J. Austral. Math. Soc. 20 (Series A) (1975), 281–289.

GENERALIZED DISCRETE l-GROUPS

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(Received 14 March 1973; revised 2 March 1974)

Communicated by G. E. Wall

1. Introduction

Let G be an abelian lattice ordered group (an l-group). If G is, in fact, totally ordered, we say that G is an 0–group. A subgroup and a sublattice of G is an l-subgroup. A subgroup C of G is called *convex* if $0 \leqq g \leqq c \in C$ and $g \in G$ imply $g \in C$, C is an *l-ideal* if C is a convex l-subgroup of G. If C is an l-ideal of G, then G/C is also an l-group under the canonical ordering inherited from G. If, in fact, G/C is an 0–group, then C is said to be a *prime subgroup* of G.

A prime subgroup C of an l-group G is an l-ideal such that $F = G_+ \backslash C$ is a prime filter; that is, F satisfies:

(1) $x \wedge y \in F$ for all $x, y \in F$,

(2) $x \in F$ if $x \geqq y$ and $y \in F$, and

(3) $x + y \in F$, for $x \geqq 0$ and $y \geqq 0$, implies $x \in F$ or $y \in F$.

If F is an ultrafilter (a maximal prime filter), then C is said to be a minimal prime subgroup of G (Conrad and McAllister (1969; page 148)).

If g is a nonzero element of G, let L_g be an l-ideal of G maximal without g. Such an l-ideal L_g is said to be a *regular* l-ideal and is called a *value* of g. It is well-known that L_g is a prime subgroup of G (Conrad, Harvey and Holland (1963; page 150)). Furthermore, if U_g is the smallest l-ideal of G containing L_g and g, then U_g/L_g is an 0-subgroup of the reals, and we say that U_g *covers* L_g. Such a pair of l-ideals (U_g, L_g) constitutes a *jump*, and the quotient group U_g/L_g is a *component* of G.

A famous theorem of Hahn (1907) states that a (divisible) 0-group G can be embedded in the lexicographic product of its components. Hill and Mott (1973) attempted to obtain a corresponding embedding theorem for the so-called generalized discrete 0–groups, that is, 0–groups all of whose components are order isomorphic to the group Z of integers. Such an embedding is not possible, in general, but, on the contrary, it is possible for countable generalized discrete 0–groups.

Here we take the definition of Hill and Mott (1973) one step further. We say that an l-group G is *generalized* discrete (abbreviated g.d. l-group) if each component of G is lattice isomorphic to Z.

We gain insight into the structure of g.d. l-groups by analyzing several well-known embedding theorems, and among the embedding theorems for l-groups (besides Hahn's theorem (1907)), two, especially, stand out: the embedding theorem of Lorenzen (1939) and the embedding theorem of Conrad, Harvey, and Holland, (1963; page 153).

These two results, together with results from Hill and Mott (1973), establish embedding theorems for g.d. l-groups. In particular, Lorenzen's theorem yields an embedding of a g.d. l-group in the cardinal product of g.d. 0-groups. Moreover, this result, with the assistance of an embedding theorem of Hill and Mott (1973), furnishes an embedding (reminiscent of the Conrad-Harvey-Holland embedding theorem) of countable g.d. l-groups in the lexicographic product $V(Z_\delta,\Lambda)$, where each Z_δ is isomorphic to Z, and where Λ is a root system.

The point of view in Hill and Mott (1973) was that generalized discrete 0-groups offered a natural extension of the order properties of the additive group Z to 0-groups of higher rank. We also investigated the algebraic properties of generalized discrete 0–groups; in particular, we showed that a countable generalized discrete 0–group is algebraically free — that is, free in the category of abelian groups. Here it is shown that the same result holds for countable generalized discrete l-groups. Moreover, subgroups and cardinal sums of g.d. l-groups are again generalized discrete, but the cardinal product of infinitely many g.d. l-groups is not generalized discrete.

One observation at this point, then, is this: the concept of a generalized discrete l-group is related to algebraically freeness — for a g.d. l-group is $\aleph_1$-free, but on the contrary, not to order freeness — for the free abelian l-groups introduced by Weinberg (1963) are not generalized discrete (except in one trivial case). On the other hand, a free abelian l-group is algebraically free, Hill (preprint).

Throughout this paper, let Z denote the additive group of integers under the natural order. Moreover, if G_λ is a partially ordered group for each $\lambda \in \Lambda$, let $\prod^*_{\lambda\in\Lambda} G_\lambda$ and $\Sigma^*_{\lambda\in\Lambda} G_\lambda$ denote the cardinal product and cardinal sum of the G_λ. If Λ is a finite set, we also use the notation, $G_1 \oplus_C G_2 \oplus_C \oplus_C \cdots \oplus_C G_n$, to denote the cardinal product.

2. Equivalent forms of the definition

Let us begin by listing some other forms of the definition of generalized discrete l-group. The proof is straightforward.

PROPOSITION 1. *Let G be an abelian l-group. The following conditions are equivalent*:

(1) *G is generalized discrete.*

(2) *For each prime subgroup H of G, G/H is generalized discrete.*

https://doi.org/10.1017/S1446788700020632 Published online by Cambridge University Press

(3) *For each regular subgroup H of G, G/H is generalized discrete.*

(4) *For each minimal prime subgroup H of G, G/H is generalized discrete.*

Of course, implicit in the above proposition is the following: If C is an l-ideal of a generalized discrete l-group G, then G/C is also a g.d. l-group.

3. Stability properties of generalized discrete l-groups

Suppose that G is a g.d. l-group, and suppose, further, that H is an l-group bearing some (as yet unspecified) connection with G. It is natural to ask: is H also generalized discrete? We have already seen that the answer is affirmative in case there is a lattice epimorphism of G onto H. We show, in this section, that the answer is affirmative in other situations, too. Let us begin with the case where H is a sublattice of G.

PROPOSITION 2. *An l-subgroup of a generalized discrete l-group is generalized discrete.*

PROOF. Let $h \in H$ and K be a value of h in H. By Conrad and McAllister (1969; page 188), there is a value M of h in G such that $M \cap H = K$. This implies that H/K is a 0–subgroup of the generalized discrete 0–group G/M, and, with the aid of Proposition 4.1 of Hill and Mott (1973), that H/K is generalized discrete. Proposition 1 gives the conclusion at once.

If $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is an exact sequence of partially ordered groups, then the sequence is *order exact* if $B_+ \cap \alpha(A) = \alpha(A_+)$ and $\beta(B_+) = C_+$ (here G_+ denotes the positive cone of a partially ordered group G). The sequence is *lexicographically exact* if $B_+ = \{b \in B \mid \beta(b) > 0 \text{ or } b \in \alpha(A_+)\}$, and, in this case, B is lexicographic extension of A by C. It is well-known that a lexicographic extension of an l-group A by an 0–group C is again an l-group. Let us consider lexicographic extensions of generalized discrete l-groups.

PROPOSITION 3. *Suppose that A is an l-group and that C is an 0–group. If A and C are generalized discrete, then any lexicographic extension of A by C is also generalized discrete.*

The proof follows, essentially, from the fact that each l-ideal of B contains, or is contained in, $\alpha(A)$.

Of course, an arbitary extension of a g.d. l-group by a g.d. l-group need not be generalized discrete; in fact, a subdirect sum of two groups will provide an example. But before describing the order properties of a subdirect sum of two l-groups, let us recall a few simple acts about the cardinal product of a collection of l-groups $\{G_\lambda\}_{\lambda \in \Lambda}$. First, $\sum^*_{\lambda \in \Lambda} G_\lambda$ is an l-ideal of $\prod^*_{\lambda \in \Lambda} G_\lambda$, second, l-ideals of $\sum^*_{\lambda \in \Lambda} G_\lambda$ are of the form $C = \sum^*_{\lambda \in \Lambda} C_\lambda$, where each C_λ is an l-ideal of G, and, third, $\sum^*_{\lambda \in \Lambda} G_\lambda/C$ is lattice isomorphic to $\sum^*_{\lambda \in \Lambda}(G_\lambda/C_\lambda)$. Conclude, therefore, that the prime subgroups of $\sum^*_{\lambda \in \Lambda} G_\lambda$ are of the form $C = \sum^*_{\lambda \in \Lambda} C_\lambda$, where $C_\lambda = G_\lambda$ for all $\lambda \neq \lambda_0$, and C_{λ_0} is a prime subgroup of G_{λ_0}.

PROPOSITION 4. *The cardinal sum of generalized discrete l-groups is generalized discrete.*

The proof is easy from Proposition 1 and the above remarks.

To be sure, one could ask: is the cardinal product of g.d. l-groups again generalized discrete? This question has a negative answer; in fact, we show in Proposition 8 that the cardinal product of infinitely many copies of Z is not generalized discrete.

For the moment, however, let us discuss the order properties of a subdirect sum of two l-groups G and H. Suppose that σ and τ are, respectively, lattice epimorphisms of G and H onto the l-group K. A *subdirect sum* of G and H determined by σ and τ is the subgroup S of the cardinal sum of G and H consisting of all elements (g, h) such that $\sigma(g) = \tau(h)$. Clearly, S is a sublattice of $G \oplus_C H$, and the projection maps of $G \oplus_C H$ onto G and H have kernels naturally identified with B and A, the kernels of σ and τ, respectively. Also, $A \oplus_C B$ is an l-ideal of S, and $S/(A \oplus_C B) \simeq K$. Moreover, each prime subgroup of S contains A or B, and any totally ordered epimorphic image of S is, therefore, either an image of G or of H.

The following proposition is immediate.

PROPOSITION 5. *A subdirect sum of l-groups G and H is generalized discrete if and only if G and H are generalized discrete.*

In particular, if B and G are g.d. l-groups, while H is not generalized discrete, then S is an extension of B by G, but S is not generalized discrete.

REMARK. In Mott (to appear), I defined the dimension of an l-group G in the following manner. First, if G is an 0–group, then $\dim G = n$ if G contains exactly n distinct convex subgroups ($\neq G$). Second, if G is an l-group, then $\dim G = \sup\{\dim G/H \mid H \text{ is a prime subgroup of } G\}$. Thus, if S is the subdirect sum of the l-groups G and H, then $\dim S = \max\{\dim G, \dim H\}$.

4. Some embedding theorems

Lorenzen's Embedding Theorem states that an l-group G can be embedded as a sublattice in the cardinal product of 0–groups of the form G/P_λ, where P_λ is a (minimal) prime subgroup of G. A conclusion follows at once:

PROPOSITION 6. *A generalized discrete l-group can be embedded in the cardinal product of generalized discrete 0–groups.*

Recall: an l-group is *algebraically free* if it is free as an unordered abelian group — that is — if it is free in the category of abelian groups. The following result is a direct generalization of Corollary 4.2 in Hill and Mott (1973); it follows immediately from the cited result for 0–groups, Proposition 2, and Specker's Theorem (Fuchs (1960; page 1968)).

PROPOSITION 7. *A countable generalized discrete l-group is algebraically free.*

Now let us answer the question raised after Proposition 4; to wit, is the cardinal product of g.d. l-groups again generalized discrete? The answer to this question will also show that the converse of Proposition 6 is invalid.

PROPOSITION 8. *The cardinal product of infinitely many copies of Z is not generalized discrete.*

PROOF. It suffices to show that the cardinal product P of countably many copies of Z is not generalized discrete. If B is the subgroup of all bounded elements in P, then it is not difficult to argue that P/B is divisible. Alternately, if S is the cardinal sum of countably many copies of Z, then P/S contains a divisible subgroup (Griffith (1970; page 44)). Thus, in either case, P has an order homomorphic image L which contains a divisible subgroup. We conclude: L is not generalized discrete, and neither is P.

In Proposition 6 we proved an embedding theorem for g.d. l-groups, and Lorenzen's embedding provided the essential clue to the proof. After this success, we turn our attention to another well-known embedding theorem.

Conrad, Harvey, and Holland (1963) extend Hahn's embedding theorem to include all l-groups. They define the lexicographic product V of partially ordered groups G_δ, indexed by a partially ordered set Δ.

Now comes the essential point. If the set Δ is a *root system* and if each G_δ is an 0-group, then V is, in fact, an l-group. (A tacit assumption in Hahn's theorem for 0-groups is that Δ is linearly ordered.) The concept of a root system, then, is the major clue to understanding the embedding theorem of Conrad, Harvey, and Holland.

DEFINITION. A partially ordered set Δ is a root system if for each $\delta \in \Delta$, the set $\{\alpha \in \Delta \mid \alpha \geqq \delta\}$ is totally ordered, or equivalently, if noncomparable elements do not have a lower bound.

Let $V = V(G_\delta, \Delta)$ be the set of all those elements in the cartesian product of $\{G_\delta\}_{\delta \in \Delta}$ whose support satisfies the ascending chain condition. Thus, an element $v = (\cdots, v_\delta, \cdots)$ is in V if and only if $\{\delta \in \Delta \mid v_\delta \neq 0\}$ contains no infinite ascending chains. We say that v_δ is a *maximal component* of v if $v_\delta \neq 0$ and $v_\alpha = 0$ for all $\alpha \in \Delta$ such that $\delta < \alpha$. Define $v \in V$ to be positive if each maximal component is positive.

Against this general background, let us analyze one isolated situation that will become the focus of our attention. Two root systems X and Y contained in Δ are *mutually unrelated* if for each $x \in X$ and $y \in Y$, x and y do not compare under the order relation on Δ. Then if Δ is the union of disjoint mutually unrelated root systems Δ_α, $V = V(G_\delta, \Delta)$ is order isomorphic to the cardinal product of $V_\alpha = V(G_\delta, \Delta_\alpha)$. In particular, if Δ is trivially ordered, then $V(G_\delta, \Delta)$ is the cardinal product of $\{G_\delta\}_{\delta\in\Delta}$. On the other hand, if each Δ_α is linearly ordered, then $V_\alpha = V(G_\delta, \Delta_\alpha)$ is an 0–group, and, in this case, V_α is the traditional lexicographic product of $\{G_\delta\}_{\delta\in\Delta_\alpha}$ in Hahn's theorem.

Here, then, we have arrived at the key to the argument of the next proposition. In brief, if the root system Δ is the union of a family $\{\Delta_\alpha\}$ of linearly ordered, mutually unrelated root systems Δ_α, then $V = V(G_\delta, \Delta)$ is l-isomorphic isomorphic to the cardinal product of the 0–groups $V(G_\delta, \Delta_\alpha)$.

PROPOSITION 9. *A countable generalized discrete l-group G is a sublattice of a lexicographic product of copies of Z.*

PROOF. By Proposition 6, G can be embedded as a sublattice in $\prod^*_{\lambda\in\Lambda} G_\lambda$, where Λ is an index set for the minimal prime subgroups M_λ of G, and where $G_\lambda = G/M_\lambda$ is a countable generalized discrete 0–group. By Theorem 5.1 of Hill and Mott (1973) each G_λ can be embedded in the lexicographic product (in fact, sum) of copies of Z over a linearly ordered set Λ_λ. Let Λ be the disjoint union of the Λ_λ and let Λ be partially ordered in such a way that the subsets Λ_λ are mutually unrelated. Then G can be embedded in $V(Z_\delta, \Lambda)$, where each Z_λ is 0–isomorphic to Z.

REMARKS. The countability assumption is necessary in Proposition 9, for in Hill and Mott (1973) an example is given of a generalized discrete 0–group G that cannot be embedded in a cartesian product of copies of Z. Moreover, it is actually proved in Hill and Mott (1973) that a countable generalized discrete 0–group can be embedded in a lexicographic *sum* of copies of Z. That is: each G_λ is embedded in $V_f(Z_\delta, \Delta_\lambda)$, the 0–subgroup of $V(Z_\delta, \Delta_\lambda)$ consisting of all elements with finite support. We now ask: can we prove an analogous result for countable generalized discrete l-groups? Or, in other words: in the embedding of Proposition 9, can each element of G be identified with an element of $V(Z_\delta, \Delta)$ with finite support? In short: can G be embedded in $\Sigma^*_{\lambda\in\Lambda} V_f(Z_\delta, \Delta_\lambda)$? The answer to this question can be found in example I of section 6. This example shows that even if the group G has a basis, there need not exist an embedding into $\Sigma^*_{\lambda\in\Lambda} V_f(Z_\delta, \Delta_\lambda)$.

If a group G is a sublattice of a cardinal product of generalized discrete 0–groups, Proposition 8 shows that G need not be generalized discrete. The group G must, therefore, be analyzed more closely. Proposition 10 will be useful in this respect. First, let us recall the definition of subcardinal product. If G and G_λ are partially ordered groups with positive cone P and P_λ, respectively, then (G, P) is a *subcardinal product* of $\prod^*_\lambda (G_\lambda, P_\lambda)$ if $P = G \cap \prod^*_\lambda P_\lambda$ and if for each projection map Π_λ onto G_λ, $\Pi_\lambda(P) = P_\lambda$.

PROPOSITION 10. *Suppose that H_λ is a convex subgroup of a generalized discrete 0–group G_λ. Moreover, suppose that G is a sublattice and a subcardinal product of the groups G_λ such that $\Sigma^*_\lambda H_\lambda \subseteq G$. If $G/\Sigma^*_\lambda H_\lambda$ is generalized discrete, then so is G.*

https://doi.org/10.1017/S1446788700020632 Published online by Cambridge University Press

PROOF. The proof is essentially the same as that of Proposition 3.1 of Mott (to appear) but we include it here for completeness.

Let K be a prime subgroup of G and let σ be the natural homomorphism onto the 0–group $T = G/K$. If $K \supseteq \sum_{\lambda}^{*} H_{\lambda}$, then T is an l-homomorphic image of the generalized discrete group $G/\sum_{\lambda}^{*} H_{\lambda}$. If $K \not\supseteq \sum_{\lambda}^{*} H_{\lambda}$, then $H_{\lambda_0} \not\subseteq K$ for some λ_0. Let H be the subgroup of $\prod_{\lambda}^{*} G_{\lambda}$ consisting of all elements $\{g_{\lambda}\}$ with $g_{\lambda_0} = 0$. If $h_{\lambda_0} \in H_{\lambda_0} \setminus K$, and if h_{λ_0} is positive, then $h \wedge h_{\lambda_0} = 0$ for each $h \in H \cap G$. Since K is a prime subgroup, $H \cap G \subseteq K$. We conclude, therefore, that T is an l–homomorphic image of $G/G \cap H$. But G a subcardinal product of the groups G_{λ} implies that $G/G \cap H$ is 0–isomorphic to $(\prod_{\lambda}^{*} G_{\lambda})/H$. Or, in other words: $G/G \cap H \simeq G_{\lambda_0}$. Thus, T is an image of G_{λ_0}, and we conclude: T is generalized discrete.

5. Order freeness

Up to this point we have been considering certain similarities between generalized discrete l-groups and algebraically free groups. But few comparisons are entirely exact, and the present one is no exception. For instance, there are two features fundamental to the concept of algebraically free groups, the existence of a basis and the universal mapping property, but even though these features have their counterparts in ordered groups, they are not necessarily satisfied by generalized discrete l-groups.

Let us first discuss the notion of a basis for a partially ordered group. A positive element of a partially ordered group G is *basic* if the set $\{x \in G \mid 0 \leqq x \leqq b)$ is totally ordered. Conrad introduces the concept of basis for an l–group G in Conrad (1961), and then proves that G has a basis if and only if each positive element of G exceeds a basic element.

This question, then, arises: does a generalized discrete l-group necessarily have a basis? Example II gives a negative answer. We turn our attention, therefore, to a class of groups characterized by a universal mapping property — the free abelian l-groups of Weinberg (1963).

If F is an algebraically free group, then a free abelian l-group over F is an l-group L, together with an embedding $\alpha : F \to L$, such that for each l-group G and each homomorphism $\gamma : F \to G$, there is an l-homomorphism $\beta : L \to G$ such that $\beta\alpha = \gamma$. The traditional model for L is obtained by taking all possible total orders T_{λ} on F and letting L be the l-subgroup of the cardinal product of the 0–groups (F, T_{λ}) generated by the diagonal (Conrad and McAllister (1969)) .In this setting, the elements of F are identified with the diagonal elements of $\prod_{\lambda}^{*} (F, T_{\lambda})$. Weinberg (1963) was the first to discuss free abelian l-groups, and he showed in Weinberg (1965) that a free abelian l-group L can be embedded in a cardinal product of copies of Z.

Despite the fact that any free abelian l-group L is algebraically free (Hill (preprint)), L is not, for all practical purposes, generalized discrete.

https://doi.org/10.1017/S1446788700020632 Published online by Cambridge University Press

PROPOSITION 11. *A free abelian l-group L over a free abelian group F is generalized discrete if and only if F is isomorphic to Z.*

PROOF. Let F be a free abelian group generated by A. If card $A \geqq \aleph_0$, then Q, the additive group of rational numbers, is a homomorphic image of F. But then the totally ordered group Q is the image of L under an l-map. Thus, L is not generalized discrete.

Thus, card A is finite. If card $A = n > 1$, then F can be embedded in the group R of real numbers. With this ordering, F is clearly not generalized discrete, and since the identity map on F can be lifted to an l-map of L onto F we see that L is not generalized discrete in this case either.

Therefore, card $A = 1$.

6. Some examples

We conclude with some examples. Examples I and II are significant because they answer questions raised in the paper. Example III is included as an application of Proposition 10.

I. Bounded integer-valued sequences. Let G be the sublattice of all bounded sequences in $\prod^*_{i\in\omega} Z_i$, the cardinal product of countably many copies of Z. Then G is a generalized discrete group of dimension one. The fact that G is one dimensional (epi-archimidean in Conrad's terminology (preprint)) follows since if $f > 0$ and $g > 0$, there is an integer n such that $nf(i) > g(i)$, for all i such that $f(i) \neq 0$. Moreover, for each proper prime subgroup of C of G, G/C is isomorphic to Z and is generated by the constant function h, where $h(i) = 1$ for each i (Conrad (preprint)). Each positive element $g \in G$ exceeds a basic element — namely, for some integer i, g exceeds the characteristic function of the set $\{i\}$. Therefore, G has a basis, but G cannot be embedded in a cardinal sum of copies of Z since G does not satisfy the descending chain condition on positive elements.

II. Periodic integer valued functions. The group G above contains the sublattice H of all periodic integer-valued sequences. Thus, H is also generalized discrete. But H contains no basic elements (Conrad, Harvey and Holland (1963; page 165)), thus, this example answers the question raised in Section 5.

III. Let $Z \oplus_L Z$ denote the lexicographic product over the linearly ordered set $\Delta = \{1,2\}$. Sheldon (to appear) considers the group S of all functions f from the positive integers into $Z \oplus_L Z$ for which there exists integers a and b such that $f(n) = (0, an + b)$ for all positive integers n outside a finite set. Observe that S is a subcardinal product of $(Z \oplus_L Z)_i$ for $i \in \omega$, and that $S \supseteq \sum^*_{i\in\omega}(Z \oplus_L Z)_i$. Moreover, the map $\sigma : S \to Z \oplus_L Z$ defined by $\sigma(f) = (a, b)$ is an l-map with kernel $\sum^*_{i\in\omega}(Z \oplus_L Z)_i$. We conclude: S is generalized discrete.

IV. An example of Sheldon. For each $b \in [0, 1]$, let $h_b(x) = [1/(x-b)^2]$, where [ ] denotes the greatest integer function. Define $h_b(b) = 0$. Consider the group generated by all finite step functions and all functions h_b. Sheldon (to appear) observes that G is a sublattice of $\prod^*_{\alpha \in [0,1]} Z_\alpha$.

If $f = g + \sum c_b h_b$, where g is a step function and $c_b \in Z$, let $c_b(f) = c_b$. Then the only prime subgroups of $G' = G/\sum^*_\alpha Z_\alpha$ are of the form

$$K(b) = \{f \in G' \mid c_b(f) = 0\},$$

$$K_r(b) = \{f \in G' \mid f = 0 \text{ on } (b, c) \text{ for some } b < c\},$$

$$K_l(b) = \{f \in G' \mid f = 0 \text{ on } (c, b) \text{ for some } c < b\}.$$

Clearly, the group G_s of step functions is contained in $K(b)$ for each $b \in [0,1]$, and $G'/G_s \simeq \sum^*_{\alpha \in [0,1]} Z_\alpha$ under the map that takes $f = g + \sum c_b h_b$ to the function $s \in \sum^*_{\alpha \in [0,1]} Z_\alpha$, where $s(b) = c_b$. Thus, $G'/K(b) \simeq Z$. Moreover, $G'/K_r(b') \simeq Z \oplus_L Z$ under the map that takes $f = g + \sum c_b h_b$ to $(v(g), c_{b'})$, where $v(g)$ is the constant value of g on an interval (b, c).

Examples I and II are one-dimensional, whereas examples III and IV are two-dimensional.

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