 8·6 imply that M is almost geodesically complete. Let $\mu > 0$ be a number given by that definition[8]. For every $y \in gx \in Y_n$ we choose an endpoint $e = e(y)$, with the property that the geodesic ray $\gamma = \gamma_y$ from $\gamma(0) = h(y)$ to $e = \gamma(\infty)$, passes the point $b \in M$ at distance $< \mu$. We put $\psi(y) := gc(g^{-1}e(y), g^{-1}\varphi^k(g\partial x)) \in g\Psi(x)$, noting that $\partial\psi(gx) = g\partial c(g^{-1}e(gx), g^{-1}\varphi^k(g\partial x)) = \varphi^k(\partial gx)$, hence $\partial\psi = \varphi^k\partial$, as required.

It remains to show that ψ pushes towards $b \in M$. Let $R_1 = R(||\Phi^k\partial||, \delta)$ be the radius given by Lemma 8·7 where $\delta > 0$ is arbitrary. Then Lemma 8·7 yields

$$|\mathrm{sh}_{\varphi^k\partial, e(y)}(y) - \mathrm{sh}_{\varphi^k\partial, b}(y)| < \delta, \ \text{ for all } y = gx \in Y_n \text{ with } D_b(y)) \geq R_1. \tag{8·6}$$

Now, we take R_2 to be an event radius for $\varphi^k\partial : \mathbf{F}^{(n)} \to \mathbf{F}^{(n)}$. For every $y = gx \in Y_n$, with $D_b(y) \geq R_2$ and $\gamma = \gamma_y$ the ray from $h(y)$ to $e = e(y)$, we find

$$\begin{aligned}
v_\gamma(\psi(y)) - v_\gamma(y) &= v_\gamma(gc(g^{-1}e, g^{-1}\varphi^k(\partial y))) - v_\gamma(y) \\
&= v_{g^{-1}\gamma}(c(g^{-1}e, g^{-1}\varphi^k(\partial y))) - v_\gamma(y), && \text{by Lemma 3·1}, \\
&> v_{g^{-1}\gamma}(g^{-1}\varphi^k(\partial y)) - v_\gamma(y) - \lambda, && \text{by (8·5)}, \\
&= v_\gamma(\varphi^k\partial y) - v_\gamma(y) - \lambda, && \text{by Lemma 3·1}, \\
&= \mathrm{sh}_{\varphi^k\partial, e}(y) - \lambda, \\
&\geq \mathrm{sh}_{\varphi^k\partial, b}(y) - \delta - \lambda, && \text{by (8·6)} \\
&\geq \mathrm{gsh}_b(\varphi^k\partial) - \delta - \lambda, \\
&\geq 2\delta. && \text{by (8·4)}.
\end{aligned}$$

Hence $\mathrm{sh}_{\psi, e(y)}(y) \geq 2\delta$, and therefore, by Lemma 8·7 there exists $R_3(||\Psi||, \delta)$ such that $\mathrm{sh}_{\psi, b}(y) \geq \delta$ when $D_b(y) > R_3$. Thus we find $\mathrm{gsh}_b(\psi) > 0$, i.e. ψ pushes F_n towards b.

THEOREM 8·10. *Let M be a proper $CAT(0)$ space. Let $\mathbf{F} \twoheadrightarrow A$ be an augmented G-free resolution with finitely generated n-skeleton, and $h : \mathbf{F} \to fM$ a control map. The following are equivalent*:

(i) $\Sigma^n(M; A) = \partial M$;

(ii) *there are positive numbers (R, α) with the property that for every $b \in M$ there is G-finitary chain map $\varphi_b : \mathbf{F}^{(n)} \to \mathbf{F}^{(n)}$, inducing id_A and pushing all of the n-skeleton towards $b \in M$, with guaranteed shift α and event radius R*;

(iii) *$\mathbf{F} \twoheadrightarrow A$ is controlled $(n-1)$-acyclic over M*;

(iv) *$\mathbf{F} \twoheadrightarrow A$ is controlled $(n-1)$-acyclic over M with a constant lag.*

Proof. We begin with (i) $\Rightarrow$ (ii). From Theorem 7·4 we know that there is a volley $\Phi : F^{(0)} \to f\mathbf{F}^{(0)}$ inducing the identity on A and satisfying (i) of Proposition 8·8. Thus

[8] i.e. for every a and b there is a geodesic ray starting at a and passing within μ of b.

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Proposition 8·8 applied to this volley gives a chain map $\varphi_b : \mathbf{F}^{(0)} \to \mathbf{F}^{(0)}$ pushing $\mathbf{F}^{(0)}$ towards b. From the proof of Proposition 8·8 we see that the event radius and the guaranteed shift of φ_b depend only on Φ and not on b. This starts an induction, the inductive step being given by Proposition 8·9.

(ii) $\Rightarrow$ (iv). This follows from Proposition 8·2(i).

(iv) $\Rightarrow$ (i). Let the geodesic ray γ define an end point e and let $HB_{\gamma(t)}$ be a horoball. A cycle over this horoball also lies over some ball centered along γ. If the constant lag in the hypothesis is λ then this cycle bounds a chain over the ball obtained by increasing the previous ball's radius by λ. That ball lies in the horoball $HB_{\gamma(t)-\lambda}$.

(iii) $\Rightarrow$ (iv). This follows from Proposition 8·2(ii). (iv) $\Rightarrow$ (iii) is trivial.

For $b \in M$ we write G_b for the subgroup of G fixing b. For any other b' the group $G_{b'}$ is commensurable with G_b.

COROLLARY 8·11. *Assume that the action ρ is cocompact and that its orbits are discrete subsets of M. Then $\Sigma^n(M; A) = \partial M$ if and only if A has type FP_n as a G_b-module, where $b \in M$.*

Proof. Filter $\mathbf{F}$ by $h^{-1}(fB_r(b))$, where $b \in M$, $r \geq 1$, and the notation means the largest $\mathbb{Z}$-subcomplex mapped by h into $fB_r(b)$. These subcomplexes are G_b-invariant, and because the orbits are discrete these subcomplexes are finitely generated modulo G_b. According to (an obvious adaptation of) Theorem 2.2 of [**Bro87**], A has type FP_n as a G_b-module if and only if $\mathbf{F} \twoheadrightarrow A$ is CA^{n-1} over M. By Theorem 8·10 the Corollary follows.

A variant is:

COROLLARY 8·12. *Assume that the orbits of the action ρ are discrete subsets of M and that the group $\rho(G)$ acts properly discontinuously and cocompactly (aka "geometrically") on M. Then $\Sigma^n(M; A) = \partial M$ if and only if A has type FP_n as a $\mathbb{Z}[\ker(\rho)]$-module.*

Proof. The hypothesis implies that $\ker(\rho)$ and G_b are commensurable.

9. *Dispensing with lags*

We continue to assume the (finitely generated) module A is non-zero. Let $\mathbf{F} \twoheadrightarrow A$ be a controlled based free resolution with finitely generated n-skeleton with control map $h : \mathbf{F} \to fM$. In this section we show that when $\Sigma^n(M; A) = \partial M$ we can replace $\mathbf{F}$ by another such resolution $\mathbf{F}'$ and define a control map $h' : \mathbf{F}' \to fM$ so that the pre-images under h' of horoballs and of large balls are $(n-1)$-acyclic. In short, we can reduce the lags to zero.

We begin with the horoball case, and with $n = 0$. When $e \in \Sigma^0(M; A)$ then for each $x \in X_0$ and $\nu > 0$, there exists $c \in F_0$ with $\epsilon(c) = \epsilon(x)$ and

$$v_\gamma(c) - v_\gamma(x) > \nu. \tag{9·1}$$

where γ is a geodesic ray with $\gamma(\infty) = e$. We write $\mathbf{F}^{(\gamma,t)}$ for the subcomplex generated by $\{y \in Y | v_\gamma(y) \geq t\}$.

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PROPOSITION 9·1. *When $e \in \Sigma^0(M; A)$ the augmentation map ϵ takes $F_0^{(\gamma,t)}$ onto A.*

This is equivalent to saying that $\mathbf{F}$ is CA^{-1} over e.

Next, assume $\Sigma^0(M; A) = \partial M$. Just as in the $n = 0$ proof of Theorem 7·4, a compactness argument shows that for each $x \in X_0$ there is a finite set $\Phi(x)$ so that for every $e \in \partial M$ (9·1) holds for some $c \in \Phi(x)$. The resulting function $X_0 \to f F_0$ defines a G-volley $\Phi : F_0 \to f F_0$. We now alter F_1 and F_2 by performing "elementary expansions". For each $c \in \Phi(x)$ we choose $d \in F_1$ with $\partial d = x - c$. We add new generators ξ to X_1 and η to X_2, defining $\partial\xi = x - c$ and $\partial\eta = d - \xi$. We extend h to h' by $h'(\xi) = h(x) \cup h(c)$ and $h'(\eta) = h(x) \cup h(c) \cup h(d)$. The resulting enlarged chain complex $\mathbf{F}' \twoheadrightarrow A$ is again a free resolution with finitely generated n-skeleton. We note that $\xi = \xi(x, c)$ with $x \in X_0$ and $c \in \Phi(x)$. Define $\Sigma(x) := \{\xi(x, c) \mid c \in \Phi(x)\}$. This function $X_0 \to f F_1$ defines a finite degree 1 G-volley $\Sigma : F_0 \to f F_1$. The definition of $h'(\xi(x, c))$ ensures

$$v_\gamma(\xi(x, c)) = v_\gamma(x). \tag{9·2}$$

LEMMA 9·2. *For each $e \in \partial M$ there are selections φ_e from Φ and σ_e from Σ such that, for all $y \in GX_0$, $\partial\sigma_e(y) = y - \varphi_e(y)$, $v_\gamma(\varphi_e(y)) - v_\gamma(y) > \nu$, and $v_\gamma(\sigma_e(y)) = v_\gamma(y)$.*

Proof. Fix $e \in \partial M$. Let $y = gx$. Using (9·1) and (9·2) pick c and $\xi(x, c)$ so that:

(i) $v_{g^{-1}\gamma}(c) - v_{g^{-1}\gamma}(x) > \nu$;
(ii) $v_{g^{-1}\gamma}(\xi) = v_{g^{-1}\gamma}(x)$; and
(iii) $\partial\xi = x - c$.

Then $v_\gamma(gc) - v_\gamma(y) > \nu$ and $v_\gamma(g\xi) = v_\gamma(y)$. Define $\varphi_e(y) = gc$ and $\sigma_e(y) = g\xi$. Then $\partial\sigma_e(y) = y - \varphi_e(y)$.

Lemma 9·2 provides a G-finitary push φ_e towards each e and a G-finitary chain homotopy $\sigma_e : \mathrm{id} \sim \varphi_e$. We think of these chain homotopies as "monotone" because they have the property that $v_\gamma(\sigma_e(c)) \geq v_\gamma(c)$ for all chains c.

PROPOSITION 9·3. *Assume $\Sigma^0(M; A) = \partial M$. Let $e \in \Sigma^1(M; A)$ and let h' be the extended control map on $\mathbf{F}'$. Then the resolution $\mathbf{F}' \twoheadrightarrow A$ is CA^0 over e with zero lag. Equivalently, for any t, $\mathbf{F}'^{(t)}$ is 0-acyclic.*

Proof. Writing $e = \gamma(\infty)$ there is a lag $\lambda(e, t)$ as in the definition of CA^0 in Section 6. Let z be a 0-cycle over $HB_{\gamma,t}$. For k a positive integer we consider the chain homotopy $\bar{\sigma}_{e,k} := \sigma_e(\varphi_e^k + \varphi_e^{k-1} + \cdots + \varphi_e + 1)$ as in the proof of Proposition 7·2(ii). If k is large enough, $\varphi^k(z)$ bounds over $HB_{\gamma,t}$ and because σ_e is monotone $\bar{\sigma}_{e,k}$ provides a homology over $HB_{\gamma,t}$ between z and $\varphi^k(z)$. Thus z bounds over $HB_{\gamma,t}$.

Next, we repeat for $n = 1$ what we have just done for $n = 0$. Assuming $\Sigma^1(M; A) = \partial M$ we extend the G-volleys Φ and Σ by defining $\Phi : F_1 \to f F_1$ and $\Sigma : F_1 \to f F_2$, adding new generators in dimensions 2 and 3. The only difference is that in the analog of Lemma 9·2 we will have $\partial\sigma_e(y) = y - \varphi_e(y) - \sigma_e\partial y$. The pattern for higher n is now clear. We have proved:

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PROPOSITION 9·4. *When $\Sigma^{n-1}(M; A) = \partial M$ there is a controlled based free resolution $\mathbf{F} \twoheadrightarrow A$ with finitely generated n-skeleton which is CA^{n-1} with zero lag over every $e \in \Sigma^n(M; A)$.*

Essentially the same proof gives:

PROPOSITION 9·5. *Let E be a closed G-invariant subset of ∂M. When $\Sigma^{n-1}(M; A) \supseteq E$ there is a resolution $\mathbf{F} \twoheadrightarrow A$ with finitely generated n-skeleton which is CA^{n-1} with zero lag over every $e \in E \cap \Sigma^n(M; A)$.*

Proposition 9·5 applies, in particular, to a singleton set $\{e\}$ where e is a fixed point of the G-action on ∂M. For example, in the Euclidean case, where G acts by translations, every point of the boundary is fixed by G, and this recovers [**BR88**, theorem 4·2].

A straightforward adaptation of the proof of Proposition 9·4 gives the following addition to Theorem 8·10:

THEOREM 9·6. *$\Sigma^n(M; A) = \partial M$ if and only if there is a controlled based free resolution $\mathbf{F} \twoheadrightarrow A$ with finitely generated n-skeleton and a radius R with the property that $h^{-1}(fB)$ is $(n-1)$-acyclic when B is any ball of radius $\geq R$ (or any horoball, for that matter).*

10. *Openness theorems*

When $E \subseteq \partial M$ we write $\mathcal{R}_E := \mathrm{Hom}(G, \mathrm{Isom}(M, E))$, the set of all isometric actions of G on M which leave E invariant. We endow the sets $\mathrm{Isom}(M, E)$ and $\mathcal{R}_E$ with the compact-open topology[9]. The boundary ∂M carries the cone topology.

In this section, when we discuss a particular action $\rho \in \mathcal{R}_E$ we will write ${}_\rho M$ rather than M. We choose a base point $b \in M$. The canonical control map $h^\rho : F \to f({}_\rho M)$ takes the $\mathbb{Z}$-generator gx to the singleton set $\{\rho(g)b\} \subseteq M$.

THEOREM 10·1 (Openness Theorem). *Let E be a compact subset of ∂M, and let $\rho \in \mathcal{R}_E$ be such that $E \subseteq {}^\circ\Sigma^n({}_\rho M; A)$. There is a neighbourhood N of ρ in $\mathcal{R}_E$ such that for all $\rho' \in N$, $E \subseteq {}^\circ\Sigma^n({}_{\rho'} M; A)$. Moreover, we can choose N so that there is a uniform constant $\nu > 0$ and a G-volley $\Phi : \mathbf{F}^n \to f\mathbf{F}^n$ inducing id_A such that for each $e \in E$ and $\rho' \in N$ there is a selection $\varphi_{e,\rho'}$ from Φ with $\mathrm{gsh}_e \varphi_{e,\rho'} \geq \nu$.*

Of course, by Theorem 7·4 the same holds when ${}^\circ\Sigma^n$ is replaced by Σ^n.

Proof. The proof of Theorem 10·1 is by induction on n. The case $n = 0$ was proved in [**BG16**] We will use the following ([**BG16**, lemma 8·1]):

LEMMA 10·2. *For given $c \in F$, the valuation $v_{e,\rho}(c)$ is (jointly) continuous in (e, ρ).*

To keep notation simple, we prove the $n = 1$ case of the theorem in detail; the general inductive case proceeds in the same way, and is left to the reader.

So, we assume that $E \subseteq {}^\circ\Sigma^1({}_\rho M; A)$ and for all ρ' in a neighbourhood $N_0(\rho)$ of ρ that $E \subseteq {}^\circ\Sigma^0({}_{\rho'} M)$. By the previous sections we can assume more:

[9] G is of course discrete.

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(1) $E \subseteq \Sigma^1({}_\rho M; A)$ with uniform constant lag $\lambda \geq 0$ — see [**BG16**, remark 8·3];

(2) there is a G-volley $\Phi : F_0 \to f F_0$ lifting id_A and a number $\nu > 0$ such that for every $e \in E$ and every $\rho' \in N_0$ there is a selection $\varphi_{e,\rho'}$ from Φ with $\mathrm{gsh}_e(\varphi_{e,\rho'}) > \nu$ — see Theorem 7·4.

For any positive integer k, $\mathrm{gsh}_e(\varphi^k_{e,\rho'}) \geq k\nu$. We choose k so that $k\nu \geq \lambda + ||\partial \mid \mathbf{F}^{(1)}|| + \delta$ where $\delta > 0$ is arbitrary.

When $e \in E$ and $\rho' \in N_0$ the endomorphisms $\varphi^k_{e,\rho'}$ are selections from the finite G-volley $\Phi^k : \mathbf{F}^{(0)} \to f\mathbf{F}^{(0)}$.

For each $x \in X_1$ define $\Pi(x) := \{g^{-1}\varphi^k_{e,\rho'}(g\partial x) \mid g \in G, e \in E, \rho' \in N_0\}$. This is a finite set of cycles, hence of boundaries.

We fix $x \in X_1$ and $p \in \Pi(x)$ for a moment. For each $e \in E$ we choose $\bar{c}(e) \in F_1$ such that $\partial(\bar{c}(e)) = p$ and

$$v_{e,\rho}(\bar{c}(e)) > v_{e,\rho}(p) - \lambda.$$

Then there is a neighbourhood $N(e, \rho) = N(e) \times N_e(\rho)$ such that for all $(e', \rho') \in N(e, \rho)$

$$v_{e',\rho'}(\bar{c}(e)) > v_{e',\rho'}(p) - \lambda > v_{e',\rho'}(p) - k\nu + ||\partial \mid \mathbf{F}^{(n)}|| + \delta.$$

Since E is compact, a finite set of neighbourhoods $N(e_i)$ covers E. Write $N = (\bigcap_i N_{e_i}) \cap N_0$, a neighbourhood of ρ.

Still fixing x and p, for each $e \in E$ we choose i such that $e \in N(e_i)$, and we define $c(e)$ to be $\bar{c}(e_i)$. Then

$$v_{e,\rho'}(c(e)) > v_{e,\rho'}(p) - k\nu + ||\partial \mid \mathbf{F}^{(n)}|| + \delta.$$

Recall that $\partial c(e) = p$. This suggests consideration of the set $\{c(e) \mid e \in E\} \in f F_1$. Letting p vary, and writing $c(e, x, p)$ in place of $c(e)$, we define

$$\Psi(x) = \{c(e, x, p) \mid e \in E, p \in \Pi(x)\} \in f F_1.$$

We then extend Ψ to the associated canonical volley $F_1 \to f F_1$.

For $(e, \rho') \in E \times N$ the additive homomorphism $F_1 \to F_1$ defined by

$$\psi_{e,\rho'}(gx) := gc(\rho'(g^{-1})e, x, g^{-1}\varphi^k_{e,\rho'}(g\partial x))$$

is a selection from the volley Ψ. Moreover $\partial \circ \psi_{e,\rho'} = \varphi^k_{e,\rho'} \circ \partial$, so this selection extends a previous chain map selection from the volley Φ^k. A calculation shows that it has guaranteed shift $\geq \delta$.

COROLLARY 10·3. *Let ρ be an isometric action on M as above. If $\Sigma^n({}_\rho M; A) = \partial M$ then there is a neighborhood N of ρ such that $\Sigma^n({}_{\rho'} M; A) = \partial M$ for all $\rho' \in N$.*

Our other openness theorem, the second part of Theorem 1·3, is:

THEOREM 10·4 (Tits Openness). *${}^\circ\Sigma^n(M; A)$ is open in the Tits metric topology on ∂M.*

Proof. The proof of this is exactly the same as the corresponding proof for ${}^\circ\Sigma^0(M; A)$ in [**BG16**]. We briefly recall it here. The set ${}^\circ\Sigma^n(M; A)$ can be described as the union of subsets of the form

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$$\Sigma(\varphi) := \{e \mid \text{gsh}_e(\varphi) > 0\},$$

where φ runs through all G-finitary endomorphisms of $\mathbf{F}^n$ which commute with the augmentation ϵ and satisfy $\text{gsh}_e > 0$ for some $e \in {}^\circ\Sigma^n(M; A)$. The norm of a G-finitary map is always finite, so the theorem follows from [**BG16**, theorem 3·9] which asserts that under these conditions $\Sigma(\varphi)$ is an open subset of ∂M in the Tits metric topology.

11. *Connections with Novikov homology*

11·1. *The Novikov module*

Assume given: an isometric action of G on M, an end point $e \in \partial M$, and a base point $b \in M$. As we have seen, this action extends to a topological action of G on ∂M; let G_e denote the subgroup of G which fixes e.

From now on we allow any commutative ring K as ground ring[10] unless we restrict it explicitly.

The *Novikov module* $\widehat{KG}^e$ is the (KG_e, KG)-bimodule defined as follows: As a set, it consists of all finite and infinite sums $\sum_{g\in G, r_g \in K} r_g g$ such that, for any horoball HB at e, all but finitely many of the points gb for which $r_g \in R$ is non-zero lie in HB. This definition is independent of b. The abelian group structure is termwise addition. The right action of G is by termwise right multiplication; note that $\widehat{KG}^e$ is preserved under right multiplication by $g \in G$ because the effect is merely to change the base point from b to gb. Left multiplication by $g \in G$ preserves $\widehat{KG}^e$ if and only if $g \in G_e$. One thinks of $\widehat{KG}^e$ as a sort of "completion towards e" of the group algebra KG. It is a generalisation of what is often called the "Novikov ring"; however, in the present generality there is no obvious multiplication which would make $\widehat{KG}^e$ a ring.

11·2. *Novikov chains*

Starting with a controlled based free resolution $\mathbf{F} \twoheadrightarrow A$ (over M) we consider the homology of the chain complex $\widehat{KG}^e \otimes_{KG} \mathbf{F}$ of left KG_e-modules. This is the *Novikov homology of A with respect to ρ and $e \in \partial M$.*

To give this a more geometric interpretation we describe the chain complex in a different way. As before, X_k denotes the given basis for F_k and $Y_k = GX_k$ is the corresponding K-basis. A *Novikov k-chain* (with respect to e) is a (possibly) infinite k-chain of the form $c = \sum_{y \in Y_k} r_y y$ such that:

(i) for every horoball HB at e all but a finite subset of $\text{supp}_Y(c)$ lies over HB, and
(ii) there is a finite subset $X_k(c)$ of X_k such that all the members of $\text{supp}_Y(c)$ are (left) G-translates of members of $X_k(c)$; i.e. the X-support of c is finite.

When X_k is finite the second condition is redundant. Typically X_k is finite for the values $k \leq n$ of interest, but the second condition can be important in the next dimension $n+1$.

We write C_k^e for the set of all such chains, with the obvious left G_e-module structure. Thus we get a chain complex $\mathbf{C}^e$ and we write H_k^e for the corresponding homology. The map $\mathbf{C}^e \to \widehat{KG}^e \otimes_{KG} \mathbf{F}$ which rewrites $\sum_{y\in Y_k} r_y y$ as $\sum_{x \in X_k}(\sum_{g\in G} r_{g,x} g)x$ and takes it to

[10] In fact all our work up to this point goes through for such a ground ring K.

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$\sum_{x\in X_k}(\sum_{g\in G} r_{g,x}g)\otimes x$ is an isomorphism of chain complexes. Thus H_k^e is isomorphic to $\mathrm{Tor}_k(\widehat{KG}^e, A)$ as an KG_e-module, and is therefore independent of the choice of resolution of A.

11·3. *Homological characterisation of* $^\circ\Sigma^n(M; A)$

THEOREM 11·1. *Assume* $\mathbf{F}^{(n)}$ *is finitely generated over* KG. *Let* $e\in\partial M$.

$$e\in {}^\circ\Sigma^n(M; A) \textit{ if and only if } \mathrm{Tor}_k(\widehat{KG}^{e'}; A)=0 \textit{ for all } e'\in \mathrm{cl}Ge \textit{ and all } k\leq n.$$

In proving this theorem we can replace $\mathrm{Tor}_k(\widehat{KG}^{e'}; A)$ by $H_k^{e'}$. Let $\varphi:\mathbf{F}^{(n)}\to\mathbf{F}^{(n)}$ be a K-chain map which pushes $\mathbf{F}^{(n)}$ towards e. A *Lipschitz deformation* for φ is a K-chain homotopy $\sigma:\mathbf{F}^{(n)}\to\mathbf{F}$ between the identity map and φ such that there exists a function $\nu:\mathbb{R}\to\mathbb{R}$ satisfying

$$v_\gamma(\sigma(y))\geq v_\gamma(y)-\nu(v_\gamma(y)) \tag{11·1}$$

for every $y\in Y$. This suggests a new Σ-invariant, namely:

$$\widetilde{\Sigma}^n(M; A)=\{e\in\partial M \mid \text{ there is such a push and Lipschitz deformation }\}.$$

By definition $\widetilde{\Sigma}^{-1}=\partial M$.

We note that the resolution $\mathbf{F}$ is a subcomplex of $\mathbf{C}^e$. The chain map

$$\bar{\varphi}^e := 1+\varphi+\varphi^2+\cdots : (\mathbf{C}^e)^{(n)}\longrightarrow(\mathbf{C}^e)^{(n)}$$

is well defined. The valuation v_γ on $\mathbf{F}$ extends to $\mathbf{C}^e$ in the obvious way, hence also do such notions as "guaranteed shift".

LEMMA 11·2. $^\circ\Sigma^n(M; A)\subseteq\widetilde{\Sigma}^n(M; A)$

Proof. If φ is a G-finitary push of $\mathbf{F}^{(n)}$ towards e then any G-finitary chain homotopy between the identity map and φ is a Lipschitz deformation for φ.

LEMMA 11·3. *If* $e\in\widetilde{\Sigma}^n(M; A)$ *then* $e\in\widetilde{\Sigma}^{n-1}(M; A)$ *and* $H_n^e=0$.

Proof. If z is an n-cycle in $\mathbf{C}^e$ then a calculation gives $z=\partial\bar{\varphi}^e\sigma(z)$.

LEMMA 11·4. *If* $e\in\widetilde{\Sigma}^{n-1}(M; A)$ *and* $H_n^e=0$ *then* $e\in\Sigma^n(M; A)$.

Proof. The case $n=0$ is clear, so we assume $n>0$. Let $z\in F_{n-1}$ be a cycle. Define $w:=\bar{\varphi}^e\sigma(z)\in C_n^e$ where σ comes from the $\widetilde{\Sigma}$ hypothesis. Then $\partial w=z$ and (see the inequality (11·1)):

$$v_\gamma(w)\geq v_\gamma(z)-\nu(v_\gamma(z)).$$

Since $\mathbf{F}$ is acyclic in dimension $n-1$, $z=\partial c$ for some finite n-chain c. The chain $w-c$ is a cycle in C_n^e, and $H_n^e=0$, so there is a chain $u\in C_{n+1}^e$ with $\partial u=w-c$. The free module C_{n+1}^e is a direct sum of copies of $\widehat{KG}^e$. The condition (ii) in the definition of Novikov chains means that there is a finite direct summand, indexed by $X_{n+1}(u)$, such that u is a

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KG-combination of elements lying in those factors. So there is a number $\lambda \geq 0$ such that $v_\gamma(\partial u) \geq v_\gamma(u) - \lambda$. Thus we can write $u = u_1 + u_2$ where u_1 is finite and

$$v_\gamma(\partial u_2) \geq v_\gamma(z) - \nu(v_\gamma(z)).$$

Define $c' := w - \partial u_2$. Then $\partial c' = \partial w = z$, c' is a finite chain because $c' = \partial u_1 + c$, and $v_\gamma(c') \geq v_\gamma(z) - \nu(v_\gamma(z))$.

COROLLARY 11·5. *$e \in {}^\circ\Sigma^n(M; A)$ if and only if, for all $e' \in \mathrm{cl}Ge$, $e' \in \widetilde{\Sigma}^{n-1}(M; A)$ and $H_n^{e'} = 0$.*

Proof. Let $e' \in \mathrm{cl}Ge$. We use the Lemmas: $e \in {}^\circ\Sigma^n(M; A)$ implies $e' \in {}^\circ\Sigma^n(M; A)$ (by Theorem 5·2), hence $e' \in \widetilde{\Sigma}^n(M; A)$. This implies $e' \in \widetilde{\Sigma}^{n-1}(M; A)$ and $H_n^{e'} = 0$; hence $e' \in \Sigma^n(M; A)$, which implies $e \in {}^\circ\Sigma^n(M; A)$ by the Characterisation Theorem (Theorem 7·1).

Proof of Theorem 11·1. It follows from Corollary 11·5 by induction, using the fact that $\mathrm{Tor}_k(\widehat{KG}^{e'}; A) = H_k^{e'}$.

Remark 11·6. If we define $\Sigma^n_{\mathrm{Tor}}(M; A)$ to be the set of end points e such that $H_k^e = 0$ for all $k \leq n$, then the lemmas in this section establish the following containments:

$${}^\circ\Sigma^n(M; A) \subseteq \widetilde{\Sigma}^n(M; A) \subseteq \Sigma^n_{\mathrm{Tor}}(M; A) \cap \widetilde{\Sigma}^{n-1}(M; A) \subseteq \Sigma^n(M; A).$$

11·4. *Behavior of the dynamical invariant on exact sequences*

Let $A' \rightarrowtail A \twoheadrightarrow A''$ be a short exact sequence of finitely generated KG-modules. For each $e \in \partial M$ there is an exact coefficient sequence

$$\cdots \longrightarrow \mathrm{Tor}_k(\widehat{KG}^e; A') \longrightarrow \mathrm{Tor}_k(\widehat{KG}^e; A) \longrightarrow \mathrm{Tor}_k(\widehat{KG}^e; A'') \xrightarrow{\partial_*} \mathrm{Tor}_{k-1}(\widehat{KG}^e; A') \longrightarrow \cdots$$

This, together with Theorem 11·1 gives:

THEOREM 11·7. *Let A and A' be finitely generated and let $e \in {}^\circ\Sigma^{n+1}(M; A'')$. Then $e \in {}^\circ\Sigma^n(M; A')$ if and only if $e \in {}^\circ\Sigma^n(M; A)$.*

Since Tor commutes with direct sums, we have:

PROPOSITION 11·8. ${}^\circ\Sigma^n(M; A' \oplus A'') = {}^\circ\Sigma^n(M; A') \cap {}^\circ\Sigma^n(M; A'')$.

12. *Products*

This section is about the behavior of the Σ-invariants with respect to direct products of groups and tensor products of modules. In particular, we prove Theorem 1·5. The set-up is as follows: We are given:

(i) $\mathbf{F} \twoheadrightarrow A$ and $\mathbf{F}' \twoheadrightarrow A'$, admissible free resolutions of the KG-module A and the KH-module A' respectively, which are finitely generated in dimensions $\leq n$, and

(ii) isometric actions of groups G and H on proper $CAT(0)$ spaces M and M' respectively.

These define a resolution $\mathbf{F} \otimes_K \mathbf{F}' \twoheadrightarrow A \otimes_K A'$ of the $G \times H$ module $A \otimes_K A'$ and an isometric action of $G \times H$ on the proper $CAT(0)$ space $M \times M'$. Again, this resolution is finitely generated in dimensions $\leq n$.

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We begin by generalising a theorem of Meinert [**Geh98**]:

THEOREM 12·1. *Assume K has no zero-divisors.*

$$^{\circ}\Sigma^n(M \times M';\, A \otimes_K A')^c \subseteq \bigcup_{p=0}^{n} {}^{\circ}\Sigma^p(M;\, A)^c * {}^{\circ}\Sigma^{n-p}(M';\, A')^c.$$

For the proof we need a lemma:

LEMMA 12·2. *Let γ be a geodesic ray with $\gamma(\infty) = e$ where $e \in {}^{\circ}\Sigma^k(M;\, A)$ for $k \leq n$. There exists $\nu \geq 0$ such that for any $\mu \geq 0$ there is a finitary chain map $\zeta : \mathbf{F}^{(n)} \to \mathbf{F}^{(n)}$ lifting* id_A *such that* $\mathrm{gsh}_e(\zeta\,|\mathbf{F}^{(k)}) \geq \mu$, *and*

$$v_\gamma(\zeta(c)) \geq v_\gamma(c) - \nu,$$

for all $c \in \mathbf{F}^{(n)}$.

Proof. Let $\varphi : \mathbf{F}^{(k)} \to \mathbf{F}^{(k)}$ be a G-finitary chain map inducing id_A and pushing $\mathbf{F}^{(k)}$ towards e. By Lemma 2·4, φ and id are chain homotopic by a G-finitary chain homotopy $\sigma : \mathbf{F}^{(k)} \to \mathbf{F}^{(k+1)}$. Extend φ to $\mathbf{F}^{(n)}$ as follows: Define $\psi_{k+1} : \mathbf{F}^{k+1} \to \mathbf{F}^{k+1}$ by $\psi_{k+1}(c) = c - \sigma\partial c$, and define $\psi(c) = c$ when c has degree $\geq k+2$. Then ψ is a chain map, and if σ is extended by defining it to be the zero map on chains of degree $\geq k+1$, then σ is a G-finitary chain homotopy between ψ and id.

Let $r > 0$ be such that $r \cdot \mathrm{gsh}_e(\varphi) > \mu$. We will show that $\zeta := \psi^{r+1}$ and $\nu := ||\partial|| + ||\sigma||$ satisfy the requirements of the lemma. Certainly, $\mathrm{gsh}_e(\psi^{r+1}|\mathbf{F}^{(k)}) \geq \mu$.

Consider the chain homotopy

$$\tau := \sigma(\mathrm{id} + \psi + \cdots + \psi^r)$$

between id and ψ^{r+1}. Then for any $c \in \mathbf{F}^{(n)}$ we have

$$\begin{aligned} v_\gamma(\tau(c)) &= v_\gamma\sigma(\mathrm{id} + \psi + \cdots + \psi^{r-1} + \psi^r)(c) \\ &\geq \min\{v_\gamma\sigma(\psi^p\sigma(c)) \mid 0 \leq p \leq k\} \\ &\geq \min\{v_\gamma(c) + p\cdot\epsilon - ||\sigma|| \mid 0 \leq p \leq k\} \\ &= v_\gamma(c) - ||\sigma||. \end{aligned}$$

If c has degree $k+1$ then $\psi^{r+1}(c) = c - \tau\partial c$. So

$$\begin{aligned} v_\gamma(\psi^{r+1}(c)) &\geq \min\{v_\gamma(c),\, v_\gamma(\tau\partial c)\} \\ &\geq \min\{v_\gamma(c),\, v_\gamma(\partial c) - ||\sigma||\} \\ &\geq \min\{v_\gamma(c),\, v_\gamma(c) - ||\partial|| - ||\sigma||\} \\ &= v_\gamma(c) - ||\partial|| - ||\sigma||. \end{aligned}$$

And if c has degree $> k+1$ then $\psi^{r+1}(c) = c$.

Proof of Theorem 12·1. By [**BH99**, section I·5·15] there is a canonical identification of $\partial(M \times M')$ with the join $\partial M * \partial M'$. Following [**BH99**, page 266], if $e \in \partial M$ and $e' \in \partial M'$, the θ-point on the join line from e to e' is denoted by $\cos\theta\; e + \sin\theta\; e'$ where $0 \leq \theta \leq \pi/2$. Picking base points $b \in M$ and $b' \in M'$ let γ, γ' and γ'' be the geodesic rays in M, M' and $M \times M'$ determining e, e', and $\cos\theta\; e + \sin\theta\; e'$. Then $\gamma''(t) = (\gamma(t\cos\theta), \gamma'(t\sin\theta))$.

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Assuming $\cos\theta\ e + \sin\theta\ e' \notin \bigcup_{p=0}^{n} {}^{\circ}\Sigma^{p}(M;A)^{c} * {}^{\circ}\Sigma^{n-p}(M';A')^{c}$, we will show that

$$\cos\theta\ e + \sin\theta\ e' \in {}^{\circ}\Sigma^{n}(M \times M'; A \otimes_K A').$$

Case 1. $0 < \theta < \pi/2$ and $e \in {}^{\circ}\Sigma^{n}(M;A)^{c}$. Let p be the largest integer such that $e \in^{\circ} \Sigma^{p-1}(M;A)$. Thus $e \in^{\circ} \Sigma^{p}(M;A)^{c}$, so $e' \in^{\circ} \Sigma^{n-p}(M';A')$. Then

$$v_{\gamma''}(c \otimes c') = \cos\theta\ v_{\gamma}(c) + \sin\theta\ v_{\gamma'}(c').$$

(For this one needs $\mathrm{supp}(c \otimes c') = \mathrm{supp}(c) \times \mathrm{supp}(c')$ which is true because K has no zero divisors.)

Let $\epsilon > 0$ be fixed. Let ν and ν' be as in Lemma 12·2. Choose μ so that $\cos\theta\ \mu - \sin\theta\ \nu' > \epsilon$, and choose μ' so that $\sin\theta\ \mu' - \cos\theta\ \nu > \epsilon$. By Lemma 12·2 there are finitary chain maps $\zeta : \mathbf{F}^{(n)} \to \mathbf{F}^{(n)}$ lifting id_A and $\zeta' : \mathbf{F}'^{(n)} \to \mathbf{F}'^{(n)}$ lifting $\mathrm{id}_{A'}$ such that

$$v_{\gamma}(\zeta(c)) \geq v_{\gamma}(c) + \mu \text{ for all } c \in \mathbf{F}^{(p-1)} \text{and}$$
$$v_{\gamma}(\zeta(c)) \geq v_{\gamma}(c) - \nu \text{ for all } c \in \mathbf{F}^{(n)}.$$

When $c \otimes c'$ has degree $\leq n$ and c has degree $\leq p-1$ then, by [**BH99**, section II·8·24], we have

$$\begin{aligned} v_{\gamma''}(\zeta(c) \otimes \zeta'(c')) &= \cos\theta\ v_{\gamma}(\zeta(c)) + \sin\theta\ v_{\gamma'}(\zeta'(c')) \\ &\geq \cos\theta[v_{\gamma}(c) + \mu] + \sin\theta[v_{\gamma'}(c') - \nu'] \\ &= v_{\gamma''}(c \otimes c') + \cos\theta\ \mu - \sin\theta\ \nu' \\ &> v_{\gamma''}(c \otimes c') + \epsilon. \end{aligned}$$

When $c \otimes c'$ has degree $\leq n$ and c has degree $\geq p$ then a similar discussion gives

$$v_{\gamma''}(\zeta(c) \otimes \zeta'(c')) > v_{\gamma''}(c \otimes c') + \epsilon.$$

So $\mathrm{gsh}_e(\zeta \otimes \zeta') \geq \epsilon$, and thus $\cos\theta\ e + \sin\theta\ e' \in {}^{\circ}\Sigma^{n}(M \times M'; A \otimes_K A')$.

Case 2. $0 < \theta < \pi/2$ and $e \in^{\circ} \Sigma^{n}(M;A)$. If $c \otimes c'$ has degree $\leq n$, the above argument again gives

$$v_{\gamma''}(\zeta(c) \otimes \zeta'(c')) > v_{\gamma''}(c \otimes c') + \epsilon.$$

Case 3. If $\theta = 0$ then $e \in {}^{\circ}\Sigma^{n}(M;A)$, and $\mathrm{id}'_{\mathbf{F}}$ plays the role previously played by ζ'. The case $\theta = \pi/2$ is handled similarly.

We turn to the opposite inclusion "$\supseteq$", starting with the observation that it cannot hold generally in a situation where $A \neq 0 \neq A'$ while $A \otimes A' = 0$. Therefore, from now on we will assume that K is a field.

THEOREM 12·3. *Let K be a field, A a KG-module of type FP_p and A' a KH-module of type FP_q. If $\Sigma^0(M;A) = \partial M$ and $\Sigma^0(M';A') = \partial M'$ then*

$$\Sigma^{p}(M;A)^{c} * \Sigma^{q}(M';A')^{c} \subseteq \Sigma^{p+q}(M \times M'; A \otimes_K A')^{c}.$$

Remarks. (1) The statement that $\Sigma^0(M;A) = \partial M$ is equivalent to saying that the G-action on M is cocompact and A has bounded support; see [**BG16**, theorem 9·1]. When the G-action has discrete orbits, this reduces to cocompactness together with A being finitely

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generated over the point stabilizer G_b for some (equivalently, any) point $b \in M$. See also Theorem 8·10 and Corollary 8·11. (2) In [**BG10**] we established the product formula for $\Sigma^n(G \times H; K)$, i.e. the case where $M = G_{ab} \otimes \mathbb{R}$ and $M' = H_{ab} \otimes \mathbb{R}$ are Euclidean and $A = K = A'$, where the 0-dimensional assumptions discussed in the previous remark are trivially satisfied. The proof of Theorem 12·3 given below lifts the key arguments of [**BG10**] to the $CAT(0)$ case with modules A, A', but this lifting only works when those assumptions hold.

Proof of Theorem 12·3. We use the projective $K(G \times H)$-resolution $\epsilon \otimes \epsilon' : \mathbf{F} \otimes_K \mathbf{F}' \twoheadrightarrow A \otimes A'$, noting that if $p \neq 0 \neq q$ then the chain arguments used in the proof of theorem 5·2 of [**BG10**] away from $H_0(\mathbf{F} \otimes_K \mathbf{F}') = A \otimes A'$ carry over *mutatis mutandis*. Therefore, without loss of generality it only remains to consider the case $q = 0$. And, as we assume $\Sigma^0(M'; A') = \partial M'$, we need only show

$$\partial M \cap \Sigma^p(M \times M'; A \otimes A') \subseteq \Sigma^p(M; A). \tag{**}$$

For this we can ignore the H-action, choose a KG-embedding $A \rightarrowtail A \otimes A'$ with a K-splitting, and lift it to a K-split KG-embedding $s : \mathbf{F} \rightarrowtail \mathbf{F} \otimes_K \mathbf{F}'$ with corresponding projection $\pi : \mathbf{F} \otimes_K \mathbf{F}' \twoheadrightarrow \mathbf{F}$; we then have $\pi \circ s = \mathrm{id}_{\mathbf{F}}$. The horoballs of $M \times M'$ at $e \in \partial M \subseteq \partial(M \times M')$ are of the form $HB_e(M \times M') = HB_e(M) \times M'$, and the Busemann function $\beta_e : M \times M' \to \mathbb{R}$ ignores the M' contribution. Hence the valuation $v_e : \mathbf{F} \otimes_K \mathbf{F}' \to M \times M' \to \mathbb{R}$ restricts to the corresponding valuation $\mathbf{F} \to \mathbb{R}$, and this implies the inclusion (**).

To complete the proof of Theorem 1·5, i.e. to prove

$${}^\circ\Sigma^n(M \times M'; A \otimes_K A')^c = \bigcup_{p=0}^{n} {}^\circ\Sigma^p(M; A)^c * {}^\circ\Sigma^{n-p}(M'; A')^c,$$

it only remains to replace Σ by ${}^\circ\Sigma$ in Theorem 12·3. Let

$$\cos\theta\, e_0 + \sin\theta\, e_0' \in {}^\circ\Sigma^p(M; A)^c * {}^\circ\Sigma^{n-p}(M'; A')^c.$$

First assume $0 < \theta < \pi/2$. Then, by the Characterisation Theorem 7·1, $\mathrm{cl}\, Ge \cap \Sigma^p(M; A)^c \neq \emptyset$ and $\mathrm{cl}\, He' \cap \Sigma^{n-p}(M'; A')^c \neq \emptyset$. Pick $e_0 \in \mathrm{cl}\, Ge \cap\ \Sigma^p(M; A)^c$ and $e_0' \in \mathrm{cl}\, He' \cap\ \Sigma^{n-p}(M'; A')^c$. Consider $\cos\theta\, e_0 + \sin\theta\, e_0'$. By Theorem 12·3 this lies in $\Sigma^n(M \times M'; A \otimes A')^c$.

We are to show $\cos\theta\, e_0 + \sin\theta\, e_0' \in {}^\circ\Sigma^n(M \times M'; A \otimes A')^c$. Suppose not. Then

$$\mathrm{cl}[(G \times H)(\cos\theta\, e_0 + \sin\theta\, e_0')] \subseteq \Sigma^n(M \times M'; A \otimes A')).$$

Since $e_0 \in\ \mathrm{cl}\, Ge$, e_0 is the limit elements of the form ge. Similarly, e_0' is the limit elements of the form he'. So $\cos\theta\ e_0 + \sin\theta\ e_0'$ lies in $\mathrm{cl}[(G \times H)(\cos\theta\ e_0 + \sin\theta\ e_0')] \subseteq \Sigma^n(M \times M'; A \otimes A')$. This is a contradiction.

Obvious alterations of this argument cover the cases $\theta = 0$ and $\theta = \pi/2$.

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