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RANK 3 BINGO

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Abstract. We classify irreducible actions of connected groups of finite Morley rank on abelian groups of Morley rank 3.

§1. The result and its context.

1.1. The context. The present article deals with representations of groups of finite Morley rank. Morley rank is the logician's coarse approach to Zariski dimension; a good general reference on the topic is [9], where the theory is systematically developed, and the reader not too familiar with the subject may start there. Following one's algebraic intuition is another possibility, as groups of finite Morley rank behave in many respects very much like algebraic groups over algebraically closed fields do. This intuition shaped the famous Cherlin–Zilber Conjecture: *simple infinite groups of finite Morley ranks are simple algebraic groups over algebraically closed fields*.

However, since there is no rational structure around, the Cherlin–Zilber Conjecture is still an open question, and the present setting is broader than the theory of algebraic groups. On the other hand, finite groups are groups of Morley rank 0, and it had happened that methods of classification of finite simple groups could be successfully applied to the general case of groups of finite Morley rank. This became, over the last 20 years, the principal line of development and resulted, in particular, in confirmation of the Cherlin–Zilber conjecture in a number of important cases, see [1].

This paper (together with [6,8,13,14,30]) signals a shift in the direction of research in the theory of groups of finite Morley rank: instead of the study of their internal structure we focus on the study of *actions* of groups of finite Morley rank.

Indeed groups of finite Morley rank naturally arise in model theory as Galois groups of extensions of definable sets, and have an action naturally attached to them. More precisely, any uncountably categorical structure is controlled by certain definable groups of permutations which by definability have finite Morley rank. This observation leads to the concept of a *binding group* [25, Section 2.5], introduced by Zilber and developed in other contexts by Hrushovski, an important special case being that of Lie groups in the Picard–Vessiot theory of linear differential equations.

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But as another consequence of the absence of a rational structure, representations (permutation and linear) in the finite Morley rank category must be studied by elementary means. The topic being rather new we deal in this paper with a basic case: actions on a module of rank 3, for which we provide a classification.

1.2. The result. One word on terminology may be in order. We reserve the phrase G-module for a definable, *connected*, abelian group acted on definably by G. Accordingly, reducibility refers to the existence of a nontrivial, proper G-submodule W: definability and connectedness of W are therefore required. Likewise, a G-composition series $0 = V_0 < \cdots < V_\ell = V$ being a series of G-submodules of maximal length $\ell_G(V) = \ell$, the V_i's are definable and connected. If G acts irreducibly on V, one also says that V is G-minimal.

THEOREM 1.1. *Let G be a connected, nonsoluble group of finite Morley rank and V be a faithful G-module of Morley rank* 3. *Suppose that V is G-minimal. Then*:

- *either $G = \mathrm{PSL}_2(\mathbb{K}) \times Z(G)$ where $\mathrm{PSL}_2(\mathbb{K})$ acts in its adjoint action on $V \simeq \mathbb{K}_+^3$*,
- *or $G = \mathrm{SL}_3(\mathbb{K}) * Z(G)$ in its natural action on $V \simeq \mathbb{K}_+^3$*,
- *or G is a simple bad group of rank* 3, *and V has odd prime exponent*.

In the algebraic category, irreducible, three-dimensional representations are of course well-known; in particular, the only simple algebraic groups which have such representations are $\mathrm{SL}_3(\mathbb{K})$ (in its canonical action) and $\mathrm{PSL}_2(\mathbb{K})$ in its adjoint action—this follows from the classification of the simple algebraic groups and basic representation theory of $\mathrm{SL}_2(\mathbb{K})$ (the latter can be substituted by the analysis in the category of groups of finite Morley rank [13]).

But this is the whole point: to prove that the pair (G, V) lives in the algebraic group category. The principal difficulties are related to the possibility of so-called *bad groups*, on which we say more in the prerequisites.

Interestingly enough, our proof involves ideas from more or less all directions explored over almost forty years of groups of finite Morley rank. The present article is therefore the best opportunity we shall ever have to print our hearty thanks to all members of the ranked universe: Tuna, Christine, Oleg, Ayşe, Jeffrey, Gregory, Luis-Jaime, Olivier, Ursula, Ehud, the late Éric, James, Angus, Dugald, Yerulan, Ali, Anand, Bruno, Katrin, Jules, Pınar, Frank, Joshua, and Boris (with our apologies to whomever we forgot). The reader can play bingo with these names and match them against the various results we shall mention.

And of course, our special extra thanks to Ali, mayor of the Matematik Koyü at Şirince, Turkey—this is one more result proved there.

1.3. Future directions. The result of this paper deals with a configuration that arises in bases of induction (on Morley rank) in proofs of more general results on representations in the finite Morley rank category. One of the examples is the following work-in-progress result by Berkman and the first author (generic k-transitivity on a set X means that G has an orbit on X^k of the same Morley rank as X^k):

THEOREM 1.2 (Berkman and Borovik, work in progress). *Let H and V be connected groups of finite Morley rank and V an elementary abelian p-group for $p \neq 2$ of Morley rank $n > 2$. Assume that H acts on V definably, and the action is faithful and generically n-transitive.*

Then there is an algebraically closed field F such that $V \cong F^n$ and $H \cong \mathrm{GL}(V)$, and the action is the natural action.

Notice that the basic case $n = 3$ is a corollary to our Theorem, since neither a proper subgroup of $\mathrm{GL}_3(\mathbb{K})$ on $\mathbb{K}^3$ nor a simple bad group on a rank 3 module acts generically 3-transitively (the former is by inspection; in the latter case this would create an involution in the simple bad group, a contradiction we shall often invoke in Proposition 2.5 below).

The theorem by Berkman and Borovik, in its turn, is needed for confirming a conjecture that makes a bound on degree of generic k-transitivity of group actions in the finite Morley rank category [8] explicit and sharp. This is the result from [8]:

THEOREM 1.3 (Borovik and Cherlin). *There exists a function $f : \mathbb{N} \to \mathbb{N}$ with the following property. If a group G of finite Morley rank acts faithfully, definably, transitively and generically k-transitively on a set X of Morley rank n then one has*:

$$k \leq f(\mathrm{rk}(X)).$$

The following conjecture, if true, considerably clarifies the situation.

CONJECTURE 1.4. *Let G be a connected group of finite Morley rank acting faithfully, definably, transitively and generically k-transitively on a set X of Morley rank n. Then $k \leq n + 2$, and if, in addition, $k = n + 2$ then the pair (G, X) is equivalent to the projective general linear group $\mathrm{PGL}_{n+1}(F)$ acting on the projective space $\mathbb{P}^n(F)$ for some algebraically closed field F.*

(Actually, the group $V \rtimes H$ from the tentative result by Berkman and Borovik is likely to appear in G as the stabiliser of a generic point in X.)

The conjecture above is ideologically very important: it bounds the complexity (formally measured by the degree of generic transitivity) of permutation groups of finite Morley rank exactly at the level of "classical" mathematics and canonical examples. It also relates the general theory of abstract groups of finite Morley rank both to combinatorial geometry and to the classification programme.

So perhaps it should not be surprising that the present paper that looks at one of the special configurations in the basis of induction uses the total of the research on groups of finite Morley rank accumulated over 40 years.

1.4. Prerequisites. The article is far from being self-contained as we assume familiarity with a number of topics: definable closure [9, Section 5.5], connected component [9, Section 5.2], torsion lifting [9, ex.11 p.98], Zilber's Indecomposibility Theorem [9, Section 5.4], the structure of abelian and nilpotent groups [9, Section 6.2], the structure of soluble p-subgroups [9, Section 6.4], the Prüfer p-rank $\mathrm{Pr}_p(\cdot)$, p-unipotent subgroups and the $U_p(\cdot)$ radical [17, Section 2.3], Borel subgroups [17, Section 2.4], fields of finite Morley rank [9, Section 8.1], Sylow 2-subgroups [9, Section 10.3], good tori [12], torality principles [10, Corollary 3]. There are no specific prerequisites on permutation groups, but [22] can provide useful background. More subjects will be mentioned in due time; for the moment let us quote only the key results and methods.

Recall that a *bad group* is a (potential) group of finite Morley rank all of whose definable, connected, proper subgroups are nilpotent. Be careful that the condition is on *all* proper subgroups, and that one does not require simplicity. Bad groups of rank 3 were encountered by Cherlin in the very first article on groups of finite Morley rank [11]; we still do not know whether these do exist, but they have been extensively studied, in particular by Cherlin, Nesin, and Corredor.

Bad Group Analysis (from [9, Theorem 13.3 and Proposition 13.4]). *Let G be a simple bad group. Then the definable, connected, maximal, proper subgroups of G are conjugate to each other, and G has no involutions. Actually G has no definable, involutive automorphism.*

We now start talking about group actions. First recall one definition and two facts on semisimplicity. A *good torus* [12] is a definable, divisible, abelian group with the property that every definable, connected subgroup is the definable closure of its torsion subgroup.

Wagner's Torus Theorem ([29]). *Let $\mathbb{K}$ be a field of finite Morley rank of positive characteristic. Then $\mathbb{K}^\times$ is a good torus.*

Semi-Simple Actions ([17, Lemma G]). *In a universe of finite Morley rank, consider the following definable objects: a definable, soluble group T with no elements of order p, a connected, elementary abelian p-group A, and an action of T on A. Then $A = C_A(T) \oplus [A, T]$. Let $A_0 \leq A$ be a definable, connected, T-invariant subgroup. Then $C_A(T)$ covers $C_{A/A_0}(T)$ and $C_T(A) = C_T(A_0, A/A_0)$.*

This will be applied with T a cyclic group or T a good torus (in Lemma B below we shall remind the reader why a good torus acting faithfully on a module of exponent p can have no elements of order p). Parenthetically said, Tindzogho Ntsiri has obtained in his Ph.D. [28, Section 5.2] an analogue to Maschke's Theorem for subtori of $\mathbb{K}^\times$ in positive characteristic.

When the acting group is not a torus, much less is known—whence the present article. The basic case is the action on a strongly minimal set.

Hrushovski's Theorem (from [9, Theorem 11.98]). *Let G be a connected group of finite Morley rank acting definably, transitively, and faithfully on a set X with $\operatorname{rk} X = \deg X = 1$. Then $\operatorname{rk}(G) \leq 3$, and if G is nonsoluble there is a definable field structure $\mathbb{K}$ such that $G \simeq \mathrm{PSL}_2(\mathbb{K})$.*

Incidently, Wiscons pursued in this permutation-theoretic vein and could classify nonsoluble groups of Morley rank 4 acting sufficiently generically on sets of rank 2 [30, Corollary B], extending and simplifying earlier work by Gropp [19]. Then Altınel and Wiscons [4, preprint] pushed the topic even further by proving that generic 4-transitivity on a set of rank 2 can arise only from the projective action of $\mathrm{PGL}_3(\mathbb{K})$, thus covering the $k = 4, n = 2$ case of the conjecture stated in Section1.3. Although some aspects of Wiscons' work are extremely helpful in the proof below (and we suspect the recent joint work by Altınel and Wiscons would as well, but it was made public only after completion of ours), most of our configurations will be more algebraic as we shall mainly act on modules.

Zilber's Field Theorem (from [9, Theorem 9.1]). *Let $G = A \rtimes H$ be a group of finite Morley rank where A and H are infinite definable abelian subgroups and A is H-minimal. Assume $C_H(A) = 1$. Then there is a definable field structure $\mathbb{K}$ with $H \hookrightarrow \mathbb{K}^\times$ in its action on $A \simeq \mathbb{K}_+$ (all definably).*

Zilber's Field Theorem has several variants and generalisations we shall encounter in the proof of Proposition 2.1. But for the bulk of the argument, the original version we just gave suffices.

Here are two more results of repeated use; notice the difference of settings, since in the rank $3k$ analysis the group is supposed to be given explicitly. The present work extends the rank 2 analysis.

Rank 2 Analysis ([14, Theorem A]). *Let G be a connected, nonsoluble group of finite Morley rank acting definably and faithfully on a connected abelian group V of Morley rank 2. Then there is an algebraically closed field $\mathbb{K}$ of Morley rank 1 such that $V \simeq \mathbb{K}^2$, and G is isomorphic to $\mathrm{GL}_2(\mathbb{K})$ or $\mathrm{SL}_2(\mathbb{K})$ in its natural action.*

Rank $3k$ Analysis ([13]). *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$, a group $G \simeq (\mathrm{P})\mathrm{SL}_2(\mathbb{K})$, an abelian group V, and a faithful action of G on V for which V is G-minimal. Assume $\mathrm{rk}\, V \leq 3\,\mathrm{rk}\,\mathbb{K}$. Then V bears a structure of $\mathbb{K}$-vector space such that:*

- *either $V \simeq \mathbb{K}^2$ is the natural module for $G \simeq \mathrm{SL}_2(\mathbb{K})$,*
- *or $V \simeq \mathbb{K}^3$ is the irreducible 3-dimensional representation of $G \simeq \mathrm{PSL}_2(\mathbb{K})$ with* $\mathrm{char}\, \mathbb{K} \neq 2$.

In particular, $\mathrm{SL}_2(\mathbb{K})$ acting on an abelian group of rank 3 must centralise a rank 1 factor in a composition series; in characteristic not 2, composition series then split thanks to the central involution.

1.5. Two trivial generalities. Here are two principles no one cared to write down so far; they involve good tori and unipotent subgroups.

A *p-torus* τ is a divisible, abelian p-group; if τ is a subgroup of a group of finite Morley rank then it is a direct sum $\mathbb{Z}_{p^\infty}^d$ of *finitely many* [9, ex.9 p.98] copies of the quasicyclic p-group $\mathbb{Z}_{p^\infty}$; the integer $\mathrm{Pr}_p(\tau) = d$ is called the *Prüfer p-rank* of the p-torus. The notion immediately extends to the case of a good torus.

By definition a *p-unipotent subgroup* [17, Section 2.3] is a definable, connected, nilpotent p-group of bounded exponent; in the current state of knowledge nilpotence is a nonredundant requirement. If H is a soluble group of finite Morley rank it has a well-behaved *p-unipotent radical*, denoted $U_p(H)$, and which behaves as expected.

Lemma A. *Let T be a good torus acting definably and faithfully on a module V. Then $\mathrm{rk}\, T \leq \mathrm{rk}\, V$, and for any prime q with $U_q(V) = 1$:*

$$\mathrm{rk}\, T \leq \mathrm{rk}\, V + \mathrm{Pr}_q(T) - \ell_T(V).$$

Proof. We argue by induction on $\mathrm{rk}\, V$. The result is obvious if $\mathrm{rk}\, V = 0$. So let $0 \leq W < V$ be such that V/W is T-minimal, and set $\Theta = C_T(V/W)$. Notice that Θ° is a good torus and acts faithfully on W; one has $\ell_{\Theta^\circ}(W) \geq \ell_T(W) = \ell_T(V) - 1$. So by induction,

$$\mathrm{rk}(\Theta^\circ) \leq \mathrm{rk}\, W + \mathrm{Pr}_q(\Theta^\circ) - \ell_{\Theta^\circ}(W) \leq \mathrm{rk}\, W + \mathrm{Pr}_q(\Theta^\circ) - \ell_T(V) + 1$$

and therefore:

$$\mathrm{rk}\, T \leq \mathrm{rk}\,(T/\Theta) + \mathrm{rk}\, W + \mathrm{Pr}_q(\Theta^\circ) - \ell_T(V) + 1.$$

(Also bear in mind the other estimate $\mathrm{rk}\, T \leq \mathrm{rk}(T/\Theta) + \mathrm{rk}\, W$.)

By Zilber's Field Theorem there is a field structure $\mathbb{K}$ such that $T/\Theta \hookrightarrow \mathbb{K}^\times$ and $V/W \simeq \mathbb{K}_+$ definably (this is not literally true in case $\Theta = T$ as there is no field

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structure around, but this is harmless). Quickly notice that $\operatorname{rk}(T/\Theta) \leq \operatorname{rk}(\mathbb{K}^\times) = \operatorname{rk}(\mathbb{K}_+) = \operatorname{rk}(V/W)$, so $\operatorname{rk} T \leq \operatorname{rk}(V/W) + \operatorname{rk} W = \operatorname{rk} V$. If T/Θ is proper in $\mathbb{K}^\times$, then actually $\operatorname{rk}(T/\Theta) \leq \operatorname{rk} V - \operatorname{rk} W - 1$ whereas $\operatorname{Pr}_q(\Theta^\circ) \leq \operatorname{Pr}_q(T)$: we are done. If on the other hand $T/\Theta \simeq \mathbb{K}^\times$, then $\operatorname{rk}(T/\Theta) = \operatorname{rk} V - \operatorname{rk} W$ and $\operatorname{Pr}_q(\Theta^\circ) = \operatorname{Pr}_q(T) - 1$ since $\mathbb{K}$ does not have characteristic q: we are done again. ⊣

Lemma B. *Let H be a definable, connected group acting definably and faithfully on a module V of exponent p. If H is soluble, then $H = U \rtimes T$ with $U = U_p(H)$ and T a good torus with no elements of order p. Moreover, H centralises all quotients in an H-composition series of V if and only if H is p-unipotent, in which case the exponent is bounded by $q = p^k$ with $q \geq \ell_H(V)$.*

Proof. First suppose H to be soluble. By faithfulness and the structure theorem for locally soluble p-groups [9, Corollary 6.20], H contains no p-torus. Moreover, the only unipotence parameter [17, Section 2.3] which can occur in H is (p, ∞). In particular, $H/U_p(H)$ has no unipotence at all: it is a good torus. Let $T \leq H$ be a maximal good torus of H. Then T covers $H/U_p(H)$, and $T \cap U_p(H) = 1$ since T has no element of order p. Therefore $H = U \rtimes T$ for T a maximal good torus.

If H is actually p-unipotent, it clearly centralises all quotients in an H-composition series. Conversely, if H centralises all quotients in $0 = V_0 < \cdots < V_\ell = H$, then H is soluble of class $\leq \ell - 1$: induction on ℓ, the claim being obvious at $\ell = 1$. So write $H = U \rtimes T$ as above. By assumption, T centralises all quotients in the series so T centralises V; by faithfulness, $T = 0$ and $H = U$ is p-unipotent. Finally observe how for $u \in U$, $(u - 1)^\ell = 0$ in $\operatorname{End}(V)$. So for $q = p^k \geq \ell$, one has $(u - 1)^q = u^q - 1 = 0$ in $\operatorname{End}(V)$ and $u^q = 1$ in H. ⊣

In particular, when acting on a module of exponent p, decent tori [12] of automorphisms are good tori.

§**2. The proof.** We now start proving the theorem. After an initial section (Section 2.1) dealing with various aspects of linearity, we shall adopt a more abstract line. The main division is along values of the Prüfer 2-rank, which measures the size of Sylow 2-subgroups: for such subgroups, including the fundamental conjugacy theorem, [9, Section 10.3] provides all necessary material.

We first handle the pathological case of an acting group with no involutions, which we prove bad; configurations are tight and we doubt that any general lesson can be learnt from Section 2.2. Then Section 2.3 deals with the Prüfer rank 1 case where the adjoint action of $\operatorname{PSL}_2(\mathbb{K})$ is retrieved; this makes use of recent results on abstract, so-called $N_\circ^\circ$-groups. Section 2.4 is essentially different: when the Prüfer rank is 2, we can use classical group-theoretic technology, viz. strongly embedded subgroups.

Notation.

- *Let G be a connected, nonsoluble group of finite Morley rank acting definably and faithfully on an abelian group V of rank 3 which is G-minimal.*
- *Let $S \leq G$ be a Sylow 2-subgroup of G; if G has odd type, let $T \leq G$ be a maximal good torus containing S°.*

Notice that we do *not* make assumptions on triviality of $C_V(G)$; of course by G-minimality of V, the former is finite. For the same reason, V is either of prime exponent or torsion-free; the phrase "the characteristic of V" therefore makes sense.

2.1. Reductions. We first deal with a number of reductions involving a wide arsenal of methods. Model-theoretically speaking we shall use two n-dimensional versions of Zilber's Field Theorem:

- [9, Theorem 9.5] which linearises irreducible actions of non semisimple groups, in the abstract sense of the *connected soluble radical* $R^\circ(G)$ being nontrivial ($R^\circ(G)$, viz. the group generated by all definable, connected, soluble, normal subgoups of G, happens to be the largest such subgroup [10, Section 7.2]; this was first studied by Belegradek);
- [21, Theorem 4], which linearises actions on torsion-free modules.

We shall also invoke work of Poizat [24], generalised by Mustafin [23], on the structure of definably linear groups of finite Morley rank, which in characteristic p is a consequence of Wagner's Torus Theorem.

In a more group-theoretic direction, we shall rely on the classification of the simple groups of finite Morley rank of even type [1], and a theorem of Timmesfeld [27] on abstract $\mathrm{SL}_n(\mathbb{K})$-modules will play a significant part.

Proposition 2.1. *We may suppose that $C_V(G) = 0$, that $R^\circ(G) = 1$, and that V has exponent an odd prime number p. In particular every definable, connected, soluble subgroup $B \leq G$ has the form $B = Y \rtimes \Theta$ where Y is a p-unipotent subgroup and Θ is a good torus (either may be trivial).*

Proof. It follows from irreducibility of G and Macintyre's classical results on abelian groups [9, Theorem 6.7] that V is either of prime exponent or divisible; in the latter case, the structure of soluble p-groups (more accurately a property known as the rigidity of p-tori [9, Theorem 6.16]) forces torsion-freeness of V.

Claim 1. *We may suppose $C_V(G) = 0$.*

Proof of Claim 1. Let $\overline{V} = V/C_V(G)$, which clearly satisfies the assumption of our Theorem. Suppose that the desired classification holds for $\overline{V}$: then $(G, \overline{V})$ is known. If G is a simple bad group of rank 3, we are done as we assert nothing on the action. If G contains $\mathrm{PSL}_2(\mathbb{K})$, we know the structure of V by the rank $3k$ analysis, and $C_V(G) = 0$. If G contains $\mathrm{SL}_3(\mathbb{K})$ acting naturally on $\overline{V}$, we show $C_V(G) = 0$ as follows.

More generally we shall prove the following: if $\mathbb{K}$ is any field of finite Morley rank and $G \simeq \mathrm{SL}_n(\mathbb{K})$ acts definably on a definable, connected module V such that $C_V(G)$ is finite and $V/C_V(G)$ is the natural G-module, then $C_V(G) = 0$. The argument follows that of [14, Fact 2.7].

The module V is G-minimal because $V/C_V(G)$ is and $C_V(G)$ is finite. In particular, if $\mathbb{K}$ has characteristic zero then V is torsion-free and $C_V(G) = 0$. Otherwise, V has prime exponent equal to the characteristic p of $\mathbb{K}$. Set $W = C_V(G)$. Let $x \in V \setminus W$. Consider the image $\overline{x}$ in V/W. Then by inspection, $C_G(\overline{x})$ is a semidirect product $\mathbb{K}^{n-1} \rtimes \mathrm{SL}_{n-1}(\mathbb{K})$; in particular it is connected, and has rank $(n(n-1)-1) \cdot \mathrm{rk}\,\mathbb{K}$. Now by Zilber's Indecomposibility Theorem [9, Section 5.4], $[C_G(\overline{x}), x]$ is a connected subgroup of the finite group W, hence trivial: it follows that $C_G(\overline{x}) = C_G(x)$, a group we denote by H. Moreover, $O = x^G$ has rank $n \cdot \mathrm{rk}\,\mathbb{K}$ so it is generic in V. By connectedness of V, $V \setminus O$ is not generic. Fix $w_0 \in W \setminus \{0\}$. Since $\langle w_0 \rangle$ is finite there is a translate $v + \langle w_0 \rangle$ of $\langle w_0 \rangle$ contained in O. Hence there are x and y in V with $y = x + w_0$ and $y = x^g$ for some $g \in G$. Iterating, one finds $x^{g^p} = x$, so $g^p \in H \simeq \mathbb{K}^{n-1} \rtimes \mathrm{SL}_{n-1}(\mathbb{K})$. But on the other hand, since G

centralises w_0, g normalises H (the author forgot to write down this sentence in the proof of [14, Fact 2.7]). Now $g \in N_G(U_p(H))$ which is an extension of H by a torus as a computation in $\mathrm{SL}_n(\mathbb{K})$ reveals. This and $g^p \in H$ show $g \in H$, so $x = y$: a contradiction. ⊣

Claim 2. *If G is definably linear (i.e., there is a field structure $\mathbb{K}$ such that $V \simeq \mathbb{K}^n$ and $G \hookrightarrow \mathrm{GL}(V)$, all definably), then the theorem is proved.*

Proof of Claim 2. Suppose that there is a definable field structure $\mathbb{K}$ with $V \simeq \mathbb{K}^n_+$ and $G \hookrightarrow \mathrm{GL}(V)$ definably. Then clearly $\mathrm{rk}\, \mathbb{K} = 1$ and $n = 3$; hence $G \leq \mathrm{GL}_3(\mathbb{K})$ is a definable subgroup. Be careful that a field of Morley rank 1 need not be a pure field (see [20] for the most dramatic example), so there remains something to prove.

We shall show that G is a closed subgroup of $\mathrm{GL}_3(\mathbb{K})$. If $R^\circ(G) \neq 1$ then linearising again with [9, Theorem 9.5] and up to taking $\mathbb{K}$ to be the newly found field structure, $R^\circ(G) = \mathbb{K}^\times \mathrm{Id}_V$. We then go to $H = (G \cap \mathrm{SL}(V))^\circ$, which satisfies $G = H \cdot R^\circ(G)$, so that it suffices to show that H is closed. Hence we may assume $R^\circ(G) = 1$. If the characteristic is finite then by [23, Theorem 2.6], we are done. So we may assume that V is torsion-free. If the definable subgroup $G \leq \mathrm{GL}_3(\mathbb{K})$ is not closed, by [23, Theorem 2.9], we find a definable subgroup $K \leq G$ which contains only semisimple elements, in the geometric sense of the term. We may assume that K is minimal among definable, connected, nonsoluble groups: it is then a bad group. But $\mathrm{rk}\, \mathbb{K} = 1$, so any definable, connected, proper subgroup of K is actually a good torus and contains involutions: this contradicts the bad group analysis. One could also argue through the unfortunately unpublished [7].

As a consequence, G is closed and therefore algebraic. We now inspect irreducible, algebraic subgroups of $\mathrm{GL}_3(\mathbb{K})$ to conclude. ⊣

Claim 3. *If $R^\circ(G) \neq 1$ or V is torsion-free then the theorem is proved.*

Proof of Claim 3. If $R^\circ(G) \neq 1$, then we linearise the setting with [9, Theorem 9.5] and rely on Claim 2. If V is torsion-free then we use [21, Theorem 4] with the same effect. ⊣

So we may assume that V has prime exponent p. As a consequence, any definable, connected, soluble subgroup $B \leq G$ has the form $B = Y \rtimes \Theta$ where Y is a p-unipotent subgroup and Θ is a good torus.

Claim 4. *If V has exponent 2 then the theorem is proved.*

Proof of Claim 4. Here we draw the big guns: the even type classification [1]. Keep $R^\circ(G) = 1$ in mind. Let $H \leq G$ be a component, which is a quasisimple algebraic group over a field of characteristic 2; since there are finitely many components and G is connected, H is normal in G. Always by connectedness of G, notice that H acts irreducibly (for instance because of the finite Morley rank analogue of Clifford's Theorem, [9, Theorem 11.8]).

Since $\mathrm{SL}_2(\mathbb{K}) \simeq \mathrm{PSL}_2(\mathbb{K})$ has no irreducible rank 3 module in characteristic 2 by the rank $3k$ analysis, we know $H \not\simeq \mathrm{SL}_2(\mathbb{K})$. Now let $T_H \leq H$ be an algebraic torus. Then $\mathrm{rk}\, T_H \leq \mathrm{rk}\, V = 3$, so H has Lie rank at most 3. Therefore H is a simple algebraic group of one of types A_2, B_2, A_3, B_3, C_3, or G_2.

Let us prove that H has type A_2. A brief look at the extended Dynkin diagrams for these groups [18] shows that in all other cases, H contains a subgroup of type $A_1 + A_1$

(this is reflected by the presence of nonadjacent nodes in the extended diagram), that is, a direct product of two simple groups SL_2. Let us write it as $L_1 \times L_2 \leq H$ and let U_2 be a maximal unipotent subgroup in L_2. Then L_1 centralises U_2 and normalises $[V, U_2]$ which is nonzero by faithfulness, definable and connected by Zilber's Indecomposibility Theorem, and proper in V by nilpotence of $V \rtimes U_2$ (the latter follows from the structure of soluble groups [17, Section 2.3]). Also notice that by perfectness, $C_V(L_i)$ has rank at most 1, but be careful that L_i must centralise any rank 1 subquotient module it normalises. If $\mathrm{rk}[V, U_2] = 1$ then $[V, U_2] \leq C_V(L_1)$ and $C_V^\circ(L_1) = [V, U_2]$ has rank 1, showing $C_V^\circ(L_2) = C_V^\circ(L_1) = [V, U_2]$: so U_2 and therefore L_2 centralise $V/[V, U_2]$, against perfectness. If $\mathrm{rk}[V, U_2] = 2$ then L_1 centralises $V/V[V, U_2]$. Let T_1 be an algebraic torus of L_1; using semisimple actions of Section 1.4 we just saw $C_V(T_1) \neq 0$, so that $V = C_V(T_1) \oplus [V, T_1]$ is a nontrivial decomposition into nontrivial summands normalised by U_2. So $\mathrm{rk}[V, U_2] = 1$: a contradiction again. As a consequence H has type A_2.

Now T_H extends to a maximal good torus of G, still of rank at most 3, and there are therefore no other components. As a consequence, $G = H \simeq (\mathrm{P})\mathrm{SL}_3(\mathbb{K})$.

It remains to identify the action. We rely on work by Timmesfeld [27].

Let $U_0 \leq G$ be a root subgroup, say $U_0 \leq G_0 \simeq \mathrm{SL}_2(\mathbb{K})$. We claim that $\mathrm{rk}[V, U_0] = 1$: the argument is essentially like above. Let T_1 be a one-dimensional torus centralising U_0, say $T_1 \leq G_1 \simeq \mathrm{SL}_2(\mathbb{K})$. Now by the rank $3k$ analysis, G_1 cannot act irreducibly, so there is a G_1-composition series for V where G_1 centralises the rank 1 factor. Hence $C_V(T_1) \neq 0$, and as above $V = C_V(T_1) \oplus [V, T_1]$ is a nontrivial decomposition into nontrivial summands normalised by U_0. Hence $\mathrm{rk}[V, U_0] = 1$.

As a consequence, if $U_1 \leq C_G(U_0)$ is another root subgroup, then U_1 centralises the rank 1 subgroup $[V, U_0]$, meaning $[V, U_0, U_1] = 0$. So we are under the assumptions of [27] and conclude that $G \simeq \mathrm{SL}_3(\mathbb{K})$ acts on $V \simeq \mathbb{K}^3$ as on its natural module. ⊣

This concludes our series of reductions. ⊣

We finish these preliminaries with a definition and an observation.

Definition 2.2. Let G be a group of finite Morley rank and V be a G-module. A definable, connected subgroup $V_1 \leq V$ is called a *TI* subgroup (for: Trivial Intersections) if $V_1 \cap V_1^g = 0$ for all $g \notin N_G(V_1)$.

Observation 2.3. *If* $V_1 \leq V$ *is a TI-subgroup, then* $\mathrm{cork}\, N_G(V_1) = \mathrm{rk}(G/N_G(V_1)) \leq \mathrm{rk}\, V - \mathrm{rk}\, V_1$.

Proof. Consider the family $\mathcal{F} = \{V_1^g : g \in G\}$: its rank is $\mathrm{cork}\, N_G(V_1)$. The TI assumption means that elements of the family are pairwise disjoint, so $\mathrm{rk} \bigcup_{\mathcal{F}} = \mathrm{rk}\, \mathcal{F} + \mathrm{rk}\, V_1 \leq \mathrm{rk}\, V$ and $\mathrm{cork}\, N_G(V_1) = \mathrm{rk}\, \mathcal{F} \leq \mathrm{rk}\, V - \mathrm{rk}\, V_1$. ⊣

As a consequence, in our setting where $\mathrm{rk}\, V = 3$, a TI-subgroup V_1 will always have rank 1 and satisfy $\mathrm{cork}\, N_G(V_1) \leq 2$.

The rest of the proof is a case division along the Prüfer 2-rank of G. It is much more group-theoretic, and much less model-theoretic, in nature.

2.2. To have and have not (involutions). The case division on the Prüfer 2-rank starts here. We shall first deal with desperate situations: if G has no involutions,

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then it is a simple bad group of rank 3 (Proposition 2.5). If on the other hand it does have involutions, then it has a Borel subgroup of mixed nature $\beta = Y \rtimes \Theta$ (Proposition 2.6; the definition of a Borel subgroup is in a few paragraphs).

More material. The main ingredients in this section are Hrushovski's Theorem on strongly minimal actions, the analysis of bad groups, and Wiscons' analysis of groups of rank 4. But uniqueness principles in $N_\circ^\circ$-groups also play a key role.

Wiscons' Analysis (from [30, Corollary A]). *If G is a connected group of rank* 4 *with involutions then* $R^\circ(G) \neq 1$.

(Actually Wiscons states his result in terms of the *Fitting subgroup* $F^\circ(G)$ [9, Section 7.2], but for our purposes the soluble radical is enough.)

Recall from [17] that a group of finite Morley rank G is an $N_\circ^\circ$-group if for any infinite, definable, connected, abelian subgroup $A \leq G$, the connected normaliser $N_G^\circ(A)$ is soluble. Our theorem is the second application of the theory of $N_\circ^\circ$-groups after [14]. However, because of Proposition 2.1, only the rather straightforward, positive version of uniqueness principles will be used; Burdges' subtle unipotence theory [17, Section 2.3] will not.

Definition 2.4. A Borel subgroup is a definable, connected, soluble subgroup which is maximal with respect to these properties.

Uniqueness Principles in $N_\circ^\circ$-groups (from [17, Fact 8]). *Let G be an $N_\circ^\circ$-group and B be a Borel subgroup of G. Let $U \leq B$ be a nontrivial, p-unipotent subgroup of B. Then B is the only Borel subgroup of G containing U.*

It may be good to keep in mind that if G is an $N_\circ^\circ$-group and B is a Borel subgroup with $U_p(B) \neq 1$, then $U_p(B)$ is actually a *maximal* p-unipotent group of G.

2.2.1. The Prüfer Rank 0 *Analysis: Bad Groups.*

Proposition 2.5. *If G has no involutions, then G is a simple bad group of rank* 3.

Proof. By the rank 2 analysis and since there are no involutions, any definable, connected, reducible subgroup is soluble. Let $A \leq G$ be a nontrivial, definable, connected, and abelian subgroup. If $N = N_G^\circ(A) < G$ is irreducible then by induction N can only be a bad group of rank 3, a contradiction. Hence N is reducible and therefore soluble. As a consequence G is an $N_\circ^\circ$-group and we shall freely use uniqueness principles in Claims 2 and 3.

Claim 1. *For $v_0 \in V \setminus \{0\}$, $C_G^\circ(v_0)$ is a soluble group of corank* 2.

Proof of Claim 1. Observe how $\bigcap_{g\in G} C_G^\circ(v_0)^g \leq C_G(\langle v_0^G\rangle) = C_G(V) = 1$ by faithfulness. So if the corank of $H = C_G^\circ(v_0)$ is 1 we apply Hrushovski's Theorem to the action of G on G/H and find $G \simeq \mathrm{PSL}_2(\mathbb{K})$, a contradiction to the absence of involutions. If on the other hand cork $H = 3$ then v_0^G is generic in V, and so is $-v_0^G$: lifting torsion [9, ex.11 p. 98], this creates an involution in G, a contradiction again.

Now suppose that H is nonsoluble: it is therefore irreducible, so by induction it is a bad group of rank 3. In particular $\mathrm{rk}\, G = 5$; still by Hrushovski's Theorem, $N_G^\circ(H) = H$. Hence $\{H^g : g \in G\}$ has rank 2 and degree 1. Always by Hrushovski's Theorem (this time inside H), for $g \notin N_G(H)$, $H \cap H^g$ has rank 1, so $N_H(H^g)$ has

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rank 1. Therefore all orbits in the action of H on $\{H^g : g \notin N_G(H)\}$ are generic: the action is transitive. This shows that G acts 2-transitively on $\{H^g : g \in G\}$, and lifting torsion there is an involution in G: a contradiction. ⊣

Notation. *Let $B = Y \rtimes \Theta$ be a Borel subgroup, with Y a p-unipotent subgroup and Θ a good torus (either term or the action may be trivial).*

Claim 2. *At least one of Y or Θ is trivial.*

Proof of Claim 2. Since Θ is a good torus it has no elements of order p: this is the second of our general Lemmas in Section 1.5. So we are in the setting of a semisimple action (Section 1.4), and we know $V = C_V(\Theta) \oplus [V, \Theta]$; it is however not clear whether one factor may be trivial.

But suppose $Y \neq 1$ and $\Theta \neq 1$. Then Θ acts on $C_V^\circ(Y) \neq 0$ so V is not Θ-minimal. In a Θ-composition series there is therefore a Θ-invariant subquotient module of V of rank 1, say X_1. By Zilber's Field Theorem and since G has no involutions, Θ centralises X_1, and this shows $C_V(\Theta) \neq 0$. Hence $V = C_V(\Theta) \oplus [V, \Theta]$ is a nontrivial decomposition. Again with Zilber's Field Theorem and the absence of involutions, $[V, \Theta]$ has rank 2 and Θ has rank 1. Always for the same reasons, $Y \rtimes \Theta$ is now nilpotent and Θ centralises Y. So Y normalises both $C_V(\Theta)$ and $[V, \Theta]$, and it follows $\operatorname{rk} C_V^\circ(Y) \geq 2$. Then for $g \notin N_G(Y)$ the group $\langle Y, Y^g\rangle$ is reducible, therefore soluble, forcing $Y = Y^g$: a contradiction. ⊣

Claim 3. *If $Y \neq 1$, then G is a simple bad group of rank 3.*

Proof of Claim 3. Let $V_1 = C_V^\circ(Y) \neq 1$. If $\operatorname{rk} V_1 = 2$ then for $g \notin N_G(Y)$ the group $\langle Y, Y^g\rangle$ is reducible, therefore soluble, which forces $Y = Y^g$, a contradiction. Hence $\operatorname{rk} V_1 = 1$. Suppose that V_1 is not TI: there are $g \notin N_G(V_1)$ and $v_1 \in V_1 \cap V_1^g \setminus \{0\}$. Then $H = C_G^\circ(v_1) \geq \langle Y, Y^g\rangle$ is soluble by Claim 1, which yields the same contradiction.

We have just proved that V_1 is a rank 1, TI subgroup. But $Y \leq N_G^\circ(V_1)$, and equality follows since $N_G^\circ(V_1)$ is soluble and Y is a Borel subgroup by Claim 2. So $\operatorname{cork} Y \leq 2$; by uniqueness principles, $2 \operatorname{rk} Y \leq \operatorname{rk} G \leq \operatorname{rk} Y + 2$ and $\operatorname{rk} G \leq 4$. As a matter of fact, Wiscons' work [30] rules out equality; let us give a quick argument. If $\operatorname{rk} Y = 2$ and $\operatorname{rk} G = 4$, then exactly like in Claim 1, Y is transitive on $\{Y^g : g \notin N_G(Y)\}$, which creates an involution in G. Hence $\operatorname{rk} G = 3$.

It remains to prove simplicity. Observe that G has no good torus since (for instance) Borel subgroups of $G/Z(G)$ are conjugate by the bad group analysis. So torsion in G consists of p-elements. Now if $\alpha \in Z(G)$ then $C_V^\circ(\alpha) \neq 0$, contradicting G-minimality. ⊣

Claim 4. *If G has no unipotent subgroup, then G is a simple bad group of rank 3.*

Proof of Claim 4. Suppose that V is Θ-minimal: then by Zilber's Field Theorem Θ acts freely on V. Let $v_0 \in V \setminus \{0\}$. By Claim 1, $C_G^\circ(v_0)$ is soluble, therefore a good torus. By the conjugacy of maximal good tori we may assume $C_G^\circ(v_0) \leq \Theta$, against freeness of Θ.

This shows that V is not Θ-minimal. Like in Claim 2, $C_V(\Theta) \neq 0$, $\operatorname{rk}[V, \Theta] = 2$, and $\operatorname{rk} \Theta = 1$. Let $v_0 \in C_V(\Theta)$; then $\Theta \leq C_G^\circ(v_0)$ but Θ is a Borel subgroup and $C_G^\circ(v_0)$ is soluble by Claim 1: this shows $\Theta = C_G^\circ(v_0)$ and $\operatorname{rk} G = 3$.

It remains to prove simplicity. If there is $\alpha \in Z(G)$ then up to taking a power, α has prime order $q \neq p$. By torality principles [10, Corollary 3], $\alpha \in \Theta$: hence $C_V^\circ(\alpha) \neq 0$, contradicting G-minimality. $\dashv$

This concludes the Prüfer rank 0 analysis. $\dashv$

2.2.2. Good groups. From now on we shall suppose that G has involutions. It follows easily that G is not bad (this is done in the proof below); yet it is not clear at all whether G has a non-nilpotent Borel subgroup. For the moment one could imagine that all proper, nonsoluble subgroups of G are bad of rank 3, with G simple. (Recall that a bad group is defined by the condition that *all* definable, connected, proper subgroups are nilpotent: not only the soluble ones.) We nonetheless push a little further towards nonbadness. Recall that we had let $S \leq G$ be a Sylow 2-subgroup: in view of Proposition 2.1 and the current assumption, S° is a 2-torus; we had also let $T \leq G$ be a maximal good torus containing S°.

PROPOSITION 2.6. *Suppose that G has an involution. Then G has a Borel subgroup $\beta = Y \rtimes \Theta$ where $Y \neq 1$ is a nontrivial p-unipotent group and $\Theta \neq 1$ is a nontrivial good torus (but the action may be trivial). Moreover V as a T-module has length $\ell_T(V) \geq 2$.*

PROOF. We address the first claim; the second one will be proved in the final Claim 7. Suppose that G has no such Borel subgroup. Then all definable, connected, soluble subgroups are nilpotent. Therefore by the rank 2 analysis all definable, connected, nonsoluble, proper subgroups are irreducible, so by induction such subgroups are simple bad groups of rank 3. It also follows that G is an $N_\circ^\circ$-group; we shall use uniqueness principles.

CLAIM 1. *The group G has no unipotent subgroup.*

PROOF OF CLAIM 1. Suppose it does, and let $U \neq 1$ be maximal as such. By assumption, U is a Borel subgroup of G. Now for $g \notin N_G(U)$, $U \cap U^g = 1$. Otherwise, there is $x \in U \cap U^g \setminus \{1\}$. But x is a p-element, so $C_G^\circ(x)$ is reducible and therefore soluble, and it contains $Z^\circ(U)$ and $Z^\circ(U^g)$. This contradicts uniqueness principles in $N_\circ^\circ$-groups.

As a consequence, U is disjoint from its distinct conjugates and of finite index in its normaliser, therefore U^G is generic in G. By [10, Theorem 1], the definable hull $d(u)$ of the generic element $u \in U$ now contains a maximal 2-torus: a contradiction. $\dashv$

It follows that T is a Borel subgroup.

CLAIM 2. *There is a good torus $\Theta \leq T$ of rank 1 with no involutions.*

PROOF OF CLAIM 2. Quickly notice that G itself is not bad. If it is, then by the bad group analysis and since there are involutions, G is not simple: there is an infinite, proper, normal subgroup $N \lhd G$; since G is bad, N is nilpotent, against Proposition 2.1.

Hence G is not bad. By definition there is a definable, connected, non-nilpotent, proper subgroup $H < G$: H is nonsoluble, hence a bad group of rank 3. Let $\Theta < H$ be a Borel subgroup of H: since G has no unipotent elements, Θ is a good torus of rank 1, and has no involutions.

By the conjugacy of maximal good tori in G we may assume $\Theta \leq T$; inclusion is proper since T does have involutions. $\dashv$

Claim 3. *The T-module V is not T-minimal and* $\operatorname{rk} T = 2$.

Proof of Claim 3. If V is T-minimal, then by Zilber's Field Theorem T acts freely. Now for $v_0 \in V \setminus \{0\}$, $C_G^\circ(v_0)$ contains neither unipotent, nor toral subgroups: by Reineke's Theorem it is trivial and $\operatorname{rk} G = 3$. Now G is a quasisimple bad group of rank 3 but it contains an involution: against the bad group analysis. So V is not T-minimal, $\ell_T(V) \geq 2$; since $\operatorname{rk}(V) = 3$ and $\operatorname{Pr}_2(T) = 1$, we deduce $\operatorname{rk} T \leq 2$. ⊣

Claim 4. *The centraliser $V_1 = C_V(\Theta)$ has rank 1 whereas the commutator subgroup $V_2 = [V, \Theta]$ has rank 2. There is a field structure $\mathbb{L}$ with $V_2 \simeq \mathbb{L}_+$ and $\Theta < \mathbb{L}^\times$.*

Proof of Claim 4. Since V is not T-minimal, it is not Θ-minimal either. Notice that Θ having no involutions, must centralise rank 1 subquotient modules by Zilber's Field Theorem. It follows from facts on semisimple actions that $V = C_V(\Theta) \oplus [V, \Theta]$ where $V_1 = C_V(\Theta)$ has rank 1 and $V_2 = [V, \Theta]$ has rank 2 and is Θ-minimal. Apply Zilber's Field Theorem again to get the desired structure. ⊣

Claim 5. *If T does not centralise V_1, then we are done.*

Proof of Claim 5. Suppose that T does not centralise V_1, meaning $C_T^\circ(V_1) = \Theta$. By Zilber's Field Theorem there is a field structure $\mathbb{K}$ with $V_1 \simeq \mathbb{K}_+$ and $T/\Theta \simeq T/C_T(V_1) \simeq \mathbb{K}^\times$. But Θ is a nontrivial good torus, so there is a prime number $q \neq 2$ with $\operatorname{Pr}_q(\Theta) = 1$, showing $\operatorname{Pr}_q(T) \geq 2$. In particular, T does not embed into $\mathbb{L}^\times$, so $\tau = C_T^\circ(V_2)$ is infinite. Since $\tau \cap \Theta = 1$, one finds $T = \Theta \times \tau$. Finally let $v_2 \in V_2 \setminus \{0\}$ and $K = C_G^\circ(v_2) \geq \tau$. If K is non-soluble, then it is a bad group of rank 3, a contradiction since τ has involutions. So K is soluble and by the structure of Borel subgoups, $K \leq T$. Since Θ acts freely on V_2, $K = \tau$ has corank at most 3, and G has rank at most 4. By Wiscons' analysis, $R^\circ(G) \neq 1$: against Proposition 2.1. ⊣

Claim 6. *If T centralises V_1, then we are done.*

Proof of Claim 6. Now suppose instead that T centralises V_1. Observe how $C_V(T) = V_1$ and $N_G^\circ(V_1) = T$ by solubility of the former and maximality of the latter as a definable, connected, soluble group; in particular $N_G(T) = N_G(V_1)$. If V_1 is not TI, then there are $g \notin N_G(V_1)$ and $v_1 \in V_1 \cap V_1^g \setminus \{0\}$; now $K = C_G^\circ(v_1) \geq \langle T, T^g \rangle$ is nonsoluble and therefore a bad group of rank 3, a contradiction to $\operatorname{rk} T = 2$. Hence V_1 is TI, proving $\operatorname{cork} T \leq 2$ and $\operatorname{rk} G \leq 4$. Finish like in Claim 5. ⊣

We have proved the main statement; it remains to study the length of V as a T-module.

Claim 7. *Consequence: V is not T-minimal.*

Proof of Claim 7. Suppose it is. Then by Zilber's Field Theorem there is a field structure $\mathbb{L}$ with $V \simeq \mathbb{L}_+$ and $T \leq \mathbb{L}^\times$. But let $\beta = Y \rtimes \Theta$ be a Borel subgroup of mixed structure; conjugating maximal good tori we may assume $\Theta \leq T$. Consider $W = C_V^\circ(Y) \neq 0$. Then Θ normalises W and V/W, and one of them, say X_1, has rank 1. By freeness of toral elements and Zilber's Field Theorem, there is a definable field structure $\mathbb{K}$ with $X_1 \simeq \mathbb{K}_+$ and $\Theta \simeq \mathbb{K}^\times$. Hence $\mathbb{K}^\times \simeq \Theta \leq T \leq \mathbb{L}^\times$, and T/Θ is torsion-free. Now Wagner's Torus Theorem forces $T = \Theta$: so V is not T-minimal. ⊣

The proposition is proved. ⊣

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The Borel subgroup β will serve as a *deus ex machina* in Claim 3 of Proposition 2.9; be careful that for the moment it is not clear whether β need be non-nilpotent. The obstacle lies in the possibility for G to contain a "bad unipotent centraliser," we mean a bad group $K = C_G^\circ(v_0)$ of rank 3 with unipotent type, in Claim 2 of Proposition 2.6 above. The spectre of bad groups will be haunting the Prüfer rank 1 analysis hereafter (and notably Proposition 2.9), but we are done with pathologically tight configurations.

2.3. The Prüfer rank 1 analysis. This section is devoted to the adjoint representation of $\mathrm{PSL}_2(\mathbb{K})$ (Proposition 2.9); with an early interest in Section 2.4 we shall do slightly more (Proposition 2.7).

More material. The classification of $N_\circ^\circ$-groups will be heavily used throughout this section, except in Proposition 2.8 where uniqueness principles will nonetheless give the *coup de grâce*.

$N_\circ^\circ$ Analysis (from [17]). *Let G be a connected, nonsoluble, $N_\circ^\circ$-group of finite Morley rank of odd type and suppose $G \not\simeq \mathrm{PSL}_2(\mathbb{K})$. Then the Sylow 2-subgroup of G is isomorphic to that of $\mathrm{PSL}_2(\mathbb{C})$, is isomorphic to that of $\mathrm{SL}_2(\mathbb{C})$, or is a 2-torus of Prüfer 2-rank at most 2.*

Suppose in addition that for all $i \in G$, $C_G^\circ(i)$ is soluble. Then involutions are conjugate and for $i \in I(G)$, $C_G^\circ(i)$ is a Borel subgroup of G. If $i \neq j$ in $I(G)$, then $C_G^\circ(i) \neq C_G^\circ(j)$.

Of course one could imagine a more direct proof, reproving the necessary chunks of [17] in the current, particularly nice context where the structure of soluble groups is very well understood.

2.3.1. $N_\circ^\circ$-ness and bounds. We start with a proposition that will be used only in higher Prüfer rank (Section 2.4).

Proposition 2.7. *If G is an $N_\circ^\circ$-group then $\mathrm{Pr}_2(G) = 1$.*

Proof. Suppose the Prüfer rank is ≥ 2. By the $N_\circ^\circ$ analysis, the Sylow 2-subgroup of G is isomorphic to $\mathbb{Z}_{2^\infty}^2$. In particular, since the Sylow 2-subgroup of G is connected, G has no subquotient isomorphic to $\mathrm{SL}_2(\mathbb{K})$ (see [17, Lemma L] if necessary): by the rank 2 analysis, every definable, connected, reducible subgroup is soluble.

Notation. *Let $\{i, j, k\}$ be the involutions in $S = S^\circ$.*

Claim 1. *For $\ell = i, j, k$, $C_G^\circ(\ell)$ is soluble.*

Proof of Claim 1. Call a 2-element $\zeta \in G$ *meek* if $C_G^\circ(\zeta)$ is soluble; a systematic study of meek elements will be carried in the Prüfer rank 2 analysis (Proposition 2.13). Suppose there is a nonmeek involution i.

Bear in mind that for any 2-element $\zeta \in G$, $C_G^\circ(\zeta)$ has Prüfer rank 2 by torality principles. So restricting ourselves to connected centralisers of 2-elements whenever they are nonsoluble and proper, we descend to a definable, connected, nonsoluble subgroup $H \leq G$ with $\mathrm{Pr}_2(H) = 2$ and such that every 2-element in H is either meek or central in H. Since we are after a contradiction and H remains irreducible on V by nonsolubility, we may suppose $G = H$. Since i is now central in G, it inverts V; so j and k are therefore meek.

Since G is a nonsoluble $N_\circ^\circ$-group, one has $Z^\circ(G) = 1$: there are finitely many nonmeek elements in S. Take one of maximal order and $\alpha \in S$ be a square root. Notice that $\alpha^2 \neq 1$ since i is not meek. By construction α^2 is central in G, and any element of the same order as α is meek: this applies to $j\alpha$ since $\alpha^2 \neq 1$. Let us factor out $\langle \alpha^2 \rangle$ (possibly losing the action on V); let $\overline{G} = G/\langle \alpha^2 \rangle$ and denote the projection map by π. Observe that by Zilber's Indecomposibility Theorem and finiteness of $\langle \alpha^2 \rangle$, one has for any $g \in G$:

$$\pi^{-1}\left(C^\circ_{\overline{G}}(\overline{g})\right)^\circ = C^\circ_G(g).$$

In particular $\overline{j} = \overline{k}$ remains a meek involution of $\overline{G}$, and $\overline{\alpha} \neq \overline{j\alpha}$ have become meek involutions as well. So in $\overline{G}$, all involutions are meek.

By the $N_\circ^\circ$ analysis, $\overline{\alpha}$ is then $\overline{G}$-conjugate to $\overline{j}$ by some $\overline{w}$. Lifting to an element $w \in G$, one sees that $j^w = \alpha z$ for some $z \in \langle \alpha^2 \rangle$. Now $\alpha^2 = z^{-2}$ and this proves that α actually has order 2: a contradiction. ⊣

Claim 2. *Contradiction.*

Proof of Claim 2. By the $N_\circ^\circ$ analysis, $B_i = C^\circ_G(i)$ is a non-nilpotent Borel subgroup. So B_i contains some nontrivial p-unipotent subgroup $U_i = U_p(B_i)$. The involution $i \notin Z(G)$ centralises U_i, so U_i normalises $C_V(i)$ and $[V, i]$ where both are nontrivial, and this forces $\operatorname{rk} C^\circ_V(U_i) = 2$. Now for arbitrary g, $\langle U_i, U_i^g \rangle$ is reducible, hence soluble: by uniqueness principles $U_i = U_i^g$, a contradiction. ⊣

This completes the proof. ⊣

Let us repeat that Proposition 2.7 will be used only in the Prüfer rank 2 analysis, Section 2.4.

2.3.2. Bounds and $N_\circ^\circ$-ness. The next proposition is a converse to Proposition 2.7 and the real starting point of the Prüfer rank 1 analysis. Notice that it does *not* use the $N_\circ^\circ$ analysis, though uniqueness principles add the final touch.

Proposition 2.8. *Suppose that* $\Pr_2(G) = 1$. *Then any definable, connected, reducible subgroup is soluble; in particular G is an $N_\circ^\circ$-group.*

Proof. In this proof (and in others with a similar flavour) we shall use the following shorthands: for i an involutive automorphism of V, we let $V^{+_i} = C_V(i)$ and $V^{-_i} = [i, V]$.

Claim 1. *Any definable, connected, reducible, nonsoluble subgroup $H \leq G$ has the form $H = U \rtimes C$, where $C \simeq \mathrm{SL}_2(\mathbb{K})$ and the central involution $i \in C$ inverts the p-unipotent group U;* $\operatorname{rk} H \neq 4, 6$. *Moreover if H has a rank 1 submodule V_1 then $V_1 = V^{+_i} = C^\circ_V(H)$; if H has a rank 2 submodule V_2 then U centralises $V_2 = V^{-_i}$.*

Proof of Claim 1. By nonsolubility of H the length of V as an H-module is $\ell_H(V) = 2$; the argument, if necessary, is as follows. Suppose $\ell_H(V) = 3$. Then all factors in a composition series are minimal, so by [25, Proposition 3.12] for instance, H' centralises them all. Then H is clearly soluble: a contradiction.

So there is an H-composition series $0 < W < V$; let X_1 be the rank 1 factor and X_2 likewise; set $U = C_H(X_2)$. Then by the rank 2 analysis, $H/U \simeq \mathrm{SL}_2(\mathbb{K})$. Before proceeding we need to handle connectedness of U: it follows from the nonexistence of perfect central extensions of $\mathrm{SL}_2(\mathbb{K})$ [3, Theorem 1] by considering the isomorphisms $(H/U^\circ)/(U/U^\circ) \simeq H/U \simeq \mathrm{SL}_2(\mathbb{K})$. Hence $U = U^\circ$.

As a consequence of Zilber's Field Theorem, U which has no involutions must centralise X_1: $[V, U, U] = 1$ so U is abelian. Moreover, for $u \in U$ and $v \in V$ there is $w \in W$ with $v^u = v + w$. It follows $v^{u^p} = v + pw = v$ and U has exponent p.

Now let $i \in H$ be a 2-element lifting the central involution in $\mathrm{SL}_2(\mathbb{K})$: since U has no involutions, i is a genuine involution in H. Since both U and $(i \mod U) \in H/U \simeq \mathrm{SL}_2(\mathbb{K})$ centralise X_1, i centralises X_1; whereas since U centralises X_2 and $(i \mod U)$ inverts it, i inverts X_2. We then find a decomposition $V = V^+ \oplus V^-$ under the action of i, with $\mathrm{rk}\, V^+ = 1$ and $\mathrm{rk}\, V^- = 2$.

- If $W = X_1 \leq V$ then U, H/U, and therefore H as well centralise W so $V^+ = W$. For $u \in U$ and $v_- \in V^-$, there is $v_+ \in V^+$ with $v_-^u = v_- + v_+$, so $v_-^{ui} = -v_- + v_+$ and $v_-^{uiui} = v_- - v_+ + v_+ = v_-$: $uiui$ centralises $V_+ + V_- = V$. (Incidently, in this case, H centralises W; by nonsolubility of H, $\mathrm{rk}\, C_V^\circ(H) = 1$.)
- If $W = X_2 \leq V$ then U centralises W and i inverts $W = V^-$. For $u \in U$ and $v_+ \in V^+$, there is $v_- \in V^-$ with $v_+^u = v_+ + v_-$, so $v_+^{ui} = v_+ - v_-$ and $v_+^{uiui} = v_+$, so $uiui$ centralises $V_+ + V_- = V$. (Incidently, in this case, U centralises W.)

In either case, i inverts U. All involutions of H are equal modulo U and i inverts the 2-divisible group U, so for $h \in H$, $i^h \in iU = i^U$ and $H = U \cdot C_H(i) = U \rtimes C_H(i)$. Clearly $C_H(i) = C_H^\circ(i) \simeq \mathrm{SL}_2(\mathbb{K})$. Of course $\mathrm{rk}\, \mathbb{K} = 1$; if $\mathrm{rk}\, H \leq 6$ then by the rank $3k$ analysis and since i inverts U, $\mathrm{rk}\, U$ must be 0 or 2, proving $\mathrm{rk}\, H \neq 4, 6$. ⊣

We start a contradiction proof. Suppose that G contains a definable, connected, reducible, nonsoluble group: by Claim 1, G contains a subgroup $C \simeq \mathrm{SL}_2(\mathbb{K})$.

Notation. *Let $C \leq G$ be isomorphic to $\mathrm{SL}_2(\mathbb{K})$ and $i \in C$ be the central involution.*

Before we start more serious arguments, notice that a Sylow 2-subgroup of C is one of G: this is a folklore consequence of torality principles. Notice further that by the rank $3k$ analysis, $V = V^{+_i} \oplus V^{-_i}$ and the action of C on the latter is known. In particular V^{-_i} is not T-minimal, so that $\ell_T(V) = 3$, implying $\mathrm{rk}\, T = 1$. Finally C centralises V^{+_i} which has rank 1.

Notation. *Let $v_+ \in V^{+_i} \setminus \{0\}$ and $v_- \in V^{-_i} \setminus \{0\}$ (these exist by the rank $3k$ analysis). Set $H_+ = C_G^\circ(v_+)$ and $H_- = C_G^\circ(v_-)$.*

Claim 2. *Both H_+ and H_- have corank 2 or 3 but not both have corank 3. Moreover, $H_+ \simeq U_+ \rtimes C$ where U_+ is a p-unipotent group inverted by i and $\mathrm{rk}\, H_+ \neq 4, 6$; whereas H_- is a p-unipotent group.*

Proof of Claim 2. Remember that for any $v \in V \setminus \{0\}$ one has $\bigcap_{g \in G} C_G^\circ(v)^g \leq C_G(\langle v^G \rangle) = C_G(V) = 0$. So if H_+ or H_- has corank 1, then by Hrushovski's Theorem $G \simeq \mathrm{PSL}_2(\mathbb{L})$, a contradiction to G containing $\mathrm{SL}_2(\mathbb{K})$. Therefore the coranks are 2 or 3.

Since C centralises V^{+_i}, one has $H_+ \geq C$; by induction H_+ may not be irreducible, and Claim 1 yields the desired form.

On the other hand, we claim that H_- has no involutions. For if it does, say $j \in H_-$, since i normalises H_- and by a Frattini argument (see [17, Lemma B] if necessary) we may assume $[i, j] = 1$; then by the structure of the Sylow 2-subgroup of G, $i = j \in H_-$ and i centralises v_-: a contradiction. At this point it is already

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clear that v_+ and v_- may not be conjugate under G, and in particular that H_+ and H_- cannot simultaneously have corank 3 in G.

We push the analysis further. Suppose that H_- is nonsoluble. As it has no involutions, it must be irreducible by the rank 2 analysis; by induction H_- is a bad group of rank 3, and $\operatorname{rk} G \leq 6$. If $\operatorname{rk} G = 6$ then $\operatorname{rk} H_+ = 4$, a contradiction. Hence $\operatorname{rk} G \leq 5$; on the other hand i centralises H_- by the bad group analysis, but i is not central in G since it does not invert V: therefore $G > C_G^\circ(i) \geq \langle H_-, C\rangle$ and $\operatorname{rk}(H_- \cap C)^\circ \geq 2$, a contradiction to the structure of H_-. So H_- is soluble. Since it has no involutions and $\operatorname{rk}(T) = 1$, H_- is a p-unipotent group. ⊣

Claim 3. *The rank of G is at most* 6.

Proof of Claim 3. Let $x, y \in G$ be independent generic elements.

If H_- centralises a rank 2 module $V_2 \leq V$ then $(H_- \cap H_-^x)^\circ$ centralises $V_2 + V_2^x = V$, so $(H_- \cap H_-^x)^\circ = 1$ and $\operatorname{rk} H_- \leq \operatorname{cork} H_-$, proving $\operatorname{rk} G \leq 2 \operatorname{cork} H_- \leq 6$.

If H_- centralises a rank 1 module $V_1 \leq V$ then $(H_- \cap H_-^x \cap H_-^y)^\circ = 1$ and $\operatorname{rk} G \leq 3 \operatorname{cork} H_-$; if we are not done then we may suppose $\operatorname{cork} H_- = 3$, and in particular $\operatorname{cork} H_+ = 2$.

If H_+ normalises a rank 2 module $V_2 \leq V$ then we know from Claim 1 that $V_2 = V^{-i}$ is centralised by U_+. Incidently, $U_+ \neq 1$ since otherwise $\operatorname{rk} G \leq 6$ and we are done. But $I = (H_+ \cap H_+^x)^\circ$ has no involutions because one such would invert $V_2 + V_2^x = V$, against the involutions in H_+ not being central in G. In particular I is a unipotent subgroup of H_+; observe how $\operatorname{rk} I \geq 2 \operatorname{rk} H_+ - \operatorname{rk} G = \operatorname{rk} H_+ - 2 = \operatorname{rk} U_+ + 1$. Hence I is a maximal unipotent subgroup of H_+, and $U_+ \leq I$. The same applies in H_+^x: therefore $\langle U_+, U_+^x\rangle \leq I$, showing $(U_+ \cap U_+^x)^\circ \neq 1$. As the latter centralises $V_2 + V_2^x = V$, this is a contradiction.

So H_+ normalises a rank 1 module $V_1 \leq V$ and by Claim 1 again, $V_1 = V^{+i}$ is centralised by H_+. In particular $(H_+ \cap H_+^x \cap H_+^y)^\circ = 1$ and $\operatorname{rk} G \leq 3 \operatorname{cork} H_+ = 6$: we are done again. ⊣

Claim 4. *Contradiction.*

Proof of Claim 4. Since $\operatorname{rk} G \leq 6$, one has $\operatorname{rk} H_+ \leq 4$; by Claim 1, $\operatorname{rk} H_+ = 3$. On the other hand $H_+ \geq C_G^\circ(V^{+i}) \geq C$ shows that $H_+ = C$ does not depend on $v_+ \in V^{+i} \setminus \{0\}$. In particular V^{+i} is TI, implying that $N = N_G^\circ(V^{+i})$ has corank at most 2. By Hrushovski's theorem and Proposition 2.1, equality holds. But N is reducible and nonsoluble, so by Claim 1, $\operatorname{rk} N = 3$ and $\operatorname{rk} G = 5$.

Now $(V^{+i})^G$ is generic in V, so v_-^G is not. This proves that $\operatorname{rk} H_- \geq 3$. But on the other hand G is an $N_\circ^\circ$-group as easily seen in the current setting, so $\operatorname{rk} H_- \leq 2$ by uniqueness principles. ⊣

As a consequence and bearing the rank 2 analysis in mind, if $N = N_G^\circ(A)$ is nonsoluble where $A \leq G$ is an infinite abelian subgroup, then N is irreducible: induction and $\operatorname{Pr}_2(G) = 1$ yield a contradiction. This proves that G is an $N_\circ^\circ$-group. ⊣

2.3.3. Identification in Prüfer rank 1. We now identify the $N_\circ^\circ$ case.

Proposition 2.9. *If* $\operatorname{Pr}_2(G) = 1$ *then* $G \simeq \operatorname{PSL}_2(\mathbb{K})$ *in its adjoint action on* $V \simeq \mathbb{K}^3$.

Proof. By the rank $3k$ analysis it suffices to recognize $\operatorname{PSL}_2(\mathbb{K})$. We wish to apply the $N_\circ^\circ$ analysis [17]. Remember that S stands for a Sylow 2-subgroup of G.

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Notation. *Let* $\alpha \in S^\circ$ *be such that* $C_G^\circ(\alpha)$ *is soluble with* α *of minimal order.* (*Such an element certainly exists as* G *is an* $N_\circ^\circ$*-group by Proposition* 2.8.)

Claim 1. *We may suppose that* $C_G^\circ(\alpha)$ *is a Borel subgroup of* G *and* $\alpha^2 \in Z(G)$.

Proof of Claim 1. Let $H = C_G^\circ(\alpha^2)$, a nonsoluble group. By Proposition 2.8, H is irreducible. If $H < G$ then by induction $H \simeq \mathrm{PSL}_2$; one has $\alpha^2 = 1$ and $H = G$, a contradiction. So $G = H$ and $\alpha^2 \in Z(G)$. We go to the quotient $\overline{G} = G/\langle\alpha^2\rangle$, where the involution $\overline{\alpha}$ satisfies:

$$\pi^{-1}\left(C_{\overline{G}}^\circ(\overline{\alpha})\right)^\circ = C_G^\circ(\alpha),$$

meaning that $\overline{\alpha}$ has a soluble-by-finite centraliser in $\overline{G}$. By torality principles and since the Prüfer rank is 1, involutions are conjugate in $\overline{G}$. Therefore any involution in $\overline{G}$ has a soluble-by-finite centraliser, and we apply the $N_\circ^\circ$ analysis. If $\overline{G} \simeq \mathrm{PSL}_2(\mathbb{K})$ then using [3, Theorem 1], $G \simeq \mathrm{PSL}_2(\mathbb{K})$ or $G \simeq \mathrm{SL}_2(\mathbb{K})$; the rank $3k$ analysis brings the desired conclusion. Therefore we may suppose that $C_{\overline{G}}^\circ(\overline{\alpha})$ is a Borel subgroup of $\overline{G}$, so that $C_G^\circ(\alpha)$ is one of G. ⊣

Notation. *Let* $B_\alpha = C_G^\circ(\alpha)$, *a Borel subgroup of* G; *write* $B_\alpha = U_\alpha \rtimes T$.

Notice that $\alpha \in S^\circ \leq T$ where T is the maximal good torus we fixed earlier; hence B_α contains T all right. On the other hand it is not clear whether B_α is non-nilpotent, nor even whether U_α is nontrivial. By Proposition 2.6, nontrivial unipotent subgroups however exist.

Claim 2. *If* $U \leq G$ *is a maximal unipotent subgroup, then* $\operatorname{rk} U \leq 2$ *and* $\operatorname{rk} C_V^\circ(U) = 1$. *Moreover* $\operatorname{rk} U_\alpha \leq 1$; *if* $C_V(\alpha) \neq 0$ *then* $U_\alpha = 1$.

Proof of Claim 2. If $\operatorname{rk} C_V^\circ(U) = 2$ then for generic $g \in G$, $\langle U, U^g\rangle$ is reducible, hence soluble by Proposition 2.8: against maximality of U. Therefore $\operatorname{rk} C_V^\circ(U) = 1$, and let $V_1 = C_V^\circ(U)$; again, $N_G^\circ(V_1)$ is soluble, so we fix a Borel subgroup $B \geq N_G^\circ(V_1) \geq U$. Write $B = U \rtimes \Theta$ for some (possibly nonmaximal) good torus Θ of G.

If V_1 is TI then $\operatorname{cork} B \leq 2$, but conjugates of B can meet only in toral subgroups by uniqueness principles:

$$2\operatorname{rk} B - \operatorname{rk}\Theta \leq \operatorname{rk} G \leq \operatorname{rk} B + 2,$$

so $\operatorname{rk} U \leq 2$ and we are done.

If V_1 is not, then there are $g \notin N_G(V_1) \geq N_G(U)$ and $v_1 \in V_1 \cap V_1^g \setminus \{0\}$; then $G > C_G^\circ(v_1) \geq \langle U, U^g\rangle = K$ is nonsoluble, so by induction $\operatorname{rk} K = 3$ and $\operatorname{rk} U = 1$. We are done again.

Let us review the argument in the case of $U_\alpha = U_p(C_G^\circ(\alpha))$, supposing $\operatorname{rk} U_\alpha = 2$. Then $V_\alpha = C_V^\circ(U_\alpha)$ is a rank 1, TI subgroup of V, and $B_\alpha = N_G^\circ(V_\alpha)$ has corank at most 2.

Now notice that distinct conjugates of B_α, which may not intersect over unipotent elements by uniqueness principles, may not intersect in a maximal good torus either as otherwise $\alpha \in B_\alpha \cap B_\alpha^g$ and $B_\alpha = C_G^\circ(\alpha) = B_\alpha^g$. Hence $\operatorname{rk}(B_\alpha \cap B_\alpha^g)^\circ < \operatorname{rk} T$ and we refine our estimate into:

$$2\operatorname{rk} B_\alpha - (\operatorname{rk} T - 1) \leq \operatorname{rk} G \leq \operatorname{rk} B_\alpha + 2,$$

showing $\operatorname{rk} U_\alpha \leq 1$. Finally if $C_V(\alpha) \neq 0$, then U_α normalises $C_V(\alpha)$ and $[V, \alpha]$; this shows $\operatorname{rk} C_V^\circ(U_\alpha) \geq 2$ and forces $U_\alpha = 1$. ⊣

Claim 3. *The rank of T is* 1.

Proof of Claim 3. Suppose $\operatorname{rk} T > 1$. Then since $\operatorname{Pr}_2(T) = 1$ and $\ell_T(V) > 1$ by Proposition 2.6, the estimate $\operatorname{rk} T \leq \operatorname{rk} V + \operatorname{Pr}_2(T) - \ell_T(V)$ yields $\operatorname{rk} T = \ell_T(V) = 2$.

We shall construct a bad subgroup of toral type; this will keep us busy for a couple of paragraphs. In a T-composition series for V, let X_i be the rank i factor. Then $T \hookrightarrow T/C_T(X_1) \times T/C_T(X_2)$.

We first claim that T does not centralise X_1. For if it does, then $V_1 = C_V(T)$ clearly has rank 1. Now $C_V(\alpha) \neq 0$ so by Claim 2, $U_\alpha = 1$ and T is a Borel subgroup; in view of Proposition 2.8 one has $T = N_G^\circ(V_1)$. If V_1 is TI, then $\operatorname{cork} T \leq 2$ and $\operatorname{rk} G \leq 4$; by Wiscons' analysis, the presence of involutions, and Proposition 2.1, this is a contradiction. Hence V_1 is not TI: there are $g \notin N_G(V_1) = N_G(T)$ and $v_1 \in V_1 \cap V_1^g \setminus \{0\}$. Let $H = C_G^\circ(v_1) \geq \langle T, T^g \rangle$; H is not soluble so by Proposition 2.8 again, it is irreducible; induction yields a contradiction. Hence T does not centralise X_1.

We now construct a rank 1 torus with no involutions, and prove that T is a Borel subgroup. Let $\tau = C_T^\circ(X_1) < T$; by Zilber's Field Theorem, there is a field structure $\mathbb{K}$ with $T/\tau \simeq T/C_T(X_1) \simeq \mathbb{K}^\times$ in its action on X_1. Clearly τ is a good torus of rank 1. Since $\operatorname{Pr}_2(G) = 1$, τ has no involutions; since T does, τ is characteristic in T. Now let $\tau' = C_T^\circ(X_2)$. If $\tau' = 1$ then by Zilber's Field Theorem again, there is a field structure $\mathbb{L}$ with $T \simeq T/C_T(X_2) \simeq \mathbb{L}^\times$ in its action on X_2. Then the good torus $\tau \neq 1$ has no torsion, a contradiction. Hence τ' is infinite; $T = \tau \times \tau'$ and τ' does have involutions. In particular $C_V(\alpha) \neq 0$ so by Claim 2, $U_\alpha = 1$ and T is a Borel subgroup of G.

We can finally construct a bad subgroup of toral type. Let $V_1 = C_V(\tau)$; clearly V_1 has rank 1 and $N_G^\circ(V_1) = T$. Here again, if V_1 is TI then $\operatorname{rk} G \leq 4$, a contradiction as above. So V_1 is not: there are $g \notin N_G(V_1) = N_G(\tau) = N_G(T)$ and $v_1 \in V_1 \cap V_1^g \setminus \{0\}$. Let $H = C_G^\circ(v_1) \geq \langle \tau, \tau^g \rangle$. If H is soluble and contains no unipotence, then $H \leq C_G(\tau) = T$ and $T = T^g$, forcing $\tau = \tau^g$ and $V_1 = V_1^g$: a contradiction. If H is soluble it then extends to a Borel subgroup $U \rtimes \tau$ for some nontrivial p-unipotent subgroup U. By Claim 2, $\operatorname{rk} C_V^\circ(U) = 1$; so τ centralises $C_V^\circ(U) = C_V(\tau) = V_1 = V_1^g$: a contradiction again. Hence H is not soluble. By Proposition 2.8, induction, and since τ has no involutions, H is a simple bad group of rank 3 containing toral elements.

But by Proposition 2.6 there is a Borel subgroup $\beta = Y \rtimes \Theta$ where neither is trivial. Then certainly $\operatorname{rk} \Theta = 1$; moreover, by Claim 2, $W_1 = C_V^\circ(Y)$ has rank 1. If W_1 is TI then $\operatorname{cork} \beta \leq 2$, so $\operatorname{rk} G \leq \operatorname{rk} Y + \operatorname{rk} \Theta + 2 \leq 5$. By Wiscons' analysis, $\operatorname{rk} G = 5$ and $\operatorname{rk} Y = 2$, so β intersects H, necessarily in a conjugate of Θ. Hence Θ has no involutions, and therefore centralises W_1; one sees $V = W_1 \oplus [V, \Theta]$ with $W_1 = C_V(\Theta)$. Therefore T normalises W_1, so $N_G^\circ(W_1) = \beta \geq T$, a contradiction.

As a conclusion W_1 is not TI: there are $\gamma \notin N_G(W_1) = N_G(Y)$ and $w_1 \in W_1 \cap W_1^\gamma \setminus \{0\}$. Now $K = C_G^\circ(w_1) \geq \langle Y, Y^g \rangle$ is nonsoluble, hence irreducible by Proposition 2.8. By induction, K is either isomorphic to $\operatorname{PSL}_2(\mathbb{K})$ or a bad group of unipotent type. Using the rank $3k$ analysis one could directly remove the former, but in any case K cannot be a bad group of toral type. Hence $(H \cap K)^\circ = 1$, that so $\operatorname{cork} H \geq \operatorname{rk} K = 3$ and vice-versa. Therefore both v_1^G and w_1^G are generic in V: they intersect, which conjugates H to K, a contradiction. ⊣

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Always by Proposition 2.6, there is a Borel subgroup of mixed structure $\beta = Y \rtimes \Theta$. So $T = \Theta$ itself is no Borel subgroup; in particular $U_\alpha \neq 1$ and $T = d(S^\circ)$.

Claim 4. *The Sylow 2-subgroup S is connected.*

Proof of Claim 4. If $S^\circ < S$ then there is an element w inverting S°; w inverts T as well. Let V_1 be a T-minimal subgroup of V. If $V_1 = V$ then w gives rise to a finite-order field automorphism on $V_1 \rtimes T$: against [9, Theorem 8.3]. If $\operatorname{rk} V_1 = 2$ then $V_1 \cap V_1^w$ is infinite, so by T-minimality $V_1^w = V_1$; if T does not centralise V_1 then w gives rise to a field automorphism on $V_1 \rtimes T$, a contradiction. So if $\operatorname{rk} V_1 = 2$ then T centralises the rank 2 subgroup V_1, and intersecting with any distinct G-conjugate $V_1^g \neq V_1$ we contradict T-minimality of V_1. Therefore $\operatorname{rk} V_1 = 1$. If $V_1^w = V_1$ consider V_1; if not, consider $V/(V_1 + V_1^w)$. In any case T which is inverted by w acts on a rank 1, w-invariant section, and therefore centralises it.

Hence $C_V(\alpha) \neq 0$, and Claim 2 contradicts $U_\alpha \neq 1$. ⊣

The analysis of V cannot be pushed beyond a certain limit. Of course if $V_1 = C_V^\circ(U_\alpha)$ is TI we find a contradiction; but if it is not, one can imagine having inside G a bad unipotent centraliser: see the comment after the proof of Proposition 2.6. So we need to inspect the inner structure of G more closely; this will be done in the quotient $G/\langle\alpha^2\rangle$ (recall from Claim 1 that $\alpha^2 \in Z(G)$).

Claim 5. *Contradiction.*

Proof of Claim 5. We sum up the information: $\operatorname{rk} U_\alpha = \operatorname{rk} T = 1$ and the Sylow 2-subgroup is connected. We move to $\overline{G} = G/\langle\alpha^2\rangle$ where this holds as well and $\overline{\alpha}$ is an involution. By connectedness of the Sylow 2-subgroup, strongly real elements are unipotent; their set is nongeneric (for instance [10, Theorem 1]). Let f be the definable function mapping two involutions of $\overline{G}$ to their product; we just argued that the image set $\operatorname{im} f$ is nongeneric in $\overline{G}$.

Let $\overline{r} = \overline{\alpha} \cdot \overline{\beta}$ be a generic product of conjugates of $\overline{\alpha}$. Then $\overline{C} = C_{\overline{G}}^\circ(\overline{r})$ is soluble, since otherwise the preimage $(\pi^{-1}(\overline{C}))^\circ = C_G^\circ(r)$ is nonsoluble, whence irreducible by Proposition 2.8: induction applied to $C_G^\circ(r)$ yields a contradiction. If $\overline{C}$ is a good torus, then by connectedness of S and $\overline{S}$, one finds $\overline{\alpha} \in \overline{C}$, a contradiction. So $\overline{C}$ contains a nontrivial unipotent subgroup. Let $\overline{B}$ be the only Borel subgroup of $\overline{G}$ containing $\overline{C}$ (uniqueness follows from uniqueness principles); $\overline{\alpha}$ normalises $\overline{B}$. Of course $\overline{B}$ is not unipotent, as it would generically cover $\overline{G}$ by uniqueness principles, which is against [10, Theorem 1] again. So $\overline{B}$ contains a conjugate of $\overline{T}$ which we may, by a Frattini argument, assume to be $\overline{\alpha}$-invariant. Still by connectedness of $\overline{S}$, one has $\overline{\alpha} \in \overline{B}$. Hence $\overline{\alpha}$ is an involution of $\overline{B}$; such elements are conjugate over $\overline{U} = U_p(\overline{B})$.

It is then clear that the fibre $f^{-1}(\{\overline{r}\})$ over the generic strongly real element $\overline{r}$ has rank at most $m = \operatorname{rk} \overline{U}$. Since $C_{\overline{G}}^\circ(\overline{\alpha}) = \overline{B_\alpha}$, one gets the estimate:

$$2(\operatorname{rk} G - 2) - m = \operatorname{rk} \operatorname{im} f < \operatorname{rk} G,$$

that is $\operatorname{rk} G \leq m + 3 \leq 5$ by Claim 2. But $\operatorname{rk} G \neq 4$ by Wiscons' analysis and Proposition 2.1, so $\operatorname{rk} G = 5$.

Here is the contradiction concluding the analysis. We lift $\overline{B}$ to a Borel subgroup B of G; B has rank 3. But we know that $V_1 = C_V^\circ(U_\alpha)$ has rank 1 by Claim 2; moreover $B_\alpha = N_G^\circ(V_1)$ by Proposition 2.8. If V_1 is TI then $\operatorname{cork} B_\alpha = 2$ and

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rk $G = 4$: a contradiction. So V_1 is not and there are $g \notin N_G(V_1) = N_G(U_\alpha)$ and $v_1 \in V_1 \cap V_1^g \setminus \{0\}$. Then $K = C_G^\circ(v_1) \geq \langle U_\alpha, U_\alpha^g \rangle$ is nonsoluble, hence irreducible by Proposition 2.8, and we apply induction. Like in the end of the proof of Claim 3 we could use the rank $3k$ analysis to rule out $\mathrm{PSL}_2(\mathbb{K})$; here again we shall not. By Claim 4 the Sylow 2-subgroup of G is connected, so K is obviously a simple bad group of rank 3. It must intersect B nontrivially; so up to conjugacy in K, $U_\alpha \leq B$. But B contains a unipotent subgroup of rank $m = 2$: against maximality of U_α as a unipotent subgroup. ⊣

This concludes the Prüfer rank 1 analysis. ⊣

2.4. The Prüfer rank 2 analysis. We now suppose $\mathrm{Pr}_2(G) = 2$ and shall show that $G \simeq \mathrm{SL}_3(\mathbb{K})$ acts on V as on its natural module. Unfortunately we cannot rely on Altseimer's unpublished work aiming at identification of $\mathrm{PSL}_3(\mathbb{K})$ [5, Theorem 4.3] through the structure of centralisers of involutions. There also exists work by Tent [26] but as it involves BN-pairs, it is farther from our methods. Instead we shall construct a vector space structure on V for which a large subgroup of G will be linear.

More material. Technically speaking this section is quite different; the two main ingredients are strongly embedded subgroups, defined before Proposition 2.12, and the Weyl group, defined as follows: $W = N_G(S^\circ)/C_G(S^\circ)$. The Weyl group has been abundantly studied and defined in the past; this definition will suffice for our needs.

2.4.1. Central involutions.

Proposition 2.10. *Suppose that* $\mathrm{Pr}_2(G) = 2$. *If there is a central involution in G then S and* $N_G(S^\circ)$ *have degree at most* 2.

Proof. Suppose there is a central involution, say $k \in S^\circ$ by torality principles. Observe that k inverts V.

Then the other two involutions in S° do not have the same multiplicities of eigenvalues ± 1 in their actions on V: they may not be conjugate. It follows from torality principles that G has exactly three conjugacy classes of involutions, and that all elements in $N_G(S^\circ)$ centralise the involutions in S°. In particular, the Weyl group has exponent at most 2 and order at most 4 (see [16, Consequence of Fact 1] if necessary). Hence $N_G(S^\circ) = C_G(S^\circ) \cdot S$.

The argument bounding the order will resemble the one in Claim 4 of Proposition 2.9. Suppose the order of W is 4. Then by [16] again there is an element $w \in S$ inverting S°. Let $S_0 < S^\circ$ be a 2-torus of Prüfer 2-rank 1 containing k and $\Sigma = d(S_0)$. Let V_1 be a Σ-minimal subgroup of V. If $V_1 = V$ then w gives rise to a finite-order field automorphism on $V_1 \rtimes \Sigma$: a contradiction. If rk $V_1 = 2$ then $V_1 \cap V_1^w$ is infinite, so by Σ-minimality $V_1^w = V_1$; if Σ does not centralise V_1 then w gives rise to a field automorphism on $V_1 \rtimes \Sigma$; hence Σ centralises V_1, against k inverting it. Therefore rk $V_1 = 1$. If $V_1^w = V_1$ consider V_1; if not, consider $V/(V_1 + V_1^w)$. In any case Σ which is inverted by w acts on a rank 1, w-invariant section, and therefore centralises it. Hence $C_V(\Sigma) \neq 0$, a contradiction to k inverting V. ⊣

2.4.2. Removing $\mathrm{SL}_2(\mathbb{K}) \times \mathbb{K}^\times$.

Proposition 2.11. *Suppose that* $\mathrm{Pr}_2(G) = 2$. *Then G contains no definable copy of* $\mathrm{SL}_2(\mathbb{K}) \times \mathbb{K}^\times$.

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Proof. The proof will closely follow that of Proposition 2.8. There are a few differences and we prefer to replicate parts of the previous argument instead of giving one early general statement in the Prüfer rank 1 analysis. First recall that for i an involutive automorphism of V, we have let $V^{+_i} = C_V(i)$ and $V^{-_i} = [i, V]$.

Claim 1. *Any definable, connected, reducible, nonsoluble subgroup $H \leq G$ with $\Pr_2(H) \leq 1$ has the form $U \rtimes C$, where $C \simeq \mathrm{SL}_2(\mathbb{L})$ and the central involution $i \in C$ inverts the p-unipotent group U*; $\operatorname{rk} H \neq 4, 6$. *Moreover if H has a rank 1 submodule V_1 then $V_1 = V^{+_i} = C_V^\circ(H)$; if H has a rank 2 submodule V_2 then U centralises $V_2 = V^{-_i}$.*

Proof of Claim 1. This is exactly the proof of Claim 1 of Proposition 2.8 (notice the extra assumption). ⊣

We start a contradiction proof: suppose that G contains a subgroup isomorphic to $\mathrm{SL}_2(\mathbb{K}) \times \mathbb{K}^\times$.

Claim 2. *Sylow 2-subgroups of $\mathrm{SL}_2(\mathbb{K}) \times \mathbb{K}^\times$ are Sylow 2-subgroups of G. In particular, G has three conjugacy classes of involutions*; $\operatorname{rk} T = 2$ *and $C_V(S^\circ) = 0$.*

Proof of Claim 2. We first find an involution central in G. Set $K = \mathrm{SL}_2(\mathbb{K}) \times \mathbb{K}^\times$. Let i be the involution in $K' \simeq \mathrm{SL}_2(\mathbb{K})$; let j be the involution in $Z^\circ(K) \simeq \mathbb{K}^\times$ and $k = ij$. By the rank $3k$ analysis we know that $K' \simeq \mathrm{SL}_2(\mathbb{K})$ acts naturally on $V_2 = V^{-_i} \simeq \mathbb{K}^2$ and centralises $V_1 = V^{+_i}$. Now observe that by irreducibility of K' on V_2, j either centralises or inverts V_2. If j centralises V_2 and is not central then it inverts V_1: and $k = ij$ inverts $V_2 + V_1 = V$. If j inverts V_2 and is not central then it centralises V_1: and $k = ij$ centralises $V_2 + V_1 = V$, a contradiction. In either case there is a central involution.

By Proposition 2.10 the Sylow 2-subgroup of G is as described. Moreover $C_V(S^\circ) = 0$ since the central involution inverts V. Finally V^{-_i} is not T-minimal: if it is, fix some torus Θ of K'; since Θ acts nontrivially, V^{-_i} is Θ-minimal as well: a contradiction (we already gave this argument after Claim 1 of Proposition 2.8). So $\ell_T(V) = 3$ and this shows $\operatorname{rk} T = 2$. ⊣

Notation. *Let $C \leq G$ be isomorphic to $\mathrm{SL}_2(\mathbb{K})$ and $i \in C$ be the central involution.*

Notation. *Let $v_+ \in V^{+_i} \setminus \{0\}$ and $v_- \in V^{-_i} \setminus \{0\}$ (these exist by the rank $3k$ analysis). Set $H_+ = C_G^\circ(v_+)$ and $H_- = C_G^\circ(v_-)$.*

Claim 3 (cf. Claim 2 of Proposition 2.8). *Both H_+ and H_- have corank 2 or 3 but not both have corank 3. Moreover $H_+ \simeq U_+ \rtimes C$ where U_+ is a p-unipotent group inverted by i and* $\operatorname{rk} H_+ \neq 4, 6$; *whereas $H_- = U_- \rtimes \Theta$ where U_- is a p-unipotent group and Θ is a good torus of rank at most 1.*

Proof of Claim 3. Since $C_V(S^\circ) = 0$ by Claim 2, any centraliser $C_G(v)$ with $v \in V \setminus \{0\}$ has Prüfer rank at most 1. This deals with H_+ and we turn to H_-.

We claim that H_- has a connected Sylow 2-subgroup. Suppose not: say $\tau \cdot \langle w \rangle \leq H_-$ is a 2-subgroup with $w \notin \tau \simeq \mathbb{Z}_{2^\infty}$. Then by connectedness of H_- and torality principles, w inverts $\tau = [\tau, w]$. Then the structure of the Sylow 2-subgroup of G obtained in Claim 2 shows that the involution $j \in \tau$ is a G-conjugate of i. But with a Frattini argument we may assume that i normalises $\tau \cdot \langle w \rangle$, so $[i, j] = 1$. By the structure of the Sylow 2-subgroup of G, we find $i \in H_-$: a contradiction.

It follows that v_+ and v_- are not G-conjugate. Also, connectedness of the Sylow 2-subgroup of H_- easily proves solubility: otherwise use induction on irreducible subgroups on the one hand and the structure of reducible subgroups (Claim 1) on the other hand to find a contradiction. Finally, since $\operatorname{rk} T = 2$, good tori in H_- have rank at most 1. ⊣

Claim 4. *The rank of G is at most* 6.

Proof of Claim 4. Let $x, y \in G$ be independent generic elements.

If U_- centralises a rank 2 module $V_2 \leq V$ then $(U_- \cap U_-^x)^\circ$ centralises $V_2 + V_2^x = V$, so $(U_- \cap U_-^x)^\circ = 1$. In that case H_- can intersect H_-^x at most over a toral subgroup, which has rank at most 1: hence $\operatorname{rk} G \leq 2 \operatorname{cork} H_- + 1$. Notice that if we are not done then $\operatorname{cork} H_- = 3$, forcing $\operatorname{cork} H_+ = 2$.

If U_- centralises a rank 1 module $V_1 \leq V$ then H_- normalises it; so $(H_- \cap H_-^x \cap H_-^y)^\circ$ contains no unipotence and is at most a toral subgroup of rank at most 1; now $\operatorname{rk} G \leq 3 \operatorname{cork} H_- + 1$. If we are not done, then either $\operatorname{cork} H_- = 3$, in which case $\operatorname{cork} H_+ = 2$, or $\operatorname{cork} H_- = 2$ and $\operatorname{rk} G = 7$. In the latter case, $\operatorname{rk} H_+ \neq 4, 6$ forces $\operatorname{cork} H_+ = 2$ again.

The end of the argument is exactly like in Claim 3 of Proposition 2.8. ⊣

Claim 5. *Contradiction.*

Proof of Claim 5. If $\operatorname{rk} G = 6$ then $\operatorname{rk} H_+ = 3$ and $\operatorname{cork} H_+ = 3$; v_+^G is generic in V. The argument for Claim 4 of Proposition 2.8 cannot be used (we leave it to the reader to see why). But $\operatorname{rk} H_- = 4$, so for generic $x \in G$, $(H_- \cap H_-^x)^\circ$ has rank at least 2: it contains a nontrivial unipotent subgroup Y. If U_- centralises a rank 2 module W_2 then Y centralises $W_2 + W_2^x = V$, a contradiction. Hence $W_1 = C_V^\circ(U_-)$ has rank exactly 1.

Still assuming $\operatorname{rk} G = 6$, let us show that W_1 is TI. Otherwise let $w_1 \in W_1 \cap W_1^g \setminus \{0\}$ for $g \notin N_G(W_1) \geq N_G(U_-)$. Let $L = C_G^\circ(w_1) \geq \langle U_-, U_-^g \rangle$; by irreducibility and faithfulness of G one has $\bigcap_{a \in G} L^a = 1$ so by Hrushovski's Theorem L cannot have rank 5; by our choice of g, $\operatorname{rk} L > \operatorname{rk} U_-$. But from $\operatorname{rk} H_- = 4$ we find $\operatorname{rk} U_- \geq 3$. So $\operatorname{rk} L = 4$ and $\operatorname{rk} U_- = 3$; L is clearly soluble. If $L > U_p(L)$ then $U_- = U_p(L) = U_-^g$, a contradiction. If $L = U_p(L)$ then $C_V^\circ(L) \neq 0$, showing $W_1 = C_V^\circ(L) = W_1^g$, a contradiction again.

But always under the assumption that $\operatorname{rk} G = 6$, $C_G^\circ(v_+) = H_+ \simeq \mathrm{SL}_2(\mathbb{K})$ so W_1^G may not intersect v_+^G. Therefore W_1^G is not generic, showing that $N = N_G^\circ(W_1)$ has corank 1. By Hrushovski's Theorem, G has a (necessarily nonsoluble by Proposition 2.1) normal subgroup of corank 1, 2, or 3 contained in N; because G contains $H_+ \simeq \mathrm{SL}_2(\mathbb{K})$ which does not normalise W_1, the corank is 3. So G has either a normal bad subgroup of rank 3, or a normal copy of $(\mathrm{P})\mathrm{SL}_2(\mathbb{L})$. Using 2-tori of automorphisms, every case quickly leads to a contradiction.

Hence $\operatorname{rk} G \leq 5$, proving that G is an $N_\circ^\circ$-group: against Proposition 2.7. ⊣

There are therefore no definable copies of $\mathrm{SL}_2(\mathbb{K}) \times \mathbb{K}^\times$ inside G. ⊣

2.4.3. Strongly embedded methods 1: removing $\mathrm{PSL}_2(\mathbb{K}) \times \mathbb{K}^\times$. Before reading the next proposition, remember that the case where $G = \mathrm{PSL}_2(\mathbb{K}) \times \mathbb{K}^\times$ was dealt with in Proposition 2.1.

Also recall from [1, Section I.10.3] that a strongly embedded subgroup of a group G is a definable, proper subgroup $H < G$ containing an involution, but such that

$H \cap H^g$ contains no involution for $g \notin H$. By [9, Theorem 10.19] it actually suffices to check that H contains the normaliser of a Sylow 2-subgroup S of G, and that for any involution $i \in S$ one has $C_G(i) \leq H$. Moreover if $H < G$ is strongly embedded in G then all involutions are G-conjugate.

PROPOSITION 2.12. *Suppose that* $\mathrm{Pr}_2(G) = 2$. *Then* G *contains no definable copy of* $\mathrm{PSL}_2(\mathbb{K}) \times \mathbb{K}^\times$.

PROOF.

NOTATION. *Let* $H \leq G$ *be isomorphic to* $\mathrm{PSL}_2(\mathbb{K}) \times \mathbb{K}^\times$.

If $H = G$ then we contradict Proposition 2.1: hence H is proper. So $H < G$; any extension of H is irreducible; since we are after a contradiction, we may suppose G to be a minimal counter-example: H is then a definable, connected, proper, maximal subgroup. We shall prove that H is strongly embedded in G, which will be close to the contradiction.

NOTATION. *Let* $\hat{\Theta} = \Theta \rtimes \langle w \rangle$ *be a Sylow* 2*-subgroup of* $H' \simeq \mathrm{PSL}_2(\mathbb{K})$ *and* i *be the involution in* Θ; *we may assume* $\Theta \leq T$.

Since the action of H' on V is known to be the adjoint action by the rank $3k$ analysis, we note that $V^{+_i} = C_V(\Theta) \leq V^{-_w}$. Besides $\ell_T(V) = 3$ for the same reason as in Claim 2 of Proposition 2.11, so $\mathrm{rk}\, T = 2$ and $T \leq H$. Moreover, since the action of H' is irreducible, the involution in $Z(H) \simeq \mathbb{K}^\times$ inverts V and is central in G. As a consequence of Proposition 2.10, a Sylow 2-subgroup of H is one of G as well. But no subquotient of the Sylow 2-subgroup of H is isomorphic to the Sylow 2-subgroup of $\mathrm{SL}_2(\mathbb{L})$; as a consequence, G has no subquotient isomorphic to $\mathrm{SL}_2(\mathbb{L})$.

CLAIM 1. *One has* $C_G^\circ(i) = T \leq H$ *(and likewise for* w *and* iw *with another torus).*

PROOF OF CLAIM 1. By H'-conjugacy it suffices to deal with i. First suppose that $C_G^\circ(i)$ is nonsoluble. By reducibility, $C_G^\circ(i)$ has a subquotient isomorphic to $\mathrm{SL}_2(\mathbb{L})$: against our observations on the Sylow 2-subgroup. Hence $C_G^\circ(i)$ is soluble, say $C_G^\circ(i) = U \rtimes T$. Now U normalises both V^{+_i} (which has rank 1) and V^{-_i}, so $\mathrm{rk}\, C_V^\circ(U) \geq 2$ and $V^{+_i} \leq C_V^\circ(U)$. But w centralises i so it normalises U: hence w normalises $V/C_V^\circ(U)$. Since w inverts $\Theta \leq T$ and there are no field automorphisms in our setting, Θ centralises $V/C_V^\circ(U)$. This shows $V \leq C_V^\circ(U) + C_V(\Theta) = C_V^\circ(U) + V^{+_i} = C_V^\circ(U)$, and therefore $U = 1$. ⊣

NOTATION. *Let* $\alpha \in Z(H) \simeq \mathbb{K}^\times$ *have minimal order with* $\alpha \notin Z(G)$.

This certainly exists as $Z(G)$ is finite by Proposition 2.1. By maximality of H, $C_G^\circ(\alpha) = H$ and $C_G^\circ(\alpha^2) = G$; moreover $(i\alpha)^2 \neq 1$.

CLAIM 2. *One has* $C_G^\circ(i\alpha) = T$ *(and likewise for* $w\alpha$ *and* $iw\alpha$ *with another torus).*

PROOF OF CLAIM 2. By H'-conjugacy it suffices to deal with $i\alpha$. If $C_G^\circ(i\alpha)$ is nonsoluble, then by induction it must be reducible, and G has a subquotient isomorphic to $\mathrm{SL}_2(\mathbb{L})$: a contradiction. Hence $C_G^\circ(i\alpha)$ is soluble, say $C_G^\circ(i\alpha) = U \rtimes T$. Now i normalises U, so by Claim 1, i inverts U. But so do w and iw: therefore $U = 1$. ⊣

CLAIM 3. *Contradiction.*

PROOF OF CLAIM 3. Let $\overline{G} = G/\langle \alpha^2 \rangle$ and denote the image of $g \in G$ by $\overline{g}$. First, by Proposition 2.10 and the connectedness of centralisers of decent tori [2],

$N_G(S) \leq N_G(S^\circ) = C_G(S^\circ) \cdot S \subseteq C^\circ_G(i) \cdot S \subseteq H$, which goes to quotient modulo $\langle \alpha^2 \rangle$ so that $N_{\overline{G}}(\overline{S}) \leq N_{\overline{G}}(\overline{S}^\circ) \leq \overline{H}$.

By Claims 1 and 2, for any involution $\overline{\ell} \neq \overline{\alpha}$ in $\overline{S}$, one has $C^\circ_{\overline{G}}(\overline{\ell}) = \overline{T} \leq \overline{H}$; by construction, $C^\circ_{\overline{G}}(\overline{\alpha}) = \overline{H}$. Be careful that checking connected components does not suffice for strong embedding.

But by torality principles, $\overline{\ell}$ is $\overline{H}$-conjugate to an involution in $\overline{S}^\circ$, so we may assume $\overline{\ell} \in \overline{S}^\circ$; then by a Frattini argument, $C_{\overline{G}}(\overline{\ell}) \subseteq C^\circ_{\overline{G}}(\overline{\ell}) \cdot N_{\overline{G}}(\overline{S}^\circ)$; now $N_{\overline{G}}(\overline{S}^\circ) = C_{\overline{G}}(\overline{S}^\circ) \cdot \overline{S}$ by Proposition 2.10 again, so using the connectedness of centralisers of decent tori one more time:

$$C_{\overline{G}}(\overline{S}^\circ) \leq C^\circ_{\overline{G}}(\overline{i}) = \overline{T} \leq \overline{H}.$$

This shows $C_{\overline{G}}(\overline{\ell}) \leq \overline{H}$ and the whole paragraph also applies to $\overline{\ell} = \overline{\alpha}$.

Hence $\overline{H}$ is strongly embedded all right and $\overline{G}$ conjugates its involutions. This induces an element of order 3 in the Weyl group of $\overline{G}$ and of G as well: a contradiction. ⊣

There are therefore no definable copies of $\mathrm{PSL}_2(\mathbb{K}) \times \mathbb{K}^\times$ inside G. ⊣

2.4.4. Strongly embedded methods 2: classical involutions.

Proposition 2.13. *If* $\Pr_2(G) = 2$ *then all involutions in* G *satisfy* $C^\circ_G(\ell) \simeq \mathrm{GL}_2(\mathbb{K})$.

Proof. Call an involution $i \in G$ *meek* if $C^\circ_G(i)$ is soluble.

Claim 1. *If an involution* $i \in G$ *is neither meek nor central, then* $C^\circ_G(i) \simeq \mathrm{GL}_2(\mathbb{K})$.

Proof of Claim 1. Let $C = C^\circ_G(i)$. Since i is not central in G, it does not invert V: we get a decomposition $V = V_1 \oplus V_2$ where $\operatorname{rk} V_r = r$, and both are C-invariant. Set $D = C_C(V_2)$. Now D is faithful on V_1, so it is abelian-by-finite. By assumption C is not soluble, so by connectedness C/D is not either. By the rank 2 analysis, $C/D \simeq \mathrm{SL}_2(\mathbb{K})$ or $C/D \simeq \mathrm{GL}_2(\mathbb{K})$ in their natural actions on $V_2 \simeq \mathbb{K}^2$.

First suppose that $C/D \simeq \mathrm{SL}_2(\mathbb{K})$. Then $(C/D^\circ)/(D/D^\circ) \simeq C/D \simeq \mathrm{SL}_2(\mathbb{K})$ so by [3, Theorem 1], $D = D^\circ$. Notice that D° contains a 2-torus of rank 1; by Zilber's Field Theorem, $D \simeq \mathbb{L}^\times$ for some field structure $\mathbb{L}$ of rank 1 in the action on $V_1 \simeq \mathbb{L}_+$. Let $E = C^\circ_C(V_1)$; since $\operatorname{cork}_C(E) = 1 = \operatorname{rk} D$, one finds $C = E \times D$, against Proposition 2.11.

Now suppose that $C/D \simeq \mathrm{GL}_2(\mathbb{K})$. Then D° has no involutions, so it centralises V_1: D is therefore finite. Since $\mathrm{SL}_2(\mathbb{K}) \simeq (C/D)' = C'D/D \simeq C'/(C' \cap D)$, [3, Theorem 1] again forces $C' \simeq \mathrm{SL}_2(\mathbb{K})$. Moreover:

$$\mathbb{K}^\times \simeq \mathrm{GL}_2(\mathbb{K})/\mathrm{SL}_2(\mathbb{K}) \simeq (C/D)/(C'D/D) \simeq C/C'D \simeq (C/C')/(C'D/C'),$$

so a finite quotient of, and therefore C/C' itself, is definably isomorphic to $\mathbb{K}^\times$. Finally let $\Theta \leq C$ be a maximal good torus: $C = C' \cdot \Theta = C' * C_\Theta(C') = C' * Z^\circ(C)$ where the intersection is a subgroup of $Z(C') \simeq \mathbb{Z}/2\mathbb{Z}$. By Proposition 2.11, the intersection is not trivial, so that $C \simeq \mathrm{GL}_2(\mathbb{K})$. ⊣

Claim 2. *There is no central involution.*

Proof of Claim 2. Suppose there is a central involution, say k, and let i, j be the other two involutions in S°. Of course $C^\circ_G(i) = C^\circ_G(j)$. If i and j are not meek then by Claim 1, $C^\circ_G(i) = C^\circ_G(j) \simeq \mathrm{GL}_2(\mathbb{K})$, which has only one central involution, a contradiction. Hence i and j are both meek.

As a consequence, G has no definable subgroup isomorphic to $\mathrm{SL}_2(\mathbb{K})$: for if H is one such then the central involution in H cannot be meek, so it is k; but k inverts V, against the rank $3k$ analysis.

We claim that G actually has no definable subquotient isomorphic to $\mathrm{SL}_2(\mathbb{K})$. Suppose $H/K \simeq \mathrm{SL}_2(\mathbb{K})$ is one. If K has no involutions, then like in Claim 1 of Proposition 2.11, we may lift H/K to a genuine copy of $\mathrm{SL}_2(\mathbb{K})$ inside H: a contradiction. So K does have involutions; as we argued a number of times, K is connected and soluble, so we find $K = U \rtimes \Theta$ with Θ a good torus of Prüfer 2-rank 1. Now by the conjugacy of good tori in K, $H = N_H(\Theta) \cdot U$ and $N_H(\Theta)/N_K(\Theta) \simeq H/K$, so we may assume Θ to be normal, and therefore central, in H. The involution in Θ must then be k. If there is a rank 1, H-minimal module $V_1 \leq V$, then $C_H(V_1) < H$ has corank 1; we find $H = C_H(V_1) \cdot \Theta$ and $C_H^\circ(V_1)/C_K(V_1) \simeq H/K \simeq \mathrm{SL}_2$, but $C_K(V_1)$ now has no involutions: we are done. If there is a rank 2, H-minimal module $V_2 \leq V$ then we argue similarly with $C_H(V/V_2)$. If V is H-minimal then we use induction and find $H \simeq \mathrm{PSL}_2(\mathbb{L}) \times \mathbb{L}^\times$, which is against having a subquotient isomorphic to $\mathrm{SL}_2(\mathbb{K})$.

As a consequence and bearing the rank 2 analysis in mind, if $N = N_G^\circ(A)$ is nonsoluble where $A \leq G$ is an infinite abelian subgroup, then N is irreducible; of course $N < G$ by Proposition 2.1, so by induction there remains only $N \simeq \mathrm{PSL}_2(\mathbb{K}) \times \mathbb{K}^\times$, which Proposition 2.12 forbids. Hence G is an $N_\circ^\circ$-group, and Proposition 2.7 forces $\mathrm{Pr}_2(G) = 1$, a contradiction. ⊣

Claim 3. *There is (at least) one involution $k \in S^\circ$ with $C_G^\circ(k) \simeq \mathrm{GL}_2(\mathbb{K})$.*

Proof of Claim 3. This is a proper subset of the previous argument. ⊣

Claim 4. *There are (at least) two involutions $k \neq \ell \in S^\circ$ with $C_G^\circ(k) \simeq \mathrm{GL}_2(\mathbb{K})$ and $C_G^\circ(\ell) \simeq \mathrm{GL}_2(\mathbb{L})$.*

Proof of Claim 4. If there is exactly one in S°, say k, then the other two, say i and j, are meek. We shall construct a strongly embedded subgroup.

Immediately notice that the Weyl group of $C_G^\circ(k) \simeq \mathrm{GL}_2(\mathbb{K})$ gives rise to a 2-element w exchanging i and j but fixing k. Let $C_G^\circ(i) = U_i \rtimes T$ and $C_G^\circ(j) = U_j \rtimes T$. Notice that U_i normalises both V^{+_i} and V^{-_i}, so $\operatorname{rk} C_V(U_i) \geq 2$.

First suppose that $C_V(U_i) \neq C_V(U_j)$. Then k may not invert $C_V(U_i)$ since applying w, it would invert $C_V(U_j)$ as well and therefore invert all of $C_V(U_i) + C_V(U_j) = V$, against Claim 2. Since V^{+_k} has rank 1, we find $V^{+_k} \leq (C_V(U_i) \cap C_V(U_j))^\circ$ and equality follows. So $H = N_G(V^{+_k})$ contains $\langle C_G^\circ(k), C_G^\circ(i), C_G^\circ(j)\rangle$.

Now suppose that $C_V(U_i) = C_V(U_j)$. Observe from $C_G^\circ(k) \simeq \mathrm{GL}_2(\mathbb{K})$ that $\Theta = [T, w]$ contains k. Now w inverts Θ which normalises $V/C_V(U_i)$, and therefore Θ centralises $V/C_V(U_i)$. Hence $V \leq C_V(U_i) + V^{+_k}$. If $U_i \neq 1$, then k inverts $C_V(U_i)$, showing $C_V^\circ(U_i) = V^{-_k}$. In that case, $H = N_G(V^{-_k})$ contains $\langle C_G^\circ(k), C_G^\circ(i), C_G^\circ(j)\rangle$; notice that this is also true if $U_i = 1$.

We claim that H is strongly embedded in G.

Let us first show that $C_G(k)$ is connected. Let $c \in C_G(k)$; lifting torsion, we may suppose c to have finite order (as a matter of fact, by Steinberg's Torsion Theorem [15] c may be taken to be a 2-element). Then c induces an automorphism of $H_k = (C_G^\circ(k))' \simeq \mathrm{SL}_2(\mathbb{K})$, so by [9, Theorem 8.4], $c \in H_k \cdot C_G(H_k)$. Now fix any algebraic torus Θ of H_k: by connectedness of centralisers of tori

[2], $C_G(H_k) \leq C_G(\Theta) = C_G^\circ(\Theta) \leq C_G^\circ(k)$. This shows $C_G(k) = C_G^\circ(k)$. As a consequence, $N_G(S) \leq C_G(k) \leq H$.

Now let $\ell \in S$ be an involution: we show $C_G(\ell) \leq H$. Notice that $\ell \in C_G(k) = C_G^\circ(k) \simeq \mathrm{GL}_2(\mathbb{K})$, so conjugating in $C_G^\circ(k)$ we may suppose $\ell = i$ or $\ell = k$. The latter case is known since $C_G(k)$ is connected. So we may suppose $\ell = i$. But if $c \in C_G(i) \setminus C_G^\circ(i)$, lifting torsion and using Steinberg's Torsion Theorem we may suppose c to be a 2-element. By a Frattini argument, c normalises some maximal 2-torus $\Sigma^\circ \leq C_G^\circ(i)$. Let κ be the nonmeek involution in Σ°; since $S^\circ \leq C_G^\circ(i)$, κ and k are conjugate in $C_G^\circ(i) \leq H$, say $\kappa = k^h$. Now c centralises κ so $c \in C_G(\kappa) = C_G(k)^h \leq H$: we are done.

Since G has a strongly embedded subgroup, it conjugates its involutions: so i is conjugate to k, against meekness. $\dashv$

Finally let $i, j \in S^\circ$ have connected centralisers isomorphic to $\mathrm{GL}_2(\mathbb{K})$ and $\mathrm{GL}_2(\mathbb{L})$. Then $C_G^\circ(i)$ and $C_G^\circ(j)$ give rise to two distinct transpositions on the set of involutions of S°, meaning that the Weyl group is transitive on the set of involutions of S°. As a consequence, i, j, and $k = ij$ are conjugate. $\dashv$

2.4.5. Der Nibelungen Ende.

Proposition 2.14. *If* $\mathrm{Pr}_2(G) = 2$ *then* $G \simeq \mathrm{SL}_3(\mathbb{K})$ *in its natural action on* $V \simeq \mathbb{K}^3$.

Proof. As before, let i, j, k be the involutions in S°.

Claim 1. *There are a* $\mathbb{K}$*-vector space structure on* V *and a definable, connected, irreducible subgroup* $H \leq G$ *which is* $\mathbb{K}$*-linear.*

Proof of Claim 1. Let $H_i = (C_G^\circ(i))' \simeq \mathrm{SL}_2(\mathbb{K})$; define H_j and H_k similarly. We know how H_i acts on V: it centralises V^{+_i} and acts on $V^{-_i} = V^{+_j} \oplus V^{+_k}$ as on its natural module, meaning that there is a (partial) $\mathbb{K}$-vector space structure on V^{-_i}. We extend it to a global vector space structure on all of V as follows.

First let $w \in C_G^\circ(k) \simeq \mathrm{GL}_2(\mathbb{K})$ be an element of order 4 exchanging i and j while fixing k, and notice that we may actually take $w \in (C_G^\circ(k))' = H_k \simeq \mathrm{SL}_2(\mathbb{K})$; then w centralises V^{+_k} and $w^2 = k$.

Let $a_i \in V^{+_i}$. Then $a_i^{w^{-1}} \in C_V(i^{w^{-1}}) = V^{+_j}$, a $\mathbb{K}$-vector subspace of V^{-_i}, so it makes sense to define

$$\lambda \cdot a_i := \left(\lambda \cdot a_i^{w^{-1}}\right)^w.$$

This clearly maps V^{+_i} into itself; moreover it is additive in a_i and additive and multiplicative in λ. So we have extended the vector space structure to all of V.

Let $H = \langle H_i, H_i^w \rangle$; by Zilber's Indecomposibility Theorem, H is definable and connected. It clearly is irreducible on V. We show that H is linear: it suffices to prove linearity of H_i and of w. Since H_i centralises V^{+_i}, it clearly is linear on $V = V^{-_i} + V^{+_i}$. For w we argue piecewise. Let $a_i \in V^{+_i}$ and $a_j = a_i^{w^{-1}} \in V^{+_j}$. Then, bearing in mind that $w^2 = k$ inverts $V^{-_k} = V^{+_i} + V^{+_j}$:

$$\lambda \cdot a_i^w = \lambda \cdot a_j^{w^2} = \lambda \cdot (-a_j) = -\lambda \cdot a_j = (\lambda \cdot a_j)^{w^2} = (\lambda \cdot a_i)^w.$$

Let $a_j \in V^{+_j}$ and $a_i = a_j^w \in V^{+_i}$. Now:

$$\lambda \cdot a_j^w = \lambda \cdot a_i = \left(\lambda \cdot a_i^{w^{-1}}\right)^w = (\lambda \cdot a_j)^w.$$

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Finally, let $a_k \in V^{+_k} = C_V(H_k) \le C_V(w)$. Then $(\lambda \cdot a_k)^w = \lambda \cdot a_k = \lambda \cdot a_k^w$. The element w is linear. ⊣

Notice that if $H = G$ then we are done, since although we did not bother to identify H explicitly, G is then a linear group. Now semisimple, linear groups in characteristic p are known to be algebraic [23, Theorem 2.6] (which was already used in Claim 2 of Proposition 2.1), and we conclude by inspection.

So suppose $H < G$: we shall find a contradiction.

Claim 2. *Contradiction.*

Proof of Claim 2. By induction, and in view of Proposition 2.12, $H \simeq \mathrm{SL}_3(\mathbb{K})$. Up to changing the vector space structure (which should however not be necessary), V is the natural H-module.

Fix $v \in V \setminus \{0\}$ and let $K = C_G^\circ(v)$. First observe that $\mathrm{Pr}_2(K) \le 1$ since $C_V(S^\circ) = 0$ as observed from the action of $H \simeq \mathrm{SL}_3(\mathbb{K})$. Moreover $K \ge C_H^\circ(v)$ contains a copy of $\mathrm{SL}_2(\mathbb{K})$ as seen by inspection. If K is irreducible on V then by induction $K \simeq \mathrm{PSL}_2(\mathbb{L})$, a contradiction. So K is reducible; by Claim 1 of Proposition 2.11 again, write $K = U \rtimes C$ with U a unipotent group and $C \simeq \mathrm{SL}_2(\mathbb{K})$; moreover $\mathrm{rk}\, K \ne 4, 6$.

First suppose that K has a rank 2 module $V_2 \le V$. Then we know that U centralises V_2. Let $g \in G$ be generic and $v_2 \in V_2 \cap V_2^g \setminus \{0\}$. Since H is transitive on $V \setminus \{0\}$, $C_G^\circ(v_2) = Y \rtimes D$ for some conjugates Y, D of U, C respectively. Yet $Y \rtimes D \ge \langle U, U^g \rangle$, and $U \cap U^g = 1$ since the intersection centralises $V_2 + V_2^g = V$. This proves $2\,\mathrm{rk}\, U \le \mathrm{rk}\, U + \mathrm{rk}\, C$, and $\mathrm{rk}\, U \le 3$; since $\mathrm{rk}\, K \ne 6$, one finds $\mathrm{rk}\, G \le K + 3 \le 8$, against $H \simeq \mathrm{SL}_3(\mathbb{K})$ being proper.

Now suppose that K has a rank 1 module V_1. Then we know that K centralises V_1; in particular, for independent and generic $x, y \in G$, the intersection $(K \cap K^x \cap K^y)^\circ$ is trivial: it follows $\mathrm{rk}\, G \le 3\,\mathrm{cork}\, K \le 9$. So $H < G$ has corank 1; by Hrushovski's Theorem G has a normal subgroup of rank at least 6, which is certainly contained in the quasisimple group $H \simeq \mathrm{SL}_3(\mathbb{K})$. Hence H itself is normal in G; now $G = H \cdot C_G^\circ(H)$ by [9, Theorem 8.4], and $C_G^\circ(H)$ is a normal subgroup of rank 1, contradicting Proposition 2.1. ⊣

This concludes the Prüfer rank 2 analysis. ⊣

2.5. The Prüfer rank 3 analysis. This is a one-liner: [6, Theorem 1.4] settles the question. On the other hand a direct proof along the lines of the Prüfer rank 2 argument would certainly be possible. In any case our Theorem is proved. ⊣

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