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A canonical thickening of $\mathbb{Q}$ and the entropy of α-continued fraction transformations

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Abstract. We construct a countable family of open intervals contained in (0,1] whose endpoints are quadratic surds and such that their union is a full measure set. We then show that these intervals are precisely the monotonicity intervals of the entropy of α-continued fractions, thus proving a conjecture of Nakada and Natsui.

1. *Introduction*

In many areas of mathematics, the space of parameters of a family of mathematical objects is itself an object of the same type. A well-known example of this phenomenon in dynamics is the Mandelbrot set, whose local geometry reflects the geometry of the Julia set of the quadratic polynomial corresponding to a given point.

The goal of this paper is to analyse the parameter space of a family of one-dimensional dynamical systems known as α-continued fraction transformations by means of regular continued fraction (c.f.) expansions. As a consequence, we will be able to determine the intervals in parameter space where a stability condition holds, which also correspond to monotonicity intervals of the entropy function.

For each $\alpha \in (0, 1]$, the α-continued fraction transformation $T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]$ is defined as

$$T_\alpha(x) = \begin{cases} \dfrac{1}{|x|} - \left\lfloor \dfrac{1}{|x|} + 1 - \alpha \right\rfloor & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Notice that T_1 is the usual Gauss map, and all maps in the family are expanding interval maps with infinitely many branches. Moreover, every T_α admits a unique invariant measure

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absolutely continuous with respect to Lebesgue, hence it makes sense to study the metric entropy $h(T_\alpha)$ with respect to this invariant measure.

The family $\{T_\alpha\}$ was introduced by Nakada [**Nak81**], who gave an explicit formula for the entropy function $\alpha \mapsto h(T_\alpha)$ for $\alpha \in [1/2, 1]$ and showed that a phase transition occurs at $\alpha = (\sqrt{5} - 1)/2$.

Several authors have investigated the behaviour of entropy when the parameter α ranges in the interval $[0, 1/2]$: Cassa [**Cas95**] proved that the entropy is constant on $[\sqrt{2} - 1, 1/2]$ (see also the appendix of [**CMM99**]); the entropy on $[0, 1/2]$ was generally believed to be a continuous, weakly monotone function, converging to 0 as $\alpha \to 0$ ([**Cas95**] or [**BDV02**, p. 284]). In [**LM08**], however, Luzzi and Marmi produced strong numerical evidence that the entropy is not monotone; this fact was indeed rigourously confirmed by Nakada and Natsui [**NN08**], who proved that in every neighbourhood of the origin there are intervals where the entropy is strictly decreasing, as well as others where it is constant, and still others where it is strictly increasing.

The key to their proof was showing that the entropy is locally monotone on intervals I of parameters which satisfy the matching condition

$$\text{there exists } N, M \in \mathbb{N}_+ : T_\alpha^N(\alpha) = T_\alpha^M(\alpha - 1) \quad \text{for all } \alpha \in I \tag{1}$$

as well as some other technical conditions. Such intervals will be called *matching intervals*, and their union will be referred to as the *matching set*. Nakada and Natsui not only exhibited families of intervals where these conditions are met, but also made the following conjecture.

CONJECTURE 1.1. *The matching set has full measure in* $(0, 1]$ *(hence it is dense).*

Empirical evidence for this conjecture was obtained in [**CMPT10**]: this numerical study also revealed that the complement of the matching set, where phase transitions occur, displays a fairly complicated fractal structure.

The goal of the present paper is to prove rigorously the existence of the structures empirically observed there, thus proving Conjecture 1.1. The main tool to analyse the matching set will be regular c.f. expansions; in fact, this matching set can be perfectly described without even mentioning the dynamics of α-transformations. Let us briefly explain why.

It is well known that any rational value $r \in \mathbb{Q}$ can be expressed as a finite c.f. expansion of either even or odd length. This fact, usually perceived as a nuisance, will give us the chance to perform the following 'natural' construction.

(1) For any rational number $r \in \mathbb{Q} \cap (0, 1]$ we consider its two regular c.f. expansions, namely,

$$r = [0; a_1, \dots, a_n] = [0; a_1, \dots, a_n - 1, 1] \quad \text{where } a_n \geq 2.$$

We will associate to any such r the open interval I_r whose endpoints are the quadratic surds

$$[0; \overline{a_1, \dots, a_n}] \quad [0; \overline{a_1, \dots, a_n - 1, 1}].$$

Such an I_r will be called the *quadratic interval* generated by r.

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(2) We will consider the union of all quadratic intervals

$$\mathcal{M} := \bigcup_{r \in \mathbb{Q} \cap (0,1]} I_r.$$

The object of §2 will be to understand the structure of the open dense set $\mathcal{M}$, which can be summarized in the following theorem.

THEOREM 1.2. *The set $\mathcal{M}$ has full Lebesgue measure in* $(0, 1]$*, but its complement has Hausdorff dimension one.*

Although the family of quadratic intervals $\{I_r\}_{r \in \mathbb{Q}}$ will have substantial overlapping, there is a subfamily that covers $\mathcal{M}$ exactly. More precisely, a quadratic interval I_r will be called *maximal* if it is not properly contained in any other quadratic interval. It turns out that every quadratic interval is contained in some maximal one, and distinct maximal quadratic intervals do not intersect (Lemma 2.6): thus $\mathcal{M}$ is the disjoint union of this collection of maximal intervals. This suggests that $(0, 1] \setminus \mathcal{M}$ should have a Cantor-like structure; this is only partially true because $(0, 1] \setminus \mathcal{M}$ is not perfect. Indeed, the presence of isolated points is a consequence of the *period-doubling* phenomenon (see §3.3): if $r := [0; a_1, \ldots a_n] \in \mathbb{Q}$ with n odd and I_r is a maximal quadratic interval, then $r' := [0; a_1, \ldots, a_n, a_1, \ldots a_n] < r$ generates $I_{r'}$ which is maximal as well, and the quadratic surd $\alpha := [0; \overline{a_1, \ldots, a_n}]$ is a common endpoint, which is obviously not contained in any quadratic interval.

In the second part of the paper (§3) we prove that that this set $\mathcal{M}$ is closely connected to the matching intervals. More precisely, we prove the following theorem.

THEOREM 1.3. *Let $a \in \mathbb{Q} \cap (0, 1]$ such that I_a is maximal. Then there exist positive integers N, M such that*

$$T_\alpha^N(\alpha) = T_\alpha^M(\alpha - 1) \quad \text{for all } \alpha \in I_a.$$

Moreover, the entropy function $\alpha \mapsto h(T_\alpha)$ is monotone on I_a.

The proof of the theorem relies on the fact that an *algebraic matching condition* stronger than (1) holds everywhere on $\mathcal{M}$; by Theorem 1.2, this condition holds for almost every parameter.

Moreover, the set defined by the algebraic matching condition contains the matching set defined by Nakada and Natsui and the difference between them is countable (see Appendix A), hence they have the same measure and Conjecture 1.1 follows.

Our method also gives us an explicit control over the combinatorics of matchings: given any rational number, we are able to determine which maximal interval it belongs to and the *matching exponents* (N, M), hence the local behaviour of entropy (constant, increasing or decreasing). Conversely, one can use such knowledge to produce families of matching intervals with prescribed properties.

Finally, §4 contains a few technical tools which we use throughout the paper, including a criterion for comparing purely periodic quadratic surds (string Lemma 2.12) and an explicit characterization of either of the finite c.f. expansions which generate a maximal quadratic interval (Lemma 2.13).

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It is worth noting that the phenomenon we describe is strongly reminiscent of the theory of circle maps (see [**Sch05**, Ch. 7.2]): in that case, around each rational rotation number, in the parameter space there is a region ('Arnold tongue') where the dynamics is still periodic ('mode-locking'), in such a way that on the critical line the complement of the union of all Arnold tongues has measure zero (even though its Hausdorff dimension is strictly smaller than one, differently from our case [**GS96**]).

Recently Katok and Ugarcovici have studied another family of transformations, called (a, b)-continued fractions, which seem to share various features with the transformation T_α (see [**KU10**]): it would be worth investigating more closely the connection between these systems in order to see whether the two different approaches can lead to a deeper understanding of both.

2. *Thickening* $\mathbb{Q}$

Let $S = (s_1, \ldots, s_n)$ be a finite string of positive integers: we will use the notation

$$[0; S] := [0; s_1, \ldots, s_n] = \cfrac{1}{s_1 + \cfrac{1}{\ddots + \frac{1}{s_n}}}.$$

Moreover, $\overline{S}$ will be the periodic infinite string $SSS\ldots$ and $[0; \overline{S}]$ the quadratic surd with purely periodic continued fraction $[0; \overline{s_1, \ldots, s_n}]$. The symbol $|S|$ will denote the length of the string S. We will denote the denominator of the rational number r as $\mathrm{den}(r)$.

2.1. *Pseudocentres.*

Let us start out by defining a useful tool in our analysis of intervals defined by continued fractions.

LEMMA 2.1. *Let $J = (\alpha, \beta)$, $\alpha, \beta \in \mathbb{R}$, $|\alpha - \beta| < 1$. Then there exists a unique rational $p/q \in J$ such that $q = \min\{q' \geq 1 : p'/q' \in J\}$.*

Proof. Let $d := \min\{q \geq 1 : p/q \in J\}$. If $d = 1$ we are done. Let $d > 1$, and assume by contradiction that c/d and $(c + 1)/d$ both belong to J. Then there exists $k \in \mathbb{Z}$ such that $k/(d - 1) < c/d < (c + 1)/d < (k + 1)/(d - 1)$, hence $cd - c - 1 < kd < cd - c$, which is a contradiction since kd is an integer. □

Definition 2.1. The number p/q which satisfies the properties of the previous lemma will be called the *pseudocentre* of J.

LEMMA 2.2. *Let $\alpha, \beta \in (0, 1)$ be two irrational numbers with c.f. expansions $\beta = [0; S, b_0, b_1, b_2, \ldots]$ and $\alpha = [0; S, a_0, a_1, a_2, \ldots]$, where S stands for a finite string of positive integers. Assume that $b_0 > a_0$. Then the pseudocentre of the interval J with endpoints α and β is*

$$r = [0; S, a_0 + 1] (= [0; S, a_0, 1]).$$

Proof. Suppose that there exists $s \in \mathbb{Q} \cap J$ with $\mathrm{den}(s) < \mathrm{den}(r)$. Since $s \in J$, then $s = [0; S, s_0, s_1, \ldots, s_k]$ with $a_0 \leq s_0 \leq b_0$ and $k \geq 0$. The choice $s_0 \geq a_0 + 1$ gives rise to $\mathrm{den}(s) \geq \mathrm{den}(r)$, so $s_0 = a_0$. On the other hand, $[0; S, a_0]$ does not belong to the interval, so $k \geq 1$ and $s_1 \geq 1$, still implying that $\mathrm{den}(s) \geq \mathrm{den}(r)$. □

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2.2. *Quadratic intervals.*

Definition 2.2. Let $0 < a < 1$ be a rational number with c.f. expansion

$$a = [0; a_1, \ldots, a_N] = [0; a_1, \ldots, a_N - 1, 1] \quad \text{where } a_N \geq 2.$$

We define the *quadratic interval* I_a associated to a to be the open interval with endpoints

$$[0; \overline{a_1, \ldots, a_{N-1}, a_N}] \quad \text{and} \quad [0; \overline{a_1, \ldots, a_{N-1}, a_N - 1, 1}]. \tag{2}$$

Moreover, we define $I_1 := ((\sqrt{5} - 1)/2, 1]$ (recall that $(\sqrt{5} - 1)/2 = [0; \overline{1}]$).

Note that the ordering of the endpoints in (2) depends on the parity of N: given $a \in \mathbb{Q}$, we will denote by A^+ and A^- the two strings of positive integers which represent a as a continued fraction, with the convention that A^+ is the string of *even* length and A^- the string of *odd* length, so that

$$I_a = ([0; \overline{A^-}], [0; \overline{A^+}]), \quad a = [0; A^+] = [0; A^-].$$

Example. If $a = 1/3 = [0; 3] = [0; 2, 1]$, $[0; \overline{A^+}] = [0; \overline{2, 1}]$, $[0; \overline{A^-}] = [0; \overline{3}]$, $I_a = ((\sqrt{13} - 3)/2, (\sqrt{3} - 1)/2)$.

Note that a is the pseudocentre of I_a, hence $I_a = I_{a'} \Leftrightarrow a = a'$.

LEMMA 2.3.
(1) *If* $\xi \in \overline{I}_a$, *then* a *is a convergent to* ξ.
(2) *If* $I_a \cap I_b \neq \emptyset$, *then either* a *is a convergent to* b *or* b *is a convergent to* a.
(3) *If* $I_a \subsetneq I_b$ *then* b *is convergent to* a, *hence* $\mathrm{den}(b) < \mathrm{den}(a)$.

Proof. (1) Since $\xi \in I_a$, either $\xi = [0; a_1, \ldots, a_N, \ldots]$ or $\xi = [0; a_1, \ldots, a_N - 1, \ldots]$. In the first case the claim holds; in the second case one has to notice that neither $[0; a_1, \ldots, a_N - 1]$ nor all elements of the form $[0; a_1, \ldots, a_N - 1, k, \ldots]$ with $k \geq 2$ belong to I_a, so $k = 1$ and a is a convergent of ξ.

(2) Fix $\xi \in I_a \cap I_b$. By the previous point, both a and b are convergents of ξ, hence the rational with the shortest expansion is a convergent of the other.

(3) From (1) since $a \in I_a \subseteq I_b$. □

Definition 2.3. A quadratic interval I_a is *maximal* if it is not properly contained in any I_b with $b \in \mathbb{Q} \cap (0, 1]$.

The interest in maximal quadratic intervals lies in the following proposition.

PROPOSITION 2.4. *Every quadratic interval* I_a *is contained in a unique maximal quadratic interval.*

A good way to visualize the family of quadratic intervals is to plot, for any rational a, the geodesic γ_a on the hyperbolic upper half plane with the same endpoints as I_a, as in Figure 1; one can see the maximal intervals corresponding to the highest geodesics, in such a way that every γ_a has some maximal geodesic (possibly itself) above it and no two maximal γ_a intersect.

The proof of Proposition 2.4 will be given in two lemmas.

LEMMA 2.5. *Every quadratic interval* I_a *is contained in some maximal quadratic interval.*

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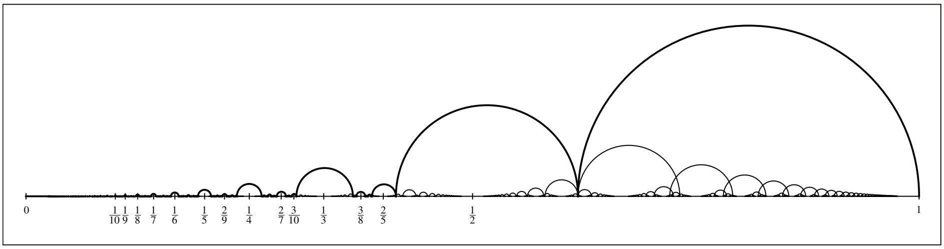

FIGURE 1. Quadratic intervals. Every I_a is represented by a geodesic landing on its endpoints. Maximal intervals are shown as thick curves. The rational values displayed are the pseudocentres of maximal intervals with denominator less than 10.

Proof. If I_a were not contained in any maximal interval, then there would exist an infinite chain $I_a \subsetneq I_{a_1} \subsetneq I_{a_2} \subsetneq \cdots$ of proper inclusions, hence by the lemma every a_i is a convergent of a, but rational numbers can only have a finite number of convergents. □

LEMMA 2.6. *If I_a is maximal then, for all $a' \in \mathbb{Q} \cap (0, 1)$,*

$$I_a \cap I_{a'} \neq \emptyset \Rightarrow I_{a'} \subset I_a,$$

and equality holds if and only if $a = a'$. In particular, distinct maximal intervals do not intersect.

Proof. We need the following lemma, which we will prove in §4. □

LEMMA 2.7. *If $I_a \cap I_b \neq \emptyset$, $I_a \backslash I_b \neq \emptyset$ and $I_b \backslash I_a \neq \emptyset$, then either I_a or I_b is not maximal.*

Let I_{a_0} be the maximal interval which contains $I_{a'}$. Since $I_a \cap I_{a_0} \neq \emptyset$, by Lemma 2.7 either $I_a \subseteq I_{a_0}$ or $I_{a_0} \subseteq I_a$, hence by maximality $I_a = I_{a_0}$ and $I_{a'} \subseteq I_a$. Since a is the pseudocentre of I_a, $I_a = I_{a'} \Rightarrow a = a'$. □

2.3. *Hausdorff dimension.* In this section we prove Theorem 1.2, which states that the exceptional set $\mathcal{E} :=]0, 1] \backslash \mathcal{M}$ has zero Lebesgue measure but Hausdorff dimension equal to one. The key tool of the proof is the following lemma, which establishes a connection between $\mathcal{E}$ and numbers of bounded type.

LEMMA 2.8.
(i) *Let $\xi \in \mathcal{E} = (0, 1] \backslash \mathcal{M}$. Then ξ is irrational and $\xi = [0; a_1, \ldots, a_n, \ldots]$ with $a_j \leq a_1$ for all $j \in \mathbb{N}_+$.*
(ii) *Let $\xi = [0; a_1, \ldots, a_n, \ldots]$ be an irrational number such that $a_k \leq a_1 - 1$ for all $k \geq 2$. Then ξ does not belong to any I_a for any $a \in \mathbb{Q} \cap (0, 1]$.*

Proof. Since $\xi \notin \mathcal{M}$ then $\xi \notin \mathbb{Q}$. If ξ has the infinite c.f. expansion $\xi = [0; a_1, \ldots, a_n, \ldots]$ with $a_k > a_1$ for some $k \in \mathbb{N}_+$ then x lies between $r := [0; a_1, \ldots, a_{k-1}]$ and $\alpha := [0; \overline{a_1, \ldots, a_{k-1}}]$; therefore $x \in I_r \subset \mathcal{M}$. Let $a = [0; A^+] = [0; A^-]$, so that $I_a = ([0; \overline{A^-}], [0; \overline{A^+}])$. If $\xi \in I_a$, by Lemma 2.3 a is a convergent of ξ, so either

$$\xi = [0; A^+, \ldots] \quad \text{or} \quad \xi = [0; A^-, \ldots].$$

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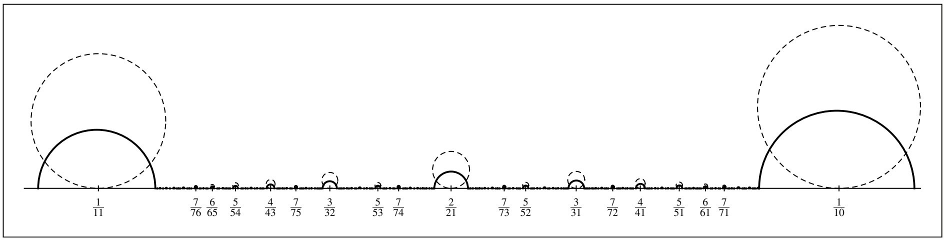

FIGURE 2. Maximal intervals (thick solid curves) versus horoballs (dashed) in the window $\left(\frac{1}{11}, \frac{1}{10}\right)$.

In the first case $\xi = [0;\, A^+, s, \ldots]$ with $s < a_1$, so $\xi > [0;\, \overline{A^+}] = [0;\, A^+, a_1, \ldots]$; in the second, $\xi = [0;\, A^-, s, \ldots]$ with $s < a_1$ and therefore $\xi < [0;\, \overline{A^-}] = [0;\, A^-, a_1, \ldots]$. □

Proof of Theorem 1.2. Lemma 2.8 implies that $\mathcal{E}$ is contained in the set of numbers of bounded type, hence it has Lebesgue measure zero.

On the other hand, let $N \geq 1$, and define

$$C_N := \{x = [0;\, a_1, \ldots] \mid a_k \leq N \ \forall k \geq 1\},$$
$$E_N := \left[\frac{1}{N+1}, \frac{1}{N}\right) \cap \mathcal{E}.$$

By Lemmas 2.10 and 2.8, $E_N \subseteq C_N$, and by Lemma 2.8, for $N \geq 2$, $E_N \supseteq \phi(C_{N-1})$ where $\phi(x) := x \mapsto 1/(N+x)$. Since ϕ is a bi-Lipschitz map, it preserves Hausdorff dimension, so

$$\dim_H C_{N-1} = \dim_H \phi(C_{N-1}) \leq \dim_H E_N \leq \dim_H C_N.$$

Since it is well known [**Jar28**] that $\sup_{N\to\infty} \dim_H C_N = 1$ and $\mathcal{E} = \bigcup_N E_N$, the claim follows. □

Remark. A similar way of stating the same result would be to say that for every $p/q \in \mathbb{Q} \cap (1/(N+1), 1/N)$,

$$B\left(\frac{p}{q}, \frac{1}{(N+2)q^2}\right) \subseteq I_{p/q} \subseteq B\left(\frac{p}{q}, \frac{1}{(N-1)q^2}\right).$$

This means that in any fixed subinterval $(1/(N+1), 1/N)$ the size of the geodesic over $I_{p/q}$ is comparable to the diameter of the horocycles $\partial B(p/q + \imath/Nq^2, 1/Nq^2)$ (which, for any fixed N, all lie in the same $\mathrm{SL}_2(\mathbb{Z})$-orbit). Figure 2 shows this comparison for $N = 10$.

2.4. *The bisection algorithm.* We will now describe an algorithmic way to produce all maximal intervals, as announced in [**CMPT10**, §4.1]. This will also provide an alternative proof of the fact the $\mathcal{M}$ has full measure.

Let $\mathcal{F}$ be a family of disjoint open intervals which *accumulate only at* 0, i.e. such that for every $\epsilon > 0$ the set $\{J \in \mathcal{F} : J \cap [\epsilon, 1] \neq \emptyset\}$ is finite, and denote $F = \bigcup_{J\in\mathcal{F}} J$. The complement $(0, 1]\backslash F$ will then be a countable union of closed disjoint intervals C_j, which we refer to as *gaps*. Note that some C_j may well be a single point. To any gap which is not

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a single point we can associate its pseudocentre $c \in \mathbb{Q}$ as defined in the previous sections, and moreover consider the interval I_c associated to this rational value. The following proposition applies.

PROPOSITION 2.9. *Let I_a and I_b be two maximal intervals such that the gap between them is not a single point, and let c be the pseudocentre of the gap. Then I_c is a maximal interval and it is disjoint from both I_a and I_b.*

Proof. Pick I_{c_0} maximal such that $I_c \subseteq I_{c_0}$, so by Lemma 2.3 $\mathrm{den}(c_0) \leq \mathrm{den}(c)$. On the other hand, since maximal intervals do not intersect, I_{c_0} is contained in the gap, and since c is pseudocentre, $\mathrm{den}(c) \leq \mathrm{den}(c_0)$, with equality holding only if $c = c_0$. □

The proposition implies that if we add to the family of maximal intervals $\mathcal{F}$ all intervals which arise as gaps between adjacent intervals then we will get another family of maximal (hence disjoint) intervals, and we can iterate the procedure.

For instance, let us start with the collection $\mathcal{F}_1 := \{I_{1/n}, n \geq 1\}$. All these intervals are maximal, since the continued fraction of their pseudocentres has only one digit (apply Lemma 2.3).

Let us construct the families of intervals $\mathcal{F}_n$ recursively as follows:

$$\mathcal{G}_n := \{C \text{ connected component of } (0, 1]\setminus F_n\},$$
$$\mathcal{F}_{n+1} := \mathcal{F}_n \cup \{I_r : r \text{ pseudocentre of } C,\ C \in \mathcal{G}_n,\ C \text{ not a single point}\}$$

where F_n denotes the union of all intervals belonging to $\mathcal{F}_n$.

It is thus clear that the union $\mathcal{F}_\infty := \bigcup \mathcal{F}_n$ will be a countable family of maximal intervals. The union of all elements of $\mathcal{F}_\infty$ will be denoted by F_∞; its complement (the set of numbers which do not belong to any of the intervals produced by the algorithm) has the following property.

LEMMA 2.10. $(0, 1)\setminus F_\infty$ *consists of irrational numbers of bounded type; more precisely, the elements of* $(1/(n+1), 1/n]\setminus F_\infty$ *have partial quotients bounded by n.*

Proof. Let $\gamma = [0; c_1, c_2, \ldots, c_n, \ldots] \notin F_\infty$; we claim that $c_k \leq c_1$ for all $k \in \mathbb{N}$. Since $\gamma \notin F_\infty$, for all $n \geq 1$ we can choose $J_n \in \mathcal{G}_n$ such that $\gamma \in J_n$. Clearly, $J_{n+1} \subseteq J_n$. Furthermore, γ cannot be contained in either I_{1/c_1} or $I_{1/(c_1+1)}$, so all J_n are produced by successive bisection of the gap $([0; \overline{c_1, 1}], [0; \overline{c_1}])$, hence by Lemma 2.2, for every n, the endpoints of J_n are quadratic surds with c.f. expansion bounded by c_1. It may happen that there exists n_0 such that $J_n = \{\gamma\}$ for all $n \geq n_0$, so γ is an endpoint of J_{n_0}, hence it is irrational and c_1-bounded. Otherwise, let p_n/q_n be the pseudocentre of J_n; by uniqueness of the pseudocentre, $\mathrm{diam}\, J_n \leq 2/q_n$, and $q_{n+1} > q_n$ since $J_{n+1} \subseteq J_n$. This implies that γ cannot be rational, since the minimum denominator of a rational sitting in J_n is $q_n \to +\infty$. Moreover, $\mathrm{diam}\, J_n \to 0$, so γ is limit point of endpoints of the J_n, which are c_1-bounded, hence γ is also c_1-bounded. □

PROPOSITION 2.11. *The family $\mathcal{F}_\infty$ is precisely the family of all maximal intervals; hence $F_\infty = \mathcal{M}$.*

Proof. If a maximal interval I_c does not belong to $\mathcal{F}_\infty$, then its pseudocentre belongs to the complement of F_∞, but the previous lemma asserts that this set does not contain any rational. □

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Note that Proposition 2.11 and Lemma 2.10 provide another way of seeing that the complement of $\mathcal{M}$ consists of numbers of bounded type, hence it has full measure.

2.5. *Maximal intervals and strings.* In order to get a finer control on the maximality properties of quadratic intervals, we introduce a systematic description of the continued fraction expansions in terms of strings and develop a few tools in order to characterize the expansions of those rational numbers which give rise to maximal intervals.

Let us start with some notation. If $S = (s_1, \ldots, s_n)$ is a finite string of positive integers and x a real number, we will denote

$$[0; S] := \cfrac{1}{s_1 + \cfrac{1}{\ddots + \frac{1}{s_n}}} \quad \text{and} \quad [0; S + x] := \cfrac{1}{s_1 + \cfrac{1}{\ddots + \frac{1}{s_n + x}}}.$$

We will also introduce a total ordering on the space of finite strings of given length: given two distinct finite strings S and T of equal length, let $l := \min\{i : S_i \neq T_i\}$. We will set

$$S < T := \begin{cases} S_l < T_l & \text{if } l \equiv 0 \bmod 2, \\ S_l > T_l & \text{if } l \equiv 1 \bmod 2. \end{cases}$$

The exact same definition also gives a total ordering on the space of infinite strings. Note that if S and T have equal length $L \in \mathbb{N} \cup \{\infty\}$,

$$S < T \Leftrightarrow [0; S] < [0; T],$$

i.e. this ordering can be obtained by pulling back the order structure on $\mathbb{R}$, via identification of a string with the value of the corresponding c.f.

The following lemma is the essential tool used to compare two purely periodic infinite strings.

LEMMA 2.12. *Let S, T be two non-empty, finite strings. Then the pair of infinite strings $\overline{S}$, $\overline{T}$ is ordered in the same way as the pair ST, TS, namely,*

$$ST \gtreqless TS \iff \overline{S} \gtreqless \overline{T}.$$

Finally, we can give an explicit characterization of the c.f. expansion of those rationals which are pseudocentres of maximal intervals.

PROPOSITION 2.13. *Let $a = [0; A] \in \mathbb{Q} \cap (0, 1]$. The following are equivalent:*

(i) *I_a is maximal;*

(ii) *if $A = ST$ with S, T finite non-empty strings, then either $ST < TS$ or $ST = TS$ with $T = S$, $|S|$ odd.*

Moreover, if $[0; ST]$ is maximal, then $[0; T] > [0; ST]$.

For the sake of readability, we postpone the proofs of these results to §4.

3. *Application to α-continued fractions*

Having investigated the properties of the maximal set itself, this section will be devoted to studying its relation with the parameter space of α-continued fractions.

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3.1. *Matching intervals.* Let $\alpha \in (0, 1]$. Recall that the α-continued fraction expansion is given by the map $T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]$ defined by $T_\alpha(0) = 0$ and

$$T_\alpha(x) = \frac{\epsilon_\alpha(x)}{x} - c_\alpha(x) \quad \text{for } x \neq 0$$

with

$$\epsilon_\alpha(x) := \mathrm{Sign}(x), \qquad c_\alpha(x) := \left\lfloor \frac{1}{|x|} + 1 - \alpha \right\rfloor.$$

Moreover, one can represent the encoding with the matrices in $\mathrm{GL}(2, \mathbb{Z})$,

$$M_{\alpha,x,n} = \begin{pmatrix} 0 & \epsilon_\alpha(x) \\ 1 & c_\alpha(x) \end{pmatrix} \cdots \begin{pmatrix} 0 & \epsilon_\alpha(T_\alpha^{n-1}(x)) \\ 1 & c_\alpha(T_\alpha^{n-1}(x)) \end{pmatrix} = \begin{pmatrix} p_{n-1,\alpha}(x) & p_{n,\alpha}(x) \\ q_{n-1,\alpha}(x) & q_{n,\alpha}(x) \end{pmatrix},$$

so that

$$x = \frac{p_{n-1,\alpha}(x)x_n + p_{n,\alpha}(x)}{q_{n-1,\alpha}(x)x_n + q_{n,\alpha}(x)} \quad \text{with } x_n = T_\alpha^n(x). \tag{3}$$

We will be interested in the metric entropy $h(T_\alpha)$ of these transformations as a function of α; in [**NN08**], a series of *matching conditions* were introduced in order to define intervals in the parameter space where the entropy function $\alpha \mapsto h(T_\alpha)$ is monotone. In the same spirit, we give the following definition.

Definition 3.1. The value $\alpha \in (0, 1]$ is said to satisfy an *algebraic matching condition* of order (N, M) when the following matrix identity, denoted by $(N, M)_{\mathrm{alg}}$, holds:

$$M_{\alpha,\alpha,N} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} M_{\alpha,\alpha-1,M} \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}.$$

We will be interested in the set

$$\mathcal{M}_{\mathrm{alg}} = \{\alpha \in (0, 1] \text{ s.t. } \exists N, M \in \mathbb{N} : \alpha \text{ satisfies } (N, M)_{\mathrm{alg}}\}.$$

To get some intuition of what this condition means from a dynamic point of view, one should note that $(N, M)_{\mathrm{alg}}$ implies that

$$T_\alpha^{N+1}(\alpha) = T_\alpha^{M+1}(\alpha - 1).$$

The formal proof of this result is given in the Appendix A, together with a thorough discussion of the relationship between our algebraic matching condition and the conditions originally considered by Nakada and Natsui.

Our main result will be the following theorem.

THEOREM 3.1. *Let $a \in \mathbb{Q} \cap (0, 1]$ such that I_a is maximal, and let $a = [0; a_1, \ldots, a_n]$, n even. If we define*

$$N := \sum_{j \text{ even}} a_j, \qquad M := \sum_{j \text{ odd}} a_j,$$

then for every $x \in I_a$, the algebraic matching condition $(N, M)_{\mathrm{alg}}$ holds.

COROLLARY 3.2. *$\mathcal{M}_{\mathrm{alg}}$ has full Lebesgue measure in* $(0, 1]$.

Proof. By Theorem 3.1, $\mathcal{M}_{\mathrm{alg}}$ contains $\mathcal{M}$, which has full measure by Theorem 1.2. □

Since it can be proved (see Appendix A) that the difference between $\mathcal{M}_{\mathrm{alg}}$ and the matching set defined by Nakada and Natsui is countable, this also establishes Conjecture 1.1.

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3.2. *Anatomy of maximal orbits.* The first step in the proof of Theorem 3.1 will be to describe explicitly the first few steps of the orbit of any point inside a maximal interval I_a: we will start by establishing the following lemma.

LEMMA 3.3. *Let $a \in \mathbb{Q} \cap (0, 1]$ be the pseudocentre of a maximal $I_a = (\alpha^-, \alpha^+)$.*

(1) *Let $a \le x < \alpha^+$, so that we can write $x = [0; a_1, \ldots, a_n + y]$ with $0 \le y < \alpha^+$, $a = [0; a_1, \ldots, a_n]$ with $n \equiv 0 \bmod 2$. Then*

$$[-1; b, a_{k+1}, \ldots, a_n + y] > \alpha^+ - 1 \quad \textit{for all } 1 \le b \le a_k,\ 1 < k \le n.$$

(2) *Let $\alpha^- < x \le a$, so that $x = [0; a_1, \ldots, a_n + y]$ with $0 \le y < \alpha^-$, $a = [0; a_1, \ldots, a_n]$ with $n \equiv 1 \bmod 2$ (note that this is the representation of a in c.f.* other *than the one given in the previous point). Then*

$$[-1; b, a_k, \ldots, a_n + y] > \alpha^+ - 1 \quad \textit{for all } 1 \le b \le a_k,\ 1 < k \le n.$$

Proof. (1) Let $S := (a_1, \ldots, a_{k-1})$, $T := (a_k, \ldots, a_n)$ and $c := [0; T]$. By Proposition 2.13 and Lemma 2.12,

$$TS \ge ST \Rightarrow \overline{TS} \ge \overline{ST} \Rightarrow [0; \overline{TS}] \ge [0; \overline{ST}].$$

Moreover,

$$TS \ge ST \Rightarrow TST \ge STT \Rightarrow \overline{T} \ge \overline{ST}.$$

Now, $I_c \cap I_a = \emptyset$ since I_a is maximal and the denominator of c is smaller than the denominator of a, hence $[0; T] > [0; \overline{ST}]$. Since $b \le a_k$ and $0 \le y < \alpha^+$, for k even we have

$$\begin{aligned}[-1; b, a_{k+1}, \ldots, a_n + y] &\ge [-1; T, y] > [-1; T, \alpha^+] \\ &= [-1; \overline{TS}] \ge [-1; \overline{ST}] = \alpha^+ - 1,\end{aligned}$$

and for k odd,

$$[-1; b, a_{k+1}, \ldots, a_n + y] \ge [-1; T, y] \ge [-1; T] > [-1; \overline{ST}] = \alpha^+ - 1.$$

(2) Let $S := (a_1, \ldots, a_{k-1})$, $T := (a_k, \ldots, a_n)$ and $c := [0; T]$. If k is odd,

$$TS \ge ST \Rightarrow TTS \le TST \Rightarrow \overline{T} \le \overline{TS}.$$

Moreover, $\overline{T} \ge \overline{ST}$ as in the previous point, and since $I_a \cap I_c = \emptyset$, then $[0; \overline{T}] \ge \alpha^+$, so $[0; \overline{TS}] \ge [0; \overline{T}] \ge \alpha^+$; hence,

$$[-1; b, a_{k+1}, \ldots, a_n + y] \ge [-1; T, y] > [-1; T, \alpha^-] = [-1; \overline{TS}] \ge \alpha^+ - 1.$$

For k even, by the last point of Proposition 2.13, $[0; T] > [0; ST]$, and since $I_a \cap I_c = \emptyset$, $[0; T] > \alpha^+$; thus,

$$[-1; b, a_{k+1}, \ldots, a_n + y] \ge [-1; T, y] \ge [-1; T] > \alpha^+ - 1. \qquad \square$$

An immediate corollary is the explicit description of the orbit of the pseudocentre which explains an empirical rule given in [**CMPT10**].

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COROLLARY 3.4. *Let $a := [0; a_1, a_2, \ldots a_n]$ $(n \geq 1)$ and let I_a be maximal; then the orbits of a and $a-1$ are as follows:*

$$\begin{array}{ll} a = [0; a_1, a_2, \ldots a_n] & a-1 = [-1; a_1, a_2, \ldots a_n] \\ T_a(a) = [-1; a_2, \ldots a_n] & T_a(a-1) = [-1; a_1 - 1, a_2, \ldots a_n] \\ \ldots & \ldots \\ T_a^{a_2}(a) = [-1; 1, a_3, \ldots a_n] & T_a^{a_1-1}(a-1) = [-1; 1, a_2, \ldots a_n] \\ T_a^{a_2+1}(a) = [-1; a_4, \ldots a_n] & T_a^{a_1}(a-1) = [-1; a_3, \ldots a_n] \\ \ldots & \ldots \\ T_a^N(a) = 0 & T_a^M(a-1) = 0. \end{array}$$

where (see also [**CMPT10**, p. 23]*)*

$$\begin{array}{lll} N = \sum_{j \text{ even}} a_j, & M = \sum_{j \text{ odd}} a_j & \text{if } n \text{ is even,} \\ N = 1 + \sum_{j \text{ even}} a_j, & M = -1 + \sum_{j \text{ odd}} a_j & \text{if } n \text{ is odd.} \end{array}$$

We will now prove that an algebraic matching condition holds for any pseudocentre of a maximal interval.

PROPOSITION 3.5. *Let $a \in \mathbb{Q} \cap (0, 1]$ so that I_a is maximal, and let N and M be given by the previous corollary. Then a satisfies the algebraic matching condition $(N, M)_{\mathrm{alg}}$ of Definition 3.1.*

Proof. We will make use of the following lemma.

LEMMA 3.6. *For $\alpha < (\sqrt{5}-1)/2$, one has $q_{n+1,\alpha}(x) > q_{n,\alpha}(x) \geq 1$ for every $n \geq 0$ and every $x \in [\alpha - 1, \alpha]$.*

Proof. By definition, $q_{0,\alpha}(x) = 1$ and $q_{1,\alpha}(x) = c_{1,\alpha}(x) \geq 2$ (the latter only for $\alpha < (\sqrt{5} - 1)/2$). By induction,

$$q_{n+1,\alpha}(x) = c_{n+1,\alpha}(x) q_{n,\alpha}(x) + \epsilon_{n+1,\alpha}(x) q_{n-1,\alpha}(x) \geq 2q_{n,\alpha}(x) - q_{n-1,\alpha}(x) > q_{n,\alpha}(x).$$

□

Since it is easy to see that all values of $\alpha > (\sqrt{5}-1)/2$ satisfy a matching condition of order $(1, 2)$, we can restrict our attention to the case in which we can apply Lemma 3.6. We will denote $p_k := p_{k,\alpha}(\alpha)$ and $p'_k := p_{k,\alpha}(\alpha - 1)$. Let (N, M) be given by Corollary 3.4, such that

$$T_a^N(a) = 0 \quad \text{and} \quad T_a^M(a-1) = 0.$$

By equation (3),

$$a = p_N/q_N, \quad a - 1 = p'_M/q'_M,$$

and since $\gcd(p_N, q_N) = \gcd(p'_M, q'_M) = 1$ (because $\det M_{a,x,k} = \pm 1$),

$$q_N = q'_M, \quad p_N = p'_M + q'_M. \tag{4}$$

Now Corollary 3.4 implies that $\epsilon_a(T_a^i(a)) = \epsilon_a(T_a^j(a-1)) = -1$ for $1 \leq i \leq N-1$, $1 \leq j \leq M-1$, hence

$$\det M_{a,a,N} = -1, \quad \det M_{a,a-1,M} = 1,$$

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by writing out the two determinants and summing up

$$p_{N-1}q_N - p_N q_{N-1} + p'_{M-1}q'_M - p'_M q'_{M-1} = 0,$$

and by using (4)

$$q'_M(p_{N-1} + p'_{M-1} - q_{N-1}) = p'_M(q'_{M-1} + q_{N-1}).$$

Now, q'_M and p'_M are coprime, hence $q'_M | (q'_{M-1} + q_{N-1})$. Moreover, by Lemma 3.6, $0 < q'_{M-1} + q_{N-1} < 2q'_M$, so

$$q'_M = q'_{M-1} + q'_{N-1}, \quad p'_M = p_{N-1} + p'_{M-1} - q_{N-1},$$

which yields precisely the algebraic matching condition

$$M_{a,a,N} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} M_{a,a-1,M} \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}. \qquad \square$$

The final step will be to prove that all points in I_a have the same convergents as the pseudocentre.

LEMMA 3.7. *Let I_a be maximal, and $x \in I_a$, N, M as in Corollary 3.4. Then*

$$\begin{aligned} M_{x,x,k} &= M_{a,a,k} && \textit{for all } 1 \le k \le N, \\ M_{x,x-1,h} &= M_{a,a-1,h} && \textit{for all } 1 \le h \le M. \end{aligned}$$

Proof. If $x \ge a$, we can write $x = [0; A + y]$ with $|A| \equiv 0 \bmod 2$ and $0 \le y < \alpha^+$; from Corollary 3.4,

$$\begin{array}{ll} x = [0; a_1, a_2, \ldots a_n + y] & x - 1 = [-1; a_1, a_2, \ldots a_n + y] \\ M^{-1}_{a,a,1}(x) = [-1; a_2, \ldots a_n + y] & M^{-1}_{a,a-1,1}(x-1) = [-1; a_1 - 1, a_2, \ldots a_n + y] \\ \ldots & \ldots \\ M^{-1}_{a,a,a_2}(x) = [-1; 1, a_3, \ldots a_n + y] & M^{-1}_{a,a-1,a_1-1}(x-1) = [-1; 1, a_2, \ldots a_n + y] \\ M^{-1}_{a,a,a_2+1}(x) = [-1; a_4, \ldots a_n + y] & M^{-1}_{a,a-1,a_1}(x-1) = [-1; a_3, \ldots a_n + y] \\ \ldots & \ldots \\ M^{-1}_{a,a,N}(x) = [-1; 1 + y] & M^{-1}_{a,a-1,M}(x-1) = y, \end{array}$$

and again from the lemma,

$$M^{-1}_{a,a,k}(x) \in (\alpha^+ - 1, 0) \subseteq (x - 1, 0) \quad 1 \le k \le N,$$
$$M^{-1}_{a,a-1,h}(x) \in (\alpha^+ - 1, 0) \subseteq (x - 1, 0) \quad 1 \le h \le M - 1,$$

hence

$$\begin{aligned} M_{x,x,k} &= M_{a,a,k} && 1 \le k \le N, \\ M_{x,x-1,h} &= M_{a,a-1,h} && 1 \le h \le M - 1. \end{aligned}$$

To prove the claim it remains to consider

$$M^{-1}_{a,a-1,M}(x-1) = y.$$

Since

$$0 < [0; A] < [0; AA] < \cdots < [0; A^k] < \cdots < [0; A^{k+1}] < \cdots$$

there exists $k \ge 0$ such that

$$[0; A^k] \le y < [0; A^{k+1}];$$

hence, $y < [0; A^{k+1}] \le [0; A + y] = x$ and $M^{-1}_{a,a-1,M}(x-1) \in (0, x)$, so $M_{a,a-1,M} = M_{x,x-1,M}$.

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The case $x \leq a$ is similar: the only non-negative element of the orbit this time is

$$M_{a,a,N}^{-1}(x) = y \quad \text{with } 0 \leq y < \alpha^-$$

which, since $\alpha^- < x$, still implies that $M_{a,a,N} = M_{x,x,N}$. □

Proof of Theorem 3.1. Let $x \in I_a$, a maximal. By Proposition 3.5, $M_{a,a,N}$ and $M_{a,a-1,M}$ are related by the identity $(N, M)_{\text{alg}}$. Since by Lemma 3.7, $M_{x,x,N} = M_{a,a,N}$ and $M_{x,x-1,M} = M_{a,a-1,M}$, the algebraic matching condition $(N, M)_{\text{alg}}$ also holds for x. □

In order to complete the proof of Theorem 1.3, it remains to prove that the entropy is monotone on every maximal I_a.

PROPOSITION 3.8. *Let I_a be a maximal quadratic interval, and let N and M be as in Theorem 3.1: then the function $\alpha \mapsto h(T_\alpha)$ is:*
(i) *strictly increasing if $N < M$;*
(ii) *constant if $N = M$;*
(iii) *strictly decreasing if $N > M$*
on the whole interval I_a.

The proof is just an adaptation of the one given in [**NN08**] (see Appendix A): let us just remark that we are able to establish explicit bounds for the domain of validity of their entropy formula, which was previously just claimed to work locally. Moreover, N and M are now given in terms of the c.f. expansion of a, so one can immediately establish which of the cases (i)–(iii) holds in a neighbourhood of any given rational number.

3.3. *Period doubling.* Another feature observed in [**CMPT10**, §4.2] was the production of infinite chains of adjacent matching intervals via *period doubling*. More formally, we have the following proposition.

PROPOSITION 3.9. *Let a be the pseudocentre of a maximal interval I_a, and write $a = [0; A^-] = [0; A^+]$ with $|A^-| \equiv 1 \bmod 2$. Then $a' := [0; A^-A^-]$ is the pseudocentre of a maximal interval.*

The proposition follows immediately from Lemma 4.4, which will be proved in the next section. By applying the proposition repeatedly, one gets the following corollary.

COROLLARY 3.10. *Let I_a be a maximal (hence matching) interval. Then there is a countable chain of matching intervals*

$$\cdots < I_{a_{n+1}} < I_{a_n} < \cdots < I_{a_1} = I_a$$

such that I_{a_n} and $I_{a_{n+1}}$ are adjacent, and $\lim_{n\to\infty} a_n := a_\infty > 0$.

Note that the proposition also gives a recursive algorithm to generate the c.f. expansion of the limit point a_∞: an explicit computation for the chain generated by $I_{1/2}$ is given in [**CMPT10**, §4.2].

4. *String techniques*

This section contains the proofs of a few technical lemmas about the string ordering mentioned in the rest of the paper.

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4.1. *String formalism.* To prove our results we shall need to fix some notation to manipulate the strings of partial quotients. If A, B are two finite strings composed with the alphabet $\mathbb{N}_+$, we denote:

- A' the *twin string* of the finite string A, i.e. the string such that the finite continued fractions $[0,\ A]$ and $[0,\ A']$ represent the same rational number;
- AB the concatenation of A and B; Ab will denote the concatenation of the finite string A with the one-letter string (b);
- A^n the concatenation of n copies of A (A^0 is the empty string);
- $\overline{A}$ the endless concatenation of A;
- $|A|$ the length of A;
- $(A)_i^j$ the substring of A going from the ith figure to the jth figure of A; to indicate the jth character of the string A we shall usually write $(A)_j$ instead of $(A)_j^j$;
- $A \subseteq B$ A is a *prefix* of B, i.e. there exists B_1 such that $B = AB_1$.

We will be interested in the alternating lexicographic order structure on the space of finite or infinite strings as defined in §2.5. Note that the set of finite strings $\mathcal{S}$ is a semigroup for the operation of concatenation. Associating a finite string S to the fractional map $x \mapsto [0;\ S + x]$ yields a natural action of the semigroup $\mathcal{S}$ on $\mathbb{R}_+$. Let us also recall that the map $x \mapsto [0;\ S + x]$ is increasing if $|S|$ is even and decreasing if $|S|$ is odd; in particular, odd convergents of any x are greater than x while even convergents are smaller. Moreover, if $x := [0;\ S,\ a + x']$ and $y := [0;\ S,\ b + y']$ with $a > b \in \mathbb{N}_+$, $x',\ y' \in [0,\ 1)$, then $x > y$ if $|S|$ is even and $x < y$ if $|S|$ is odd.

In the following we shall need some effective criterion to compare infinite periodic strings $\overline{S}$, $\overline{T}$: as soon as $|S| \neq |T|$ this becomes a non-trivial task. The next section will deal with this issue.

4.2. *String lemma.*

LEMMA 4.1. *Let S, T be two non-empty strings. Then the pair of infinite strings $\overline{S}$, $\overline{T}$ is ordered in the same way as the pair ST, TS, i.e.*

$$ST \gtreqqless TS \iff \overline{S} \gtreqqless \overline{T}.$$

Proof. If $ST = TS$ we can prove that there exist another string P and integers $k,\ h \in \mathbb{N}$ such that $S = P^k$, $T = P^h$, hence $\overline{S} = \overline{T}$. In fact, we proceed by induction on $n := \max\{S,\ T\}$. For $n = 1$ the claim is obviously true. Assume now that we have proved this claim for all pairs of strings of length strictly less than n, and let S, T be a pair of strings of maximal length n. We may assume that $0 < |T| < |S| \leq n$, the cases $|T| = 0$ and $|T| = |S|$ being trivial. The hypothesis $TS = ST$ implies that T is a prefix of S, namely $S = TS_1$ therefore $TS = ST$ translates into $TS_1 = S_1T$. Since $\max\{|T|,\ |S_1|\} < |S| \leq n$, we use the inductive hypothesis to conclude that $T = P^k$, $S_1 = P^h$, and therefore $S = P^{h+k}$.

If $ST \neq TS$, then $d := \min\{j \in \mathbb{N} : (ST)_j \neq (TS)_j\} \leq s + t$. By Lemma 4.2 with $n = d - 1$ one has

$$(ST)_1^d = (\overline{S})_1^d, \quad (TS)_1^d = (\overline{T})_1^d,$$

hence the pair $(\overline{S},\ \overline{T})$ is ordered in the same way as $(ST,\ TS)$. □

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LEMMA 4.2. *Let S, T be two non-empty strings, $s := |S|$, $t := |T|$, $n \in \mathbb{N}$, $0 \le n < s + t$. If $(ST)_1^n = (TS)_1^n$ then*

$$\begin{cases} (\overline{S})_1^{n+1} = (ST)_1^{n+1} & (*) \\ (\overline{T})_1^{n+1} = (TS)_1^{n+1}. & (**) \end{cases}$$

Proof of Lemma 4.2. We can assume that $|T| \le |S|$. We can split the proof into three cases, depending on the relation between n and the lengths data t and s.

Case 1: $0 \le n < t$. In this case both $(*)$ and $(**)$ trivially hold.

Case 2: $n < s$, $kt \le n < (k+1)t$ for some $k \ge 1$. $(ST)_1^n = (TS)_1^n$ implies that T^k is a prefix of S, i.e. $S = T^k S_1$. On the other hand:

– $\overline{S}$ coincides with ST on the first s figures $\overset{n<s}{\Longrightarrow}$ $(*)$ holds;

– $\overline{T}$ coincides with TS on the first $(k+1)t$ figures $\overset{n<(k+1)t}{\Longrightarrow}$ $(**)$ holds;

Case 3: $s \le n < s + t$. $(ST)_1^n = (TS)_1^n$ implies that S is a prefix of T^k (with $k = \lceil s/t \rceil$), i.e. $S = T^{k-1}T_0$, $T = T_0T_1$. Thus

$$(\overline{S})_1^{s+t} = T^{k-1}T_0T_0T_1 = ST, \quad (\overline{T})_1^{s+t} = T^kT_0 = TS.$$

So $(*)$ and $(**)$ are again both verified. □

The following remark will be useful later.

Remark. Let T, S be two non-empty strings and set $a := [0; ST]$, $b := [0; S]$, $I_a := (\alpha^-, \alpha^+)$ and $I_b := (\beta^-, \beta^+)$. Then:

(i) if $|S|$ is even then $b < a$ and $\beta^- < \alpha^-$;

(ii) if $|S|$ is odd and $T \ne (1)$, then $b > a$ and $\beta^+ > \alpha^+$.

LEMMA 4.3. *If $I_a \cap I_b \ne \emptyset$, $I_a \backslash I_b \ne \emptyset$ and $I_b \backslash I_a \ne \emptyset$, then either I_a or I_b is not maximal.*

Proof. By Lemma 2.3, without loss of generality, we may assume that a is a convergent of b; hence we can write $a = [0; A]$, $b = [0; A^\ell A_0]$, where $A_0 \ne \emptyset$ is a proper prefix of A. Let $a_0 := [0; A_0]$; we claim that the interval I_{a_0} contains either I_a or I_b. There are several cases to be examined; in all cases the proof that the two intervals are nested, one inside the other, amounts to checking two inequalities: one of the two inequalities will be a trivial consequence of the previous remark while the other is harder, but it will follow from the string Lemma 2.12. We treat just one case in detail, and provide Table 1 to explain how to get the 'hard' inequality for all the other cases. Let $|A| \equiv 0$, $|A_0| \equiv 0$, $\alpha^+ = [0; \overline{A}]$, $\beta^+ = [0; \overline{A^\ell A_0}]$. Then

$$\alpha^+ < \beta^+ \Leftrightarrow \overline{A} < \overline{A^\ell A_0} \Leftrightarrow AA_0 < A_0A \Leftrightarrow \overline{A} < \overline{A_0}$$

so $\alpha^+ < \alpha_0^+$ and, by the remark between Lemmas 4.2 and 4.3, $\alpha_0^- < \alpha^-$ so that $I_a \subseteq I_{a_0}$ where $a_0 := [0; A_0]$. □

LEMMA 4.4. *Let $a_1 = [0; P]$, $a_\ell = [0; P^\ell]$. The following are equivalent:*

(i) *I_{a_ℓ} is maximal;*

(ii) *I_{a_1} is maximal and*

$$\ell = 1 \quad \textit{if } |P| \textit{ is even},$$
$$\ell \le 2 \quad \textit{if } |P| \textit{ is odd}.$$

TABLE 1.

Cases		Hypotheses used	Hard inequalitiy		Aim
$\|A\|$ even $a < b$ $\alpha^+ := [0; \overline{A}]$	$\|A_0\|$ even $a_0 < a < b < \alpha^+$	$\overline{A^\ell A_0} > \overline{A}$ $\beta^+ > \alpha^+$	$\overline{A} < \overline{A_0}$ $\alpha^+ < \alpha_0^+$		$I_a \subset I_{a_0}$
$\|A\|$ even $a < b$ $\alpha^+ := [0; \overline{A}]$	$\|A_0\|$ odd $\alpha^+ < b < a_0$	$\overline{A^\ell A_0} < \overline{A}$ $\beta^- < \alpha^+$	$\overline{A_0} < \overline{A}$ $\alpha_0^- < \beta^-$		$I_b \subset I_{a_0}$
$\|A\|$ odd	$\|A^\ell A_0\|$ even	$\overline{A^\ell A_0} > \overline{A}$	$\begin{cases} \overline{A_0} > \overline{A^\ell A_0} \\ \alpha_0^+ > \beta^+ \end{cases}$	if $\begin{cases} \|A_0\| & \text{even} \\ \ell & \text{even} \end{cases}$	$I_b \subset I_{a_0}$
$b < a$ $\alpha^- := [0; \overline{A}]$	$b < \alpha^-$	$\beta^+ > \alpha^-$	$\begin{cases} \overline{A_0} < \overline{A} \\ \alpha_0^- < \alpha^- \end{cases}$	if $\begin{cases} \|A_0\| & \text{odd} \\ \ell & \text{odd} \end{cases}$	$I_a \subset I_{a_0}$
$\|A\|$ odd	$\|A^\ell A_0\|$ odd	$\overline{A^\ell A'_0} > \overline{A}$	$\begin{cases} \overline{A'_0} > \overline{A^\ell A'_0} \\ \alpha_0^+ > \beta^+ \end{cases}$	if $\begin{cases} \|A'_0\| & \text{even} \\ \ell & \text{odd} \end{cases}$	$I_b \subset I_{a_0}$
$b < a$ $\alpha^- := [0; \overline{A}]$	$\alpha^- < b$	$\beta^+ > \alpha^-$	$\begin{cases} \overline{A'_0} < \overline{A} \\ \alpha_0^- < \alpha^- \end{cases}$	if $\begin{cases} \|A'_0\| & \text{odd} \\ \ell & \text{odd} \end{cases}$	$I_a \subset I_{a_0}$

Proof. (i) $\Rightarrow$ (ii). If $|P|$ even, $\ell > 1$ and $a_{\ell-1} = [0; P^{\ell-1}]$, then $I_{a_{\ell-1}} \supsetneq I_{a_\ell}$ so that I_{a_ℓ} cannot be maximal. If $|P|$ is odd and $\ell > 2$, setting $a_{\ell-2} = [0; P^{\ell-2}]$ then $I_{a_{\ell-2}} \supsetneq I_{a_\ell}$ so, again, I_{a_ℓ} cannot be maximal. To conclude the proof we just need to prove that I_{a_1} is maximal. Let I_{a_*} be the maximal interval containing I_{a_1}, so that $a := [0; P_*]$ is a convergent of a_1. The function $\phi(x) := [0; P + x]$ is injective, $\phi : I_{a_*} \xrightarrow{\sim} \phi(I_{a_*}) = I_{\phi(a_*)}$, with $\phi(a_*) := [0; PP_*]$; moreover, $\phi(I_{a_1}) = I_{\phi(a_1)} = I_{a_2}$. So

$$I_{a_1} \subset I_{a_*}, \quad \phi(I_{a_1}) \subset \phi(I_{a_*}) = I_{\phi(a_*)}, \quad I_{a_2} = \phi(I_{a_1}) \subset \phi(I_{a_*}) = I_{\phi(a_*)}.$$

Since I_{a_2} is maximal, $I_{a_2} = I_{\phi(a_*)}$ and hence $I_{a_1} = I_{a_*}$ is maximal.

(ii) $\Rightarrow$ (i). Let $|P|$ be odd and $\ell = 2$ (otherwise there is nothing to prove); we have to show that I_{a_2} is maximal (if I_{a_1} is). Let $I_{a_j} := (\alpha_j^-, \alpha_j^+)$ $(j = 1, 2)$ and observe that, since a_1 is an odd convergent of α_1^- and a_2 is an even convergent of α_2^+,

$$a_2 := [0; PP] < [0; \overline{P}] = \alpha_2^+ = \alpha_1^- - < a_1.$$

If I_a is the maximal interval containing I_{a_2}, $a := [0; A]$, $|A|$ even, we have that $I_a \cap I_{a_1} = \emptyset$ and so $\overline{A} = \overline{P}$. Therefore A and P have a common period Q: $A = Q^m$, $P = Q^\ell$; on the other hand, by virtue of the implication (i) $\Rightarrow$ (ii), already proved, we get $\ell = 1$ ($\ell = 2$ is impossible, since $|P|$ is odd) and therefore $m = 2$, so $I_{a_2} = I_a$ is maximal. □

Let S, T be two non-empty strings and

$$\begin{aligned} a &:= [0; ST], & b &:= [0; S], & c &:= [0; T]; \\ I_a &:= (\alpha^-, \alpha^+), & I_b &:= (\beta^-, \beta^+), & I_c &:= (\gamma^-, \gamma^+). \end{aligned} \tag{5}$$

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PROPOSITION 4.5. *Let* $a = [0; A] \in \mathbb{Q} \cap (0, 1]$.

(1) *The following are equivalent:*
 (i) I_a *is maximal;*
 (ii) *if* $A = ST$ *with* S, T *finite non-empty strings, then either* $ST < TS$ *or* $ST = TS$ *with* $T = S$, $|S|$ *odd.*

(2) *Moreover, if* $a = [0; ST]$ *and* I_a *is maximal,* $[0; T] > [0; ST]$ *(i.e.* $a < c$*).*

Proof. (1) To prove (i) $\Rightarrow$ (ii), let us use the notation introduced above in (5); by maximality of I_a we immediately get that $b \notin I_a$ ($a, b \in \mathbb{Q}$ and $\mathrm{den}(b) < \mathrm{den}(a)$—see also Definition 2.2); since $b \in I_b \setminus I_a \neq \emptyset$, maximality of I_a and Lemma 2.6 also imply that $I_a \cap I_b = \emptyset$.

Case 0. If b is an even convergent of a (i.e. if $|S|$ is even and $b < a$) then I_b lies to the left of I_a and hence $\beta^+ \leq \alpha^-$; since $[0; \overline{S}] = \beta^+$ and $[0; \overline{ST}] \in \{\alpha^\pm\}$, by Lemma 2.12 we get $SST \leq STS$ and, since $|S|$ is even, $ST \leq TS$. Lemma 4.4 tells us that, since $|S|$ is even and I_a is maximal, equality cannot hold.

Case 1. If b is an odd convergent (i.e. if $|S|$ is odd and $b > a$) by the previous argument $\alpha^+ \leq \beta^-$. If $[0; \overline{ST}] = \alpha^+ = \beta^- = [0; \overline{S}]$ then, by Lemma 4.4, $T = S$. If not, then $[0; \overline{ST}] < [0; \overline{S}]$; by Lemma 2.12, $STS < SST$ and, since $|S|$ is odd, this implies that $TS > ST$ (which is the same conclusion as the previous case).

The first implication is thus proved.

We now prove (ii) $\Rightarrow$ (i). Assume that I_a *is not* maximal; then there exist two non-empty strings such that $a := [0; ST]$, $b := [0; S]$, I_b is maximal and $I_b \supset I_a$ (which, in particular, implies that if $|S|$ is odd then $S \neq T$). Then $\alpha^+ \leq \beta^+$ and $\alpha^- \geq \beta^-$. Let us take a quick glance at the cases that can occur:

$\lvert S\rvert$	$\lvert T\rvert$	$[0; \overline{ST}]$	$[0; \overline{S}]$	Consequence of string lemma	Conclusion
Even	Even	α^+	β^+	$STS \leq SST$	$ST \geq TS$
Even	Odd	α^-	β^+	$STS < SST$	$ST > TS$
Odd	Even	α^-	β^-	$STS \leq SST$	$ST \geq TS$
Odd	Odd	α^+	β^-	$STS < SST$	$ST > TS$

It is thus easy to realize that condition (ii) never holds.

(2) Let us now prove the second statement of the previous proposition. Since our claim concerns rational values, we may assume that $|ST|$ is even (so that $\alpha^+ = [0; \overline{ST}]$). Let us rule out the 'period doubling case' (i.e. $|S|$ odd and $S = T$): in this case $a < c$ because c is an odd convergent of a. In all other cases the strict inequality $ST < TS$ holds and hence $STT < TST$.

Moreover, we know that:

- $\gamma := [0; \overline{T}]$ is an endpoint of I_c;
- $\gamma > \alpha^+$ (because $STT \leq TST$);
- $I_a \cap I_c = \emptyset$ because I_c must contain points which are not in I_a, and I_a is maximal (recall Lemma 2.6).

Therefore $c > a$ (and in fact $\alpha^+ \leq \gamma^-$ since $I_a \cap I_c = \emptyset$). □

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Proposition 4.5 provides an effective algorithm to decide whether or not a string defines the pseudocentre of a maximal interval: it is sufficient to check that all its cyclical permutations produce strings which are strictly bigger (unless the exceptional case of period doubling occurs).

A. *Appendix. Matching conditions and entropy*

A.1. *Comparison between matching conditions.*

Let us recall the matching conditions given in [**NN08**]:

(c-1) $\{T_\alpha^n(\alpha) : 0 \le n < k_1\} \cap \{T_\alpha^m(\alpha - 1) : 0 \le m < k_2\} = \emptyset$;

(c-2) $M_{\alpha,\alpha,k_1} = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) M_{\alpha,\alpha-1,k_2} (\Rightarrow T_\alpha^{k_1}(\alpha) = T_\alpha^{k_2}(\alpha - 1))$;

(c-3) $T_\alpha^{k_1}(\alpha)(= T_\alpha^{k_2}(\alpha - 1)) \notin \{\alpha, \alpha - 1\}$.

The matching set is therefore

$$\tilde{\mathcal{M}} := \{\alpha \in (0, 1) : \text{(c-1), (c-2), (c-3) hold for some } (k_1, k_2)\}.$$

PROPOSITION A.1. *If α satisfies the algebraic matching condition $(N, M)_{\text{alg}}$ of Definition 3.1, then $T_\alpha^{N+1}(\alpha) = T_\alpha^{M+1}(\alpha - 1)$.*

Proof. Writing the identity $(N, M)_{\text{alg}}$ in terms of Möbius transformations and evaluating it at α,

$$T_\alpha^N(\alpha) + T_\alpha^M(\alpha - 1) = -T_\alpha^N(\alpha) T_\alpha^M(\alpha - 1),$$

which implies that $T_\alpha^N(\alpha) = 0 \Leftrightarrow T_\alpha^M(\alpha - 1) = 0$. If both are zero, the claim follows trivially since $T_\alpha(0) = 0$; if they are non-zero, one can write

$$\frac{1}{T_\alpha^N(\alpha)} + \frac{1}{T_\alpha^M(\alpha - 1)} = -1. \tag{A.1}$$

Now suppose that $\epsilon_\alpha(T_\alpha^N(\alpha)) = \epsilon$ and $c_\alpha(T_\alpha^N(\alpha)) = c$ so that

$$\frac{\epsilon}{T_\alpha^N(\alpha)} - c \in [\alpha - 1, \alpha).$$

The fact that $|T_\alpha^N(\alpha)| < 1$ and (A.1) imply that $\epsilon_\alpha(T_\alpha^M(\alpha - 1)) = -\epsilon$, hence

$$-\frac{\epsilon}{T_\alpha^M(\alpha - 1)} - c - \epsilon \in [\alpha - 1, \alpha)$$

so $c_\alpha(T_\alpha^M(\alpha - 1)) = c + \epsilon$ and $T_\alpha^{N+1}(\alpha) = T_\alpha^{M+1}(\alpha - 1)$. □

PROPOSITION A.2. *Let I_a be a maximal quadratic interval, and let the two c.f. expansions of a be $a = [0; A^+] = [0; A^-]$. Let N and M be as in Theorem 3.1 and*

$$\tilde{I}_a := \{\alpha \in I_a \text{ s.t. (c-1), (c-2), (c-3) hold with } k_1 = N + 1, k_2 = M + 1\}.$$

Then

$$I_a \setminus \tilde{I}_a \subseteq \{a\} \cup \{\alpha = [0; \overline{A^+, k}], k \in \mathbb{N}\} \cup \{\alpha = [0; \overline{A^-, k}], k \in \mathbb{N}\}.$$

Proof. By the proof of the previous proposition, (c-2) holds for $\alpha \in I_a \setminus \{a\}$. By using the explicit description of the orbits as in Corollary 3.4 and Lemma 3.7, one can check that (c-1) holds for every $\alpha \in I_a \setminus \{a\}$. Exceptions to (c-3) precisely correspond to $\alpha = [0; \overline{A^-, k}]$ or $\alpha = [0; \overline{A^+, k}]$. □

COROLLARY A.3. *$\mathcal{M} \setminus \tilde{\mathcal{M}}$ is a countable set.*

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A.2. *The entropy is monotone on maximal intervals.* Let us now prove Proposition 3.8.

LEMMA A.4. *Let* $a \in \mathbb{Q} \cap (0, 1]$ *such that* $I_a = (\alpha^-, \alpha^+)$ *is maximal, let* $a = [0; A]$ *be its c.f. expansion with* $|A| \equiv 0 \bmod 2$*, and choose* α, α' *such that* $\alpha^- < \alpha < \alpha' < \alpha^+$ *and* $\alpha' \leq [0; A + \alpha]$. *Then*

$$\frac{h(T_{\alpha'})}{h(T_\alpha)} = 1 + (M - N)\mu_{\alpha'}([\alpha, \alpha']),$$
$$\frac{h(T_\alpha)}{h(T_{\alpha'})} = 1 + (N - M)\mu_\alpha([\alpha - 1, \alpha' - 1]),$$

where μ_α *and* $\mu_{\alpha'}$ *are the invariant densities of* T_α *and* $T_{\alpha'}$*, respectively.*

Proof. Choose $x \in (\alpha, \alpha')$. The proof proceeds exactly as in [**NN08**, Theorem 2], once we show that

$$M^{-1}_{\alpha',x,k}(x) \notin (\alpha, \alpha') \quad 1 \leq k \leq N,$$
$$M^{-1}_{\alpha,x-1,h}(x - 1) \notin (\alpha - 1, \alpha' - 1) \quad 1 \leq h \leq M.$$

This follows directly from Lemma 3.3, except for two cases: one in which $h = M$ and $x \geq a$, and the other in which and $k = N$ and $x \leq a$. In the first case one can write $x = [0; A + y]$ with $a = [0; A]$, $|A| \equiv 0 \bmod 2$, $0 \leq y < \alpha^+$. Then $M^{-1}_{\alpha,x-1,M}(x - 1) = y < \alpha$ because

$$[0; A + y] = x < \alpha' \leq [0; A + \alpha] \Rightarrow y < \alpha.$$

The second case is handled similarly. □

Proof of Proposition 3.8. Given $\alpha, \alpha' \in I_a$, $\alpha < \alpha'$, let $\alpha_k := [0; A^k + \alpha]$ and $k_0 := \max\{k > 0 \text{ s.t. } \alpha_k < \alpha'\}$. One can apply the lemma to each consecutive pair of the chain $\alpha < \alpha_1 < \cdots < \alpha_{k_0} < \alpha$. □

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