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On the direct problem singularities of a class of 3-DOF parallel manipulators

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SUMMARY
The 3-PS structure features one rigid body (platform) connected to another rigid body (base) by means of three kinematic chains (limbs) of type PS (P and S stand for prismatic pair and spherical pair, respectively). All the 3-degree-of-freedom parallel manipulators with three connectivity-5 limbs, each one constituted of one passive (i.e. not actuated) prismatic pair, one passive spherical pair and one actuated kinematic pair of any type, become 3-PS structures when the actuated pairs are locked. Direct kinematics of this class of manipulators is tied to the properties of the 3-PS structure. In particular, the direct position analysis is tied to the assembly modes of the 3-PS structure; whereas the determination of the singularities of the direct instantaneous problem is tied to the determination of the singular geometries of the 3-PS structure, where instantaneous relative motions between platform and base are possible. The solution of these two problems is necessary both for designing the manipulators and for controlling them during motion. This paper deal with the determination of the singular geometries of the 3-PS structure.

KEYWORDS: Direct kinematics; Parallel manipulators; Singularities.

1. INTRODUCTION

Parallel manipulators with less than six degrees of freedom (dof) have recently attracted the attention of the industrial and academic world. Such an interest is mainly due to the fact that a lot of specialized manipulation tasks require less than six dof.

The use of less-than-six-dof manipulators in those manipulation tasks is cheaper than the use of general purpose 6-dof manipulators, since they have simpler architectures and control systems. Moreover, less-than-six-dof manipulators, in general, exhibit a wider workspace because of the reduced number of links they have.

Among less-than-six-dof manipulators, 3-dof manipulators constitute an important family. In fact, planar manipulators,[1] spherical manipulators[2–7] and translational manipulators[8–12] belong to this family. Moreover, in the literature, 3-dof manipulators of mixed type[13,14] have been presented too.

Three-dof parallel manipulators can be obtained by connecting the end effector to the frame by means of three kinematic chains (limbs) with connectivity 5 (i.e. each limb leaves 5 dof to the relative motion of end effector and frame) and only one actuated joint. Such 3-dof manipulators become isostatic structures, where one rigid body (hereafter called platform) is connected to another rigid body (hereafter called base) by means of three connectivity-4 limbs, when the actuated joints are locked. The platform and the base of these structures are the end effector and the frame, respectively, of the manipulators they derive from.

Three-dof manipulators with different topology and three connectivity-5 limbs can yield isostatic structures with the same topology, when the actuated joints are locked. Therefore, studying the properties of the structures that are obtained by locking the actuators allows general conclusions to be found which are applicable to the whole class of manipulators they derive from.

In particular, the direct position analysis reduces itself to the determination of all the possible assembly modes of the structure obtained by locking the actuators.[15,16] Moreover, the determination of the singularities of the instantaneous direct problem (i.e. the determination of the manipulator configurations where the end effector can perform infinitesimal motions even if the actuators are locked[17]) reduces itself to the determination of the singular geometries of the structure, obtained by locking the actuators, where infinitesimal relative motions are possible between platform and base.

One particular structure where one rigid body (platform) is connected to another rigid body (base) by means of three connectivity-4 limbs is the 3-PS structure (Fig. 1). The 3-PS structure has the platform connected to the base by means of three limbs of type PS (P and S stand for prismatic pair and spherical pair, respectively). All the 3-dof manipulators with three connectivity-5 limbs, each one constituted of one passive (i.e. not actuated) prismatic pair, one passive spherical pair and one actuated kinematic pair of any type, become 3-PS structures when the actuated pairs are locked. By denoting the generic actuated joint, which is present in each limb, with the letter X, the limbs of these 3-dof manipulators belong to one out of the following three types: XPS, PXS and PSX. In the literature, manipulators of this type have been presented in Ceccarelli[13] and Di Gregorio et al.[14]

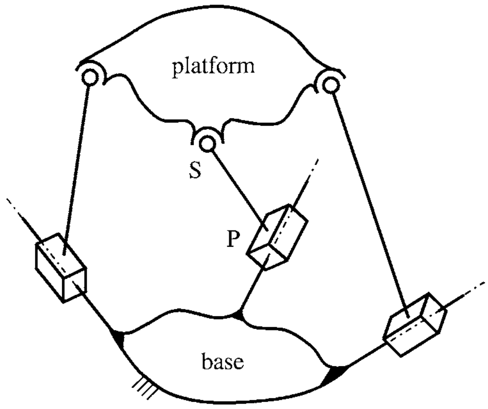

Fig. 1. The 3-PS structure (P and S stand for prismatic pair and spherical pair respectively).

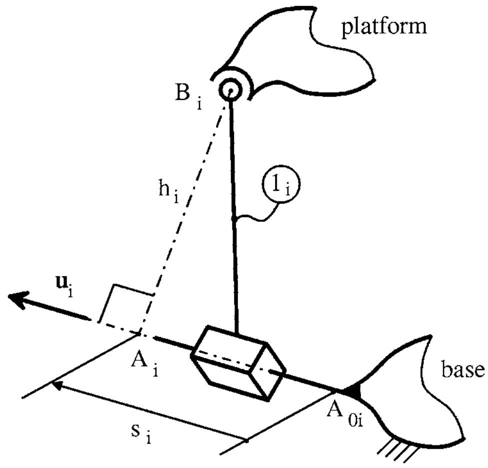

Fig. 2. The i-th limb of the 3-PS structure.

Direct kinematics of this class of manipulators is tied to the properties of the 3-PS structure. In particular, the direct position analysis is tied to the assembly modes of the 3-PS structure; whereas the determination of the singularities of the direct instantaneous problem is tied to the determination of the singular geometries of the 3-PS structure, where instantaneous relative motions between platform and base are possible. The solution of these two problems is necessary both for designing the manipulators and for controlling them during motion.

The determination of the assembly modes of the 3-PS structure was presented in Parenti-Castelli et al.,[15] where it has been reduced to the solution of a system of three quadratic equations. Such a system admits at most eight real solution that, in Parenti-Castelli et al.,[15] have been determined in analytic form by determining the general expression of the Sylvester's eliminant of the system.

The singular geometries of the 3-PS structure have been investigated by using static reasoning in Ebert-Uphoff et al.[18] and by using kinematic reasoning in Kong et al.[19] and Yang et al.[20] Both Ebert-Uphoff et al.[18] and Kong et al.[19] conclude that "if four particular planes (one plane located by the centers of the three spherical pairs and other three planes perpendicular to the slide direction of one limb's prismatic pair and passing through the center of the spherical pair of the same limb) have at least one point as common intersection, then the 3-PS is singular"; whereas, the analytic results reported in Yang et al.[20] are the same as those reported in Kong et al.,[19] but their geometric interpretation is less general.

This paper reconsiders the kinematic point of view for searching the singular geometries of the 3-PS structure, and writes the analytic form of its singularity condition in a way which is different from the one presented in Kong et al.[19] and Yang et al.[20] The new form makes it possible the exhaustive enumeration of the geometric conditions, which identify the singular geometries, and reveals that there are some geometric conditions that were not identified in Ebert-Uphoff et al.[18] and Kong et al.[19]

Finally, in this paper, the determination of the analytic expression of the singularity loci of 3-dof manipulators which generate 3-PS structures, when the actuators are locked, will be discussed.

2. BACKGROUND

This section recalls some analytic results, reported in Kong et al.[19] and Yang et al.,[20] that will be used in the next sections.

Figure 2 shows the i-th limb of the 3-PS structure. With reference to Fig. 2, point B_i is the center of the spherical pair. Point A_{0i} is fixed to the base. $\mathbf{u}_i$ is the unit vector of the sliding direction of the prismatic pair. Point A_i is the foot of the perpendicular through B_i to the straight line, fixed to the base, which passes through A_{0i} and has the direction of $\mathbf{u}_i$. h_i is the length of the segment A_iB_i. s_i is the signed distance from A_{0i} to A_i and is the joint variable of the prismatic pair. Hereafter, if it is not otherwise specified, all the vectors are measured in a reference system fixed to the base, and a capital bold letter will indicate the position vector of the point the capital letter refers to.

With these notations, the closure equations of the 3-PS structure can be written as follows

$$[\mathbf{m}_i + s_i\mathbf{u}_i + \mathbf{A}_{0i} - \mathbf{m}_j - s_j\mathbf{u}_j - \mathbf{A}_{0j}]^2 = c_{ij}^2, \quad i, j \in \{1, 2, 3 | i \neq j\} \tag{1}$$

with

$$\mathbf{m}_k = (\mathbf{B}_k - \mathbf{A}_k) \qquad k = 1, 2, 3 \tag{2a}$$

$$c_{ij} = \|\mathbf{B}_i - \mathbf{B}_j\| \qquad i, j \in \{1, 2, 3 | i \neq j\} \tag{2b}$$

In Eqs. (1), the vectors $\mathbf{m}_k$, $\mathbf{u}_k$ and $\mathbf{A}_{0k}$, $k = 1, 2, 3$, and c_{ij} are geometric constants of the 3-PS structure.

If the platform moves, only the joint variables s_i and s_j may vary in Eq. (1). Therefore, the first time derivative of Eqs. (1) are:

$$\mathbf{u}_3 \cdot (\mathbf{B}_3 - \mathbf{B}_2)\,\dot{s}_3 - \mathbf{u}_2 \cdot (\mathbf{B}_3 - \mathbf{B}_2)\,\dot{s}_2 = 0 \tag{3a}$$

$$\mathbf{u}_2 \cdot (\mathbf{B}_2 - \mathbf{B}_1)\,\dot{s}_2 - \mathbf{u}_1 \cdot (\mathbf{B}_2 - \mathbf{B}_1)\,\dot{s}_1 = 0 \tag{3b}$$

$$\mathbf{u}_3 \cdot (\mathbf{B}_3 - \mathbf{B}_1)\,\dot{s}_3 - \mathbf{u}_1 \cdot (\mathbf{B}_3 - \mathbf{B}_1)\,\dot{s}_1 = 0 \tag{3c}$$

where

$$\mathbf{B}_k = \mathbf{m}_k + s_k\mathbf{u}_k + \mathbf{A}_{0k} \qquad k = 1, 2, 3 \tag{4}$$

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Equations (3), in matrix form, become

$$\mathbf{H}\,\dot{\mathbf{s}} = 0 \tag{5}$$

where

$$\dot{\mathbf{s}} = \{\dot{s}_1,\ \dot{s}_2,\ \dot{s}_3\}^T \tag{6a}$$

$$\mathbf{H} = \begin{bmatrix} \mathbf{u}_3\cdot(\mathbf{B}_3-\mathbf{B}_2) & -\mathbf{u}_2\cdot(\mathbf{B}_3-\mathbf{B}_2) & 0 \\ 0 & \mathbf{u}_2\cdot(\mathbf{B}_2-\mathbf{B}_1) & -\mathbf{u}_1\cdot(\mathbf{B}_2-\mathbf{B}_1) \\ \mathbf{u}_3\cdot(\mathbf{B}_3-\mathbf{B}_1) & 0 & -\mathbf{u}_1\cdot(\mathbf{B}_3-\mathbf{B}_1) \end{bmatrix} \tag{6b}$$

Relative motions of platform and base will be possible if and only if system (5) admits solutions different from zero. Since system (5) is linear and homogenous, it will admit solutions that are not trivial if and only if matrix **H** is singular. The singular geometries of the 3-PS structure are the ones that make matrix **H** singular.

Matrix **H** is singular when

$$\det(\mathbf{H}) = 0 \tag{7}$$

where

$$\det(\mathbf{H}) = -[\mathbf{u}_3\cdot(\mathbf{B}_3-\mathbf{B}_2)][\mathbf{u}_2\cdot(\mathbf{B}_2-\mathbf{B}_1)][\mathbf{u}_1\cdot(\mathbf{B}_3-\mathbf{B}_1)] + [\mathbf{u}_3\cdot(\mathbf{B}_3-\mathbf{B}_1)][\mathbf{u}_2\cdot(\mathbf{B}_3-\mathbf{B}_2)][\mathbf{u}_1\cdot(\mathbf{B}_2-\mathbf{B}_1)]. \tag{8}$$

Equation (7) is the analytic form of the singularity condition that has been reported in Kong et al.[19] and Yang et al.[20]

3. ALTERNATIVE FORM OF THE SINGULARITY CONDITION

If the terms appearing in expression (8) are collected in a suitable way and the vector relationship

$$(\mathbf{B}_3-\mathbf{B}_2) = (\mathbf{B}_3-\mathbf{B}_1) - (\mathbf{B}_2-\mathbf{B}_1),$$

is used, expression (8) becomes

$$\begin{aligned}\det(\mathbf{H}) = \mathbf{u}_1\cdot\{(\mathbf{u}_3\cdot(\mathbf{B}_3-\mathbf{B}_1))[(\mathbf{B}_2-\mathbf{B}_1)(\mathbf{u}_2\cdot(\mathbf{B}_3-\mathbf{B}_1)) \\ -(\mathbf{B}_3-\mathbf{B}_1)(\mathbf{u}_2\cdot(\mathbf{B}_2-\mathbf{B}_1))] - (\mathbf{u}_2\cdot(\mathbf{B}_2-\mathbf{B}_1)) \\ \times[(\mathbf{B}_2-\mathbf{B}_1)(\mathbf{u}_3\cdot(\mathbf{B}_3-\mathbf{B}_1)) - (\mathbf{B}_3-\mathbf{B}_1) \\ \times(\mathbf{u}_3\cdot(\mathbf{B}_2-\mathbf{B}_1))]\}\end{aligned} \tag{9}$$

In addition, the vector identity

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c},$$

where **a**, **b** and **c** are any vectors, leads to the conclusion that

$$(\mathbf{B}_2-\mathbf{B}_1)(\mathbf{u}_2\cdot(\mathbf{B}_3-\mathbf{B}_1)) - (\mathbf{B}_3-\mathbf{B}_1)(\mathbf{u}_2\cdot(\mathbf{B}_2-\mathbf{B}_1)) = \mathbf{u}_2\times\mathbf{g}_1 \tag{10a}$$

$$(\mathbf{B}_2-\mathbf{B}_1)(\mathbf{u}_3\cdot(\mathbf{B}_3-\mathbf{B}_1)) - (\mathbf{B}_3-\mathbf{B}_1)(\mathbf{u}_3\cdot(\mathbf{B}_2-\mathbf{B}_1)) = \mathbf{u}_3\times\mathbf{g}_1 \tag{10b}$$

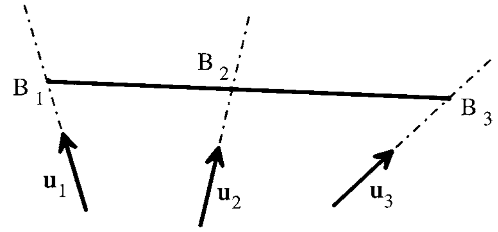

Fig. 3. Singular geometry of type (a.1).

where

$$\mathbf{g}_1 = (\mathbf{B}_2-\mathbf{B}_1)\times(\mathbf{B}_3-\mathbf{B}_1) \tag{11}$$

Finally, the introduction of expressions (10) into relationship (9) yields

$$\det(\mathbf{H}) = \mathbf{u}_1\cdot\mathbf{f}_1\times\mathbf{g}_1 \tag{12}$$

where

$$\mathbf{f}_1 = [\mathbf{u}_3\cdot(\mathbf{B}_3-\mathbf{B}_1)]\mathbf{u}_2 - [\mathbf{u}_2\cdot(\mathbf{B}_2-\mathbf{B}_1)]\mathbf{u}_3 \tag{13}$$

Using relationship (12) and the properties of the mixed product, singularity condition (7) becomes

$$\mathbf{v}_1\cdot\mathbf{f}_1 = 0 \tag{14}$$

where

$$\mathbf{v}_1 = \mathbf{g}_1\times\mathbf{u}_1 \tag{15}$$

Vector $\mathbf{v}_1$ has the direction of the common line between the plane of the triangle $B_1B_2B_3$ and the plane perpendicular to $\mathbf{u}_1$ and passing through B_1; whereas vector $\mathbf{f}_1$ belongs to the span of the vector set $\{\mathbf{u}_2, \mathbf{u}_3\}$.

Since the numbering of the limbs is arbitrary, Eq. (14) can be written in other two different ways by permuting the indices {1, 2, 3} of the limbs. These three expressions of the singularity condition are completely equivalent and are satisfied by the same singular geometries. Therefore, without losing generality, only expression (14) will be considered in the following discussion.

4. GEOMETRIC INTERPRETATION OF THE SINGULARITY CONDITION

The singularity condition (14) is satisfied if and only if the scalar product $\mathbf{v}_1\cdot\mathbf{f}_1$ vanishes. Such a scalar product vanishes if and only if one out of the following conditions occurs: (a) $\mathbf{v}_1$ vanishes; (b) $\mathbf{f}_1$ vanishes; (c) $\mathbf{v}_1$ is perpendicular to $\mathbf{f}_1$.

Condition (a) (i.e. $\mathbf{v}_1$ vanishes) occurs if and only if one out of the following geometric conditions is satisfied:

(a.1) the three points B_i, $i = 1, 2, 3$, are aligned (Fig. 3) (*proof*: the magnitude of the vector $\mathbf{g}_1$ (see Eq. (11)) is equal to the area of the triangle $B_1B_2B_3$; therefore, when this condition occurs, $\mathbf{g}_1$ (see Eq. (11)) vanishes and makes $\mathbf{v}_1$ (see Eq. (15)) vanish);

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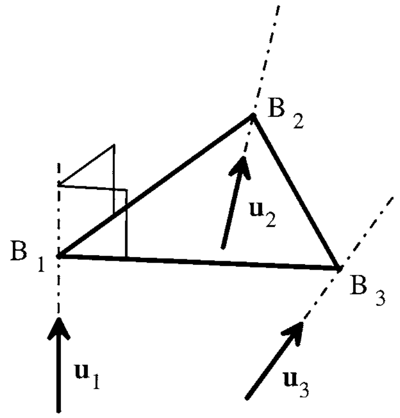

Fig. 4. Singular geometry of type (a.2).

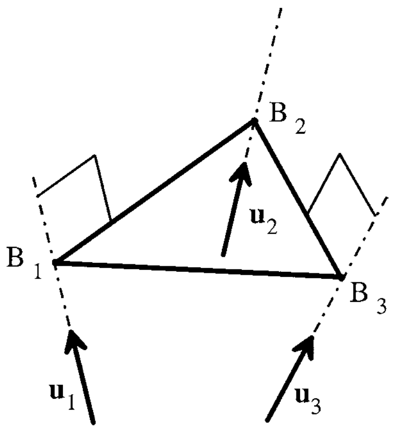

Fig. 5. Singular geometry of type (b.1).

(a.2)′ the unit vector $\mathbf{u}_1$ is perpendicular to the plane of the triangle $B_1B_2B_3$ (Fig. 4) (*proof*: when this condition occurs, $\mathbf{u}_1$ and $\mathbf{g}_1$ are parallel and make $\mathbf{v}_1$ vanish).

Condition (b) (i.e. $\mathbf{f}_1$ vanishes) occurs if and only if one out of the following geometric conditions is matched:

(b.1)′ $\mathbf{u}_2$ and $\mathbf{u}_3$ are perpendicular to $(\mathbf{B}_2 - \mathbf{B}_1)$ and $(\mathbf{B}_3 - \mathbf{B}_1)$ respectively (*proof*: when this condition is satisfied, in Eq. (13), the coefficients of $\mathbf{u}_2$ and $\mathbf{u}_3$ vanish and make $\mathbf{f}_1$ vanish);

(b.2)′ $\mathbf{u}_2$ and $\mathbf{u}_3$ are parallel to one another and perpendicular to $(\mathbf{B}_3 – \mathbf{B}_2)$ (*proof*: this condition can be verified by direct substitution in Eq. (13)).

Since the numbering of points B_i, $i = 1, 2, 3$, is arbitrary the condition (a.2)′, (b.1)′ and (b.2)′ can be generalized as follows:

(a.2) a unit vector $\mathbf{u}_i$, $i = 1, 2, 3$, is perpendicular to the plane of the triangle $B_1B_2B_3$;

(b.1) $\mathbf{u}_j$ and $\mathbf{u}_k$ are perpendicular to $(\mathbf{B}_j - \mathbf{B}_i)$ and $(\mathbf{B}_k - \mathbf{B}_i)$, respectively $(i, j, k \in \{1, 2, 3 | i \neq j; i \neq k; j \neq k\})$ (Fig. 5);

(b.2) $\mathbf{u}_j$ and $\mathbf{u}_k$ $(j, k \in \{1, 2, 3 | j \neq k\})$ are parallel to one another and perpendicular to $(\mathbf{B}_k - \mathbf{B}_j)$.

The discussion of condition (c) (i.e. $\mathbf{v}_1$ is perpendicular to $\mathbf{f}_1$) must consider the fact that $\mathbf{f}_1$ may be (see Eq. 13)

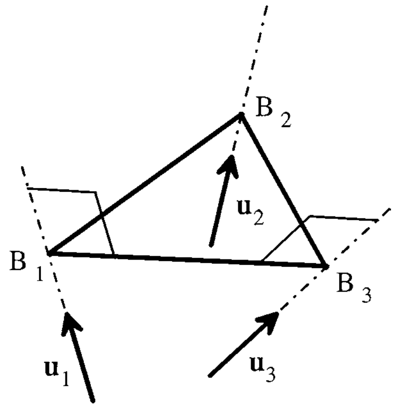

Fig. 6. Singular geometry of type (c.1).

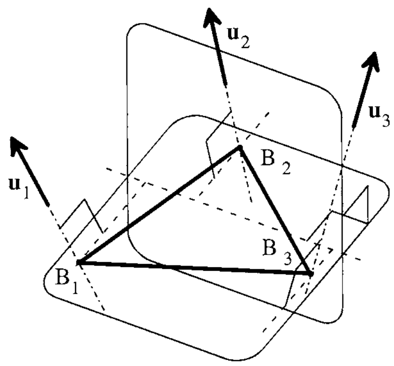

Fig. 7. Singular geometry of type (c.2).

(i) parallel to $\mathbf{u}_2$ (which implies that $\mathbf{u}_3$ is perpendicular to $(\mathbf{B}_3 - \mathbf{B}_1)$), (ii) parallel to $\mathbf{u}_3$ (which implies that $\mathbf{u}_2$ is perpendicular to $(\mathbf{B}_2 - \mathbf{B}_1)$ and (iii) a linear combination of $\mathbf{u}_2$ and $\mathbf{u}_3$.

The analysis of condition (c) in the cases (i) and (ii) brings to singular geometries that are special cases of (a.2) or of the following case

(c.1) $\mathbf{u}_j$ and $\mathbf{u}_k$ $(j, k \in \{1, 2, 3 | j \neq k\})$ are perpendicular to $(\mathbf{B}_k – \mathbf{B}_j)$ (Fig. 6) (note that condition (b.2) is a special case of this condition).

Condition (c.1) was also found in Yang et al.[20]

In the case (iii), if the expressions of the coefficients of $\mathbf{u}_2$ and $\mathbf{u}_3$, appearing in Eq. (13) are not considered, condition (c) implies that $\mathbf{v}_1$ is contemporarily perpendicular to $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$. Therefore, by noting that v_1 is always parallel to the plane of the triangle $B_1B_2B_3$, the following geometric condition results

(c.2) the three unit vectors $\mathbf{u}_i$, $i = 1, 2, 3$, are parallel to one plane which is perpendicular to the plane of the triangle $B_1B_2B_3$. (Fig. 7)

Finally, in the case (iii), if the special expressions of the coefficients of $\mathbf{u}_2$ and $\mathbf{u}_3$ in Eq. (13) are considered, the demonstration, reported in Kong et al.[19] (which starts from expression (8)), holds. Such a demostration brings to the conclusion that, in this case, condition (c) brings to the following geometric condition

(c.3) at least one point is in common among the plane of the triangle $B_1B_2B_3$ and three planes π_i, $i = 1, 2, 3$,

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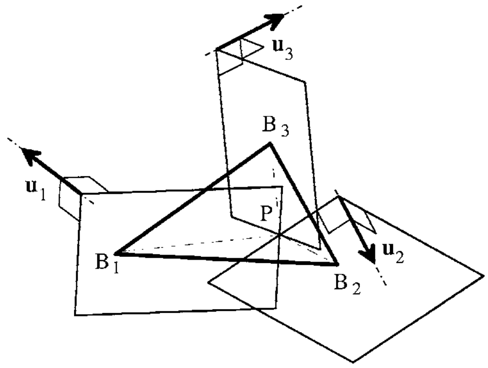

Fig. 8. Singular geometry of type (c.3).

defined as follows: π_i is the plane passing through B_i and perpendicular to $\mathbf{u}_i$; (Fig. 8) (note that conditions (a.2), (b.1) and (c.1) are special cases of this condition).

Since the above-reported discussion is exhaustive, the conclusion is that all the singular geometries of the 3-PS structure must belong to at least one out of the cases (a.1), (c.2) and (c.3). The geometric conditions (a.1) and (c.2) were not identified in Ebert-Uphoff et al.[18] and Kong et al.[19] and are new.

5. USE OF THE SINGULARITY CONDITION OF THE 3-PS STRUCTURE

From an analytic point of view, condition (14) is one equation which contains only vectors embedded either in the base or in the platform. As a consequence, when these vectors are all written in a unique reference system fixed either in the base or in the platform, only the geometric constants, and the parameters which identify the relative orientation between platform and base appear in Eq. (14).

In order to apply condition (14) to the study of the 3-dof manipulators which become 3-PS structures when the actuators are locked, the following procedure can be used

(a.1) three parameters which identify the relative orientation between end effector and frame are chosen as generalized coordinates of the manipulator;
(a.2) by solving the inverse position analysis of the manipulator with assigned relative orientation of end effector and frame, the actuated joint variables are explicitly expressed as function of the orientation parameters, chosen in the step (a.1);
(a.3) the geometric parameters which depend on the actuated joint variables, and become geometric constant of the 3-PS structure when the actuated joint are locked, are explicitly expressed as function of the orientation parameters, chosen in the step (a.1), by exploiting the results obtained in step (a.2);
(a.4) the expressions obtained in step (a.3) are substituted for the corresponding geometric constants in Eq. (14), and the resulting equation, which, now, contains only the geometric constant of the manipulator and the orientation parameters, and is the analytic expression of the direct-problem-singularity locus of the manipulator, is used to draw the singularity locus of the manipulator in a three-dimensional diagram whose coordinates are the orientation parameters.

If the geometric constants of the manipulator are assigned, the equation obtained at the end of the above procedure is an scalar equation in three unknowns (the orientation parameters). Thus, it is the analytic expression of a surface of the three-dimensional space whose coordinates are the orientation parameters. This surface is the geometric locus of the direct problem singularities of the manipulator. The same equation can be used to visualize the effect of variations in the geometric constant of the manipulator on the singularity locus.

6. CONCLUSIONS

Three-degree-of-freedom parallel manipulators with three limbs, each one constituted of one passive prismatic pair, one passive spherical pair and one actuated kinematic pair of any type, become 3-PS structures when the actuated pairs are locked.

Direct kinematics of this class of manipulators is tied to the properties of the 3-PS structure. In particular, the direct position analysis is tied to the assembly modes of the 3-PS structure; whereas the determination of the singularities of the direct instantaneous problem is tied to the determination of the singular geometries of the 3-PS structure.

This paper has studied the singular geometries of the 3-PS structure.

In particular, the kinematic point of view, presented in Kong et al.[19] and Yang et al.,[20] for searching the singular geometries of the 3-PS structure has been reconsidered. The analytic form of the 3-PS structure's singularity condition has been transformed into a new form. The new form has made it possible the exhaustive enumeration of the geometric conditions, which identify the singular geometries, and has revealed that there are some geometric conditions which were not identified in the literature.

Finally, the determination of the analytic expression of the singularity loci of 3-dof manipulators which generate 3-PS structures, when the actuators are locked, has been discussed.

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