

Predicate transformers for extended probability and non-determinism

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Received 20 February 2007; revised 28 November 2008

We investigate laws for predicate transformers for the combination of non-deterministic choice and (extended) probabilistic choice, where predicates are taken to be functions to the extended non-negative reals, or to closed intervals of such reals. These predicate transformers correspond to state transformers, which are functions to conical powerdomains, which are the appropriate powerdomains for the combined forms of non-determinism. As with standard powerdomains for non-deterministic choice, these come in three flavours – lower, upper and (order-)convex – so there are also three kinds of predicate transformers. In order to make the connection, the powerdomains are first characterised in terms of relevant classes of functionals.

Much of the development is carried out at an abstract level, a kind of domain-theoretic functional analysis: one considers d-cones, which are deposes equipped with a module structure over the non-negative extended reals, in place of topological vector spaces. Such a development still needs to be carried out for probabilistic choice *per se*; it would presumably be necessary to work with a notion of convex space rather than a cone.

1. Introduction

In this paper we characterise predicate transformers combining non-determinism and (general) valuations, as a contribution to the programme of giving a domain-theoretic account of the combination of ordinary and probabilistic non-determinism. The problem of finding such a characterisation was raised, but left open, in Tix *et al.* (2008). The problem is, in fact, threefold, as there are three such natural combinations, corresponding to the three classical domain-theoretic powerdomains: lower, upper and (order-)convex.

It would be more natural, from the point of view of computer science applications, to restrict to subprobability valuations, rather than allowing all of them. There has already been work done along these lines for discrete domains (McIver and Morgan 2001a; McIver and Morgan 2001b; McIver *et al.* 1996; Ying Minsheng 2003), and there has been interest in statistics in using spaces of sets of probability measures in the area of ‘imprecise probabilities’: see Huber (1981) for early work and Walley (1991) for later developments.

However, the mathematics seems to be more natural if we take all the valuations, since one can then work with notions of linearity rather than convexity. Indeed, in Tix

[‡] This work was done with the support of EPSRC grant GR/S86372/01, a Royal Society–Wolfson Research Merit Award and APPSEM II.

et al. (2008) it was possible to work in a rather abstract way by considering d-cones and lower, upper and convex powercone constructions. The unrestricted valuations on a domain form the free d-cone over it, and the required, combined, *conical powerdomains* of a domain can be found by taking the powercones of the d-cone of its valuations, restricting to coherent domains in the convex case.

We therefore first consider predicate transformers for powercones and then specialise to the powerdomains. We would certainly also like to have corresponding results for the probabilistic case, and we hope that the present work, together with that of Tix *et al.* (2008) will prove helpful to that end.

There is an illuminating relationship between predicate transformers and functional representations of monads. Dijkstra’s classical ‘healthy’ predicate transformers (Dijkstra 1976) on a given set of states S are strict, continuous, binary meet-preserving maps

$$\mathcal{P}(S) \longrightarrow \mathcal{P}(S).$$

This generalises to strict, continuous, binary meet-preserving maps

$$\mathcal{O}(Q) \longrightarrow \mathcal{O}(P)$$

where, for any dcpo P , $\mathcal{O}(P)$ is the dcpo of open subsets of P , and, provided Q is a domain, such maps are in bijective correspondence with continuous functions

$$P \longrightarrow \mathcal{S}(Q_{\perp})$$

where \mathcal{S} is the upper powerdomain monad, and $(-)\perp$ is the lifting construction. (One can show that for any domain Q , $\mathcal{S}(Q_{\perp})$ is the free lower semilattice over Q with a least element.) The connection between Dijkstra’s predicate transformers and Smyth’s powerdomains of flat dcpos was given in Plotkin (1980); the above generalisation to arbitrary domains was, essentially, first given in Smyth (1983), in the even more general setting of sober spaces. The relationship between suitable notions of predicate transformer for the lower and order-convex powerdomains was considered in Bonsangue (1998).

To see the relationship to a functional representation of the upper powerdomain, note first that, by transposition, one has a bijective correspondence of continuous functions

$$\frac{P \longrightarrow \mathcal{O}^{\mathcal{O}^Q}}{\mathcal{O}(Q) \longrightarrow \mathcal{O}(P)}$$

as $\mathcal{O}(P)$ is isomorphic to the dcpo \mathcal{O}^P of all continuous functions from P to Sierpinski space. This correspondence evidently cuts down to one between predicate transformers, as defined above, and continuous functions to the sub-dcpo of $\mathcal{O}^{\mathcal{O}^Q}$ of those functionals that are strict and preserve binary meets. However, if Q is a domain, $\mathcal{S}(Q_{\perp})$ is isomorphic to the dcpo of these functionals, and this gives us the above general characterisation. This functional characterisation of $\mathcal{S}(Q_{\perp})$ was, essentially, given in Heckmann (1993), and it follows from the Hofmann–Mislove theorem (Gierz *et al.* 2003); the relation between this theorem, functional representations and continuous universal quantifiers was presented in Escardo (2004, Chapter 11).

Let us take another example, which is closer to our present concerns and illustrates the fact that the notion of predicate will, in general, vary. There is a ‘Riesz’ representation

theorem (see Kirch (1993), Tix (1995) and Tix *et al.* (2008, Chapter 2)) for the dcpo of all valuations $\mathcal{V}(P)$ of a dcpo P :

$$\mathcal{V}(P) \cong \mathcal{L}(P)^* .$$

Here $\mathcal{L}(P)$ is the collection of all continuous functions to $\overline{\mathbb{R}}_+$ the dcpo of the non-negative reals extended by a point at infinity, the latter having an evident semiring structure, and, then, $\mathcal{L}(P)^*$ consists of the *linear* functionals in $\mathcal{L}(\mathcal{L}(P))$. (We say that a functional is linear if it preserves the operations of addition and multiplication by a positive real (*scalar* multiplication), with these operations being defined in the natural pointwise way on $\mathcal{L}(P)$.) We therefore have a bijective correspondence between continuous functions

$$P \longrightarrow \mathcal{V}(Q)$$

and predicate transformers if we now take these to be linear continuous functions

$$\mathcal{L}(Q) \longrightarrow \mathcal{L}(P) .$$

In both examples, a functional representation theorem gives rise to a predicate transformer characterisation. Notice that the converse also holds: the characterisation implies the representation (take $P = \mathbb{1}$ in the above). Our strategy is to find the functional representation first, as that seems simpler and more direct than beginning with the predicate transformer characterisation.

The two examples follow a certain pattern, with the ‘object of truthvalues’ $\mathbb{0}$, $\overline{\mathbb{R}}_+$ being, respectively, the free $\mathcal{S}(-)_\perp$ or \mathcal{V} -algebra on $\mathbb{1}$ (the terminal object of the category of dcpos); furthermore, the requirement to be strict and to preserve binary meets or to preserve addition and scalar multiplication is the same as requiring the relevant algebra structure to be preserved. Our case will be similar, but we will not be able to require all the algebra structure to be preserved, the essential obstacle being that the monads we consider are not commutative. However, there are more subtle requirements, such as sublinearity, that do allow functional representation theorems and consequent appropriate notions of healthy predicate transformers.

All our functional representation theorems deal with the representation of certain convex sets by functionals with characteristic properties. Functional representations have a long history in convex analysis, going back to the seminal paper Minkowski (1903), where an order-preserving bijection between compact convex subsets of \mathbb{R}^3 and their *support functionals* is established; the latter are characterised as sublinear functionals on \mathbb{R}^3 . The book Bonnesen and Fenchel (1934) contains an extension of these results to \mathbb{R}^n . There the bijection is further shown to be an isomorphism of topological cones, where the compact convex subsets are endowed with the Hausdorff metric and the Minkowski sum, and the sublinear functionals with the compact-open topology and pointwise addition. These results were extended in Rockafellar (1972, Chapter 13) to possibly unbounded closed convex sets and sublinear functionals that admit the value $+\infty$.

The literature contains many generalisations to topological vector spaces. In particular, Hörmander (1955) is noteworthy. There, among other results, a bijection was established between the closed convex subsets of a locally convex topological vector space V and those sublinear functionals on its topological dual V^* that are lower semicontinuous

for the weak* topology. The more subtle question of characterising the support functionals of compact convex sets in this generality was treated in Tolstogonov (1976). The survey paper Kutateladze and Rubinov (1972) gives a very complete account of the classical theory. To some extent, our representation theorems follow the classical patterns, but in the quite different domain theoretical setting. We do not know of any classical functional representation results corresponding to those for our convex lenses in Section 6.

Functional representation theorems have also been given by workers in the area of imprecise probabilities. Huber gives a theorem (Huber 1981, Proposition 10.2.1) characterising the functionals generated by non-empty closed convex sets of probability measures over a finite set; Maaß (Maaß 2002) gives a theorem generalising both that theorem and Walley (1991, Theorem 3.6.1), and refers to Bonsall (1954) for a yet more general functional analytic theorem.

In Section 2, we present the required technical background for our results and introduce a useful general notion, that of a d-cone semilattice. This is followed, in Section 3, by some further development of the powercones introduced in Tix *et al.* (2008), including some abstract discussion of powercones at the level of d-cone semilattices. In Section 4, we consider generalities on functional representations of powercones and powerdomains. This enables the efficient presentation of the form and elementary properties of these representations in the cases of the lower and upper powercones. However, deriving the corresponding information for the convex powercone is a rather complex affair, involving, among other things, the crucial condition (*) introduced below to define the so-called canonically \subseteq -sublinear maps. An analogous condition arises in the case of order-convex powerdomains in the treatment of predicate transformers in Bonsangue (1998) and, less directly, in the definition of the basis of the Vietoris locale (Johnstone 1985).

Section 5 gives theorems on sublinear and superlinear functions as sups or infs of linear ones, concluding with Theorem 5.9 characterising canonically \subseteq -sublinear maps as unions of linear ones, thereby casting some light on property (*). These results enable us to prove our functional representation theorems in Section 6: at the level of d-cones, they are given by Theorems 6.2, 6.5 and 6.8; at the level of conical powerdomains, they are given by Corollaries 6.3, 6.6 and 6.9. It is worth noting that in the upper and order-convex cases, we make use of the domain-theoretic Banach–Alaoglu theorem established in Plotkin (2006); indeed that theorem was proved in order to make such representation theorems possible. Finally, in Section 7, we use our representation theorems to characterise the predicate transformers corresponding to state transformers. We again begin the development at a suitably general level. At the level of d-cones, the predicate transformer characterisations are given by Theorems 7.2, 7.4 and 7.7; at the level of conical powerdomains, they are given by Corollaries 7.3, 7.5 and 7.9.

We remark, finally, that a small imperative language was given in Tix *et al.* (2008) that had both ordinary and probabilistic non-determinism, together with three semantics, using the three conical powerdomains. It is straightforward using our results to give this language three further corresponding predicate transformer semantics, and to show that each pair of semantics is isomorphic, with the isomorphism being given by the appropriate functor W of Section 7.

2. Technical preliminaries

2.1. Dcpo and domains

See Gierz *et al.* (2003) for a detailed discussion of *dcpo* (directed complete partially ordered sets) and *domains* (continuous dcpo) – we will just recall some notation and definitions here. Let X be a subset of a dcpo. If it is directed, we write $\bigvee^\uparrow X$ for its least upper bound. We write $\uparrow X$ for the set of all elements of the dcpo dominating some element of X , and $\downarrow X$ is defined dually. Upper sets, which are also called *saturated* sets, are characterised by the property that $\uparrow X = X$. We say that X is *order-convex* if and only if $X = \downarrow X \cap \uparrow X$, and write $\text{conv}_{\leq}(X)$ for the least order-convex set containing X , *viz.* $\downarrow X \cap \uparrow X$. The way-below relation is written \ll , and $\uparrow X$ is the set of all elements of the dcpo way-above some element of X . Topological notions on dcpo like continuity, open, closed, compact, and so on, always refer to the Scott topology, unless indicated otherwise; we write \overline{X} for the closure of a subset of a topological space. A domain is *coherent* if the intersection of any two compact saturated subsets is also compact. The product of two coherent domains is again coherent: this can be shown using, for example, Jung and Tix (1998, Lemma 18). A continuous map $f : P \rightarrow Q$ is an *order-embedding* if $fx \leq fy$ implies $x \leq y$ for all x, y in P . Finally, we write Dom for the category of domains and continuous maps, and Dom^c for the full subcategory of the coherent domains.

2.2. d-Cones

The central concept in this paper is that of a *d-cone*. This concept has been introduced by Kirch and by Tix (Kirch 1993; Tix 1995) as a slight modification of the *abstract probabilistic domains* due to S. Graham and Claire Jones (Graham 1988; Jones 1990; Jones and Plotkin 1989; Heckmann 1994). See Tix *et al.* (2008) for information about d-cones, but we will give all the required definitions here.

A d-cone C has an order structure and an algebraic structure. The order structure is that of a dcpo. The algebraic structure is that of a *cone*, that is, there is an addition $(x, y) \mapsto x + y : C \times C \rightarrow C$, which is required to be associative and commutative and to have a neutral element 0, and a scalar multiplication $(r, x) \mapsto rx : \mathbb{R}_+ \times C \rightarrow C$, which satisfies the same equational laws as in vector spaces except that the scalars are restricted to the set \mathbb{R}_+ of non-negative reals.

The order and the algebraic structure are linked by the requirement that addition and scalar multiplication are continuous in both variables. The notion of continuity employed here is that of Scott continuity in *bounded* directed complete partially ordered sets (bdcpo for short), which are defined to be those partial orders with lubs of bounded directed sets. A function between bdcpo is Scott continuous if it is monotonic and preserves suprema of bounded directed sets; this reduces to the usual notion of Scott continuity in the case of dcpo. Note that the non-negative reals \mathbb{R}_+ endowed with the usual order form a bdcpo rather than a dcpo; adding an element $+\infty$ to \mathbb{R}_+ , we obtain the extended non-negative reals $\overline{\mathbb{R}}_+$, which form a dcpo, and even a d-cone. Scalar multiplication extends uniquely to a continuous function on $\overline{\mathbb{R}}_+ \times C$.

Let X be a subset of a d-cone. It is *convex* if $rx + (1 - r)y \in X$ whenever $x, y \in X$ and $r \in [0, 1]$. We write $\text{conv}(X)$ for the least convex set containing X . A d-cone is said to be *locally convex*, if every point has a neighbourhood basis of Scott-open convex sets. Continuous d-cones are always locally convex. Consider a function $f: C \rightarrow D$ between d-cones. If it is always true that

$$f(rx) = rf(x), f(x + y) \leq f(x) + f(y), f(x + y) \geq f(x) + f(y),$$

then f is said to be *homogeneous*, *subadditive* and *superadditive*, respectively. We say that f is *sublinear* (*superlinear*) if it is homogeneous and subadditive (superadditive). A *linear function* is one that is both sublinear and superlinear.

We will work in the category **Cone** of d-cones and linear continuous maps. We will use two full subcategories **CCone** and **CCone^c**, the objects of which are the *continuous* d-cones and the *coherent* continuous d-cones, respectively. The way-below relation on a continuous d-cone is *additive* if whenever $a \ll b$ and $a' \ll b'$, we have $a + a' \ll b + b'$.

Given dcpos P and Q , we write Q^P for the dcpo of all continuous maps from P to Q . If D is a d-cone, then D^P is also a d-cone when it is endowed with the pointwise operations. A special case was mentioned in the introduction: $\mathcal{L}(P) =_{\text{def}} \overline{\mathbb{R}}_+^P$ denotes the d-cone of all continuous functionals $f: P \rightarrow \overline{\mathbb{R}}_+$ (functions with range $\overline{\mathbb{R}}_+$ are often termed ‘functionals’); $\mathcal{L}(P)$ is a domain if P is also, and then its way-below relation is additive if and only if P is coherent (Tix *et al.* 2008, Proposition 2.28). Recall here that we are using the Scott topology on $\overline{\mathbb{R}}_+$, the only open sets of which are the intervals $]r, +\infty]$, and not the usual Hausdorff topology.

Given d-cones C and D , we write $[C, D]$ for the sub-d-cone of D^C of linear continuous maps from C to D . It can be shown that **Cone** is a symmetric monoidal closed category with unit $\overline{\mathbb{R}}_+$ and exponential $[-, -]$ (the tensor is harder to describe). A special case of this function space was mentioned in the introduction: $C^* =_{\text{def}} [C, \overline{\mathbb{R}}_+]$ denotes the d-cone of all linear continuous functionals on C ; it is called the *dual* d-cone of C . Every element a of a d-cone C defines a linear continuous functional $a^{**} = (f \mapsto f(a))$ on C^* , yielding a natural linear continuous map $a \mapsto a^{**}: C \rightarrow C^{**}$. If C is a continuous d-cone, this map is an order-embedding, see Tix *et al.* (2008, Corollary 3.5). When it is also surjective, and so an isomorphism of d-cones, we say that C is *reflexive*.

The evaluation functional $\text{ev}: C^* \times C \rightarrow \overline{\mathbb{R}}_+$ gives rise to two topologies of interest. The *weak* Scott topology* on C^* has all sets of the form $W_{x,r} =_{\text{def}} \{f \in C^* \mid f(x) > r\}$ as a subbasis, where $x \in C$ and $r \in \mathbb{R}_+$, and the *weak Scott topology* on C has all sets of the form $W_{f,r} =_{\text{def}} \{x \in C \mid f(x) > r\}$ as a subbasis, where $f \in C^*$ and $r \in \mathbb{R}_+$.

For convenience, we will quote Theorems 3.2, 3.4, 3.8 in Tix *et al.* (2008) and Corollary 2 of the Banach–Alaoglu Theorem in Plotkin (2006) – they are of a functional analytic flavour and are used several times in this paper. In each of the theorems we suppose that C is a continuous d-cone.

Theorem 2.1 (Sandwich Theorem). Let $p: C \rightarrow \overline{\mathbb{R}}_+$ be sublinear and $q: C \rightarrow \mathbb{R}_+$ be superlinear and Scott-continuous with $q \leq p$. Then there is a Scott-continuous linear functional $f: C \rightarrow \overline{\mathbb{R}}_+$ such that $q \leq f \leq p$.

Theorem 2.2 (Separation Theorem). Let A and B be two disjoint non-empty convex subsets of C . If, in addition, B is Scott-open, there exists a Scott-continuous linear functional $f : C \rightarrow \overline{\mathbb{R}}_+$ such that $f(a) \leq 1 < f(b)$ for all $a \in A$ and all $b \in B$.

Theorem 2.3 (Strict Separation Theorem). Let K be a Scott-compact convex set and A be a non-empty Scott-closed convex set disjoint from B . Then there is a Scott-continuous linear functional f and an r in \mathbb{R}_+ such that $f(x) > r > 1 \geq f(y)$ for all x in K and all y in A .

The following is immediate from Corollary 2 of the Banach–Alaoglu Theorem of Plotkin (2006).

Theorem 2.4. If the way-below relation is additive and p is a continuous superlinear functional on C , then the set B of all continuous linear functionals f on C with $p \leq f$ is weak* Scott-compact in C^* .

2.3. Extended probabilistic powerdomain

We will be particularly interested in the *extended probabilistic powerdomain*, that is, the d-cone $\mathcal{V}(P)$ of all continuous valuations of a dcpo P mentioned in the introduction. A *valuation* on P is a strict monotonic modular function $\mu : \mathcal{O}(P) \rightarrow \overline{\mathbb{R}}_+$, with modularity meaning that for all open subsets U, V of P ,

$$\mu(U) + \mu(V) = \mu(U \cup V) + \mu(U \cap V).$$

The ordering of $\mathcal{V}(P)$ is defined pointwise, as are its addition and scalar multiplication. See Tix *et al.* (2008) for details of this construction and its properties – we will just recall the main points here. There is a bilinear continuous integration functional $\int : \mathcal{L}(P) \times \mathcal{V}(P) \rightarrow \overline{\mathbb{R}}_+$, which yields the Riesz-type isomorphism $\mathcal{V}(P) \cong \mathcal{L}(P)^*$ mentioned in the introduction, sending μ to $f \mapsto \int f d\mu$. The inverse of this isomorphism sends φ to $U \mapsto \varphi(\chi_U)$ where χ_U is the characteristic function of U .

When P is a domain, $\mathcal{V}(P)$ is also, and it is the free d-cone over P (Kirch 1993; Gierz *et al.* 2003). The unit map $\eta : P \rightarrow \mathcal{V}(P)$ is given by $\eta(x) = U \mapsto \chi_U(x)$. The extension of a continuous map $P \rightarrow \overline{\mathbb{R}}_+$ to a linear continuous map $\mathcal{V}(P) \rightarrow \overline{\mathbb{R}}_+$ is given by integration and yields an isomorphism of d-cones $\mathcal{L}(P) \cong \mathcal{V}(P)^*$.

Putting all this together, we get that $\mathcal{L}(P)$ and $\mathcal{V}(P)$ are both reflexive continuous d-cones for any domain P . Finally, let us remark that, again in the case where P is a domain, the weak Scott topology on $\mathcal{V}(P)$ coincides with its Scott topology (Kirch 1993; Tix 1995) and if P is also coherent, then $\mathcal{V}(P)$ is too (Tix *et al.* 2008, 2.10). Thus, for every coherent domain P , the extended probabilistic powerdomain $\mathcal{V}(P)$ is a *convenient d-cone* in the following sense: it is continuous and reflexive; its weak Scott topology coincides with its Scott topology; and, furthermore, its dual C^* is continuous and has an additive way-below relation. This rather strong notion is useful for the formulation of our results in Section 6.

2.4. d-cone semilattices

Powercones have as extra structure, a continuous semilattice operation \cup , that is, an associative, commutative and idempotent binary operation satisfying properties, which

we will now briefly consider abstractly. A *d-cone semilattice* is a d-cone together with a continuous semilattice ‘union’ operation \cup over which addition and scalar multiplication distribute, the latter meaning that the equations $x + (y \cup z) = (x + y) \cup (y + z)$ and $r(x \cup y) = rx \cup ry$ both hold. We write ConeSL for the category of all d-cone semilattices and \cup -preserving linear continuous functions.

The partial order associated with the semilattice operation is written as \subseteq , where $x \subseteq y$ holds if and only if $x \cup y = y$. It is closed under directed sups, scalar multiplication and addition. In all the powercones it turns out that \subseteq is the ordinary subset relation, but \cup is not the ordinary union operation, but rather ordinary union followed by the application of a suitable closure operation. The cone \mathbb{R}_+ can be viewed as a d-cone semilattice in precisely two ways: either as a d-cone join-semilattice, meaning that $\cup = \vee$; or as a d-cone meet-semilattice, meaning that $\cup = \wedge$. However, for a general cone, \cup need not be the join or the meet with respect to the d-cone ordering \leq .

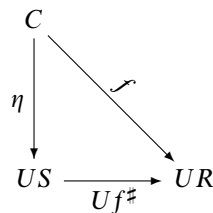
If S is a d-cone semilattice, then so is S^C when it is equipped with the pointwise union. It is important to note that this is not true of $[C, S]$ as the pointwise union of two additive functions need not be additive. For example, taking $S = \mathbb{R}_+$ and $C = \mathcal{L}(P)$, we have $C^* \cong \mathcal{V}(P)$, but the latter does not need to have either binary sups or binary meets.

However, we can at least define a pointwise partial order \subseteq on $[C, S]$, and that is closed under all the d-cone operations. If C is also a d-cone semilattice, then the \subseteq -monotonic functions in $[C, S]$ (those preserving \subseteq) form a sub-d-cone, as is straightforwardly verified using the closure properties of \subseteq on S . The pointwise union $h = f \cup g$ of two maps $f, g \in [C, S]$ is \subseteq -sublinear, that is, it is homogeneous and \subseteq -additive, where the latter means that $f(x + y) \subseteq f(x) + f(y)$ for all $x, y \in C$.

3. Powercone and conical powerdomain constructions

Tix *et al.* (2008, Chapter 4) presents three convex powercone and corresponding conical powerdomain constructions. We begin by developing some of their common properties in an abstract setting. We suppose we have full subcategories \mathbf{K} and \mathbf{L} of the categories Cone of d-cones and ConeSL of d-cone semilattices, respectively, and we write $U : \mathbf{L} \rightarrow \text{Cone}$ for the evident forgetful functor. We assume that \mathbf{L} is closed under binary products and sub-d-cone semilattices.

For a d-cone C , we say that a linear continuous map $\eta : C \rightarrow US$, with S in \mathbf{L} , is *universal* if for any other such map $f : C \rightarrow UR$ there is a unique \cup -preserving linear continuous map $f^\# : S \rightarrow R$ such that the following diagram commutes



In other words, S is the free \mathbf{L} -d-cone semilattice over C , with unit map η .

We now suppose that for any C in \mathbf{K} there is a free L-d-cone semilattice FC in \mathbf{L} with UFC also in \mathbf{K} and with universal map $\eta_C : C \rightarrow UFC$. This allows us to define a monad T on \mathbf{K} , setting $TC =_{\text{def}} UFC$ and $Tf =_{\text{def}} (\eta_D \circ f)^\#$, for any linear continuous $f : C \rightarrow D$, and with unit η and multiplication $\mu_C =_{\text{def}} (\text{id}_{TC})^\#$.

We now consider the properties of extension $f \mapsto f^\#$ considered as a function from $[C, S]$ to $[TC, S]$ for a given choice of C in \mathbf{K} and S in \mathbf{L} .

Proposition 3.1. For objects C in \mathbf{K} and S in \mathbf{L} , the extension $f \mapsto f^\#$, considered as a map $[C, S] \rightarrow [TC, S]$, is continuous, \sqsubseteq -monotonic and \sqsubseteq -sublinear.

Proof. We begin by proving that it is monotonic. Suppose that $f \leq g$ for f, g in $[C, S]$. The set $\Delta_S = \{(y, z) \in S^2 \mid y \leq z\}$ is a sub-d-cone semilattice of S^2 and, by presupposition, belongs to \mathbf{L} . Since $f \leq g$, we can define a linear continuous map $h : C \rightarrow \Delta_S$ by putting $h(x) = (fx, gx)$. We have $\pi_0 h^\# \eta_C = \pi_0 h = f$, where π_0 is the restriction to Δ_S of the first projection on S . It follows, by universality, that $f^\# = \pi_0 h^\#$ and, similarly, that $g^\# = \pi_1 h^\#$, with π_1 the corresponding restriction of the second projection. But $\pi_0 \leq \pi_1$, so $f^\# \leq g^\#$, as required.

To finish the proof of continuity, let $f_\lambda : C \rightarrow S$ be a directed family. Then we have $(\bigvee^\uparrow f_\lambda^\#) \eta_C = \bigvee^\uparrow f_\lambda^\# \eta_C = \bigvee^\uparrow f_\lambda$, and thus, by universality, $(\bigvee^\uparrow f_\lambda)^\# = \bigvee^\uparrow f_\lambda^\#$. The proof of homogeneity is similar to the last part of the proof of continuity. The proof of \sqsubseteq -monotonicity is just like that of monotonicity, but now we need the fact that $\{(y, z) \in S^2 \mid y \leq z\}$ is a sub-d-cone semilattice of S^2 . Finally, for the proof of \sqsubseteq -subadditivity, one shows that $\{(u, v, w) \in S^3 \mid u \leq v + w\}$ is a sub-d-cone semilattice of S^3 , and then to show that $(f + g)^\# \sqsubseteq f^\# + g^\#$ one takes $h(x) = (fx + gx, fx, gx)$. \square

Note that it follows from this proposition that each action $T : [C, D] \rightarrow [TC, TD]$ is continuous and homogeneous.

We now recall the three powercone and powerdomain constructions, but note a slight change of terminology with respect to Tix *et al.* (2008): in the case of cones, we will drop the word *convex*, and will say *lower powercone* instead of *lower convex powercone*, and so on. Next, as seen above, we use the word ‘conical’ in the case of powerdomains, speaking of the *lower or upper conical powerdomains* or the *(order)-convex conical powerdomain* to distinguish these powerdomains from the standard powerdomains for (non-probabilistic) non-determinism; when it is clear from the context which is meant, we may simply speak of powerdomains rather than conical powerdomains. We also present some additional material giving explicit formulas for extensions, particularly Kleisli extensions, which are extensions of maps with codomain of the form TD , and also for monad multiplications.

3.1. The lower powercone and lower conical powerdomain

Let C be a d-cone. Then its *lower powercone* \mathcal{HC} is formed from the set of all its non-empty closed convex subsets. The lower powercone is ordered by inclusion, with directed sups being given by the closure of the union; addition and scalar multiplication are defined by $X +_H Y =_{\text{def}} \overline{X + Y}$ (the closure of $X + Y$) and $r \cdot_H X =_{\text{def}} rX$. If C is continuous, so

is $\mathcal{H}C$. This defines the object part of a functor on \mathbf{Cone} that cuts down to a functor on \mathbf{CCone} ; its action on morphisms is given by $\mathcal{H}(f)(X) =_{\text{def}} \overline{f(X)}$.

We also have that $\mathcal{H}C$ is a join-semilattice, with $X \vee Y =_{\text{def}} \overline{\text{conv}(X \cup Y)}$, and, indeed, it is characterised by a universal property (Tix *et al.* 2008, Theorem 4.10): through the map $\eta_C : C \rightarrow \mathcal{H}C$, where $\eta_C(c) = \downarrow\{c\}$, it is the free d-cone join-semilattice over C . More precisely, we have the following proposition.

Proposition 3.2. For every continuous linear map f from a d-cone C to a d-cone join-semilattice S there is a unique join-preserving continuous linear map $f^\# : \mathcal{H}C \rightarrow S$ such that $f^\# \circ \eta_C = f$. The extension $f^\#$ is defined by

$$f^\#(X) = \bigvee_{x \in X} f(x).$$

The above framework and Proposition 3.1 therefore apply, taking \mathbf{K} to be either \mathbf{Cone} or \mathbf{CCone} and \mathbf{L} to be the full subcategory of \mathbf{ConeSL} of all d-cone join-semilattices. We will need some additional information on the Kleisli extension and the monad multiplication for \mathcal{S} . For this, we prove the following lemma.

Lemma 3.3. If \mathcal{X} is a closed subset of $\mathcal{H}C$, then $A =_{\text{def}} \bigcup_{X \in \mathcal{X}} X$ is a closed subset of C .

Proof. Let $y \leq x \in A$. There is an $X \in \mathcal{X}$ such that $x \in X$. As X is closed, we have $y \in X$, too, and thus $y \in A$.

Let (x_i) be directed in A . First note that $\downarrow x_i \in \mathcal{X}$, as x_i is contained in some member X_i of \mathcal{X} , and thus $\downarrow x_i \subseteq X_i$, which implies $\downarrow x_i \in \mathcal{X}$ as \mathcal{X} is a lower set. As \mathcal{X} is closed, it follows that $\bigvee^\uparrow x_i \in \bigcup_{\downarrow x_i} \downarrow x_i = \bigvee_{\mathcal{H}C} \downarrow x_i \in \mathcal{X}$. □

Proposition 3.4. Let C and D be d-cones.

- (a) The Kleisli extension $f^\# : \mathcal{H}C \rightarrow \mathcal{H}D$ of a linear continuous map $f : C \rightarrow \mathcal{H}D$ is given by $f^\#(X) = \overline{\bigcup_{x \in X} f(x)}$.
- (b) The monad multiplication is given by $\mu_C(\mathcal{X}) = \bigcup_{X \in \mathcal{X}} X$.

Proof.

- (a) $\bigvee_{x \in X} f(x) = \overline{\bigcup_{x \in X} f(x)}$ by the characterisation of arbitrary sups in $\mathcal{H}D$ given in Tix *et al.* (2008). However, $\bigcup_{x \in X} f(x)$ is convex since if c is a convex combination $ra + (1 - r)b$ of elements a, b , then there are $x, y \in X$ such that $a \in f(x)$ and $b \in f(y)$, and it follows that $c \in f(rx + (1 - r)y)$. We therefore have $f^\#(X) = \overline{\bigcup_{x \in X} f(x)}$, as required.
- (b) Since $\mu_C = (\text{id}_{\mathcal{H}C})^\#$, we have $\mu_C(\mathcal{X}) = \overline{(\bigcup_{X \in \mathcal{X}} X)} = \bigcup_{X \in \mathcal{X}} X$ with the last equality holding because of Lemma 3.3. □

Combining the extended probabilistic powerdomain functor \mathcal{V} and \mathcal{H} , we obtain the lower conical powerdomain $\mathcal{H}\mathcal{V}(P)$ of a dcpo P . This is a domain if P is, and it is then the free d-cone join-semilattice over P .

3.2. The upper powercone and upper conical powerdomain

Let C be a continuous d-cone. (Continuity of the d-cone will be needed for the universal property of this powercone construction.) Then its *upper powercone* $\mathcal{S}C$ is formed from the set of all its non-empty compact saturated convex subsets. The upper powercone is ordered by reverse inclusion, with directed sups being given by intersection; addition and scalar multiplication are defined by $X \overset{+}{\underset{s}{\smile}} Y =_{\text{def}} \uparrow(X + Y)$ and $r \overset{\cdot}{\underset{s}{\smile}} X =_{\text{def}} \uparrow(rX)$. The d-cone $\mathcal{S}C$ is itself continuous, and we have $X \ll Y$ if and only if Y is contained in the interior of X . We now have the object part of a functor on \mathbf{CCone} , the category of continuous d-cones. Its action on morphisms is given by $\mathcal{S}(f)(X) =_{\text{def}} \uparrow f(X)$.

We also have that $\mathcal{S}C$ has continuous binary meets, with $X \wedge Y =_{\text{def}} \uparrow \text{conv}(X \cup Y)$ and, indeed, it is characterised by a universal property: *via* the map $\eta_C : C \rightarrow \mathcal{S}C$, where $\eta_C(c) = \uparrow\{c\}$, the d-cone $\mathcal{S}C$ is the free continuous d-cone meet-semilattice over C . More generally, we have the following proposition.

Proposition 3.5. For every continuous linear map f from a continuous d-cone C to a d-cone meet-semilattice S' that is embeddable in a continuous d-cone meet-semilattice S , there is a unique meet-preserving continuous linear map $f^\# : \mathcal{S}C \rightarrow S'$ such that $f^\# \circ \eta_C = f$. The extension $f^\#$ is given by

$$f^\#(X) = \bigwedge f(X).$$

Proof. We may assume that S' is a sub-d-cone semilattice of the continuous d-cone meet-semilattice S . Given a continuous linear map $f : C \rightarrow S'$, by Tix *et al.* (2008, Theorem 4.4.13), there is a unique meet-preserving continuous linear map $f^\# : \mathcal{S}C \rightarrow S$ such that $f^\# \circ \eta_C = f$ given by $f^\#(X) = \bigwedge f(X)$ for every non-empty compact convex saturated subset X of C . The proposition is proved if we show that $f^\#(X) \in S'$. As S' is supposed to be a sub-d-cone semilattice in S , it suffices to show that $\bigwedge f(X) = \bigvee^\uparrow \{ \inf f(F) \mid F \text{ finite and } X \subseteq \text{int}(\uparrow \text{conv}(F)) \}$. To show this, we choose any $a \ll \bigwedge f(X)$ in S . The set U of all $x \in C$ such that $a \ll f(x)$ is an open neighbourhood of X . By Tix *et al.* (2008, Proposition 3.11), X is the intersection of a filtered family of sets of the form $\uparrow \text{conv}(F)$ such that $X \subseteq \text{int}(\uparrow \text{conv}(F))$. One of these sets, say F_0 , has to be contained in the open neighbourhood U . It then follows that $\inf f(F_0) \geq a$. \square

Hence, both the general framework and Proposition 3.1 apply, taking \mathbf{K} to be \mathbf{CCone} and \mathbf{L} to be the full subcategory in \mathbf{ConeSL} of d-cone meet-semilattices that are embeddable in continuous ones.

We will need an explicit formula for the Kleisli extension and for the multiplication of the monad \mathcal{S} . For this we prove the following lemma.

Lemma 3.6. Let C be a continuous d-cone and \mathcal{X} be a compact convex subset of $\mathcal{S}C$. Then $A =_{\text{def}} \bigcup_{X \in \mathcal{X}} X$ is a compact saturated convex subset of C .

Proof. First, A is convex, the argument being the same as that of Part (a) of Proposition 3.4; furthermore, A is saturated, as all members of \mathcal{X} are saturated. It remains to prove that A is compact. To show this, let U_i be a directed family of open sets covering A . Then, for every $X \in \mathcal{X}$, there is an index i_X such that $X \subseteq U_{i_X}$. By Tix

et al. (2008), U_{i_X} contains some compact convex saturated set Y_X that is a neighbourhood of X (so $Y_X \ll_{\mathcal{S}C} X$). Thus $\uparrow_{\mathcal{S}C} Y_X$ is a neighbourhood of X in $\mathcal{S}C$. As \mathcal{X} is a compact subset of $\mathcal{S}C$, there are finitely many $X_1, \dots, X_n \in \mathcal{X}$ such that $\mathcal{X} \subseteq \uparrow_{\mathcal{S}C} Y_{X_1} \cup \dots \cup \uparrow_{\mathcal{S}C} Y_{X_n}$. Thus, for all $X \in \mathcal{X}$, there is an index j such that $Y_{X_j} \ll_{\mathcal{S}C} X$. We conclude that X is in the interior of Y_{X_j} and, *a fortiori*, $X \subseteq U_{X_j}$. We conclude that $A \subseteq U_{X_1} \cup \dots \cup U_{X_n}$. \square

Proposition 3.7. Let C and D be continuous d-cones.

- (a) The Kleisli extension $f^\# : \mathcal{S}C \rightarrow \mathcal{S}D$ of a continuous linear map $f : C \rightarrow \mathcal{S}D$ is given by $f^\#(X) = \bigwedge_{x \in X} f(x) = \bigcup_{x \in X} f(x)$.
- (b) The monad multiplication is given by $\mu_C(\mathcal{X}) = \bigcup_{X \in \mathcal{X}} X$.

Proof.

- (a) Let $X \in \mathcal{S}C$. We have seen before that $f^\#(X) = \bigwedge_{x \in X} f(x)$. Since X is compact and convex and f is continuous and linear, $\mathcal{X} = \{f(x) \mid x \in X\}$ is compact and convex. It follows from Lemma 3.6 that $\bigcup_{x \in X} f(x)$ is compact and convex. As it is saturated, it is a member of $\mathcal{S}D$. As this d-cone is ordered by reverse inclusion, we must therefore have $\bigwedge_{x \in X} f(x) = \bigcup_{x \in X} f(x)$ and the conclusion follows.
- (b) This part of the proposition follows immediately from the proof of part (a), using the fact that $\mu_C = (\text{id}_{\mathcal{S}C})^\#$. \square

Combining the extended probabilistic powerdomain functor \mathcal{V} and \mathcal{S} , we obtain the *upper conical powerdomain* $\mathcal{S}\mathcal{V}(P)$ of a domain P . This is also a domain, and it is the free continuous d-cone meet-semilattice over P .

3.3. The convex powercone and the order-convex conical powerdomain

Let C be a coherent continuous d-cone. (The additional hypothesis of coherence will again be needed for the universal property of this powercone construction.) Then the *convex powercone* $\mathcal{P}C$ is formed from its *convex lenses*, which, by definition, are those subsets that are non-empty intersections of a closed convex set with a compact saturated convex set. The convex powercone is ordered by the Egli–Milner ordering $X \sqsubseteq_{EM} Y$ if and only if $X \subseteq \downarrow Y$ and $\uparrow X \supseteq Y$, and addition and scalar multiplication are defined by $X \vdash_p Y =_{\text{def}} (X + Y)^\ell$, where $Z^\ell =_{\text{def}} \overline{Z} \cap \uparrow Z$, and $r \cdot X =_{\text{def}} rX$. Note that Z^ℓ is a convex lens when Z is a compact convex set. The d-cone $\mathcal{P}C$ is itself coherent and continuous and this defines the object part of a functor on CCone^c , the category of coherent continuous d-cones. Its action on morphisms is given by $\mathcal{P}(f)(X) =_{\text{def}} f(X)^\ell$.

We also have that $\mathcal{P}C$ has a continuous semilattice operation, given by $X \cup Y =_{\text{def}} (\text{conv}(X \cup Y))^\ell$ and, indeed, is characterised by a universal property: *via* the map $\eta_C : C \rightarrow \mathcal{P}C$, where $\eta_C(c) = \{c\}$, the d-cone $\mathcal{P}C$ is the free coherent continuous d-cone semilattice over C . More generally, we have the following proposition.

Proposition 3.8. For any continuous linear map f from a coherent continuous d-cone C to a d-cone semilattice S' that is embeddable in a coherent continuous d-cone

semilattice S , there is a unique \cup -preserving continuous linear map $f^\# : \mathcal{P}C \rightarrow S'$ such that $f^\# \circ \eta_C = f$.

Proof. We may suppose that S' is a sub-d-cone semilattice of the continuous coherent d-cone semilattice S . We know from Tix *et al.* (2008, Theorem 4.37) that there is a unique \cup -preserving continuous linear map $f^\# : \mathcal{P}C \rightarrow S$ such that $f^\# \circ \eta_C = f$. It remains to show that $f^\#(X) \in S'$ for all $X \in \mathcal{P}C$. This is proved from the fact that $f^\#(X) = \bigvee^\uparrow \{ \bigcup f(F) \mid F \text{ finite and } k(F) \ll X \}$ in a similar way to the proof of Proposition 3.5. (Here $k(F)$ denotes the convex lens generated by the finite set F , that is, $k(F) = \downarrow \text{conv}(F) \cap \uparrow \text{conv}(F)$). □

The general framework and Proposition 3.1 both apply, taking \mathbf{K} to be \mathbf{CCone}^c and \mathbf{L} to be the full subcategory of \mathbf{ConeSL} consisting of all d-cone semilattices embeddable in coherent continuous ones.

In order to find Kleisli extension formulas, we first look at the relationship between the convex powercone and the other two, beginning with the lower one. Some of this material has already appeared in Tix *et al.* (2008) following the statement of Theorem 4.24, in particular, in Lemmas 4.25 and 4.26.

By the universal property of \mathcal{P} , for every coherent continuous d-cone C , there is a unique \cup -preserving, linear, continuous map $\downarrow_C : \mathcal{P}C \rightarrow \mathcal{H}C$ extending the unit $\eta_C : C \rightarrow \mathcal{H}C$, and one then has that \downarrow_- is a map of monads; one can show that $\downarrow_C(X) = \downarrow X$.

Lemma 3.9. Let C, D be coherent continuous d-cones and $f : C \rightarrow \mathcal{P}D$ be a linear continuous map. Then the following diagram commutes

$$\begin{array}{ccc}
 \mathcal{P}C & \xrightarrow{f^\#} & \mathcal{P}D \\
 \downarrow \downarrow_C & & \downarrow \downarrow_D \\
 \mathcal{H}C & \xrightarrow{(\downarrow_D \circ f)^\#} & \mathcal{H}D
 \end{array}$$

Proof. First show that $\downarrow_D f^\#$ and $(\downarrow_D f)^\# \downarrow_C$ are both \cup -preserving linear continuous maps extending $\downarrow_D f$ along the unit, then apply the universal property of $\mathcal{P}C$. □

There is a \cup -preserving linear continuous map $l_C : \mathcal{H}C \rightarrow \mathcal{P}C$ in the other direction, where $l_C(X) = X$ (but l_- is not a natural transformation, let alone a map of monads); the proof that this map is monotonic relies on the fact that every d-cone has a least element. Note that l_C is right-inverse to \downarrow_C , and also that $\text{id}_{\mathcal{P}C} \geq l_C \downarrow_C \geq \text{id}_{\mathcal{P}C}$.

Turning now to the relationship with the upper powercone, by the universal property of \mathcal{P} , for every coherent continuous d-cone C , there is a unique \cup -preserving linear continuous map $\uparrow_C : \mathcal{P}C \rightarrow \mathcal{S}C$ extending the unit $\eta_C : C \rightarrow \mathcal{S}C$ (we then have that \uparrow_- is a map of monads); we can show that $\uparrow_D(X) = \uparrow X$. We then have the following proposition, whose proof is analogous to that of the preceding one.

Lemma 3.10. Let C, D be coherent continuous d-cones and $f : C \rightarrow \mathcal{P}D$ be a linear continuous map. Then the following diagram commutes

$$\begin{array}{ccc}
 \mathcal{P}C & \xrightarrow{f^\#} & \mathcal{P}D \\
 \uparrow_C \downarrow & & \downarrow \uparrow_D \\
 \mathcal{S}C & \xrightarrow{(\uparrow_D \circ f)^\#} & \mathcal{S}D
 \end{array}$$

There is a \cup -preserving linear continuous map $u_C : \mathcal{S}C \rightarrow \mathcal{P}C$ in the other direction, where $u_C(X) = X$ (but u_- is not a natural transformation, let alone a map of monads); the proof that this map is monotonic relies on the fact that every d-cone has a greatest element. Note that u_C is right-inverse to \uparrow_C , and also that $\text{id}_{\mathcal{P}C} \leq u_C \uparrow_C \cong \text{id}_{\mathcal{P}C}$.

Proposition 3.11. Let C and D be coherent continuous d-cones.

- (a) The Kleisli extension $f^\# : \mathcal{P}C \rightarrow \mathcal{P}D$ of a linear continuous map $f : C \rightarrow \mathcal{P}D$ is given by $f^\#(X) = \bigcup_{x \in X} f(x) =_{\text{def}} (\bigcup_{x \in X} f(x))^\ell$.
- (b) The monad multiplication is given by $\mu_C(\mathcal{X}) = \text{conv}_{\leq}(\bigcup_{X \in \mathcal{X}} X)$.

Proof.

- (a) The proof consists of two calculations, relating to the lower and upper powercones, respectively:
 - (i) We have

$$\begin{aligned}
 \downarrow f^\#(X) &= \downarrow_D f^\#(X) \\
 &= (\downarrow_D f)^\#(\downarrow_C(X)) \quad (\text{by Lemma 3.9}) \\
 &= \overline{\bigcup_{x \in \downarrow X} \downarrow f(x)} \quad (\text{by Proposition 3.4}) \\
 &= \overline{\bigcup_{x \in X} f(x)}.
 \end{aligned}$$

- (ii) We have

$$\begin{aligned}
 \uparrow f^\#(X) &= \uparrow_D f^\#(X) \\
 &= (\uparrow_D f)^\#(\uparrow_C(X)) \quad (\text{by Lemma 3.10}) \\
 &= \bigcup_{x \in \uparrow X} \uparrow f(x) \quad (\text{by Proposition 3.7}) \\
 &= \uparrow(\bigcup_{x \in X} f(x)).
 \end{aligned}$$

Putting these together we have

$$f^\#(X) = \downarrow f^\#(X) \cap \uparrow f^\#(X) = \left(\bigcup_{x \in X} f(x) \right)^\ell.$$

So $(\bigcup_{x \in X} f(x))^\ell$ is the smallest convex lens containing $f(x)$ for all $x \in X$, and thus it equals $\bigcup_{x \in X} f(x)$.

- (b) As $\mu_C = (\text{id}_{\mathcal{P}C})^\#$, we have

$$\begin{aligned}
 \downarrow \mu_C(\mathcal{X}) &= \overline{\bigcup_{X \in \downarrow \mathcal{X}} \downarrow X} \quad (\text{following the proof of part (a)}) \\
 &= \bigcup_{X \in \downarrow \mathcal{X}} \downarrow X \quad (\text{by Lemma 3.3}) \\
 &= \downarrow(\bigcup_{X \in \mathcal{X}} X).
 \end{aligned}$$

As we also have $\uparrow \mu_C(\mathcal{X}) = \uparrow(\bigcup_{X \in \mathcal{X}} X)$ following the proof of part (a), the conclusion follows. \square

Combining the extended probabilistic powerdomain functor \mathcal{V} and \mathcal{P} , we obtain the *order-convex conical powerdomain* $\mathcal{PV}(P)$ of a coherent domain P . This is also a coherent domain, and is the free coherent continuous d-cone semilattice over P . We remark that $\mathcal{PV}(P)$ is even the free d-cone semilattice over P ; the proof will appear elsewhere.

4. Functional representations

We begin with some general properties of functional representations, and then consider the three powercones: lower, upper and convex. To this end, we return to the framework at the beginning of Section 3. Let C and D be d-cones in \mathbf{K} . By Proposition 3.1, extension

$$f \mapsto f^\# : [C, TD] \rightarrow [TC, TD]$$

is continuous, \subseteq -monotonic and \subseteq -sublinear. Composing this map with the evaluation at $\gamma \in TC$ yields the following corollary.

Corollary 4.1. For every $\gamma \in TC$, the map

$$\Lambda_\gamma = f \mapsto f^\#(\gamma) : [C, TD] \rightarrow TD$$

is continuous, \subseteq -monotonic and \subseteq -sublinear.

We regard Λ_γ as the *functional representation* of γ relative to the choice of D . Assuming that $\overline{\mathbb{R}}_+$ is an object of \mathbf{K} , the natural *standard* choice is $D = \overline{\mathbb{R}}_+$.

Furthermore, the representation map

$$\Lambda = (\gamma \mapsto \Lambda_\gamma) : TC \longrightarrow TD^{[C,TD]}$$

is itself a morphism of d-cone semilattices as every $f^\#$ is.

We assume that, with respect to the order \subseteq on TD , the set $\{f(x) \mid \eta_C(x) \subseteq \gamma\}$ always has a least upper bound $\bigcup_{\eta_C(x) \subseteq \gamma} f(x)$ and that

$$\Lambda_\gamma(f) = f^\#(\gamma) = \bigcup_{\eta_C(x) \subseteq \gamma} f(x). \tag{U}$$

Propositions 3.4, 3.7, and 3.11 assure us that this is satisfied in our three special cases. The formula looks even simpler in these cases as then the elements γ of TD are subsets of D and $\eta_C(x) \subseteq \gamma$ if and only if $x \in \gamma$.

For every x , the evaluation map $f \mapsto f(x) : [C, TD] \rightarrow TD$ is linear and \subseteq -monotonic. (Recall that the d-cone $[C, TD]$ carries the pointwise defined partial order \subseteq .) Thus, formula (U) above shows the following proposition.

Proposition 4.2. Under the above hypotheses, Λ_γ is the pointwise \bigcup of the continuous \subseteq -monotonic linear maps $f \mapsto f(x)$, $\eta_C(x) \subseteq \gamma$.

We now turn to the three special cases of interest to us.

4.1. *The lower powercone*

Here the monad T is

$$\mathcal{H} : \text{Cone} \rightarrow \text{Cone},$$

and we can simplify the standard representation a little. The free d-cone join-semilattice over $\overline{\mathbb{R}}_+$ is $\overline{\mathbb{R}}_+$ itself with the usual supremum as semilattice operation. So, using $\overline{\mathbb{R}}_+$ in place of the standard choice $\mathcal{H}(\overline{\mathbb{R}}_+)$, the cone $[C, \overline{\mathbb{R}}_+]$ is the dual cone C^* and we obtain an equivalent functional representation $\Lambda : \mathcal{H}C \rightarrow \overline{\mathbb{R}}_+^{C^*}$ where, by Proposition 3.4,

$$\Lambda_X(f) = \sup_{x \in X} f(x).$$

We see that each Λ_X is the pointwise supremum of continuous linear functionals, and hence continuous and sublinear. Since \subseteq and \leq coincide in the case of d-cone join-semilattices, this can be viewed as a special case of Proposition 4.2 and Corollary 4.1.

4.2. *The upper powercone*

Here the monad T is

$$\mathcal{S} : \text{CCone} \rightarrow \text{CCone},$$

and we can again simplify the standard representation a little. The free d-cone meet-semilattice over $\overline{\mathbb{R}}_+$ is $\overline{\mathbb{R}}_+$ itself with the usual infimum as semilattice operation. So we obtain a functional representation $\Lambda : \mathcal{S}C \rightarrow \overline{\mathbb{R}}_+^{C^*}$ equivalent to the standard one where, by Proposition 3.7,

$$\Lambda_X(f) = \inf_{x \in X} f(x).$$

We see that each Λ_X is the pointwise infimum of continuous linear functionals. Since \subseteq and \supseteq coincide in the case of d-cone meet-semilattices, Λ_X is continuous and sublinear by Proposition 4.2 and Corollary 4.1.

4.3. *The convex powercone*

Here the monad T is

$$\mathcal{P} : \text{CCone}^c \rightarrow \text{CCone}^c,$$

and with $S = \mathcal{P}\overline{\mathbb{R}}_+$, we have the standard representation

$$\Lambda : \mathcal{P}C \rightarrow \mathcal{P}\overline{\mathbb{R}}_+^{[C, \mathcal{P}\overline{\mathbb{R}}_+]}$$

where, by Proposition 3.11,

$$\Lambda_X(f) = \bigcup_{x \in X} f(x) = \left(\bigcup_{x \in X} f(x) \right)^\ell.$$

From Proposition 4.2 and Corollary 4.1, we know that each Λ_X is the pointwise \bigcup of continuous \subseteq -monotonic linear maps, and hence continuous, \subseteq -monotonic and \subseteq -sublinear.

To be more specific, recall that $\mathcal{P}\overline{\mathbb{R}}_+$ is the collection of all closed intervals $a = [\underline{a}, \overline{a}]$, $\underline{a} \leq \overline{a}$, in $\overline{\mathbb{R}}_+$. The cone operations on $\mathcal{P}\overline{\mathbb{R}}_+$ are

$$\begin{aligned} [\underline{a}, \overline{a}] + [\underline{b}, \overline{b}] &= [\underline{a} + \underline{b}, \overline{a} + \overline{b}] \\ r[\underline{a}, \overline{a}] &= [r\underline{a}, r\overline{a}] \end{aligned}$$

and the Egli–Milner order is given by

$$[\underline{a}, \overline{a}] \sqsubseteq_{EM} [\underline{b}, \overline{b}] \iff \underline{a} \leq \underline{b}, \overline{a} \leq \overline{b}.$$

The semilattice operation \cup gives the convex hull of two intervals and the associated order is subset inclusion

$$\begin{aligned} [\underline{a}, \overline{a}] \cup [\underline{b}, \overline{b}] &= [\underline{a} \wedge \underline{b}, \overline{a} \vee \overline{b}] \\ [\underline{a}, \overline{a}] \subseteq [\underline{b}, \overline{b}] &\iff \underline{b} \leq \underline{a} \leq \overline{a} \leq \overline{b}. \end{aligned}$$

We need some notation and some facts about maps into $\mathcal{P}\overline{\mathbb{R}}_+$. Let D be a d-cone the elements of which will be denoted by f, f' , and so on. For a function $F : D \rightarrow \mathcal{P}\overline{\mathbb{R}}_+$, the image of any $f \in D$ is an interval $F(f) = [F(f), \overline{F(f)}]$; picking the endpoints of these intervals, we obtain a pair of functions $\underline{F}, \overline{F} : D \rightarrow \overline{\mathbb{R}}_+$ such that $F(f) = [\underline{F}(f), \overline{F}(f)]$. Thus, the functions $F : D \rightarrow \mathcal{P}\overline{\mathbb{R}}_+$ correspond, in a one-to-one way, to pairs of functions $\underline{F}, \overline{F} : D \rightarrow \overline{\mathbb{R}}_+$ with $\underline{F} \leq \overline{F}$. We employ the notation $F = [\underline{F}, \overline{F}]$ and make the following observations.

Remark 4.3.

- (1) The map F is continuous and linear, respectively, if and only if both \underline{F} and \overline{F} are. Thus, we have the following d-cone isomorphisms and inclusions

$$\begin{aligned} (\mathcal{P}\overline{\mathbb{R}}_+)^D &\cong \{[\underline{F}, \overline{F}] \mid \underline{F} \leq \overline{F}\} \subseteq \overline{\mathbb{R}}_+^D \times \overline{\mathbb{R}}_+^D \\ [D, \mathcal{P}\overline{\mathbb{R}}_+] &\cong \{[\underline{F}, \overline{F}] \mid \underline{F} \leq \overline{F}\} \subseteq D^* \times D^*. \end{aligned}$$

- (2) F is \subseteq -sublinear if and only if \underline{F} is superlinear and \overline{F} is sublinear.
- (3) If F is the pointwise \cup of linear maps $F_i = [\underline{F}_i, \overline{F}_i] : D \rightarrow \mathcal{P}\overline{\mathbb{R}}_+$, then

$$\underline{F}(f) = \inf_i \underline{F}_i(f) \text{ and } \overline{F}(f) = \sup_i \overline{F}_i(f),$$

and the following condition holds:

$$\underline{F}(f + f') \leq \underline{F}(f) + \overline{F}(f') \leq \overline{F}(f + f'). \tag{*}$$

Proof. These assertions are all straightforward except for condition (*), which we will now verify. The linearity of \underline{F}_i yields the first inequality in condition (*):

$$\begin{aligned} \underline{F}(f + f') &= \inf_i \underline{F}_i(f + f') \\ &= \inf_i (\underline{F}_i(f) + \underline{F}_i(f')) \\ &\leq \inf_i (\underline{F}_i(f) + \sup_i \overline{F}_i(f')) \\ &= \underline{F}(f) + \overline{F}(f'). \end{aligned}$$

The second inequality is proved similarly. □

We will say that a \subseteq -sublinear map $F : D \rightarrow \mathcal{P}\overline{\mathbb{R}}_+$ or, equivalently, a pair $\underline{F}, \overline{F} : D \rightarrow \overline{\mathbb{R}}_+$ of superlinear and sublinear maps, respectively, is *canonical*, if it satisfies condition (*). Note that condition (*) implies $\underline{F} \leq \overline{F}$ (consider, for example, the case $f' = 0$).

We apply these considerations to the case where $D = [C, \mathcal{P}\overline{\mathbb{R}}_+]$ for a coherent continuous d-cone C and $F = \Lambda_X$. As above, the functions $f : C \rightarrow \mathcal{P}\overline{\mathbb{R}}_+$ correspond in a one-to-one way to the pairs of functions $\underline{f}, \overline{f} : C \rightarrow \overline{\mathbb{R}}_+$ with $\underline{f} \leq \overline{f}$, the correspondence being given by $f(x) = [\underline{f}(x), \overline{f}(x)]$, and, as before, we use the notation $f = [\underline{f}, \overline{f}]$. If $f \in [C, \mathcal{P}\overline{\mathbb{R}}_+]$, that is, if f is continuous and linear, \underline{f} and \overline{f} are too, that is, $\underline{f}, \overline{f} \in C^*$. In our general considerations we have seen that for every $X \in \mathcal{P}C$, the map $\Lambda_X : [C, \mathcal{P}\overline{\mathbb{R}}_+] \rightarrow \mathcal{P}\overline{\mathbb{R}}_+$ is continuous, \subseteq -monotonic, and pointwise the \bigcup of the linear maps $f \mapsto f(x), x \in X$, and hence \subseteq -sublinear. Together with Remark 4.3, this yields the following proposition.

Proposition 4.4. For every $X \in \mathcal{P}C$, the functional $\Lambda_X : [C, \mathcal{P}\overline{\mathbb{R}}_+] \rightarrow \mathcal{P}\overline{\mathbb{R}}_+$ representing X is continuous, \subseteq -monotonic, and canonically \subseteq -sublinear; equivalently, the functionals $\underline{\Lambda}_X, \overline{\Lambda}_X : [C, \mathcal{P}\overline{\mathbb{R}}_+] \rightarrow \overline{\mathbb{R}}_+$ are, respectively, superlinear and sublinear and satisfy condition (*) for all $f, f' \in [C, \mathcal{P}\overline{\mathbb{R}}_+]$. Moreover,

$$\underline{\Lambda}_X(f) = \inf_{x \in X} \underline{f}(x) \text{ and } \overline{\Lambda}_X(f) = \sup_{x \in X} \overline{f}(x).$$

4.4. *The diagonal representation*

We now exhibit another functional representation $\Lambda' : \mathcal{P}C \rightarrow \mathcal{P}\overline{\mathbb{R}}_+^{C^*}$, where C is a continuous coherent d-cone, and C^* is its dual. As in the previous subsection, the functions $f : D \rightarrow \mathcal{P}\overline{\mathbb{R}}_+$ correspond in a one-to-one way to the pairs of functions $\underline{f}, \overline{f} : D \rightarrow \overline{\mathbb{R}}_+$ with $\underline{f} \leq \overline{f}$, the correspondence being given by $f(x) = [\underline{f}(x), \overline{f}(x)]$ and, as before, we use the notation $f = [\underline{f}, \overline{f}]$.

We restrict every continuous map $F : [C, \mathcal{P}\overline{\mathbb{R}}_+] \rightarrow \mathcal{P}\overline{\mathbb{R}}_+$ to the ‘diagonal’ of the f in $[C, \mathcal{P}\overline{\mathbb{R}}_+]$ with $\underline{f} = \overline{f}$. Better, we compose every F with the linear continuous map $\Delta_C : (g \mapsto [g, g]) : C^* \rightarrow [C, \mathcal{P}\overline{\mathbb{R}}_+]$, thereby obtaining a d-cone semilattice morphism $R_C : \mathcal{P}\overline{\mathbb{R}}_+^{[C, \mathcal{P}\overline{\mathbb{R}}_+]} \rightarrow \mathcal{P}\overline{\mathbb{R}}_+^{C^*}$ where

$$R_C(F) = F \circ \Delta_C,$$

which assigns to $F = [\underline{F}, \overline{F}]$ the pair $F' = [\underline{F}', \overline{F}']$ defined by $\underline{F}'(g) = \underline{F}[g, g]$ and $\overline{F}'(g) = \overline{F}[g, g]$. It is crucial that, if F is \subseteq -monotonic, it is already completely determined by F' ; the following lemma allows us to say even more.

Lemma 4.5.

(1) Let $F : [C, \mathcal{P}\overline{\mathbb{R}}_+] \rightarrow \mathcal{P}\overline{\mathbb{R}}_+$ be continuous and \subseteq -monotonic. Then,

$$F(f) = F[\underline{f}, \overline{f}] = [\underline{F}[\underline{f}, \underline{f}], \overline{F}[\overline{f}, \overline{f}]] = [\underline{F}'(\underline{f}), \overline{F}'(\overline{f})].$$

(2) The map R_C restricts to a d-cone semilattice isomorphism between the sub-d-cone semilattice of the \subseteq -monotonic functionals in $\mathcal{P}\overline{\mathbb{R}}_+^{[C, \mathcal{P}\overline{\mathbb{R}}_+]}$ and $\mathcal{P}\overline{\mathbb{R}}_+^{C^*}$. Its inverse is given by $R_C^{-1}(F')(f) = [\underline{F}'(\underline{f}), \overline{F}'(\overline{f})]$.

Proof.

- (1) Let $g = \underline{f}$ and $h = \overline{f}$. As $g \leq h$, we have $[g, g] \leq [g, h] \leq [h, h]$, so, as F is monotonic, we have

$$\underline{F}[g, g] \leq \underline{F}[g, h] \text{ and } \overline{F}[g, h] \leq \overline{F}[h, h].$$

We also have $[g, g] \subseteq [g, h]$ and $[h, h] \subseteq [g, h]$, so, as F is \subseteq -monotonic, we have

$$\underline{F}[g, g] \supseteq \underline{F}[g, h] \text{ and } \overline{F}[h, h] \subseteq \overline{F}[g, h],$$

and the conclusion follows.

- (2) As $R_C^{-1}[G, H]$ is, evidently, \subseteq -monotonic, and R_C^{-1} is continuous, it is only necessary to prove that R_C and R_C^{-1} are inverses. To show that R_C is the right inverse of R_C^{-1} , we calculate as follows:

$$\begin{aligned} R_C^{-1}(R_C(F))(f) &= R_C^{-1}(g \mapsto [\underline{F}[g, g], \overline{F}[g, g]])(f) \\ &= [\underline{F}[\underline{f}, \underline{f}], \overline{F}[\overline{f}, \overline{f}]] \\ &= F(f). \end{aligned} \tag{by Part 1}$$

The proof that it is the left inverse is similar, but does not require the use of Part 1. □

Applying the above to our functional representation Λ , we get the *diagonal* representation $\Lambda' : \mathcal{P}C \rightarrow \mathcal{P}\overline{\mathbb{R}}_+^{C^*}$ given by

$$\Lambda'_X(g) = [\underline{\Lambda}_X[g, g], \overline{\Lambda}_X[g, g]]$$

for all $g \in C^*$. From Lemma 4.5, Λ' inherits from Λ the properties of being continuous, linear and \cup -preserving. From Proposition 4.4, we obtain the following proposition.

Proposition 4.6. For every $X \in \mathcal{P}C$, the functional $\Lambda'_X : C^* \rightarrow \mathcal{P}\overline{\mathbb{R}}_+$ representing X is continuous and canonically \subseteq -sublinear, that is, the functionals $\underline{\Lambda}'_X, \overline{\Lambda}'_X : C^* \rightarrow \overline{\mathbb{R}}_+$ are superlinear and sublinear, respectively, and they satisfy condition (*). Moreover,

$$\underline{\Lambda}'_X(g) = \inf_{x \in X} g(x) \text{ and } \overline{\Lambda}'_X(g) = \sup_{x \in X} g(x).$$

As Λ_X is \subseteq -monotonic, Lemma 4.5 allows us to recover Λ from Λ' , as follows:

$$\Lambda_X(f) = [\underline{\Lambda}'_X(\underline{f}), \overline{\Lambda}'_X(\overline{f})].$$

Thus the diagonal representation can be considered to be equivalent to the standard one, Λ . The last part of Proposition 4.6 shows that it combines the functional representations of the lower and upper powercones $\mathcal{H}C$ and $\mathcal{S}C$. For $g \in C^*$, we indeed get

$$\begin{aligned} \Lambda'_X(g) &= [\inf_{x \in X} g(x), \sup_{x \in X} g(x)] \\ &= [\inf_{x \in \uparrow X} g(x), \sup_{x \in \downarrow X} g(x)] \\ &= [\Lambda_{\uparrow X}(g), \Lambda_{\downarrow X}(g)]. \end{aligned}$$

How should we view condition (*)? Suppose you have a concave function \underline{F} and a convex function \overline{F} of one real variable with $\underline{F} \leq \overline{F}$. The above condition expresses the fact that from every point on one of the two curves you can see every point of the other curve. In other words, if we draw the line segment from a point on the lower curve to a

point on the upper curve, this line segment lies between the two curves. Our interest in condition (*) stems from the fact that a \subseteq -sublinear map from a continuous d-cone with an additive way-below relation to $\mathcal{P}\overline{\mathbb{R}}_+$ is pointwise the \bigcup of continuous linear maps if and only if it is canonical, as we shall see at the end of the next section.

5. Continuous sublinear and superlinear functionals

In order to characterise the functionals representing the objects constituting our three powercones we need the following information about sublinear and superlinear functionals.

Lemma 5.1 (Main Lemma). Let $G, H : D \rightarrow \overline{\mathbb{R}}_+$ be continuous superlinear and sublinear functionals, respectively, on a continuous d-cone D . Let L be the set of all linear continuous functionals f on D with $G \leq f \leq H$.

(1) Suppose that the following condition is satisfied

$$G(u + v) \leq G(u) + H(v) \text{ for all } u, v \in D. \tag{*_1}$$

Then,

$$G(x) = \inf\{f(x) \mid f \in L\} \text{ for all } x \in D.$$

(2) If the condition

$$G(u) + H(v) \leq H(u + v) \text{ for all } u, v \in D \tag{*_2}$$

is satisfied and the way below relation is additive on D , then

$$H(x) = \sup\{f(x) \mid f \in L\} \text{ for all } x \in D.$$

In the special case $G = 0$, the hypothesis of the additivity of the way-below relation is superfluous.

Before proving the Main Lemma, we state a corollary.

Corollary 5.2. Let D be a continuous d-cone.

(1) For every continuous superlinear functional $G : D \rightarrow \overline{\mathbb{R}}_+$, we have

$$G(x) = \inf\{f(x) \mid f \in D^*, f \geq G\} \text{ for all } x \in D.$$

(2) For every continuous sublinear functional $H : D \rightarrow \overline{\mathbb{R}}_+$, we have

$$H(x) = \sup\{f(x) \mid f \in D^*, f \leq H\} \text{ for all } x \in D.$$

Item (1) in the corollary follows from item (1) in the Main Lemma by choosing H to be the functional with value $+\infty$ for all $x \neq 0$. Similarly, item (2) in the corollary follows from item (2) in the Main Lemma, if we choose G to be the zero functional.

The proof of the Main Lemma is carried out in several steps. We will attack part (1) first.

Lemma 5.3. Let b be an element in an arbitrary d-cone D . Then $P : D \rightarrow \overline{\mathbb{R}}_+$ defined by

$$P(x) = \inf\{r \mid rb \geq x\}$$

is a sublinear continuous functional with $P(b) \leq 1$.

Proof. Clearly, $P(rx) = rP(x)$. For all $r > P(x)$ and all $s > P(y)$, we have $rb \geq x$ and $sb \geq y$, and thus $(r + s)b \geq x + y$, and, consequently, $P(x) + P(y) \geq P(x + y)$. Thus, P is sublinear. Clearly, P is monotonic. For continuity, let $x = \sup_i x_i$ for a directed family (x_i) . Choose any $r > \sup_i P(x_i)$. Then $rb \geq x_i$ for all i , and thus $rb \geq \sup x_i = x$, and we conclude that $r \geq P(x)$. We then conclude that $\sup_i P(x_i) \geq P(x)$. As the converse inequality follows from monotonicity, continuity is proved. \square

Lemma 5.4. Let P and H be sublinear functionals on a d-cone D . By defining $J : D \rightarrow \overline{\mathbb{R}}_+$ by

$$J(x) = \inf\{P(y) + H(z) \mid x \leq y + z\},$$

we get the greatest monotonic sublinear functional minorising P and H .

Proof. Clearly, J is monotonic and satisfies $J(rx) = rJ(x)$. To prove $J(x + x') \leq J(x) + J(x')$, choose arbitrary $r > J(x)$ and $r' > J(x')$. Then there are $y, z \in D$ such that $y + z \geq x$ and $r \geq P(y) + H(z)$, and there are $y', z' \in D$ such that $y' + z' \geq x'$ and $r' \geq P(y') + H(z')$. We then conclude that $y + y' + z + z' \geq x + x'$ and $r + r' \geq P(y) + P(y') + H(y) + H(y') \geq P(y + y') + H(y + y')$ by the sublinearity of H and P . We can conclude that $J(x) + J(x') \geq J(x + x')$.

Clearly, J is below H and P . Now let E be any monotonic sublinear functional minorising H and P . For all y, z such that $y + z \geq x$, we then have $P(y) + H(z) \geq E(y) + E(z) \geq E(y + z) \geq E(x)$. We conclude that $J(x) \geq E(x)$. \square

For the proof of part (1) of the Main Lemma (Lemma 5.1), we consider continuous superlinear and sublinear functionals G and H satisfying condition $(*_1)$ on a continuous d-cone D . Note that condition $(*_1)$ implies $G \leq H$; it suffices to consider the case $u = 0$. As in the Main Lemma, we use L to denote the set of all linear continuous functionals f on D such that $G \leq f \leq H$. Choose any $b \in D$. If $G(b) = H(b)$, there is a linear functional $f \in L$ with $G(b) = f(b) = H(b)$ by the Sandwich Theorem (Theorem 2.1). So from now on we assume that $G(b) < H(b)$. Let r be any real number such that $G(b) < r < H(b)$. Claim (1) is a direct consequence of the following lemma.

Lemma 5.5. There is a linear continuous functional $f \in L$ such that $f(b) \leq r$.

Proof. Without loss of generality, we may suppose $r = 1$. For the given b , we first define the continuous sublinear functional P as in Lemma 5.3. We first show that $G(x) \leq P(x)$ for all x . For every $r > P(x)$, we do indeed have $rb \geq x$, hence $G(rb) \geq G(x)$; as $G(b) \leq 1$, we conclude that $r \geq rG(b) = G(rb) \geq G(x)$.

We then form the sublinear functional J as in Lemma 5.4. We have $J \leq H$ and $J \leq P$, and thus $J(b) \leq P(b) \leq 1$. For all x and all y, z such that $x \leq y + z$, we have

$$G(x) \leq G(y + z) \leq G(y) + H(z) \leq P(y) + H(z),$$

where we have used hypothesis $(*_1)$ for the inequality in the middle. We conclude that $G(x) \leq J(x)$ by the definition of J .

We can now apply the Sandwich Theorem (Theorem 2.1) to G and J to find a linear continuous functional f lying in between them, and this has the desired properties. \square

We now proceed in a similar way for the proof of part (2) of the Main Lemma.

Lemma 5.6. Let a, b be elements of a continuous d-cone D with $a \ll b$. Then there is a continuous superlinear functional $Q : D \rightarrow \overline{\mathbb{R}}_+$ such that $Q(b) \geq 1$ and $Q(x)a \leq x$ for all $x \in D$.

Proof. By local convexity, there is a convex open neighbourhood V of b contained in $\uparrow a$. We look at the Minkowski functional of V :

$$Q(x) = \sup\{r > 0 \mid x \in rV\}.$$

By Plotkin (2006, Lemma 3(1)), Q is continuous and superlinear. Clearly, $Q(b) \geq 1$ as $b \in 1 \cdot V$. Consider any x . Whenever $0 < r < Q(x)$, we have $x \in rV \subseteq r\uparrow a$, and thus $ra \ll x$. So $Q(x)a = \sup\{r \mid 0 < r < Q(x)\} \cdot a = \sup\{ra \mid 0 < r < Q(x)\} \leq x$, which establishes the two inequalities. \square

Lemma 5.7. Let G and Q be monotonic superlinear functionals on a d-cone D . Then $E : D \rightarrow \overline{\mathbb{R}}_+$ defined by

$$E(x) = \sup\{G(y) + Q(z) \mid y + z \leq x\}$$

is the least monotonic superlinear functional majorising G and Q . If D is a continuous d-cone with an additive way-below relation and if G and Q are continuous, then E is continuous too.

Proof. The proof of the first claim is the same as the proof of Lemma 5.4 with the order turned upside down. Suppose now that D is a continuous d-cone with an additive way-below relation. For the continuity of E , suppose that x is the supremum of a directed family (x_i) . Consider any $r < E(x)$. Then there are y, z with $y + z \leq x$ and $r < G(y) + Q(z)$. If G and Q are continuous, we may find $y' \ll y$ and $z' \ll z$ such that $r \leq G(y') + Q(z')$. By additivity of the way-below relation, $y' + z' \ll y + z \leq x = \sup_i x_i$. Thus $y' + z' \leq x_i$ for some i . We conclude that $r \leq E(x_i)$ for some i . As this holds for every $r < E(x)$, we conclude that $E(x) \leq \sup_i E(x_i)$. As the converse inequality follows from the monotonicity of E , continuity of E is proved. \square

For the proof of part (2) of the Main Lemma (Lemma 5.1), we consider continuous superlinear and sublinear functionals G and H satisfying condition $(*_2)$ on a continuous d-cone D . Note that condition $(*_2)$ implies $G \leq H$; it suffices to consider the case $v = 0$. Choose any $b \in D$. If $G(b) = H(b)$, there is a linear functional $f \in L$ with $G(b) = f(b) = H(b)$ by the Sandwich Theorem (Theorem 2.1). So from now on we assume that $G(b) < H(b)$. Let r be any real number such that $G(b) < r < H(b)$. Claim (2) is a direct consequence of the following Lemma.

Lemma 5.8. There is a linear continuous functional $f \in L$ such that $r \leq f(b)$ provided the way-below relation is additive on D . The latter hypothesis is superfluous if G is the zero functional.

Proof. Without loss of generality, we may suppose $r = 1$. By the continuity of H and the continuity of D , there is an $a \ll b$ such that $1 < H(a)$. For these elements a

and b , define the continuous superlinear functional Q as in Lemma 5.6, so it satisfies $Q(b) \geq 1$ and $Q(z)a \leq z$ for all z . For G and this Q , we now define the continuous superlinear functional E as in Lemma 5.7. We clearly then have $G \leq E$ and $1 \leq E(b)$. We further prove that $E \leq H$. For every $z \in D$, we do indeed have $Q(z)a \leq z$. We deduce $Q(z)H(a) = H(Q(z)a) \leq H(z)$. On the other hand, $Q(z) \leq Q(z)H(a)$, as $1 \leq H(a)$. Thus $Q(z) \leq H(z)$ for every z . For arbitrary elements y, z with $y + z \leq x$, we now have

$$G(y) + Q(z) \leq G(y) + H(z) \leq H(y + z) \leq H(x),$$

where we have used hypothesis $(*_2)$ for the inequality in the middle. We then conclude that $E(x) \leq H(x)$.

We can now apply the Sandwich Theorem (Theorem 2.1) to E and H to find a linear continuous functional f lying in between them, and this has the desired properties.

The additivity of the way-below relation is only needed in this proof when we use Lemma 5.7. If G is the zero functional, we do not need this lemma, as we may choose $E = Q$. □

Having finished the proof of the Main Lemma, we proceed to our crucial theorem. In particular, part (3) of this theorem clarifies the significance of the strange-looking conjunction $(*)$ of the conditions $(*_1)$ and $(*_2)$ in the Main Lemma.

Theorem 5.9. Let D be a continuous d-cone, whose way-below relation is additionally assumed to be additive for the claims (2) and (3) below.

- (1) A functional $H : D \rightarrow \overline{\mathbb{R}}_+$ is continuous and sublinear if and only if it is pointwise the supremum of continuous linear functionals, that is, $H(x) = \sup_{f \in A} f(x)$ for some subset $A \subseteq D^*$. We may choose $A = \{f \in D^* \mid f \leq H\}$ that is convex and weak* Scott-closed in D^* .
- (2) A functional $G : D \rightarrow \overline{\mathbb{R}}_+$ is continuous and superlinear if and only if it is pointwise the infimum of a weak* Scott-compact set of continuous linear functionals, that is, $G(x) = \inf_{f \in B} f(x)$ for some weak* Scott-compact subset $B \subseteq D^*$. We may choose $B = \{f \in D^* \mid G \leq f\}$ that is convex, saturated and weak* Scott-compact.
- (3) A map $F = [E, \overline{F}] : D \rightarrow \mathcal{P}\overline{\mathbb{R}}_+$ is continuous and canonically \subseteq -sublinear if and only if it is pointwise the \bigcup of a weak* Scott-compact set of continuous linear maps $f : D \rightarrow \mathcal{P}\overline{\mathbb{R}}_+$ if and only if $F(x) = \bigcup_{f \in L} [f(x), f(x)]$ for some weak* Scott-compact subset $L \subseteq D^*$. The set L may be chosen to be the convex lens obtained as the intersection of the weak* Scott-closed convex subset $A = \{f \in D^* \mid f \leq \overline{F}\}$ and the weak* Scott-compact convex saturated subset $B = \{f \in D^* \mid E \leq f\}$.

Proof.

- (1) The pointwise supremum $H(x) = \sup_{f \in A} f(x)$ of any set A of continuous functions $f : D \rightarrow \overline{\mathbb{R}}_+$ is continuous. If all $f \in A$ are linear, the pointwise supremum is sublinear. Conversely, if H is continuous and sublinear, it is the pointwise supremum of the set A of those $f \in D^*$ with $f \leq H$ by Corollary 5.2 (1) following the Main Lemma. Clearly, A is weak* Scott closed and convex in D^* .

(2) The pointwise infimum of a set B of linear functionals is superlinear. In general, when the functionals $f \in B$ are all continuous, the pointwise infimum need not be continuous. But this is true if B is weak* Scott-compact in D^* . Indeed, the map $(f, x) \mapsto f(x) : D^* \times D \rightarrow \overline{\mathbb{R}}_+$ is separately continuous in each argument, where on D^* we take the weak* Scott topology. As D is continuous, this map is automatically jointly continuous. Keimel and Gierz (1982, Corollary (9)) tells us that if X is a T_0 -space and Y a locally compact space, then, for every continuous map g from $X \times Y$ into a continuous lattice and every compact subset B of X , the pointwise infimum $\inf_{x \in B} g(x, y)$ is continuous on Y . This allows us to conclude that for each weak* Scott-compact subset $B \subseteq D^*$, the pointwise infimum is continuous.

Conversely, if G is a superlinear continuous functional on C , then, by Corollary 5.2 (2) following the Main Lemma, it is pointwise the infimum of the set B of continuous linear functionals $f \geq G$. Clearly, B is convex and saturated. As we suppose here that the way-below relation is additive on C , the Banach–Alaoglu Theorem (Theorem 2.4) tells us that B is weak* Scott-compact.

(3) We begin by clarifying that when we say ‘the weak* Scott topology on the d-cone of continuous linear maps $f = [f, \bar{f}]$ from D to $\mathcal{P}\overline{\mathbb{R}}_+$ ’, we mean the weakest topology making the evaluations $f \mapsto \bar{f}(x)$ continuous for all $x \in D$. This is equivalent to requiring that all the maps $f \mapsto \underline{f}(x)$ and $f \mapsto \bar{f}(x)$ are continuous.

Let $F(x) = \bigcup_{f \in L} f(x)$ for some set L of continuous linear maps $f : D \rightarrow \mathcal{P}\overline{\mathbb{R}}_+$. By Remark 4.3, we know that F is canonically \subseteq -sublinear, that $\underline{F}(x) = \inf_{f \in L} \underline{f}(x)$ and that $\bar{F}(x) = \sup_{f \in L} \bar{f}(x)$. Then \bar{F} is continuous as it is the pointwise supremum of the continuous linear functionals \bar{f} for $f \in L$ (see item (1)). If L is weak* Scott-compact, then \underline{F} is continuous as it is the pointwise infimum of the weak* Scott-compact set of continuous linear functionals \underline{f} for $f \in L$ (see item (2)). It follows that $F = [\underline{F}, \bar{F}]$ is also continuous.

Next, if $F(x) = \bigcup_{f \in L} [f(x), f(x)]$ for some weak* Scott-compact subset $L \subseteq D^*$, then $\{[f, \bar{f}] \mid f \in L\}$ is also weak* Scott-compact as the function $f \mapsto [f, \bar{f}] : [D, \overline{\mathbb{R}}_+] \rightarrow [D, \mathcal{P}\overline{\mathbb{R}}_+]$ is easily seen to be weak* Scott-continuous. So F is pointwise the \bigcup of a weak* Scott-compact set of continuous linear maps $f : D \rightarrow \mathcal{P}\overline{\mathbb{R}}_+$.

Conversely, let $F : D \rightarrow \mathcal{P}\overline{\mathbb{R}}_+$ be a canonical \subseteq -sublinear continuous map. The set A of all $f \in D^*$ with $f \leq \bar{F}$ is weak* Scott-closed and convex in D^* by item (1), and the set B of all $f \in D^*$ with $f \geq \underline{F}$ is weak* Scott-compact, convex and saturated by item (2). Using condition (*), the Main Lemma (Lemma 5.1) tells us that $\underline{F}(x) = \inf_{f \in L} \underline{f}(x)$ and $\bar{F}(x) = \sup_{f \in L} \bar{f}(x)$, and, consequently, $F(x) = [\underline{F}(x), \bar{F}(x)] = [\inf_{f \in L} \underline{f}(x), \sup_{f \in L} \bar{f}(x)]$. Thus F is pointwise the \bigcup of the linear maps $[f, f] : D \rightarrow \mathcal{P}\overline{\mathbb{R}}_+$ with $f \in A \cap B = L$. □

6. The functional representation theorems

We are going to characterise the functionals Λ_X for our three types of powercones. In all cases we have to restrict ourselves to reflexive continuous d-cones. This strong hypothesis is satisfied by our main example, the extended probabilistic powercone $\mathcal{V}(X)$ of all continuous valuations on a domain X (see Subsection 2.3).

6.1. The lower powercone

From Section 4 we know that for every d-cone C , the representation function

$$\Lambda : \mathcal{H}C \rightarrow \overline{\mathbb{R}}_+^{C^*}$$

is a morphism of d-cone join-semilattices that transforms every closed convex subset X of C into the continuous sublinear functional $\Lambda_X : C^* \rightarrow \overline{\mathbb{R}}_+$ defined by

$$\Lambda_X(f) = \sup_{x \in X} f(x).$$

We want to show that, under appropriate additional hypotheses, the sublinear continuous functionals on C^* form a sub-d-cone join-semilattice of the d-cone join-semilattice of all the continuous functionals on C^* and, furthermore, that Λ is then a d-cone join-semilattice isomorphism of $\mathcal{H}C$ and the sublinear continuous functionals. This will follow from the above remarks if we can show that Λ is an order-embedding and that its range includes all of the sublinear continuous functionals.

Proposition 6.1. For a continuous d-cone C , the map Λ is an order embedding, that is, for $X, Y \in \mathcal{H}C$, we have $\Lambda_X \leq \Lambda_Y$ if and only if $X \subseteq Y$.

Proof. As Λ is monotonic by the general considerations in Section 4, it remains to show that, if $\Lambda_X \leq \Lambda_Y$, then $X \subseteq Y$. To do this, we suppose $X \not\subseteq Y$, and choose an element $a \in X \setminus Y$. As Y is closed, and as a continuous d-cone is locally convex, there is a convex open neighbourhood U of a disjoint from Y . By the Separation Theorem (Theorem 2.2), there is a linear continuous functional $f : C \rightarrow \overline{\mathbb{R}}_+$ such that $f(a) > 1$, but $f(y) \leq 1$ for all $y \in Y$. Hence $\Lambda_X(f) = \sup f(X) \geq f(a) > 1$, but $\Lambda_Y(f) = \sup f(Y) \leq 1$, which implies $\Lambda_X \not\leq \Lambda_Y$. □

Suppose that C is a reflexive continuous d-cone whose dual d-cone C^* is also continuous, and let H be a continuous sublinear functional on C^* . We may apply Theorem 5.9 (1) with $D = C^*$ and $D^* = C$ to find a convex weak Scott-closed subset $X \subseteq C$ such that $H(f) = \sup_{x \in X} f(x) = \Lambda_X(f)$, and thus $\Lambda_X = H$. Note that weak Scott-closed sets are closed. Taken together with the previous proposition, this yields the following theorem.

Theorem 6.2. Let C be a reflexive continuous d-cone whose dual C^* is also continuous. Then the sublinear functionals form a sub-d-cone join-semilattice of $\overline{\mathbb{R}}_+^{C^*}$ and Λ cuts down to a d-cone join-semilattice isomorphism between $\mathcal{H}C$ and the continuous sublinear functionals.

Proof. Since Λ is a morphism of d-cone join-semilattices, its range is a sub-d-cone join-semilattice of $\overline{\mathbb{R}}_+^{C^*}$. As its range consists of the continuous sublinear functionals, the first assertion follows. Then, as Λ is an order-embedding as well as a morphism of d-cone join-semilattices, it cuts down to an isomorphism of d-cone join-semilattices as asserted. □

Let us now consider d-cones of the form $\mathcal{V}(P)$ for a domain P . Here we have a representation function

$$\mathcal{H}\mathcal{V}(P) \rightarrow \overline{\mathbb{R}}_+^{\mathcal{V}(P)^*} \cong \overline{\mathbb{R}}_+^{\mathcal{L}(P)},$$

which we also call Λ and which is given by

$$\Lambda_X(f) = \sup_{\mu \in X} \int f d\mu.$$

As $\mathcal{V}(P)$ is a reflexive continuous d-cone when P is a domain and as the dual cone $\mathcal{V}(P)^* \cong \mathcal{L}(P)$ is continuous too, we may apply Theorem 6.2.

Corollary 6.3. Let P be a domain. Then the sublinear functionals form a sub-d-cone join-semilattice of $\overline{\mathbb{R}}_+^{\mathcal{L}(P)}$ and Λ cuts down to a d-cone join-semilattice isomorphism between $\mathcal{H}\mathcal{V}(P)$ and the continuous sublinear functionals $H : \mathcal{L}(P) \rightarrow \overline{\mathbb{R}}_+$.

6.2. The upper powercone

From Section 4.2 we know that for every continuous d-cone C , the representation function

$$\Lambda : \mathcal{S}C \rightarrow \overline{\mathbb{R}}_+^{C^*}$$

is a morphism of d-cone meet-semilattices that transforms every compact convex saturated subset X of C into the continuous superlinear functional $\Lambda_X : C^* \rightarrow \overline{\mathbb{R}}_+$ defined by

$$\Lambda_X(f) = \inf_{x \in X} f(x).$$

Analogously to the previous case, we want to show that under appropriate additional hypotheses, Λ is a d-cone meet-semilattice isomorphism between $\mathcal{S}C$ and the sub-d-cone meet-semilattice of the superlinear continuous functionals, and to this end we again need only further show that it is an order embedding whose range includes the superlinear continuous functionals.

Proposition 6.4. For a continuous d-cone C , the map Λ is an order embedding, that is, for $X, Y \in \mathcal{S}C$, we have $\Lambda_X \leq \Lambda_Y$ if and only if $X \supseteq Y$.

Proof. As Λ is monotonic by the general considerations in Section 4.1, it just remains to show that, if $\Lambda_X \leq \Lambda_Y$, then $X \supseteq Y$. To do this, we suppose $X \not\supseteq Y$ and choose an element $b \in Y \setminus X$. As X is a compact convex saturated set, by the Strict Separation Theorem (Theorem 2.3), there is a linear continuous functional $f : C \rightarrow \overline{\mathbb{R}}_+$ such that $f(b) \leq 1$, but $f(x) \geq r > 1$ for some r and all $x \in X$. It follows that $\Lambda_X(f) = \inf f(X) \geq r > 1$, but $\Lambda_Y(f) = \inf f(Y) \leq f(b) \leq 1$, which implies $\Lambda_X \not\leq \Lambda_Y$. \square

Now let G be a continuous superlinear functional on C^* . Supposing that C is a convenient d-cone – that is, it is continuous, reflexive and its weak Scott topology coincides with its Scott topology and, furthermore, the dual cone C^* is continuous and has an additive way-below relation – we may apply Theorem 5.9(2) for $D = C^*$ and $D^* = C$, and find a compact convex saturated subset $X \subseteq C$ such that $G(f) = \sup_{x \in X} f(x) = \Lambda_X(f)$, and thus $\Lambda_X = G$. Taking this together with the previous proposition, we get the following theorem.

Theorem 6.5. Let C be a convenient d-cone. Then the superlinear functionals form a sub-d-cone meet-semilattice of $\overline{\mathbb{R}}_+^{C^*}$ and Λ cuts down to a d-cone meet-semilattice isomorphism between $\mathcal{S}C$ and the continuous superlinear functionals.

Let us now consider d-cones of the form $\mathcal{V}(P)$ for a domain P . Here we have a representation function

$$\mathcal{S}\mathcal{V}(P) \rightarrow \overline{\mathbb{R}}_+^{\mathcal{V}(P)^*} \cong \overline{\mathbb{R}}_+^{\mathcal{L}(P)},$$

which we again also call Λ and which is given by

$$\Lambda_X(f) = \inf_{\mu \in X} \int f d\mu.$$

For a coherent domain P , the extended probabilistic powerdomain $\mathcal{V}(P)$ is a convenient d-cone, so we may apply Theorem 6.5.

Corollary 6.6. Let P be a coherent domain. Then the superlinear functionals form a sub-d-cone meet-semilattice of $\overline{\mathbb{R}}_+^{\mathcal{L}(P)}$ and Λ cuts down to a d-cone meet-semilattice isomorphism between $\mathcal{S}\mathcal{V}(P)$ and the continuous superlinear functionals $H : \mathcal{L}(P) \rightarrow \overline{\mathbb{R}}_+$.

6.3. The convex powercone

For every coherent continuous d-cone C , we have two representations according to Sections 4.3 and 4.4: the standard representation

$$\Lambda : \mathcal{P}C \rightarrow \mathcal{P}\overline{\mathbb{R}}_+^{[C, \mathcal{P}\overline{\mathbb{R}}_+]}$$

and the diagonal representation

$$\Lambda' : \mathcal{P}C \rightarrow \mathcal{P}\overline{\mathbb{R}}_+^{C^*}.$$

Both representations are morphisms of d-cone semilattices. Every convex lens $X \subseteq C$ is represented by a pair of continuous real valued functionals $\Lambda_X = [\underline{\Lambda}_X, \overline{\Lambda}_X]$ defined on the d-cone $[C, \mathcal{P}\overline{\mathbb{R}}_+]$ in the case of Λ , and by a pair of continuous real valued functionals $\Lambda'_X = [\underline{\Lambda}'_X, \overline{\Lambda}'_X]$ defined on the dual cone C^* in the case of Λ' . The latter are defined by

$$\underline{\Lambda}'_X(g) = \inf_{x \in X} g(x)$$

and

$$\overline{\Lambda}'_X(g) = \sup_{x \in X} g(x),$$

and the two representations are related by the formulas

$$\begin{aligned} \Lambda'_X(g) &= \Lambda_X[g, g] \\ \Lambda_X[g, h] &= [\underline{\Lambda}'_X(g), \overline{\Lambda}'_X(h)] \end{aligned}$$

for $g \in C^*$ and $[g, h] \in [C, \mathcal{P}\overline{\mathbb{R}}_+]$. For each $X \in \mathcal{P}\overline{\mathbb{R}}_+$, the functional $\Lambda_X : [C, \mathcal{P}\overline{\mathbb{R}}_+] \rightarrow \mathcal{P}\overline{\mathbb{R}}_+$ and, similarly, the functional $\Lambda'_X : C^* \rightarrow \overline{\mathbb{R}}_+$ is continuous and canonically \subseteq -sublinear.

Proposition 6.7. For a coherent continuous d-cone C , the maps Λ and Λ' are order embeddings, that is, for $X, Y \in \mathcal{PC}$, we have $\Lambda_X \leq \Lambda_Y$ and $\Lambda'_X \leq \Lambda'_Y$, respectively, if and only if $X \sqsubseteq_{EM} Y$.

Proof. By the general considerations in Section 4.3, Λ and Λ' are monotonic. Conversely, if $\Lambda_X \leq \Lambda_Y$, then $\Lambda'_X \leq \Lambda'_Y$ by the definition of Λ' . From $\Lambda'_X \leq \Lambda'_Y$, we deduce:

- (1) $\Lambda_{\downarrow X} \leq \Lambda_{\downarrow Y}$, and thus $\downarrow X \subseteq \downarrow Y$ by Proposition 6.1, where now Λ is the lower powercone functional representation; and
- (2) $\Lambda_{\uparrow X} \leq \Lambda_{\uparrow Y}$, so $\uparrow X \subseteq \uparrow Y$ by Proposition 6.4, where now Λ is the upper powercone functional representation.

So we have $X \sqsubseteq_{EM} Y$, as required. □

Now let C be a convenient d-cone. Then $D =_{\text{def}} C^*$ is continuous and has an additive way-below relation and the weak* Scott topology on $D^* \cong C$ is identical to the Scott topology. Theorem 5.9(3) allows us to conclude that for every continuous canonically \sqsubseteq -sublinear functional $F : C^* \rightarrow \mathcal{P}\overline{\mathbb{R}}_+$, there is a convex lens $X \subseteq D^* \cong C$ such that $\Lambda'_X = F$.

We now consider a continuous \sqsubseteq -monotonic canonically \sqsubseteq -sublinear functional $F : [C, \mathcal{P}\overline{\mathbb{R}}_+] \rightarrow \mathcal{P}\overline{\mathbb{R}}_+$. By Lemma 4.5, the map $F' =_{\text{def}} R_C(F) : C^* \rightarrow \mathcal{P}\overline{\mathbb{R}}_+$ is continuous and canonically \sqsubseteq -sublinear. By the preceding, there is a convex lens $X \subseteq C$ such that $\Lambda'_X = F'$. As, again by Lemma 4.5, F can be recovered from F' in the same way as Λ_X from Λ'_X by applying R_C^{-1} , we conclude that $F = \Lambda_X$.

Together with the previous proposition, we have the following theorem.

Theorem 6.8. Let C be a convenient coherent d-cone. Then Λ and Λ' cut down to isomorphisms between the continuous coherent d-cone semilattice \mathcal{PC} and sub-d-cone semilattices of $\mathcal{P}\overline{\mathbb{R}}_+^{[C, \mathcal{P}\overline{\mathbb{R}}_+]}$ and $\mathcal{P}\overline{\mathbb{R}}_+^{C^*}$, respectively, consisting of all canonically \sqsubseteq -sublinear functionals that, in the first case, are also \sqsubseteq -monotonic.

Let us now specialise to d-cones of the form $C = \mathcal{V}(P)$ for a domain P . We recall that $\mathcal{V}(P)$ is a convenient d-cone and that the dual cone $\mathcal{V}(P)^*$ is naturally isomorphic to $\mathcal{L}(P)$. Similarly, the cone $[\mathcal{V}(P), \mathcal{P}\overline{\mathbb{R}}_+]$ is naturally isomorphic to the cone $\mathcal{P}\overline{\mathbb{R}}_+^P$ of all continuous functions $f : P \rightarrow \mathcal{P}\overline{\mathbb{R}}_+$ that can be represented as pairs $[g, h]$ of functions $g, h \in \mathcal{L}(P)$ with $g \leq h$. We therefore have representation functions

$$\begin{aligned} \mathcal{PV}(P) &\longrightarrow \mathcal{P}\overline{\mathbb{R}}_+^{[\mathcal{V}(P), \mathcal{P}\overline{\mathbb{R}}_+]} \cong (\mathcal{P}\overline{\mathbb{R}}_+)^{\mathcal{P}\overline{\mathbb{R}}_+^P} \\ \mathcal{PV}(P) &\longrightarrow \mathcal{P}\overline{\mathbb{R}}_+^{\mathcal{V}(P)^*} \cong \mathcal{P}\overline{\mathbb{R}}_+^{\mathcal{L}(P)}, \end{aligned}$$

which we again also call Λ and Λ' , respectively; Λ' is given by the formulas

$$\underline{\Lambda}'_X(g) = \inf_{\mu \in X} \int g d\mu$$

and

$$\overline{\Lambda}'_X(g) = \sup_{\mu \in X} \int g d\mu.$$

And Λ can be calculated from Λ' as above:

$$\Lambda_X[g, h] = [\underline{\Lambda}'_X(g), \overline{\Lambda}'_X(h)].$$

We may now apply Theorem 6.8.

Corollary 6.9. Let P be a coherent domain. Then Λ and Λ' cut down to isomorphisms between the continuous d-cone semilattice $\mathcal{PV}(P)$ and sub-d-cone semilattices of $(\mathcal{P}\overline{\mathbb{R}}_+)^{\mathcal{P}\overline{\mathbb{R}}_+}$ and $(\mathcal{P}\overline{\mathbb{R}}_+)^{\mathcal{L}(P)}$, respectively, consisting of all canonically \subseteq -sublinear maps that, in the first case, are also \subseteq -monotonic.

7. Predicate transformers

As in the case of functional representations, a certain amount of the development can be carried out at a general level. We place ourselves in the framework of Sections 3 and 4, assume the category \mathbf{K} of d-cones contains $\overline{\mathbb{R}}_+$, and work with the standard representation

$$\Lambda : TC \longrightarrow T\overline{\mathbb{R}}_+^{[C, T\overline{\mathbb{R}}_+]}$$

given by

$$\Lambda_\gamma = f \mapsto f^\#(\gamma).$$

We take ‘predicates’ on a d-cone C to be linear continuous maps from C to $T\overline{\mathbb{R}}_+$ and predicate transformers from one d-cone D to another C to be continuous maps $\Phi : [D, T\overline{\mathbb{R}}_+] \rightarrow [C, T\overline{\mathbb{R}}_+]$. The general question then concerns the relation between such predicate transformers and ‘state transformers’ from C to D , which we take to be linear continuous maps from C to TD .

There is an evident isomorphism of d-cones

$$t : [C, T\overline{\mathbb{R}}_+^{[D, T\overline{\mathbb{R}}_+]}] \cong [C, T\overline{\mathbb{R}}_+]^{[D, T\overline{\mathbb{R}}_+]}$$

defined by transposition: $t(m')(f)(x) =_{\text{def}} m'(x)(f)$. Composing with Λ and applying t , we then get a linear continuous map

$$W_{C,D} : [C, TD] \longrightarrow [C, T\overline{\mathbb{R}}_+]^{[D, T\overline{\mathbb{R}}_+]}$$

where $W_{C,D}(m) =_{\text{def}} t(\Lambda \circ m)$. More explicit formulas for this map are

$$W_{C,D}(m)(f)(x) = \Lambda_{mx}(f) = f^\#(mx).$$

Using the last formula, it is easy to verify that this defines the morphism part of a locally linear and continuous functor

$$W : \mathbf{K}_T \longrightarrow \mathbf{PT}^{\text{op}}$$

that acts as the identity on objects. Here \mathbf{K}_T is, as usual, the Kleisli category of T , and is our category of state transformers; \mathbf{PT} is the category with the same objects as \mathbf{K} and with the morphisms from C to D being the predicate transformers from C to D .

It further follows from the second formula for $W_{C,D}$, together with Corollary 4.1, that every predicate transformer in the range of W is \subseteq -monotonic and \subseteq -sublinear. The

collection of such predicate transformers from a given C to a given D forms a sub-d-cone of the d-cone of all predicate transformers from C to D .

Proposition 7.1. If Λ is an order-embedding, so is W (locally).

Proof. By the assumption, the map $W_{C,D}$ consists of a composition with an order-embedding, which is itself an order-embedding, followed by an isomorphism. \square

The converse also holds, but we do not need it.

We now specialise the discussion to free d-cones on domains. Suppose that J is a full subcategory of the category of domains and that we have an adjunction

$$\mathcal{V} \dashv G : K \longrightarrow J$$

where G is the evident forgetful functor and \mathcal{V} is the appropriate restriction of the valuation functor, and suppose further that the natural transformation

$$\psi_{P,D} : D^P = J(P, GD) \cong K(\mathcal{V}P, D) = [\mathcal{V}P, D]$$

is an isomorphism of d-cones. Here, and below, we neglect to write the forgetful functor G and consider \mathcal{V} to be a left adjoint or a monad, as convenient.

With these assumptions, we have a monad on J that may be written as $T\mathcal{V}$. We then take state transformers at the level of domains to be continuous functions $P \rightarrow T\mathcal{V}Q$ in J , and so take the category of state transformers to be $J_{T\mathcal{V}}$. We can define a full and faithful functor $\mathcal{V}_T : J_{T\mathcal{V}} \longrightarrow K_T$, which is locally an isomorphism of d-cones, by putting

$$\mathcal{V}_T(P) = \mathcal{V}(P)$$

on objects, and

$$\mathcal{V}_T(m) = \psi_{P,T\mathcal{V}Q}(m)$$

on morphisms; showing functoriality is a straightforward, if tedious, calculation.

We take predicates on a domain P in J to be continuous maps from P to $GT\overline{\mathbb{R}}_+$ and predicate transformers from another such Q to P to be continuous maps $\Phi : GT\overline{\mathbb{R}}_+^Q \rightarrow GT\overline{\mathbb{R}}_+^P$, yielding the category PT_d of predicate transformers. We can define a useful functor $\mathcal{V}_p : PT_d \rightarrow PT$ by putting

$$\mathcal{V}_p(P) = \mathcal{V}(P)$$

on objects, and

$$\mathcal{V}_p(\Phi) = (\psi_{P,T\overline{\mathbb{R}}_+}) \circ \Phi \circ (\psi_{Q,T\overline{\mathbb{R}}_+})^{-1}$$

on morphisms.

Next, since $W\mathcal{V}_T(P) = \mathcal{V}(P)$, we can define a functor $W_d : J_{T\mathcal{V}} \longrightarrow PT_d^{op}$ that is the identity on objects, and on morphisms $m : P \rightarrow GT\mathcal{V}Q$ is given by

$$W_d(m) = (\mathcal{V}_p)_{Q,P}^{-1}(W \circ \mathcal{V}_T) = (\psi_{P,T\overline{\mathbb{R}}_+})^{-1} \circ W(\mathcal{V}_T(m)) \circ \psi_{Q,T\overline{\mathbb{R}}_+}.$$

Note that $\mathcal{V}_p \circ W_d = W \circ \mathcal{V}_T$. We then have

$$\begin{aligned} W_d(m)(\theta)(x) &= W(\mathcal{V}_T(m))(\psi_{Q,T\overline{\mathbb{R}}_+}(\theta))(\eta x) \\ &= \Lambda_{\mathcal{V}_T(m)(\eta x)}(\psi_{Q,T\overline{\mathbb{R}}_+}(\theta)) \\ &= \Lambda_{mx}(\psi_{Q,T\overline{\mathbb{R}}_+}(\theta)) \\ &= \psi_{Q,T\overline{\mathbb{R}}_+}(\theta)^\#(mx). \end{aligned}$$

Note that W_d is locally the composition of W with isomorphisms of d-cones, viz. \mathcal{V}_T and $(\psi_{P,T\overline{\mathbb{R}}_+})^{-1} \circ \psi_{Q,T\overline{\mathbb{R}}_+}$. So W_d is locally continuous and linear since W is; it also preserves \cup since $\psi_{Q,T\overline{\mathbb{R}}_+}(\theta)^\#$ does.

7.1. The lower powercone and conical powerdomain

Here we follow Section 4.1 in simplifying from the d-cone join-semilattice $\mathcal{H}\overline{\mathbb{R}}_+$ to the isomorphic one on $\overline{\mathbb{R}}_+$, yielding the functional representation

$$\Lambda : \mathcal{H}C \longrightarrow \overline{\mathbb{R}}_+^{C^*}$$

where

$$\Lambda_X(f) = \sup_{x \in X} f(x).$$

Predicates on a d-cone C are now linear continuous maps from C to $\overline{\mathbb{R}}_+$, and the predicate transformers from one d-cone D to another C are continuous maps $\Phi : D^* \rightarrow C^*$. The latter maps provide the morphisms of the category PT, which retains the same objects as before. The morphism part of the locally linear and continuous functor $W : \text{Cone}_{\mathcal{J}C} \rightarrow \text{PT}^{\text{op}}$ is given by the calculation

$$W_{C,D}(m)(f)(x) = f^\#(mx) = \sup_{y \in mx} f(y),$$

and, as \subseteq and \leq coincide in the join-semilattice case, all predicate transformers in the range of W are sublinear and, locally, form a sub-d-cone of the d-cone of all predicate transformers.

Let PT_l be the subcategory of PT restricted to the sublinear predicate transformers, the ‘healthy’ ones. It is easily verified that the sublinear predicate transformers form a sub-d-cone of the d-cone all predicate transformers.

Theorem 7.2. The functor W cuts down to a locally linear and continuous order-embedding

$$W : \text{CCone}_{\mathcal{J}C} \rightarrow \text{PT}_l^{\text{op}}.$$

It further cuts down to an equivalence of the full subcategories of reflexive continuous d-cones with continuous duals that is locally an isomorphism of d-cones.

Proof. The first part of the theorem follows from Propositions 6.1 and 7.1. For the second part, note that, by Theorem 6.2, if D is a reflexive continuous d-cone, the maps $m' : C \rightarrow \overline{\mathbb{R}}_+^{D^*}$ whose range consists of sublinear functionals are in bijective correspondence, via composition with Λ , with the state transformers $m : C \rightarrow \mathcal{H}\overline{\mathbb{R}}_+$. So W is locally a bijection and the conclusion follows by applying the first part of the theorem. \square

Turning to powerdomains, we now take \mathbf{K} to be \mathbf{CCone} and \mathbf{J} to be \mathbf{Dom} , recalling that \mathcal{H} preserves continuity. \mathbf{PT} then has continuous d-cones as objects, and \mathbf{PT}_d has domains as objects and the morphisms from P to Q are the continuous maps $\Phi : \mathcal{L}(P) \rightarrow \mathcal{L}(Q)$, simplifying from $\mathcal{H}(\overline{\mathbb{R}}_+^P$ to $\overline{\mathbb{R}}_+^P$. The functor $W : \mathbf{CCone}_{\mathcal{H}\mathcal{C}} \rightarrow \mathbf{PT}^{\text{op}}$ is then the restriction of the W considered immediately above. The functor $W_d : \mathbf{Dom}_{\mathcal{H}\mathcal{C}\mathcal{V}} \rightarrow \mathbf{PT}_d^{\text{op}}$ is locally linear, continuous and \vee -preserving; it is also an order-embedding, since, by Theorem 7.2, the same is true of W . Its action on morphisms $m : P \rightarrow \mathcal{H}\mathcal{V}Q$ is given by the calculation

$$W_d(m)(f)(x) = (\psi_{Q, \overline{\mathbb{R}}_+} f)^{\#}(mx) = \sup_{\mu \in mx} (\psi_{Q, \overline{\mathbb{R}}_+} f)(\mu) = \sup_{\mu \in mx} \int f d\mu.$$

Now let \mathbf{PT}_{dl} be the subcategory of \mathbf{PT}_d of the sublinear predicate transformers. The following corollary is an immediate consequence of Theorem 7.2, given the relationship between W and W_d , and the fact that W_d is locally a morphism of d-cone join-semilattices

Corollary 7.3. The sublinear predicate transformers form a sub-d-cone join semilattice of the predicate transformers, and the functor W_d cuts down to an equivalence of $\mathbf{Dom}_{\mathcal{H}\mathcal{C}\mathcal{V}}$ and \mathbf{PT}_{dl} that is locally an isomorphism of d-cone join-semilattices.

The functor W_d is, essentially, the greatest pre-expectation function wlp defined in the conclusion of Tix *et al.* (2008); the difference is that in the latter, only the action on endomorphisms of a coherent domain is considered. The corollary therefore characterises the greatest liberal pre-expectation function transformers associated to state transformers of the form $P \rightarrow \mathcal{H}\mathcal{V}P$, for P a coherent domain, and, indeed, more generally.

7.2. The upper powercone and conical powerdomain

Here we follow Section 4.2 in simplifying from d-cone meet-semilattice $\mathcal{S}\overline{\mathbb{R}}_+$ to the isomorphic one on $\overline{\mathbb{R}}_+$, yielding the functional representation

$$\Lambda : \mathcal{S}\mathcal{C} \longrightarrow \overline{\mathbb{R}}_+^{C^*}$$

where

$$\Lambda_X(f) = \inf_{x \in X} f(x).$$

Predicates, predicate transformers and the category \mathbf{PT} are then as in the previous (lower powercone) case, but restricted to continuous d-cones. The morphism part of the locally linear and continuous functor $W : \mathbf{CCone}_s \rightarrow \mathbf{PT}^{\text{op}}$ is given by the calculation

$$W_{C,D}(m)(f)(x) = f^{\#}(mx) = \inf_{y \in mx} f(y).$$

Using Propositions 6.4 and 7.1, we see that W is a local order-embedding. As \subseteq coincides with \geq in the meet-semilattice case, we further see that all predicate transformers in the range of W are superlinear, and that, locally, they form a sub-d-cone of the d-cone of all predicate transformers.

Let \mathbf{PT}_u be the subcategory of \mathbf{PT} restricted to the superlinear predicate transformers, the ‘healthy’ ones. It is easily verified that the superlinear predicate transformers form a sub-d-cone of the d-cone all predicate transformers.

Theorem 7.4. The functor W cuts down to a locally continuous linear order-embedding

$$W : \mathbb{C}\text{Cone}_S \rightarrow \text{PT}_u^{\text{op}}.$$

It further cuts down to an equivalence of the full subcategories of convenient d-cones that is locally an isomorphism of d-cones.

Proof. The proof is much as in the previous (lower powercone) case, but now using the remark above concerning its being an order-embedding and Theorem 6.5 for the local order isomorphism. \square

Turning to powerdomains, we leave K as $\mathbb{C}\text{Cone}$ and again take J to be Dom ; PT and PT_d are also as in the lower case. The functor $W_d : \text{Dom}_{S\mathcal{V}} \rightarrow \text{PT}_d^{\text{op}}$ is locally linear, continuous and \wedge -preserving; it is also an order-embedding, since, by the above remarks, the same is true of W . Its action on morphisms $m : P \rightarrow S\mathcal{V}Q$ is given by the calculation

$$W_d(m)(f)(x) = (\psi_{Q, \overline{\mathbb{R}}_+} f)^{\#}(mx) = \inf_{\mu \in mx} (\psi_{Q, \overline{\mathbb{R}}_+} f)(\mu) = \inf_{\mu \in mx} \int f d\mu.$$

Now let PT_{du} be the subcategory of PT_d of the superlinear predicate transformers. The following corollary is an immediate consequence of the fact that W_d is locally a morphism of d-cone meet-semilattices together with Theorem 7.4.

Corollary 7.5. The superlinear predicate transformers form a sub-d-cone meet-semilattice of the predicate transformers, and the functor W_d cuts down to an equivalence of the full subcategories of $\text{Dom}_{S\mathcal{V}}$ and PT_{du} of coherent domains that is locally an isomorphism of d-cone meet-semilattices.

The functor W_d is, essentially, the weakest pre-expectation function w_p defined in the conclusion of Tix *et al.* (2008). The corollary therefore characterises the weakest pre-expectation function transformers associated to state transformers of the form $P \rightarrow S\mathcal{V}P$, for P a coherent domain, and, indeed, more generally.

7.3. The convex powercone and the order-convex conical powerdomain

We follow Section 4.3 and employ the standard representation

$$\Lambda : \mathcal{P}C \longrightarrow \mathcal{P}\overline{\mathbb{R}}_+^{[C, \mathcal{P}\overline{\mathbb{R}}_+]}$$

where

$$\Lambda_X(f) = [\inf_{x \in X} \underline{f}(x), \sup_{x \in X} \overline{f}(x)].$$

Predicates and PT are then as in the general approach, that is, predicates on C are elements of $[C, \mathcal{P}\overline{\mathbb{R}}_+]$, PT has as objects the coherent continuous cones and the predicate transformers from C to D are continuous maps $\Phi : [C, \mathcal{P}\overline{\mathbb{R}}_+] \rightarrow [D, \mathcal{P}\overline{\mathbb{R}}_+]$.

The morphism part of the locally linear and continuous functor $W : \mathbb{C}\text{Cone}^c_{\mathcal{P}} \longrightarrow \text{PT}^{\text{op}}$ is given by the calculation

$$W_{C,D}(m)(f)(x) = \Lambda_{mx}(f) = [\inf_{y \in mx} \underline{f}(y), \sup_{y \in mx} \overline{f}(y)],$$

and we know that, locally, the predicate transformers in the range of W are \subseteq -monotonic and \subseteq -sublinear. Using Propositions 6.7 and 7.1, we also see that W is a local order-embedding.

Following Section 4.4, it is natural to also consider a different kind of predicate transformer, *viz.* maps $\Phi' : C^* \rightarrow [D, \mathcal{P}\overline{\mathbb{R}}_+]$, with C and D coherent continuous d-cones as above. We call these ‘double’ predicate transformers, for reasons that will become clear shortly. First we introduce under- and over-lining conventions for functions of the form $f : C \rightarrow [D, \mathcal{P}\overline{\mathbb{R}}_+]$. We write \underline{f} for the function $x \mapsto \underline{f(x)} : C \rightarrow D^*$; and \overline{f} is defined similarly. This gives a bijection between functions $f : C \rightarrow [D, \mathcal{P}\overline{\mathbb{R}}_+]$ and pairs of functions $g, h : C \rightarrow D^*$ with $g \leq h$; the inverse of the bijection sends g, h to $[g, h] =_{\text{def}} x \mapsto [gx, hx]$.

So, in particular, double predicate transformers $\Phi' : C^* \rightarrow [D, \mathcal{P}\overline{\mathbb{R}}_+]$ correspond to pairs of predicate transformers $\underline{\Phi'}, \overline{\Phi'} : C^* \rightarrow D^*$ of the type considered in the lower and upper cases in the previous two subsections. With the aid of the bijection, it is also easy to see that they form a category, PT' : the identity on an object C is $[\text{id}_C, \text{id}_C]$ and composition is defined by setting $[\Psi', \overline{\Psi'}] \circ [\Phi', \overline{\Phi'}] = [\Psi' \circ \Phi', \overline{\Psi'} \circ \overline{\Phi'}]$.

Now let PT_m be the subcategory of PT of the \subseteq -monotonic predicate transformers.

Lemma 7.6.

(1) Let $\Phi : [C, \mathcal{P}\overline{\mathbb{R}}_+] \rightarrow [D, \mathcal{P}\overline{\mathbb{R}}_+]$ be \subseteq -monotonic. Then we have

$$\Phi[g, h] = [\Phi[g, g], \overline{\Phi[h, h]}].$$

(2) There is an isomorphism of categories $R : \text{PT}_m \cong \text{PT}'$, which is locally an isomorphism of d-cones. It acts as the identity on objects and on predicate transformers Φ from C to D , $R_{C,D}(\Phi) = \Phi \circ \Delta_C$. The inverse of $R_{C,D}$ is given by $R_{C,D}^{-1}(\Phi')[g, h] = [\underline{\Phi'}(g), \overline{\Phi'}(h)]$.

Proof.

(1) A predicate transformer $\Phi : [C, \mathcal{P}\overline{\mathbb{R}}_+] \rightarrow [D, \mathcal{P}\overline{\mathbb{R}}_+]$ is \subseteq -monotonic if and only if $t^{-1}(\Phi)(d)$ is for every d in D . Using this observation, we have

$$\begin{aligned} \underline{\Phi[g, h]}(d) &= t^{-1}(\Phi)(d)[g, h] \\ &= t^{-1}(\Phi)(d)[g, g] \text{ (by Lemma 4.5.1)} \\ &= \underline{\Phi[g, g]}(d). \end{aligned}$$

So $\underline{\Phi[g, h]} = \underline{\Phi[g, g]}$, and similarly for $\overline{\Phi[h, h]}$.

(2) We first check that R is a functor. Note that $R(\Phi)(f) = \Phi[f, f]$. So R evidently preserves the identity. To check it preserves composition, suppose Φ is a predicate transformer from C to D and Ψ is one from D to E . Then we have

$$\begin{aligned} \underline{R(\Psi) \circ R(\Phi)}(f) &= \underline{R\Psi(R\Phi)(f)} \\ &= \underline{\Psi[\Phi[f, f], \Phi[f, f]]} \\ &= \underline{\Psi(\Phi[f, f])} \text{ (by Part 1)} \\ &= \underline{R(\Psi \circ \Phi)(f)}, \end{aligned}$$

and similarly for $\overline{R(\Psi) \circ R(\Phi)}$. The rest of the proof follows from the second part of Lemma 4.5. □

Composing W with R (all predicate transformers in the range of W are \subseteq -monotonic), we obtain a functor $W' : \mathbb{C}Cone^c \rightarrow \mathbb{P}T'$. This acts as the identity on objects and acts on morphisms as follows:

$$W'_{C,D}(m)(f)(x) = [\inf_{y \in mx} f(y), \sup_{y \in mx} f(y)].$$

Thus W' combines the upper and lower predicate transformers. More precisely, we have

$$\underline{W'_{C,D}(m)} = W_{C,D}(\uparrow_D \circ m)$$

and

$$\overline{W'_{C,D}(m)} = W_{C,D}(\downarrow_D \circ m).$$

Turning to healthiness conditions, we consider the property for a \subseteq -monotonic predicate transformer Φ from C to D to be canonically \subseteq -sublinear, which is \subseteq -sublinearity of Φ together with the condition

$$\underline{\Phi}(f + g) \leq \underline{\Phi}(f) + \overline{\Phi}(g) \leq \overline{\Phi}(f + g). \tag{*}$$

The corresponding healthiness conditions on a double predicate transformer Φ' from C to D are superlinearity and sublinearity of the functionals $\underline{\Phi}'$ and $\overline{\Phi}'$, respectively, together with the condition

$$\underline{\Phi}'(f + g) \leq \underline{\Phi}'(f) + \overline{\Phi}'(g) \leq \overline{\Phi}'(f + g). \tag{*}$$

Using Lemma 7.6, it is straightforward to show that Φ satisfies its healthiness conditions, if and only if $\Phi' = R(\Phi)$ satisfies the corresponding properties for it. Furthermore, the healthy (double) predicate transformers, of either kind, form a subcategory of $\mathbb{P}T$ or $\mathbb{P}T'$, respectively. It is evident that the identity predicate transformer is healthy, and therefore, by the above remarks, so too is the identity double predicate transformer. Similarly, for composition it suffices to check the case of the double predicate transformer. So let Ψ' , Φ' be double predicate transformers from D to E and from C to D , respectively, and calculate

$$\begin{aligned} \underline{\Psi' \circ \Phi'}(f + g) &= \underline{\Psi'} \circ \underline{\Phi'}(f + g) \\ &\leq \underline{\Psi'}(\underline{\Phi'}(f) + \overline{\Phi'}(g)) && \text{(as } \Phi' \text{ is healthy)} \\ &\leq \underline{\Psi'}(\underline{\Phi'}(f)) + \overline{\Psi'}(\overline{\Phi'}(g)) && \text{(as } \Psi' \text{ is healthy)} \\ &= \underline{\Psi' \circ \Phi'}(f) + \overline{\Psi' \circ \Phi'}(g), \end{aligned}$$

and similarly for the second inequality.

It is straightforward to verify that the healthy predicate transformers from C to D form a sub-d-cone of all the predicate transformers from C to D , and that the same holds for the healthy double predicate transformers.

Theorem 7.7. The functor $W : \mathbb{C}Cone^c_{\mathcal{P}} \rightarrow \mathbb{P}T^{\text{op}}$ (respectively, $W' : \mathbb{C}Cone^c_{\mathcal{P}} \rightarrow \mathbb{P}T'^{\text{op}}$) cuts down to a locally linear and continuous equivalence of categories of the full subcategory of $\mathbb{C}Cone^c_{\mathcal{P}}$ of the convenient d-cones and the subcategory of $\mathbb{P}T$ (respectively, $\mathbb{P}T'$) of the convenient d-cones and the healthy predicate transformers (respectively, the healthy double predicate transformers).

Proof. First, by Theorem 6.8, Λ cuts down to a d-cone isomorphism between \mathcal{PD} and those functionals in $[D, \mathcal{P}\overline{\mathbb{R}}_+]^{\mathcal{P}\overline{\mathbb{R}}_+}$ that are canonically \subseteq -sublinear. Second, for any m' in $[C, [D, \mathcal{P}\overline{\mathbb{R}}_+]^{\mathcal{P}\overline{\mathbb{R}}_+}]$, we have $m'(x)$ is canonically \subseteq -sublinear for every x in C if and only if $t(m')$ is canonically \subseteq -sublinear on predicate transformers, so t cuts down to a d-cone isomorphism of the sub-d-cone of such an m' and the sub-d-cone of the healthy predicate transformers. The first assertion is then an immediate consequence.

The second assertion follows from the first as, by Lemma 7.6, R is locally an isomorphism of d-cones. □

Turning to powerdomains, we keep K , and so PT , as they are, and take J to be Dom^c . Then PT_d has coherent domains as objects, and the morphisms from P to Q are the continuous maps $\Phi : \mathcal{P}\overline{\mathbb{R}}_+^P \rightarrow \mathcal{P}\overline{\mathbb{R}}_+^Q$. The functor $W_d : Dom_{\mathcal{P}\mathcal{V}}^c \rightarrow PT_d^{op}$ is locally linear, continuous and \cup -preserving; it is also a local order-embedding, since W is.

We introduce under- and over-lining notation for functions of the form $f : P \rightarrow \mathcal{P}\overline{\mathbb{R}}_+^Q$ in the evident way, as well as the notation $[g, h]$ for pairs of functions $g, h : P \rightarrow \mathcal{L}(Q)$ with $g \leq h$, and the usual properties carry over. Using the naturality of ψ , we can show $\overline{\psi_{P, \mathcal{P}\overline{\mathbb{R}}_+}(f)} = \psi_{P, \overline{\mathbb{R}}_+}(\underline{f})$ and $\underline{\psi_{P, \mathcal{P}\overline{\mathbb{R}}_+}(f)} = \psi_{P, \overline{\mathbb{R}}_+}(\overline{f})$ for any f in $\mathcal{P}\overline{\mathbb{R}}_+^P$ and also that $\psi_{P, \mathcal{P}\overline{\mathbb{R}}_+}([g, h]) = [\psi_{P, \overline{\mathbb{R}}_+}(g), \psi_{P, \overline{\mathbb{R}}_+}(h)]$ for any g, h in $\mathcal{L}(P)$ with $g \leq h$.

With that, we can calculate the action of W_d on morphisms:

$$\begin{aligned} W_d(m)(f)(x) &= \Lambda_{mx}(\psi_{Q, \mathcal{P}\overline{\mathbb{R}}_+} f) \\ &= [\inf_{\mu \in mx} \underline{\psi_{Q, \mathcal{P}\overline{\mathbb{R}}_+} f}(\mu), \sup_{\mu \in mx} \overline{\psi_{Q, \mathcal{P}\overline{\mathbb{R}}_+} f}(\mu)] \\ &= [\inf_{\mu \in mx} (\psi_{Q, \overline{\mathbb{R}}_+}(\underline{f}))(\mu), \sup_{\mu \in mx} (\psi_{Q, \overline{\mathbb{R}}_+}(\overline{f}))(\mu)] \\ &= [\inf_{\mu \in mx} \int \underline{f} d\mu, \sup_{\mu \in mx} \int \overline{f} d\mu]. \end{aligned}$$

Double predicate transformers have the form $\Phi' : \mathcal{L}(P) \rightarrow \mathcal{P}\overline{\mathbb{R}}_+^Q$, and they form a category PT'_d much as before: the identity on P is $[id_P, id_P]$ and composition is defined by setting $[\Psi', \overline{\Psi'}] \circ [\Phi', \overline{\Phi'}] = [\Psi' \circ \Phi', \overline{\Psi' \circ \Phi'}]$. We can define a useful functor, which is locally an isomorphism of d-cones, $\mathcal{V}'_p : PT'_d \rightarrow PT'$ by putting

$$\mathcal{V}'_p(P) = \mathcal{V}(P)$$

on objects, and

$$\begin{aligned} \mathcal{V}'_p(\Phi') &= \psi_{Q, \mathcal{P}\overline{\mathbb{R}}_+} \circ \Phi' \circ \psi_{P, \overline{\mathbb{R}}_+}^{-1} \\ &= [\psi_{Q, \overline{\mathbb{R}}_+} \circ \underline{\Phi'} \circ (\psi_{P, \overline{\mathbb{R}}_+})^{-1}, \psi_{Q, \overline{\mathbb{R}}_+} \circ \overline{\Phi'} \circ (\psi_{P, \overline{\mathbb{R}}_+})^{-1}] \end{aligned}$$

on morphisms, with the last equation making it clear why \mathcal{V}'_p is a functor.

We write PT_{dm} to be the subcategory of PT_d of the monotonic predicate transformers. Note that $\mathcal{V}'_p(\Phi)$ is monotonic if and only if Φ is, so \mathcal{V}'_p cuts down to a functor from PT_{dm} to PT_m .

Lemma 7.8. There is an isomorphism of categories $R_d : PT_{md} \cong PT'_d$, which is locally an isomorphism of d-cone semilattices. It acts as the identity on objects and on predicate transformers Φ from P to Q , we have $R_d(\Phi) = (f \mapsto \Phi[f, f])$.

Proof. We define the action of R_d on morphisms $\Phi : \mathcal{P}\overline{\mathbb{R}}_+^P \rightarrow \mathcal{P}\overline{\mathbb{R}}_+^Q$ by

$$R_d(\Phi) = (\mathcal{V}'_p)^{-1}(R \circ \mathcal{V}_p(\Phi)).$$

As \mathcal{V}_p , R and \mathcal{V}'_p are locally d-cone isomorphisms, using Lemma 7.6, so too is R_d .

We now have

$$\begin{aligned} R_d(\Phi)(f) &= \psi_{Q, \mathcal{P}\overline{\mathbb{R}}_+}^{-1} (R(\psi_{Q, \mathcal{P}\overline{\mathbb{R}}_+} \circ \Phi \circ \psi_{P, \mathcal{P}\overline{\mathbb{R}}_+}^{-1})(\psi_{P, \overline{\mathbb{R}}_+}(f))) \\ &= \psi_{Q, \mathcal{P}\overline{\mathbb{R}}_+}^{-1} (\psi_{Q, \mathcal{P}\overline{\mathbb{R}}_+} \circ \Phi \circ \psi_{P, \mathcal{P}\overline{\mathbb{R}}_+}^{-1}([\psi_{P, \overline{\mathbb{R}}_+}(f), \psi_{P, \overline{\mathbb{R}}_+}(f)])) \\ &= \Phi \circ \psi_{P, \mathcal{P}\overline{\mathbb{R}}_+}^{-1}(\psi_{P, \mathcal{P}\overline{\mathbb{R}}_+}[f, f]) \\ &= \Phi[f, f]. \end{aligned}$$

Using this formula for $R_d(\Phi)$, we see that R_d preserves unions. □

Clearly, W_d cuts down to a functor to PT_{dm} , so, composing with R_d , we obtain a functor $W'_d : \text{Dom}^c \rightarrow \text{PT}'_d$, which is locally a morphism of d-cone semilattices. This acts as the identity on objects and on morphisms:

$$W_d(m)(f)(x) = \left[\inf_{\mu \in mx} \int f d\mu, \sup_{\mu \in mx} \int f d\mu \right].$$

Healthy predicate transformers and healthy double predicate transformers are defined analogously to before. It is straightforward to calculate that both kinds of predicate transformer are closed under (pointwise) unions.

We next check that a (double) predicate transformer is healthy if and only if its image under \mathcal{V}_p (respectively, \mathcal{V}'_p) is healthy. So both kinds of healthy predicate transformers form subcategories of their ambient categories, and, locally, form sub-d-cone semilattices of the d-cone semilattices of all predicate transformers. We also have that a predicate transformer is healthy if and only if its image under R_d is healthy.

Corollary 7.9. The functor $W_d : \text{Dom}_{\mathcal{P}\mathcal{V}}^c \rightarrow \text{PT}_d^{\text{op}}$ (respectively, $W'_d : \text{Dom}_{\mathcal{P}\mathcal{V}}^c \rightarrow \text{PT}'_d^{\text{op}}$) cuts down to an equivalence of categories, which is locally a d-cone semilattice isomorphism, of $\text{Dom}_{\mathcal{P}\mathcal{V}}^c$ and the subcategory of PT_d (respectively, PT'_d) of the coherent domains and the healthy predicate transformers (respectively, the healthy double predicate transformers).

Proof. We know that W_d is locally a morphism of d-cone semilattices and an order-embedding. Further, locally, its range consists exactly of the healthy predicate transformers because:

- $\mathcal{V}_p \circ W_d = W \circ \mathcal{V}_p$;
- by Theorem 7.7, the range of W consists exactly of the healthy predicate transformers; and
- \mathcal{V}_p preserves and reflects healthiness.

This proves the first of the assertions. The second follows from Lemma 7.8 and the fact that R_d also preserves and reflects healthiness. □

The functors W'_d and W_d are, essentially, the two forms of the weakest pre-bi-expectation function wpb defined in the conclusion of Tix *et al.* (2008). The corollary therefore

characterises the weakest pre-bi-expectation function transformers associated to state transformers of the form $P \rightarrow \mathcal{PVP}$ for P a coherent domain, and, indeed, more generally.

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