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Continuity of condenser capacity under holomorphic motions

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Abstract. We show that condenser capacity varies continuously under holomorphic motions, and the corresponding family of the equilibrium measures of the condensers is continuous with respect to the weak-star convergence. We also study the behavior of uniformly perfect sets under holomorphic motions.

1 Introduction

A condenser in the complex plane $\mathbb{C}$ is a pair (E, F) where E and F are non-empty disjoint compact subsets of $\mathbb{C}$. Let $S(E, F)$ denote the family of signed measures $\tau = \tau_E - \tau_F$, where τ_E and τ_F are Borel probability measures supported on E and F, respectively. The energy of a measure $\tau \in S(E, F)$ is defined by

$$I(\tau) := \iint \log \frac{1}{|z - w|} d\tau(z) d\tau(w).$$

We note that $I(\tau) > 0$, for every $\tau \in S(E, F)$; see *e.g.*, [11, p. 80]. The equilibrium energy of (E, F) is defined by

$$I(E, F) := \inf_{\tau \in S(E,F)} I(\tau),$$

and the capacity of (E, F) is given by

$$\mathrm{Cap}(E, F) := \frac{2\pi}{I(E, F)}.$$

When $\mathrm{Cap}(E, F) > 0$, there exists a unique measure $\sigma \in S(E, F)$, called the *equilibrium measure* of (E, F), satisfying $\mathrm{Cap}(E, F) := 2\pi/I(\sigma)$. For more information about condenser capacity, we refer the reader to [3, 11].

A classical object of study in geometric function theory is the behavior of several types of capacities (such as logarithmic, analytic, and Riesz capacity) of sets and condensers in $\mathbb{C}$, under geometric transformations such as conformal mappings, symmetrizations, and polarization. We refer the reader to the books [4, 7] and the references therein for an account of the methods and the applications of capacities

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in geometric function theory. In this note, we will study the behavior of condenser capacity under holomorphic motions.

A holomorphic motion is a holomorphically parameterized family of injective maps. Here is the precise description.

Definition 1.1 A holomorphic motion of a set $A \subset \mathbb{C}$, parameterized by a domain $D \subset \mathbb{C}$ containing 0 is a map $f : D \times A \mapsto \mathbb{C}$ satisfying:

(i) for any fixed $z \in A$, the map $\lambda \mapsto f(\lambda, z)$ is holomorphic in D;
(ii) for any fixed $\lambda \in D$, the map $z \mapsto f(\lambda, z) := f_\lambda(z)$ is an injection;
(iii) the mapping $f(0, \cdot)$ is the identity on A.

Although there is no continuity assumption of f on $D \times A$ in the above definition, the continuity (as a function of two complex variables) of holomorphic motions has been proved, among other properties, by Mañé, P. Sad, and D. Sullivan [12], who introduced the notion of holomorphic motions, in a result that is known as λ-lemma. The joint continuity will be used several times in our proofs. A fundamental result in this theory, proved by Słodkowski [21], is that a holomorphic motion of any set in the Riemann sphere parameterized by the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ can be extended to a holomorphic motion of the whole Riemann sphere parameterized by $\mathbb{D}$. For an account of the properties of holomorphic motions (such as quasiconformality, distortion, and non-extendability properties) and their applications, we refer the reader to [1, 2, 10, 13] and the references therein.

The behavior of different geometric quantities under holomorphic motions has been studied by several researchers. Some examples are the Hausdorff dimension [18, 20], the conformal modulus of doubly connected domains [5], the analytic and the logarithmic capacity [16, 19], and the capacity of condensers [15].

In this paper, we will consider the behavior of condenser capacity under holomorphic motions. Let (E, F) be a condenser with positive capacity and let f be a holomorphic motion of $E \cup F$ parameterized by a domain D containing 0. From the continuity of the injective functions $f_\lambda(\cdot)$ by the λ-lemma, it follows that $f_\lambda(E)$ and $f_\lambda(F)$ are disjoint compact subsets of $\mathbb{C}$, for every $\lambda \in D$. We will show that $T(\lambda) = \mathrm{Cap}(f_\lambda(E), f_\lambda(F))$ is a continuous subharmonic function on D. We note that, applying the methods used in [15], one can show that the function $T(\lambda)$ is upper-semicontinuous and subharmonic in D, but the continuity proved here requires different arguments. In [19], the authors showed that the logarithmic capacity of a compact set also varies continuously under holomorphic motions. Although there are estimates between condenser capacity and logarithmic capacity, the continuity of condenser capacity is not a consequence of the corresponding result for logarithmic capacity under holomorphic motions. In contrast to condenser and logarithmic capacity, the analytic capacity of a compact set may vary discontinuously (see [19]) and, in general, is neither a subharmonic nor a superharmonic function under holomorphic motions (see [16]). Also, we will show that the equilibrium measures $\{\sigma_\lambda\}$, $\lambda \in D$, of the condensers $(f_\lambda(E), f_\lambda(F))$ vary continuously with respect to the weak-star convergence. The proof is based on the properties of pointwise suprema of harmonic functions established in [19] and on Bagby's formula for condenser capacity via discrete charges [3].

Earle and Mitra proved a much stronger result for a certain class of condensers called rings. A condenser (E, F) is called a *ring* if both E and F are connected and $\mathbb{C}\setminus(E \cup F)$ is a doubly connected domain. It was proved in [5] that if (E, F) is a ring and f is a holomorphic motion of $E \cup F$ parameterized by a domain D, then the equilibrium energy of $(f_\lambda(E), f_\lambda(F))$ (which coincides with the conformal modulus of the doubly connected domain $\mathbb{C}\setminus(f_\lambda(E) \cup f_\lambda(E)))$ is a real analytic function on D. It is not known whether the equilibrium energy of $(f_\lambda(E), f_\lambda(F))$ is a real analytic function of λ for arbitrary condensers.

A notion related to rings is uniform perfectness. Let K be a compact subset of $\mathbb{C}$. We recall that a ring (E, F) is said to separate K if $K \subset E \cup F$, $K \cap E \neq \varnothing$, and $K \cap F \neq \varnothing$. We will denote by $R(K)$ the set of rings that separate K. A compact set $K \subset \mathbb{C}$ is called *uniformly perfect* if

$$P(K) := \sup\left\{I(E, F) : (E, F) \in R(K)\right\} < +\infty.$$

A uniformly perfect set is thick, in the above sense, close to each of its points; in particular, it does not have isolated points. Uniform perfectness can be characterized using several other quantities such as the logarithmic capacity or the density of the hyperbolic metric. For more information, we refer the reader to [9, 14].

In our last result, we will give an estimate of the quantity $P(\cdot)$ for compact sets moving under holomorphic motions, involving the Harnack distance.

In the following section we describe some tools needed for the proofs of our results. In Section 3, we prove the continuity of condenser capacity, and in Section 4, we prove the continuity of the equilibrium measures with respect to weak-star convergence. The estimate of the quantity $P(\cdot)$ under holomorphic motions is proved in Section 5.

2 Background Material

2.1 Bagby's Formula

Let (E, F) be a condenser and suppose that both sets E and F contain infinitely many points. That holds for all condensers (E, F) having positive capacity. For any integer $n \geq 2$, let

$$L_n(E, F) := \{(a_1, \ldots, a_n, b_1, \ldots, b_n) \in E^n \times F^n : a_i \neq a_j \text{ and } b_i \neq b_j, i \neq j\},$$
$$W_n(E, F) = \frac{1}{n(n-1)} \inf\left\{\sum_{1 \leq i < j \leq n} \log\left(\frac{|a_i - b_j||a_j - b_i|}{|a_i - a_j||b_i - b_j|}\right)\right\},$$

where the infimum is taken over all $(a_1, \ldots, a_n, b_1, \ldots, b_n) \in L_n(E, F)$. Although every discrete signed measure in $S(E, F)$ has infinite energy, the above sum can be considered as a discrete version of the energy of a discrete measure having point masses at the points a_i and b_i, $i = 1, \ldots, n$. Bagby [3] proved the following theorem relating the equilibrium energy with the discrete energies $W_n(E, F)$ of a condenser.

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Theorem 2.1 [3] *Let (E, F) be a condenser and suppose that both sets E and F contain infinitely many points. Then the sequence $\{W_n(E, F)\}$ is increasing and*

$$I(E, F) = \lim_{n\to\infty} W_n(E, F).$$

2.2 Pointwise Infima of Harmonic Functions

The following family of functions was introduced and its main properties were studied in [19].

Definition 2.2 Let $D \subset \mathbb{C}$ be a domain. A function $u : D \mapsto (-\infty, +\infty]$ is said to belong to the class $\mathcal{H}^{\downarrow}(D)$ if the following properties hold:

(i) u is locally bounded below on D;
(ii) u is the pointwise infimum of a family of harmonic functions on D.

The following proposition summarizes some properties of the members of the family $\mathcal{H}^{\downarrow}(D)$ that will be needed in the proofs of our results.

Proposition 2.3 ([19, Propositions 2.4 and 2.5]) *Let D be a domain in $\mathbb{C}$. Then*

(i) *if u_n is an increasing sequence of functions in $\mathcal{H}^{\downarrow}(D)$ and $u = \lim_{n\to\infty} u_n$, then $u \in \mathcal{H}^{\downarrow}(D)$.*
(ii) *if $u \in \mathcal{H}^{\downarrow}(D)$ and $u \not\equiv +\infty$, then $u < +\infty$ in D and u is a continuous superharmonic function in D.*

Remark 2.4 In [19], the dual family $\mathcal{H}^{\uparrow}(D)$ of pointwise suprema of harmonic functions is considered, and the corresponding results stated above for $\mathcal{H}^{\downarrow}(D)$ follow by standard modifications.

3 Continuity of Condenser Capacity

In this section, we will state and prove our first result concerning the continuity of condenser capacity under holomorphic motions.

Theorem 3.1 *Let (E, F) be a condenser with positive capacity and let f be a holomorphic motion of $E \cup F$ parameterized by a domain $D \subset \mathbb{C}$ containing 0. Then*

$$T(\lambda) = \operatorname{Cap}\big(f_\lambda(E), f_\lambda(F)\big)$$

is a continuous subharmonic function on D.

Proof We note that since $\operatorname{Cap}(E, F) > 0$, both E and F contain infinitely many points. We will first show that the functions $R_n(\lambda) := W_n(f_\lambda(E), f_\lambda(F))$ are in $\mathcal{H}^{\downarrow}(D)$. Let $n \geq 2$ and note that since f_λ is injective,

$$L_n(f_\lambda(E), f_\lambda(F)) = \{(f_\lambda(a_1), \ldots, f_\lambda(a_n), f_\lambda(b_1), \ldots, f_\lambda(b_n)) : \\ : (a_1, \ldots, a_n, b_1, \ldots, b_n) \in L_n(E, F)\}.$$

Let $(a_1, \ldots, a_n, b_1, \ldots, b_n) \in L_n(E, F)$. Since $f_\lambda(\cdot)$ is injective and depends holomorphically on λ, the functions

$$\lambda \longmapsto \frac{(f_\lambda(a_i) - f_\lambda(b_j))(f_\lambda(a_j) - f_\lambda(b_i))}{(f_\lambda(a_i) - f_\lambda(a_j))(f_\lambda(b_i) - f_\lambda(b_j))}, \qquad i, j \in \{1, \ldots, n\},\ i \neq j,$$

are holomorphic and have no zeros on D. Therefore, the function

$$\lambda \longmapsto \frac{1}{n(n-1)} \sum_{1 \le i < j \le n} \log\left(\frac{|f_\lambda(a_i) - f_\lambda(b_j)||f_\lambda(a_j) - f_\lambda(b_i)|}{|f_\lambda(a_i) - f_\lambda(a_j)||f_\lambda(b_i) - f_\lambda(b_j)|} \right)$$

is harmonic in D, since it is a finite sum of harmonic functions. We obtain that

$$R_n(\lambda) = \inf\left\{ \frac{1}{n(n-1)} \sum_{1 \le i < j \le n} \log\left(\frac{|f_\lambda(a_i) - f_\lambda(b_j)||f_\lambda(a_j) - f_\lambda(b_i)|}{|f_\lambda(a_i) - f_\lambda(a_j)||f_\lambda(b_i) - f_\lambda(b_j)|} \right) \right\},$$

where the infimum is taken over all $(a_1, \ldots, a_n, b_1, \ldots, b_n) \in L_n(E, F)$, is a pointwise infimum of harmonic functions of λ in D. Let $\lambda_0 \in D$. Since $f(\lambda, z)$ is jointly continuous in $D \times (E \cup F)$, there exist an open neighborhood V of λ_0 such that

$$\inf_{\lambda \in V} \operatorname{dist}(f_\lambda(E), f_\lambda(F)) > 0,$$
$$\sup_{\lambda \in V} \operatorname{diam}(f_\lambda(E)) < +\infty,$$
$$\sup_{\lambda \in V} \operatorname{diam}(f_\lambda(F)) < +\infty.$$

It follows that R_n is locally bounded below on D. We conclude that $R_n \in \mathcal{H}^{\downarrow}(D)$. By Theorem 2.1, R_n is an increasing sequence of functions in $\mathcal{H}^{\downarrow}(D)$ and

$$\lim_{n \to +\infty} R_n(\lambda) = I(f_\lambda(E), f_\lambda(F)), \qquad \lambda \in D.$$

Since $I(E, F) < +\infty$, by Proposition 2.3, the function $\lambda \mapsto I(f_\lambda(E), f_\lambda(F))$ belongs to $\mathcal{H}^{\downarrow}(D)$ and is a continuous superharmonic function in D. Finally, from [17, p. 43], it follows that $T(\lambda) = 2\pi / I(f_\lambda(E), f_\lambda(F))$ is a continuous subharmonic function on D. ■

4 Weak-star Continuity of Equilibrium Measures

In this section, we show that the equilibrium measures of condensers vary continuously with respect to weak-star convergence under holomorphic motions. We recall that a sequence of Borel probability measures μ_n on a compact set $K \subset \mathbb{C}$ converges weak-star to a Borel probability measure μ on K if

$$\lim_{n \to \infty} \int \phi(z) d\mu_n(z) = \int \phi(z) d\mu(z),$$

for every continuous function ϕ on K.

Theorem 4.1 *Let (E, F) be a condenser with positive capacity and let f be a holomorphic motion of $E \cup F$ parameterized by a domain $D \subset \mathbb{C}$ containing 0. Let σ_λ be the*

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equilibrium measure of the condenser $(f_\lambda(E), f_\lambda(F))$, $\lambda \in D$. *Then* σ_λ *converges to* σ_{λ_0} *in the weak-star sense, whenever* $\lambda \to \lambda_0$ *in* D.

Proof From Theorem 3.1, it follows that $\operatorname{Cap}(f_\lambda(E), f_\lambda(F)) > 0$; therefore, the condenser $(f_\lambda(E), f_\lambda(F))$ has a unique equilibrium measure $\sigma_\lambda = \sigma_E^\lambda - \sigma_F^\lambda$, for every $\lambda \in D$.

Let $\lambda_n \to \lambda_0$ in D. From the Riesz Representation Theorem and the sequential version of Alaoglu's Theorem (see *e.g.*, [6, pp. 169 and 223]), we obtain that there exist a subsequence $\sigma_E^{\lambda_{n_m}}$ of $\sigma_E^{\lambda_n}$ and a measure ν_E^0 such that $\sigma_E^{\lambda_{n_m}} \overset{w^*}{\to} \nu_E^0$. Applying Alaoglu's Theorem and passing to a subsequence if needed, we can assume that there exists a measure ν_F^0 such that $\sigma_F^{\lambda_{n_m}} \overset{w^*}{\to} \nu_F^0$. Then ν_E^0 and ν_F^0 are Borel probability measures, and from the joint continuity of $f(\lambda, z)$, it follows that they are supported on $f_{\lambda_0}(E)$ and $f_{\lambda_0}(F)$, respectively. So $\nu_0 := \nu_E^0 - \nu_F^0 \in S(f_{\lambda_0}(E), f_{\lambda_0}(F))$ and

$$I(f_{\lambda_0}(E), f_{\lambda_0}(F)) \le I(\nu_0). \tag{4.1}$$

A simple computation (see *e.g.* [3, p. 318]) shows that for every condenser (K, L) and for every $\tau = \tau_K - \tau_L \in S(K, L)$,

$$I(\tau) = \iiiint \log\left(\frac{|z-v||w-u|}{|z-w||u-v|}\right) d\tau_K(z)\, d\tau_K(w)\, d\tau_L(u)\, d\tau_L(v).$$

The function

$$(z, w, u, v) \longmapsto \log\left(\frac{|z-v||w-u|}{|z-w||u-v|}\right)$$

is lower-semicontinuous and bounded below on $K \times K \times L \times L$. Since $\sigma_E^{\lambda_{n_m}} \overset{w^*}{\to} \nu_E^0$ and $\sigma_F^{\lambda_{n_m}} \overset{w^*}{\to} \nu_F^0$, it follows that (see *e.g.*, [8, p. 224])

$$\sigma_E^{\lambda_{n_m}} \times \sigma_E^{\lambda_{n_m}} \times \sigma_F^{\lambda_{n_m}} \times \sigma_F^{\lambda_{n_m}} \overset{w^*}{\longrightarrow} \nu_E^0 \times \nu_E^0 \times \nu_F^0 \times \nu_F^0.$$

Therefore (see *e.g.* [11, pp. 78–79]), from the lower-semicontinuity of energy integrals with respect to weak-star convergence, we get that

$$I(\nu_0) \le \liminf_{m\to\infty} I(\sigma_E^{\lambda_{n_m}} - \sigma_F^{\lambda_{n_m}}). \tag{4.2}$$

Also, from Theorem 3.1 it follows that

$$\lim_{m\to\infty} I(\sigma_E^{\lambda_{n_m}} - \sigma_F^{\lambda_{n_m}}) = \lim_{m\to\infty} I\big(f_{\lambda_{n_m}}(E), f_{\lambda_{n_m}}(F)\big) = I\big(f_{\lambda_0}(E), f_{\lambda_0}(F)\big). \tag{4.3}$$

From equations (4.1), (4.2), and (4.3), we conclude that $I(f_{\lambda_0}(E), f_{\lambda_0}(F)) = I(\nu_0)$ and ν_0 is an equilibrium measure of $(f_{\lambda_0}(E), f_{\lambda_0}(F))$. From the uniqueness of equilibrium measure, we get that $\nu_0 = \sigma_{\lambda_0}$. Since the w^*-convergent subsequences $\sigma_E^{\lambda_{n_m}}$ and $\sigma_F^{\lambda_{n_m}}$ considered above were arbitrary, we conclude that the sequence σ_{λ_n} has a unique w^*-accumulation point, the measure σ_{λ_0}. Since the space of Borel probability measures on a compact set equipped with the weak-star convergence is metrizable and by Alaoglu's Theorem is w^*-compact, $\sigma_{\lambda_n} \overset{w^*}{\to} \sigma_{\lambda_0}$. Given that the sequence $\lambda_n \to \lambda_0$ was arbitrary, the conclusion follows. ∎

Remark 4.2 Similarly, it can be shown that the equilibrium measure (for the logarithmic capacity) of a compact plane set is w^*-continuous under a holomorphic motion of the compact set.

5 Uniformly Perfect Sets

In this section, we will study the change of the quantity $P(\cdot)$, measuring the thickness of a uniformly perfect compact set, under holomorphic motions. Our estimates use the Harnack distance, which is defined as follows.

Let $D \subset \mathbb{C}$ be a domain and let $z, w \in D$. The Harnack distance on D between z and w is the smallest number $\tau_D(z,w)$ such that

$$\frac{h(w)}{\tau_D(z,w)} \leq h(z) \leq \tau_D(z,w) h(w),$$

for every positive harmonic function h on D. Moreover, if D is a bounded domain, then $\log \tau_D(z,w)$ is a complete metric on D. See [17, pp. 14–21] for more information about the Harnack distance.

Theorem 5.1 *Let $K \subset \mathbb{C}$ be a uniformly perfect compact set and let f be a holomorphic motion of $\mathbb{C}$ parameterized by a bounded domain $D \subset \mathbb{C}$ containing 0. Then*

$$\frac{P(K)}{\tau_D(\lambda,0)} \leq P(f_\lambda(K)) \leq \tau_D(\lambda,0) P(K), \tag{5.1}$$

for every $\lambda \in D$.

Proof Let $(E,F) \in R(K)$ such that $I(E,F) < +\infty$. Let $\nu = \nu_E - \nu_F$ be any signed measure in $S(E,F)$ having finite energy. We note that, due to the singularity of the logarithmic kernel, we must have $\nu \times \nu((E \cup F)^2 \backslash A(E,F)) = 0$, where

$$A(E,F) := \{(z,w) \in (E \cup F)^2 : z \neq w\}.$$

For every $\lambda \in D$, let $\nu_\lambda := \nu_E \circ f_\lambda^{-1} - \nu_F \circ f_\lambda^{-1}$. Since the injective functions f_λ are continuous and therefore Borel measurable on $E \cup F$ by the λ-lemma, $\nu_\lambda \in S(f_\lambda(E), f_\lambda(F))$, $\lambda \in D$, and

$$I(\nu_\lambda) = \iint_{A(E,F)} \log \frac{1}{|f_\lambda(z) - f_\lambda(w)|} d(\nu \times \nu)(z,w).$$

Since $f_\lambda(z)$ is a holomorphic function of λ and an injective function of z, we obtain that the function $\lambda \mapsto \log[1/(|f_\lambda(z) - f_\lambda(w)|)]$ is harmonic on D, whenever $(z,w) \in A(E,F)$. It follows (see *e.g.*, [8, p. 16]) that $u_\nu(\lambda) = I(\nu_\lambda)$ is a positive harmonic function on D. Therefore,

$$\frac{I(\nu)}{\tau_D(\lambda,0)} \leq I(\nu_\lambda) \leq \tau_D(\lambda,0) I(\nu), \qquad \lambda \in D, \tag{5.2}$$

where $\tau_D(\lambda,0) > 1$, for all $\lambda \in D \backslash \{0\}$, since the domain D is assumed to be bounded. Noting that $S(f_\lambda(E), f_\lambda(F)) = \{\nu_\lambda : \nu \in S(E,F)\}$ and taking the infimum in (5.2) over all $\nu \in S(E,F)$ having finite energy, we obtain that

$$\frac{I(E,F)}{\tau_D(\lambda,0)} \leq I(f_\lambda(E), f_\lambda(F)) \leq \tau_D(\lambda,0) I(E,F), \qquad \lambda \in D. \tag{5.3}$$

Also, $R(f_\lambda(K)) = \{(f_\lambda(E), f_\lambda(F)) : (E,F) \in R(K)\}$. We conclude that (5.1) follows by taking the supremum in (5.3) over all $(E,F) \in R(K)$. ∎

In the case that the parameterizing domain of the holomorphic motion is the unit disc, a precise estimate follows from Theorem 5.1.

Corollary 5.2 *Let $K \subset \mathbb{C}$ be a uniformly perfect compact set and let f be a holomorphic motion of K parameterized by the unit disc $\mathbb{D}$. Then*

$$\frac{1-|\lambda|}{1+|\lambda|} P(K) \leq P(f_\lambda(K)) \leq \frac{1+|\lambda|}{1-|\lambda|} P(K), \tag{5.4}$$

for every $\lambda \in \mathbb{D}$.

Proof From Słodkowski's Theorem [21], f can be extended to a holomorphic motion of the Riemann sphere parameterized by $\mathbb{D}$. The conclusion follows from Theorem 5.1 and the formula (see [17, p. 14])

$$\tau_{\mathbb{D}}(z,0) = \frac{1+|z|}{1-|z|}, \qquad z \in \mathbb{D},$$

of the Harnack distance for $\mathbb{D}$. ∎

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