J. Fluid Mech. (2016), *vol.* 809, R2, doi:10.1017/jfm.2016.699

JFM RAPIDS
journals.cambridge.org/rapids

Velocity statistics inside coherent vortices generated by the inverse cascade of 2-D turbulence

I. V. Kolokolov[1,2,†] and V. V. Lebedev[1,2]

[1]Landau Institute for Theoretical Physics, RAS, 142432, Ak. Semenova 1-A, Chernogolovka, Moscow region, Russia

[2]NRU Higher School of Economics, 101000, Myasnitskaya 20, Moscow, Russia

(Received 8 September 2016; revised 22 October 2016; accepted 23 October 2016; first published online 15 November 2016)

We analyse velocity fluctuations inside coherent vortices generated as a result of the inverse cascade in the two-dimensional (2-D) turbulence in a finite box. As we demonstrated in Kolokolov & Lebedev (*Phys. Rev.* E, vol. 93, 2016, 033104), the universal velocity profile, established in Laurie *et al.* (*Phys. Rev. Lett.*, vol. 113, 2014, 254503), corresponds to the passive regime of the flow fluctuations. This property enables one to calculate correlation functions of the velocity fluctuations in the universal region. We present the results of the calculations that demonstrate a non-trivial scaling of the structure function. In addition the calculations reveal strong anisotropy of the structure function.

Key words: general fluid mechanics, turbulence theory

1. Introduction

The effects of the counteraction of (relatively fast) turbulence fluctuations with a coherent (relatively slow) flow are one of the central problems of turbulence theory (Townsend 1976). Usually the fluid energy is transferred from the slow large-scale flow to turbulent pulsations (Frisch 1995). However, in some cases the energy can go from small-scale fluctuations to the large-scale ones that can lead to the formation of a non-trivial mean flow; see Boffetta & Ecke (2012). Even some basic problems, such as to determine at what mean velocity turbulent fluctuations are sustained, are still under intense investigations (Avila *et al.* 2011). There is still no consistent theory for the mean (coherent) profile coexisting with turbulent fluctuations, so that even the celebrated logarithmic law for the turbulent boundary layer is a subject of controversy (Buschmann & Gad-el-Hak 2003). Here, we consider an important case: two-dimensional (2-D) turbulence in a restricted box where large-scale coherent

† Email address for correspondence: igor.kolokolov@gmail.com

structures are generated from small-scale fluctuations excited by pumping. This process occurs because in two dimensions the nonlinear hydrodynamic interaction favours the energy transfer to larger scales (Kraichnan 1967; Leith 1968; Batchelor 1969).

Already, the first experiments on 2-D turbulence (Sommeria 1986) have shown that, in a finite box with small bottom friction, the energy transfer to large scales leads to the formation of coherent vortices. The first numerical simulations (Smith & Yakhot 1993, 1994; Borue 1994) have also shown that coherent vortices appear in 2-D turbulence. Subsequently, more detailed numerical simulations (Chertkov *et al.* 2007) and experiments (Xia *et al.* 2009) have demonstrated that these vortices have well-defined and reproducible mean velocity (vorticity) profiles. These profiles are quite isotropic with a power-law radial decay of vorticity inside the coherent vortex. In that region the profile depends neither on the boundary conditions (no-slip in experiments, periodic in numerical simulations) nor on the type of the forcing (random in numerical simulations versus parametric excitation or electromagnetic forcing in experiments). The same flow profile is formed both in the statistically stationary case where the mean flow level is stabilized by the bottom friction and in the case where the average flow is still not stabilized and increases as time passes.

Laurie *et al.* (2014) reported results of intensive simulations of 2-D turbulence. They demonstrated that the polar velocity profile of the vortex was flat over some range of distances from the vortex centre, which we call the universal interval. The mean vorticity in the interval is inversely proportional to the distance r from the vortex centre. In the same paper a theoretical scheme based on conservation laws and symmetry arguments was proposed to explain the flat velocity profile. The proposed scheme predicted the value of the polar velocity $U=\sqrt{3\epsilon/\alpha}$ (where ϵ is the energy production rate and α is the bottom friction coefficient), which is found to be in excellent agreement with the numerical simulations (Laurie *et al.* 2014).

In our previous work (Kolokolov & Lebedev 2016) we performed an analytical investigation of the coherent vortex in the universal interval. As a result, we established that the flat velocity profile corresponds to the passive regime of the flow fluctuations where their self-interaction can be neglected. The passive regime admits consistent analytical calculations that confirm the validity of the value $U=\sqrt{3\epsilon/\alpha}$ for the polar velocity. Besides, we have found expressions for the viscous core radius of the vortex and for the border of the universal region where the flat velocity profile is realized. The results reported in the work by Kolokolov & Lebedev (2016) explain why no flat velocity profile was observed in early simulations (Smith & Yakhot 1993, 1994; Borue 1994) and imply that in some conditions a large number of coherent vortices could appear instead of a few vortices in numerical simulation (Chertkov *et al.* 2007; Laurie *et al.* 2014) and experiment (Xia *et al.* 2009).

It is worth noting that our passive approach developed here and in Kolokolov & Lebedev (2016) is closely similar to the quasi-linear approximation that is widely used in plasma theory and hydrodynamics; see, e.g. the textbook by Sturrock (1996) and the paper by Srinivasan & Young (2012). In our case the successive analytic derivations are supported by the large value of the mean velocity gradient in comparison with its fluctuating counterpart.

In this paper we examine the spatial structure of the flow fluctuations. The passive nature of the fluctuations admits a detailed analytical analysis. We find the pair correlation functions of the velocity fluctuations in the universal interval at scales less than the distance r from the vortex centre and larger than the pumping length. There the correlation function possesses a definite scaling, and that scaling is strongly anisotropic. The structure function of the velocity in the range is a linear function of the separation between the points. If the dissipation is strong enough, then it can restrict this region of the linear profile from above. At the end of the paper we discuss applicability conditions of the results and possible extensions of our scheme.

2. General relations

We consider the case where 2-D turbulence is excited in a finite box of size L by an external forcing. It is assumed to be a random quantity with statistical properties that are homogeneous in time and space. We assume also that correlation functions of the pumping force are isotropic. The main object of our investigation is the stationary (in the statistical sense) turbulent state caused by such forcing. To excite turbulence the forcing should be stronger than dissipation related both to the bottom friction and to the viscosity at the pumping scale. That implies that the characteristic velocity gradient of the fluctuations produced by the forcing should be much larger than the flow damping at the pumping scale. The velocity gradient is estimated as $\epsilon^{1/3}k_f^{2/3}$, where ϵ is the energy flux (energy production rate per unit mass) and k_f is the absolute value of the characteristic wavevector of the pumping force. Thus we arrive at the inequalities

$$\epsilon^{1/3}k_f^{2/3} \gg \alpha, \quad \nu k_f^2. \tag{2.1}$$

Here α is the bottom friction coefficient and ν is the kinematic viscosity coefficient, therefore νk_f^2 is the viscous damping rate at the pumping scale k_f^{-1}. In simulations, hyperviscosity is often used. In the case the inequalities (2.1) are still obligatory for exciting turbulence, where νk_f^2 has to be replaced by the hyperviscous damping rate at the pumping scale k_f^{-1}.

If the inequalities (2.1) are satisfied then turbulence is excited in the box and random pulsations of different scales are formed due to nonlinear hydrodynamic interaction. The pumped energy flows to larger scales whereas the pumped enstrophy flows to smaller scales (Kraichnan 1967; Leith 1968; Batchelor 1969). Thus two cascades are formed: the energy cascade (inverse cascade) realized at scales larger than the forcing scale k_f^{-1} and the enstrophy cascade realized at scales smaller than that scale. In an unbound 2-D system the inverse energy cascade is terminated by the bottom friction at the scale

$$L_\alpha = \epsilon^{1/2}\alpha^{-3/2}, \tag{2.2}$$

where a balance between the energy flux ϵ and the bottom friction is achieved. The enstrophy cascade is terminated by viscosity (or hyperviscosity) (Boffetta & Ecke 2012).

In a finite box the above two-cascade picture is realized if the box size L is larger than L_α. Here we consider the opposite case $L < L_\alpha$. Then the energy, transferred by the nonlinearity to the box size L by the inverse cascade, is accumulated there, giving rise to a mean (coherent) flow. We analyse the statistically stationary case where the mean flow is already formed and stabilized by the bottom friction. To describe the flow, we use the Reynolds decomposition, that is the flow velocity is presented as the sum $\boldsymbol{V}+\boldsymbol{v}$ where $\boldsymbol{V}$ is the velocity of the coherent flow and $\boldsymbol{v}$ represents velocity fluctuations on the background of the coherent flow. Let us stress that $\boldsymbol{V}$ is an average over time; it possesses a complicated spatial structure.

As numerical simulation and experiment show, the coherent flow contains some vortices separated by a hyperbolic flow. The characteristic velocity V of the coherent motion can be estimated as $V \sim \sqrt{\epsilon/\alpha}$. This estimate is a consequence of the energy balance: in the stationary case the incoming energy rate ϵ is equal to the bottom friction rate. The characteristic mean vorticity in the hyperbolic region is estimated as $\Omega \sim L^{-1}\sqrt{\epsilon/\alpha}$. However, inside the coherent vortices the mean vorticity Ω is much higher than the estimate (Chertkov *et al.* 2007; Xia *et al.* 2009; Laurie *et al.* 2014). The maximal value of the mean vorticity Ω is achieved in the viscous core of the vortex. The radius of the core can be estimated as $(\nu/\alpha)^{1/2}$ (Kolokolov & Lebedev 2016).

3. Coherent vortex

Here we examine the flow inside the coherent vortex. We attach the origin of our reference system to the vortex centre, which is determined as the point of maximum vorticity. This definition corresponds to the procedures used in the works by Chertkov *et al.* (2007), Xia *et al.* (2009) and Laurie *et al.* (2014) to establish the mean vortex profile. The position of the vortex centre fluctuates: in the laboratory experiments it fluctuates near a fixed position determined by the cell geometry. For the periodic set-up (used in the numerical simulations) the vortex centre can shift significantly from its initial position, and only the average relative position of the vortices is fixed. The reference system is not inertial, and the velocity of the vortex centre is subtracted from the flow velocity in the system. However, the flow vorticity in the reference system coincides with that in the laboratory reference system.

As was established experimentally and numerically (Chertkov *et al.* 2007; Xia *et al.* 2009; Laurie *et al.* 2014), in the chosen reference system the mean flow possesses axial symmetry. Such flow can be characterized by the polar velocity U, which depends on the distance r from the vortex centre. Then the mean vorticity is calculated as $\Omega = \partial_r U + U/r$. To obtain an equation for the profile $U(r)$, one has to use the complete Navier–Stokes equation. Assuming that the average pumping force is zero, one finds the Reynolds equation after averaging (Monin & Yaglom 1971). Outside the viscous core where the viscous term is irrelevant we arrive at

$$\alpha U = -\left(\partial_r + \frac{2}{r}\right)\langle uv\rangle, \tag{3.1}$$

where v and u are the radial and polar components of the velocity fluctuations, and angular brackets mean averaging over time.

To analyse the flow fluctuations inside the vortex, it is convenient to start from the equation for the fluctuating vorticity ϖ,

$$\partial_t \varpi + (U/r)\partial_\varphi \varpi + v\partial_r \Omega + \nabla(\boldsymbol{v}\varpi - \langle \boldsymbol{v}\varpi\rangle) = \phi - \hat{\Gamma}\varpi, \tag{3.2}$$

which is obtained from the same Navier–Stokes equation. Here φ is the polar angle, ϕ is the curl of the pumping force, $\boldsymbol{v}$ is the fluctuating velocity, and the operator $\hat{\Gamma}$ represents dissipation including some terms. Among the terms are the bottom friction α and the viscosity term, $-\nu\nabla^2$. For the case of hyperviscosity the last contribution to $\hat{\Gamma}$ is replaced by $(-1)^{p+1}\nu_p(\nabla^2)^p$ where p is an integer. An additional contribution to $\hat{\Gamma}$ is related to the nonlinear interaction of the fluctuations. Though the interaction is weak, it could be larger than α and $-\nu\nabla^2$ because of the smallness of the contributions.

4. Universal interval

Further we consider the region outside the vortex core where the coherent velocity gradient is large enough,

$$U/r \gg \epsilon^{1/3} k_f^{2/3}. \tag{4.1}$$

In this case fluctuations in the interval of scales between the pumping scale k_f^{-1} and the radius r are strongly suppressed by the coherent flow. The inequality (4.1) means that the mean velocity gradient U/r is larger than the gradient of the velocity fluctuations in the region at all scales larger than k_f^{-1}. Therefore the passive regime is realized there, that is, the self-interaction of the velocity fluctuations is weak. The interval of scales outside the vortex core where the inequality (4.1) is satisfied will be called hereafter the universal interval of scales.

Moreover, the passive regime is realized for scales smaller than the pumping scale k_f^{-1}. Indeed, in the direct cascade the velocity gradients can be estimated as $\epsilon^{1/3} k_f^{2/3}$, up to logarithmic factors that depend weakly on scale; see Kraichnan (1971, 1975) and Falkovich & Lebedev (1994*a*,*b*, 2011). Therefore the inequality (4.1) means domination of the coherent velocity gradient in the interval of scales where the direct cascade would be realized. The passive regime can be consistently analysed. Then one neglects the nonlinear term in (3.2), retaining a linear equation for the vorticity fluctuation ϖ. The equation enables one to express ϖ in terms of the pumping ϕ and then to calculate correlation functions of ϖ via the correlation functions of ϕ.

Further we focus on the case where the pumping ϕ is short-correlated in time and has Gaussian statistics. Direct calculations (Kolokolov & Lebedev 2016) show that in this case

$$\langle uv \rangle = \epsilon / \Sigma, \tag{4.2}$$

where Σ is the local shear rate of the coherent flow:

$$\Sigma = r \partial_r (U/r) = \partial_r U - U/r. \tag{4.3}$$

Expression (4.2) is derived for the condition $\Sigma \gg \Gamma_f$, where Γ_f is the damping of the velocity fluctuations at the pumping scale. The validity of the condition is guaranteed by the inequalities (2.1) and (4.1). Some additional condition $\nu k_f^2 \gg \alpha$ is needed for validity of the expression (4.2); the inequality is assumed to be satisfied in our scheme. (Note that the inequality is satisfied in numerical simulations (Laurie *et al.* 2014).) The opposite case needs some additional analysis that is beyond the scope of our work.

Substituting expression (4.2) into (3.1), one finds a solution

$$U = \sqrt{3\epsilon/\alpha}, \quad \Sigma = -U/r, \tag{4.4a,b}$$

for the mean profile. Thus we arrive at the flat profile of the polar velocity found in Laurie *et al.* (2014) and confirmed analytically in Kolokolov & Lebedev (2016). It is characteristic of the universal region.

The left-hand side of the inequality (4.1) diminishes as r grows. Therefore it is broken at some $r \sim R_u$. Substituting expression (4.4) into (4.1), one obtains

$$R_u = L_\alpha^{1/3} k_f^{-2/3} = \epsilon^{1/6} \alpha^{-1/2} k_f^{-2/3}. \tag{4.5}$$

The scale R_u determines the boundary of the region where flow fluctuations are passive. It could be treated as the vortex size if R_u is less than the box size, as in numerical simulations (Laurie *et al.* 2014) where the universal region is well separated from the outer region, which is not completely passive. The case $R_u \gtrsim L$ is characteristic of the numerical simulations (Chertkov *et al.* 2007; Frishman, Laurie & Falkovich 2016) and the experiment (Xia *et al.* 2009); in this case the passive regime is realized everywhere in the box. Even in this case the equations determining the global structure of the condensate are essentially nonlinear. Solutions of this equations can undergo instabilities and bifurcations of various types as the friction parameter α and the geometry of the system vary. For example, it is found in Frishman *et al.* (2016) that diminishing α in the rectangular geometry leads to a rise of the new vortex. Note that the global structure of the mean flow is a subject of special investigation that is beyond the scope of our article.

5. Vorticity fluctuations

Since the flow fluctuations inside the universal region are passive we can use the linearized version of (3.2),

$$\partial_t \varpi + (U/r)\partial_\varphi \varpi + v\partial_r \Omega + \hat{\Gamma}\varpi = \phi. \tag{5.1}$$

Since the pumping is assumed to be short-correlated in time, its statistics is determined by the pair correlation function

$$\langle \phi(t, \boldsymbol{k})\phi(t', \boldsymbol{k}')\rangle = 2(2\pi)^2 \epsilon \delta(\boldsymbol{k}+\boldsymbol{k}')\delta(t-t')k^2\chi(\boldsymbol{k}) \tag{5.2}$$

for the space Fourier transform of ϕ. The function $\chi(\boldsymbol{k})$ has a profile with the characteristic pumping wavevector k_f and is normalized:

$$\int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\chi(\boldsymbol{k}) = 1. \tag{5.3}$$

Then ϵ is the energy (per unit mass, per unit time) pumped into the system, i.e. the energy flux.

We analyse the fluctuations near a radius $r = r_0$ with scales much smaller than the radius. Then the shear approximation for the mean velocity can be used. We pass to the reference system rotating with the angular velocity $\Omega(r_0)$ and expand all terms in (5.1) in $x_r = r - r_0$ and $x_\varphi = r_0\varphi$. We assume that the parameter $(k_f r)^{-1}$ is small. Then the term $v\partial_r\Omega$ in (5.1) can be discarded and we end up with the following equation:

$$\partial_t \varpi + \Sigma x_r \partial_2 \varpi + \hat{\Gamma}\varpi = \phi. \tag{5.4}$$

Here Σ is the local shear rate (4.4) in the universal region $\Sigma \propto r^{-1}$. Let us rewrite the evolution equation (5.4) for the spatial Fourier components of the vorticity $\varpi_{\boldsymbol{k}}$:

$$\partial_t \varpi(\boldsymbol{k}) - \Sigma k_\varphi \partial \varpi(\boldsymbol{k})/\partial k_r + \Gamma(k)\varpi(\boldsymbol{k}) = \phi(t, \boldsymbol{k}), \tag{5.5}$$

where the components k_r, k_φ of the wavevector $\boldsymbol{k}$ correspond to the variables x_r, x_φ. Solving the evolution equation (5.5), one obtains a formal solution

$$\begin{aligned}\varpi(t, \boldsymbol{k}) = & \int^t \mathrm{d}\tau\, \phi[\tau, k_r + \Sigma(t-\tau)k_\varphi, k_\varphi] \\ & \times \exp\left\{-\int_\tau^t \mathrm{d}\tau'\, \Gamma\left[\sqrt{(k_r + \Sigma(t-\tau')k_\varphi)^2 + k_\varphi^2}\right]\right\}. \end{aligned} \tag{5.6}$$

Now we can find the simultaneous pair vorticity correlation function for the Fourier transform from (5.2):

$$\langle \varpi(t,\boldsymbol{k})\varpi(t,\boldsymbol{k}')\rangle = 2(2\pi)^2\epsilon\delta(\boldsymbol{k}+\boldsymbol{k}')\int_0^\infty \mathrm{d}\tau\, q^2\chi(\boldsymbol{q})\exp\left[-2\int_0^\tau \mathrm{d}\tau'\,\Gamma(\boldsymbol{q}')\right]. \tag{5.7}$$

Here

$$\boldsymbol{q} = (k_r + \Sigma\tau k_\varphi, k_\varphi), \tag{5.8}$$

and $\boldsymbol{q}'$ differs from $\boldsymbol{q}$ by a substitution $\tau\to\tau'$. The factor $\delta(\boldsymbol{k}+\boldsymbol{k}')$ in the expression (5.7) reflects space homogeneity that is not destroyed by a shear flow.

Let us consider scales larger than k_f^{-1}, that is wavevectors $k \ll k_f$. With this condition the main contribution to the integral (5.7) is gained from times $\tau\sim k_f/(\Sigma k_\varphi)$. In the case $|k_\varphi| \gg \Gamma_f k_f/\Sigma$ the dissipation is irrelevant and the last exponential factor in (5.7) can be replaced by unity. Here, as above, Γ_f is the flow damping at the pumping scale. Passing then to the integration over the wavevector (5.8), one obtains

$$\langle \varpi(t,\boldsymbol{k})\varpi(t,\boldsymbol{k}')\rangle = \delta(\boldsymbol{k}+\boldsymbol{k}')\frac{2(2\pi)^2\epsilon q_f}{\Sigma|k_\varphi|}, \tag{5.9}$$

$$q_f = \int_0^\infty \mathrm{d}q_1\, q^2\chi(q). \tag{5.10}$$

Here, we replaced the lower integration limit $|k_r|$ in the integral (5.10) by zero, since the main contribution to the integral comes from $q\sim k_f \gg k_r$. The wavevector q_f is of the order of the inverse pumping length.

There can exist an interval of the wavevectors $r^{-1} < |k_\varphi| < \Gamma_f k_f/\Sigma$ where the dissipation is relevant. Then the exponential factor in (5.7) related to dissipation is relevant. The factor can be estimated as $\exp(-\Gamma_f\tau)$, where the time τ is needed to enhance k_φ to k_f, that is $\tau \sim k_f/(\Sigma k_\varphi)$. Thus we come to the suppression factor $\exp(-A)$, $A\sim \Gamma_f k_f/(\Sigma|k_\varphi|)$. Therefore the vorticity correlations are strongly suppressed in the region of wavevectors $|k_\varphi| < \Gamma_f k_f/\Sigma$. In real space, that means decay $\exp(-B)$ at $|x_\varphi| > \Sigma/(\Gamma_f k_f)$ where $B\sim \Gamma_f^{1/2}k_f^{1/2}|x_\varphi|^{1/2}\Sigma^{-1/2}$.

6. Velocity correlation functions

Knowing the vorticity correlation function, one can calculate the velocity correlation functions using the relation

$$v_\alpha(\boldsymbol{k}) = \mathrm{i}\epsilon_{\alpha\beta}\frac{k_\beta}{k^2}\varpi(\boldsymbol{k}), \tag{6.1}$$

valid for the Fourier transforms. If $k_f \gg |k_\varphi| \gg \Gamma_f k_f/\Sigma$ and $k_f \gg |k_r|$, then one finds from (5.9), (6.1)

$$\langle v(\boldsymbol{k})v(\boldsymbol{k}')\rangle = 2(2\pi)^3\delta(\boldsymbol{k}+\boldsymbol{k}')\frac{q_f\epsilon}{\Sigma}\frac{|k_\varphi|}{\boldsymbol{k}^4}, \tag{6.2}$$

$$\langle u(\boldsymbol{k})u(\boldsymbol{k}')\rangle = 2(2\pi)^3\delta(\boldsymbol{k}+\boldsymbol{k}')\frac{q_f\epsilon}{\Sigma}\frac{k_r^2}{\boldsymbol{k}^4|k_\varphi|}, \tag{6.3}$$

$$\langle v(\boldsymbol{k})u(\boldsymbol{k}')\rangle = -2(2\pi)^3\delta(\boldsymbol{k}+\boldsymbol{k}')\frac{q_f\epsilon}{\Sigma}\frac{k_r k_\varphi}{\boldsymbol{k}^4|k_\varphi|}. \tag{6.4}$$

If $r^{-1} < |k_\varphi| < \Gamma_f k_f/\Sigma$ then the expressions are strongly suppressed due to dissipation.

It follows from the expressions (6.2), (6.4) that the averages $\langle v^2\rangle$ and $\langle u^2\rangle$ are determined by the infrared integrals. Therefore

$$\langle v^2\rangle,\, \langle u^2\rangle \sim \frac{k_f\epsilon}{\Sigma} r \quad \text{if } \Gamma_f k_f r \ll \Sigma, \tag{6.5}$$

$$\langle v^2\rangle,\, \langle u^2\rangle \sim \frac{\epsilon}{\Gamma_f} \quad \text{if } \Gamma_f k_f r \gg \Sigma. \tag{6.6}$$

The average $\langle uv\rangle$ needs an additional analysis (Kolokolov & Lebedev 2016). This analysis shows that the quantity is determined by expression (4.2).

Note that we work in the reference system attached to the (moving) vortex centre. Therefore, comparing the expressions (6.5), (6.6) with the numerical simulations (Laurie *et al.* 2014), one has to take into account the global soft mode of fluctuations related to translations of the vortex. As a result $\langle u^2\rangle$, $\langle v^2\rangle$ extracted from the numerical simulations do not go to zero as $r\to 0$. Note that the soft mode does not contribute to the structure function inside the vortex.

A special problem is calculation of $\langle u_0^2\rangle$ where u_0 is the zeroth angular harmonic of the fluctuating polar velocity. There is no advection term related to the average flow in the equation for u_0. Therefore the quantity $\langle u_0^2\rangle$ is determined solely by the damping. Strictly speaking, the calculation of $\langle u_0^2\rangle$ is outside our shear approximation. However, our logic can be easily extended to this case to obtain

$$\langle u_0^2\rangle \sim \frac{\epsilon}{k_f r \Gamma_f}. \tag{6.7}$$

An explanation of this expression is based on the expression

$$\langle u_0^2\rangle = \int \frac{\mathrm{d}\varphi}{2\pi} \langle u(\boldsymbol{r}_1) u(\boldsymbol{r}_2)\rangle, \tag{6.8}$$

where the points 1 and 2 are separated by the same distance r from the vortex centre and φ is the angle between the vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. The factor ϵ/Γ_f is the contribution to $\langle u_0^2\rangle$ caused by the pumping that is effective if $\varphi \lesssim (k_f r)^{-1}$. The contribution (6.7) should be taken into account besides (6.5); the latter is related to the sum of non-zero angular harmonics. In the case (6.6) the contribution (6.7) is small in comparison with one related to non-zero harmonics.

The average $\langle u_0^2\rangle$ was calculated previously in the paper by Falkovich (2016), where the contribution related to the pumping was ignored and the nonlinear effects were taken into account instead. This approach is correct outside the universal region, at $r > R_u$ where R_u is determined by expression (4.5). At the border, where $r\sim R_u$, our estimate (6.7) coincides with one obtained in Falkovich (2016).

It is worthwhile to characterize scales where the expressions (6.2)–(6.4) are correct by the velocity structure functions. One finds

$$\begin{aligned} S_{11}(x_r, x_\varphi) &= \langle [v(x_r, x_\varphi) - v(0,0)]^2\rangle \\ &= \frac{2q_f\epsilon}{\Sigma\pi} \int \mathrm{d}^2\boldsymbol{k}\, \frac{|k_\varphi|}{\boldsymbol{k}^4} (1 - \mathrm{e}^{\mathrm{i}k_r x_r + \mathrm{i}k_\varphi x_\varphi}), \end{aligned} \tag{6.9}$$

which is correct if $k_f^{-1} \ll |x_r, x_\varphi| \ll r,\ k_f^{-1}\Sigma/\Gamma_f$. Infrared divergence in the integral (6.9) can be regularized by substituting $k_\varphi^2 \to k_\varphi^2 + \mu^2$, where $\mu \sim 1/r$. The result of the integration can be expressed via the function

$$\mathcal{J}(z) = \int_0^\infty \mathrm{d}q \frac{\mathrm{e}^{-z}}{q^2+\mu^2} \approx \frac{\pi}{2\mu} + z[\Gamma_f - 1 + \ln(\mu z)]. \tag{6.10}$$

Particularly, one finds

$$S_{11}=\frac{2q_f\epsilon}{\Sigma}\mathrm{Re}\left[\frac{\pi}{2\mu}-\mathcal{J}+x_r\partial_1\mathcal{J}\right], \tag{6.11}$$

where $\mathcal{J}=\mathcal{J}(x_r-\mathrm{i}x_\varphi)$, Calculating the expression (6.11), one finds

$$S_{11}\approx\frac{2q_f\epsilon}{\Sigma}\left[|x_r|+x_\varphi\arctan\left(\frac{x_\varphi}{|x_r|}\right)\right]. \tag{6.12}$$

Analogous expressions can be found for other components of the structure function:

$$S_{22}=\left\langle[u(x_r,x_\varphi)-u(0,0)]^2\right\rangle\approx\frac{2q_f\epsilon}{\Sigma}\left[x_\varphi\arctan\left(\frac{x_\varphi}{|x_r|}\right)-2|x_r|\ln\left(\mu\sqrt{x_r^2+x_\varphi^2}\right)\right], \tag{6.13}$$

and

$$S_{12}=\langle[v(x_r,x_\varphi)-v(0,0)][u(x_r,x_\varphi)-u(0,0)]\rangle\approx-\frac{2q_f\epsilon}{\Sigma}x_r\arctan\left(\frac{x_\varphi}{|x_r|}\right). \tag{6.14}$$

In the region $|x_r|,\ |x_\varphi|\gg k_f^{-1}\Sigma/\Gamma_f$ the pair correlation functions are strongly suppressed and the structure functions are determined by the single-point averages.

7. Discussion

We analysed correlations of the velocity fluctuations inside a coherent vortex generated as a result of the inverse cascade in a finite 2-D cell. Our attention was concentrated on the universal region inside the vertex where the mean velocity has a flat profile. We analysed the fluctuations at a distance r from the vortex core and with scales less than r. The amplitude of the velocity fluctuations grows as the scale grows, as in the traditional inverse cascade. However, the expressions (6.12)–(6.14) demonstrate a linear profile, which is different from the 2/3 power law in the traditional inverse cascade. Let us stress also that in our case the fluctuations are strongly anisotropic. Note also that in some conditions viscous dissipation can come into play, which leads to suppression of the fluctuations at the largest scale (below r).

We performed our calculations in the reference system where the origin is attached to the vortex centre and rotating with angular velocity Ω dependent on the radius r and coinciding with the angular velocity of the mean flow at the distance r. In this reference system the correlation time of the pumping attached to the bottom of the cell cannot be larger than Ω^{-1}. That justifies our approach (where the pumping is assumed to be short-correlated in time) since the angular velocity Ω is the largest characteristic rate in the universal region. Note also that use of the rotating reference system implies an implicit angular averaging of the correlation functions (besides the averaging over time).

The universal region is restricted from above by the radius (4.5). At larger distances from the vortex centre the flow fluctuations are not completely passive, and our scheme is, strictly speaking, incorrect. In this case the traditional inverse cascade is realized on scales smaller than l, where $l\sim\epsilon^{1/2}\Sigma^{-3/2}$ is determined by the balance between the effective shear rate Σ of the mean flow and the characteristic velocity gradient in the inverse cascade. However, the flow fluctuations are passive at scales larger than l. That is the region where our scheme is applicable. And the only difference is that the role of the pumping length is played just by the scale l.

Probably, our results can be extended to some types of three-dimensional turbulent flows. Note, as an example, the turbulence excited at the fluid surface (von Kameke *et al.* 2011; Francois *et al.* 2014) where the inverse cascade is observed. This will be the subject of future investigations.

Acknowledgements

We thank G. Boffetta and G. Falkovich for valuable discussions. The work is supported by RScF grant 14-22-00259.

References

AVILA, K., MOXEY, D., DE LOZAR, A., AVILA, M., BARKLEY, D. & HOF, B. 2011 The onset of turbulence in pipe flow. *Science* **333**, 192–196.

BATCHELOR, G. K. 1969 Computation of the energy spectrum in homogeneous two-dimensional turbulence. *Phys. Fluids* **12**, 233–239.

BOFFETTA, G. & ECKE, R. E. 2012 Two-dimensional turbulence. *Annu. Rev. Fluid Mech.* **44**, 427–451.

BORUE, V. 1994 Inverse energy cascade in stationary two-dimensional homogeneous turbulence. *Phys. Rev. Lett.* **72**, 1475–1478.

BUSCHMANN, M. & GAD-EL-HAK, M. 2003 Debate concerning the mean-velocity profile of a turbulent boundary layer. *AIAA* **41**, 4.

CHERTKOV, M., CONNAUGHTON, C., KOLOKOLOV, I. & LEBEDEV, V. 2007 Dynamics of energy condensation in two-dimensional turbulence. *Phys. Rev. Lett.* **99**, 084501.

FALKOVICH, G. 2016 Interaction between mean flow and turbulence in two dimensions. *Proc. R. Soc. Lond.* A **472**, 20160287.

FALKOVICH, G. & LEBEDEV, V. 1994*a* Nonlocal vorticity cascade in 2 dimensions. *Phys. Rev.* E **49**, R1800–R1804.

FALKOVICH, G. & LEBEDEV, V. 1994*b* Universal direct cascade in 2-dimensional turbulence. *Phys. Rev.* E **50**, 3883–3899.

FALKOVICH, G. & LEBEDEV, V. 2011 Vorticity statistics in the direct cascade of two-dimensional turbulence. *Phys. Rev.* E **83**, 045301(R).

FRANCOIS, N., XIA, H., PUNZMANN, H., RAMSDEN, S. & SHATS, M. 2014 Three-dimensional fluid motion in Faraday waves: creation of vorticity and generation of two-dimensional turbulence. *Phys. Rev.* X **4**, 021021.

FRISCH, U. 1995 *Turbulence: The Legacy of A. N. Kolmogorov*. Cambridge University Press.

FRISHMAN, A., LAURIE, J. & FALKOVICH, G. 2016 Jets or vortices – what flows are generated by an inverse turbulent cascade? arXiv:1608.04628v1 [nlin.CD].

VON KAMEKE, A., HUHN, F., FERNANDEZ-GARSIA, G., MUNUZURI, A. P. & PEREZ-MUNUZURI, V. 2011 Double cascade turbulence and Richardson dispersion in a horizontal fluid flow induced by Faraday waves. *Phys. Rev. Lett.* **107**, 074502.

KOLOKOLOV, I. V. & LEBEDEV, V. V. 2016 Structure of coherent vortices generated by the inverse cascade of two-dimensional turbulence in a finite box. *Phys. Rev.* E **93**, 033104.

KRAICHNAN, R. H. 1967 Inertial ranges in two-dimensional turbulence. *Phys. Fluids* **10**, 1417–1423.

KRAICHNAN, R. H. 1971 Inertial-range transfer in two- and three-dimensional turbulence. *J. Fluid Mech.* **47**, 525–535.

KRAICHNAN, R. H. 1975 Statistical dynamics of two-dimensional flow. *J. Fluid Mech.* **67**, 155–175.

LAURIE, J., BOFFETTA, G., FALKOVICH, G., KOLOKOLOV, I. & LEBEDEV, V. 2014 Universal profile of the vortex condensate in two-dimensional turbulence. *Phys. Rev. Lett.* **113**, 254503.

LEITH, C. E. 1968 Diffusion approximation for two-dimensional turbulence. *Phys. Fluids* **11**, 671–673.

MONIN, A. S. & YAGLOM, A. M. 1971 *Statistical Fluid Mechanics, Vol. 1: Mechanics of Turbulence* (ed. J. L. Limley), chap. 3, Dover.

SMITH, L. M. & YAKHOT, V. 1993 Bose-condensation and small-scale structure generation in a random force driven 2d turbulence. *Phys. Rev. Lett.* **71**, 352–355.

SMITH, L. M. & YAKHOT, V. 1994 Finite-size effects in forced, two-dimensional turbulence. *J. Fluid Mech.* **274**, 115–138.

SOMMERIA, J. 1986 Experimental study of the two-dimensional inverse energy cascade in a square box. *J. Fluid Mech.* **170**, 139–168.

SRINIVASAN, K. & YOUNG, W. R. 2012 Zonostrophic instability. *J. Atmos. Sci.* **69** (5), 1633–1656.

STURROCK, P. A. 1996 *Plasma Physics: An Introduction to the Theory of Astrophysical, Geophysical and Laboratory Plasmas*. Cambridge University Press.

TOWNSEND, A. A. 1976 *The Structure of Turbulent Shear Flow*. Cambridge University Press.

XIA, H., SHATS, M. & FALKOVICH, G. 2009 Spectrally condensed turbulence in thin layers. *Phys. Fluids* **21**, 125101.