Euro. Jnl of Applied Mathematics (2011), *vol.* 22, *pp.* 125–149. © Cambridge University Press 2010
doi:10.1017/S095679251000032X

On the breakup of air bubbles in a Hele-Shaw cell

VLADIMIR ENTOV[1] and PAVEL ETINGOF[2]

[1] *Deceased*
[2] *Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*
e-mail: etingof@math.mit.edu

(*Received 3 September 2008; revised 25 March 2010; accepted 26 August 2010; first published online 21 December 2010*)

We study the problem of breakup of an air bubble in a Hele-Shaw cell. In particular, we propose some sufficient conditions of breakup of the bubble, and ways to find the contraction points of its parts. We also study regulated contraction of a pair of bubbles (in which the rates of air extraction from the bubbles are controlled) and study various asymptotic questions (such as the asymptotics of contraction of a bubble to a degenerate critical point, and asymptotics of contraction of a small bubble in the presence of a big bubble)

Keywords: bubble, Hele-Shaw cell, contraction, moving boundary

1 Introduction

The problem of contraction of an air bubble in a Hele-Shaw cell filled with a Newtonian fluid under the influence of suction of air from the bubble has been intensively studied by many physicists and mathematicians for more than 25 years; see [4] and references therein. In particular, in [1], the authors suggested an analytic theory which allows one to give a complete description of the asymptotics of the bubble shape as its area goes to zero, and, in particular, to find the point of its contraction. In a number of applications of the contraction problem, for instance, in the theory of gas recovery, the following question, only briefly discussed in [1], is important: will the air bubble fall apart during contraction, or will it remain connected until all the air has been extracted? In this paper, we study this question in detail. In particular, we propose some sufficient conditions of breakup of the bubble, and ways to find the contraction points of its parts. In the theory of gas recovery, these points are interpreted as the optimal positions of the gas-producing wells.

We note that all the results of this paper (with the exception of explicit solutions) extend to flows in a curved Hele-Shaw cell, along the lines of [2, 10].

The structure of the paper is as follows. In Section 2, we describe the mathematical model of the contraction problem and recall some of the results from [1] on the connection between the dynamics of the bubble and its gravity potential. We also study the asymptotic shapes of bubbles contracting to a degenerate critical point of the potential. In Section 3, we define the points of partial contraction, which are contraction points of the bubbles which appear as a result of breaking of the initial bubble, and contract before the full contraction occurs, and extend them to the results about points of complete contraction from [1]. In Section 4, we give a sufficient condition of breakup of a symmetric bubble; this is an extension of a result from [1]. In Section 5, we consider the process of regulated

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press

contraction, which is simultaneous contraction of two or more bubbles under prescribed rates of extraction from each bubble. In Section 6, we consider the case of a bubble which breaks up into two bubbles under contraction, and discuss the question whether one can use regulated contraction to make these two bubbles contract simultaneously (development of singularities in solutions may be a problem). A strategy of extraction which allows one to do so is called a synchronizing strategy, and we study such strategies in some detail. In particular, in Section 7 we study the asymptotics of contraction under a synchronizing strategy, and show that, like in the case of a single bubble, the two bubbles contract at critical points of the potential (generically, non-degenerate local minima), and the bubble shapes generically tend to ellipses, whose axes are determined by the eigenvalues of the Hessian of the potential at the minima. In Section 8, we discuss domains that are on the boundary between those that admit a synchronizing strategy and those that do not. The potential of such a domain is expected to have a degenerate critical point, and we study the asymptotics of contraction to such a point. In Section 9, we characterize domains that are on the boundary between those that break up and those that do not. Generically, they develop an instantaneous 5/2-cusp in the process of contraction. In Section 10, we correct some computational errors in the previous publications [1, 10]. Section 11 contains the conclusion.

2 The mathematical model and its main properties

2.1 The model

Let us recall the formulation of the problem and the main results from the paper [1]. Consider contraction of an air bubble in an unbounded Hele-Shaw cell, which is filled with a Newtonian fluid, under suction of air from the bubble. Let $B(t)$ be the air domain at a time t. We assume that it is connected and bounded, with (say) a smooth boundary. Its law of evolution in time is as follows. In the fluid domain $B(t)^c$ (the complement of $B(t)$), which we assume to be connected, there is a potential vector field of fluid velocities, $v(x, y, t) = \nabla\Phi(x, y, t)$. The potential Φ is determined at any time t as a solution of the boundary value problem

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0, \quad (x, y) \in B(t)^c; \quad \Phi|_{\partial B(t)} = 0; \quad \Phi(x, y) = -\frac{q}{2\pi}\log(r) + O(1), \quad r \to \infty, \tag{2.1}$$

where $r = (x^2 + y^2)^{1/2}$, and $q > 0$ is the rate of suction. The velocity of the boundary is then equal to the velocity of the fluid particles on the boundary:

$$v_b = \frac{\partial\Phi}{\partial n}. \tag{2.2}$$

The contraction problem is to find the family of domains $B(t)$ for a given initial shape of the bubble $B(0)$ and given rate q of suction. It can be shown that for every initial domain with a smooth boundary, the solution of this problem exists and is unique on some interval of time $[0, \tau)$, $\tau > 0$. It has an obvious monotonicity property $B(t_1) \subset B(t_2)$ for $t_1 > t_2$.

In a similar way, one can define contraction of several bubbles. In this case, the domain $B(t)$ is a union of finitely many disjoint simply connected domains (bubbles).

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press

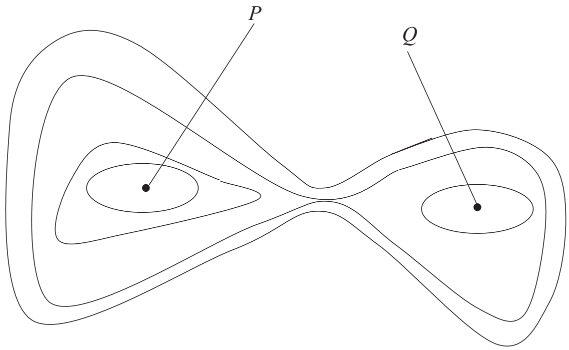

FIGURE 1. Collision of two parts of the bubble boundary.

2.2 Weak solutions

Equations (2.1) and (2.2) determine contraction as long as it reduces to continuous deformation of the boundaries of the bubbles. However, in the process of contraction, the boundary $B(t)$ may undergo topological transformations. For instance, parts of the boundary can collide (Figure 1).

In this case, the above definition of the contraction does not apply as it is, and needs clarification.

Namely, let S be the area of $B(0)$, and $t^* = S/q$ be the time of complete extraction of the air. Let us call a family of bounded domains $B(t)$, $t \in [0, t^*)$, a *weak solution* of the contraction problem if for every $t \in [0, t^*)$, $B(t)$ is a disjoint union of finitely many simply connected domains, so that

(i) $B(t_1) \subset B(t_2)$ for $t_1 > t_2$;

(ii) the area of $B(t)$ is $S - qt$; and

(iii) there exists a closed set of times $T \subset [0, t^*]$ (of topological transformations) containing 0, t^* such that for any $t \in T$, a small enough interval $(t, t+\varepsilon)$ does not intersect T, and on the intervals of time (τ_1, τ_2) not intersecting with T the domain $B(t)$ is a 'classical' solution of the contraction problem, i.e. is defined by equations (2.1) and (2.2).

In other words, a weak solution is 'glued' from usual (classical) solutions at points $\tau \in T$ where the domain $B(t)$ undergoes topological transformations. It is natural to assume that it describes the actual process of contraction in the case when the breakup of the bubble does occur.

It is known (see [4]) that the contraction problem has a unique weak solution. An example of a weak solution is given in Figure 1. From this example, one can see two types of topological transformations that can occur during contraction:

(1) breakup of a bubble into two pieces, and

(2) disappearance of a bubble.

As a result of these transformations, the number of bubbles changes in the process of contraction.

Remark 2.1 It can be shown that the set T is finite. However, a detailed proof of this would be long and we do not give it here.

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press

2.3 The gravity potential

Let us define the gravity potential of a bounded domain B to be the function

$$\Pi_B(\xi,\eta) = \frac{1}{2\pi}\int_B \log|z-\zeta|\,\mathrm{d}x\,\mathrm{d}y, \tag{2.3}$$

where $\zeta = \xi + i\eta, z = x + iy$. This function satisfies the Poisson differential equation in $\mathbb{R}^2$ with logarithmic asymptotics at infinity:

$$\Delta\Pi_B = \chi_B, \quad \Pi_B(\xi,\eta) = \frac{S}{2\pi}\log|\zeta| + O(1), \quad \zeta\to\infty, \tag{2.4}$$

where $\chi_B(\xi,\eta)$ is the characteristic function of B, and S is the area of B. It is shown in [1] that this function is closely related to the contraction problem. For example, we have the following theorem.

Theorem 2.2 *Let $B(t)$ be a weak solution to the contraction problem. Then*

(i) The gravity potential inside $B(t)$ changes by a constant under contraction:

$$\Pi_{B(0)}(\xi,\eta) - \Pi_{B(t)}(\xi,\eta) = C(t), \ (\xi,\eta)\in B(t). \tag{2.5}$$

(ii) Let us set $\Phi(x,y,t) = 0$ if $(x,y)\in B(t)$. Then we have

$$\Pi_{B(0)}(\xi,\eta) = K - \int_0^{t^*}\Phi(\xi,\eta,t)\,\mathrm{d}t. \tag{2.6}$$

Example 2.3 Let us say that a simply connected bounded domain B is algebraic of degree d if its Cauchy transform $h_B = -\bar{z} + 4\partial_z\Pi_B$ (which is analytic in B) is actually a rational function of degree $d-1$ (see [10, 3]). In this case, the same is true for $B(t)$ for $t>0$, and thus it may be shown, similar to [3], that the boundary of $B(t)$ is defined by the algebraic equation $Q(z,\bar{z}) = 0$, where Q is a polynomial of degree $2d$. The genus of this algebraic curve is thus $\leqslant(2d-1)(d-1)$, and thus the number of components of $B(t)$ is at most $(2d-1)(d-1)$.

2.4 Points of complete contraction

Let us say that a point of $\mathbb{R}^2$ is a point of complete contraction if it belongs to $B(t)$ for all $t\in[0,t^*)$. Thus, the points of complete contraction are the points of disappearance of the bubbles which 'survive' until the time of complete contraction t^*. The set of all points of complete contraction is the intersection $\cap_{t<t^*}B(t)$. In [1] we described the structure of this set.

Theorem 2.4 *(i) The set of points of complete contraction is finite.*

(ii) The points of complete contraction are the global minimum points of the gravity potential.

Remark 2.5 Note that if the boundary of a domain B is smooth, then by Hopf's strong maximum principle, a global minimum of Π_B cannot be attained on the boundary of B.

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press

2.5 Asymptotics of contraction

It is shown in [10] that when a bubble contracts completely to a point and the Hessian of the potential at that point is non-degenerate, then the boundary of the bubble has the asymptotic shape of an ellipse, whose half-axes are directed along the eigenvectors of the Hessian, and their lengths are inverse proportional to its eigenvalues. Here we would like to extend this result to the case when the Hessian may be degenerate.

For example, assume that the bubble B contracts at a point 0. In this case, $\Pi_B(0)$ is an isolated global minimum point of Π_B; [1, 10]. Let us assume that 0 is a degenerate critical point, and the kernel of the Hessian of Π_B at zero is the x-axis, i.e. $\Pi_B(x,y) = \frac{1}{2}y^2 + O(|z|^3)$ near 0. For simplicity, let us first assume that the bubble is symmetric with respect to the x-axis. Then

$$\Pi_B = \frac{1}{2}y^2 + \frac{\beta}{2n}x^{2n} + \cdots ,$$

where β is some positive number, and the dots are monomials strictly inside the Newton polygon (i.e. $x^i y^j$ with $i + nj > 2n$). Let us call n the degree of the critical point 0.

Let $B_*(t)$ be the image of the bubble at the time t under the renormalization $x \to cx$, $y \to c^{2n-1}y$, where $c = c(t)$ is chosen in such a way that the diameter of $B_*(t)$ is 2 (so $c(t)$ behaves like $(t^* - t)^{-1/2n}$ as $t \to t^*$).

Define the polynomials

$$Q_n(u) = \sum_{k=0}^{n} \frac{(2k)!}{4^k k!^2} u^{n-k}.$$

Theorem 2.6 *The boundary of the domain $B_*(t)$ tends to the curve*

$$y^2 = \beta^2(1 - x^2)Q_{n-1}^2(x^2).$$

Proof We write Π_B in the form

$$\Pi_B = -\frac{1}{8}(z - \bar{z})^2 + \beta\,\mathrm{Re}(z^{2n}/2n) + \cdots ,$$

where the dots are the terms strictly inside the Newton polygon.

The conformal map of the unit disk into the outside of $B(t)$ which maps 0 to ∞ has the form

$$f(\zeta) = A(\zeta + \zeta^{-1}) + \phi(\zeta),$$

where one may assume that $A > 0$, and ϕ is an odd holomorphic function in the unit disk. Here, $\phi = \phi_t$, $A = A(t)$, $A \to 0$ as $t \to t^*$, and $\phi = o(A)$ (which corresponds to the fact that in the first approximation the bubble tends to a segment).

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press

Let f^* denote the function obtained from f by conjugating the coefficients of the Taylor series, and let $h_B(z)$ be the Cauchy transform of B:

$$h_B = \bar{z} - 4\partial_z \Pi_B = z - 2\beta z^{2n-1} + O(z^{2n}).$$

Then, by Richardson's theorem (see [10, 9]), the function

$$u(\zeta) := f^*(\zeta^{-1}) - h_B(f(\zeta))$$

extends holomorphically from the boundary of the unit disk to its interior and vanishes at 0. Therefore, we obtain, for $|\zeta| = 1$:

$$\begin{aligned} \operatorname{Im} \phi(\zeta) &= \frac{1}{2i}(\phi(\zeta) - \phi^*(\zeta^{-1})) = \frac{1}{2i}(f(\zeta) - f^*(\zeta^{-1})) \\ &= \frac{1}{2i}(f(\zeta) - h_B(f(\zeta)) - u(\zeta)). \end{aligned}$$

Since this function is real and odd, it equals twice the real part of the sum of the negative degree terms of its Fourier expansion. Thus, we get

$$\begin{aligned} \operatorname{Im} \phi(\zeta) &= 2\beta A^{2n-1} \operatorname{Im}(\zeta + \zeta^{-1})_-^{2n-1} + o(A^{2n-1}) \\ &= -2\beta A^{2n-1} \operatorname{Im}(\zeta + \zeta^{-1})_+^{2n-1} + o(A^{2n-1}), \end{aligned}$$

where the subscripts $-, +$ stand for the negative (respectively, positive) degree terms of the Taylor series.

Now, we claim that

$$\operatorname{Im}(\zeta + \zeta^{-1})_+^{2n-1} = 4^{n-1} Q_{n-1}(\cos^2\theta) \sin\theta,$$

where $\zeta = e^{i\theta}$. This is easily proved by induction in n. Also, if $|\zeta| = 1$, then $\zeta + \zeta^{-1}$ is real. Thus, we get for $|\zeta| = 1$:

$$y = \operatorname{Im} f(\zeta) = -\beta(2A)^{2n-1} Q_{n-1}(\cos^2\theta) \sin\theta + o(A^{2n-1}).$$

On the other hand, we have

$$x = \operatorname{Re} f(\zeta) = 2A\cos\theta + o(A).$$

Therefore, upon rescaling $x \to x/2A, y \to y/(2A)^{2n-1}$, we will obtain the following equations:

$$\begin{aligned} y &= -\beta Q_{n-1}(\cos^2\theta) \sin\theta + o(1), \\ x &= \cos\theta + o(1), \end{aligned}$$

Thus, when A goes to zero (i.e. for $t \to t^*$), we get the limiting shape

$$y^2 = \beta^2(1 - x^2) Q_{n-1}^2(x^2),$$

as desired. □

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press

Now let us consider the general case, i.e. a bubble which is not necessarily symmetric with respect to the x-axis. In this case,

$$\begin{aligned} \Pi_B &= \frac{1}{2}y^2 + \alpha y x^n + \frac{\beta}{2n}x^{2n} + \cdots \\ &= -\frac{1}{8}(z-\bar{z})^2 + \mathrm{Re}\left(i\alpha\frac{z^{n+1}}{n+1} + \beta\frac{z^{2n}}{2n}\right) + \cdots, \end{aligned}$$

where α, β are real and $\beta - n\alpha^2 > 0$ (to ensure that 0 is an isolated minimum of Π_B), and

$$h_B = z + 2i\alpha z^n - 2\beta z^{2n-1} + O(z^{2n}).$$

Applying a method similar to that used in the symmetric case, we obtain the following result.

Let $B_*(t)$ be the image of the bubble at the time t under the map $z \to z + i\alpha z^n$, followed by the renormalization $x \to cx$, $y \to c^{2n-1}y$, where $c = c(t)$ is chosen in such a way that the diameter of $B_*(t)$ is 2.

Theorem 2.7 *The boundary of the domain $B_*(t)$ tends to the curve*

$$y^2 = (\beta - n\alpha^2)^2(1-x^2)Q_{n-1}^2(x^2).$$

The most interesting case for applications is $n = 2$. This case corresponds to contraction of (symmetric) domains that are on the boundary between those that break up and those that do not.

Example 2.8 Assume that $\Pi_B = \frac{y^2}{2} + \beta\,\mathrm{Re}\,z^4/4$. This potential corresponds to the contracting bubble whose conformal map from the unit disk to the outside region has the form

$$f_t(\zeta) = A\zeta^{-1} + \frac{A}{1+6\beta A^2}\zeta - 2\beta A^3\zeta^3,$$

where $A = A(t) > 0$ is some function. (This map is found from the singularity correspondence; [10].) For small enough A, this function is univalent and defines a bubble. Contraction of the bubble corresponds to decreasing A to 0, leaving β fixed. Then, the bubble contracts to the origin with the asymptotic shape given by the above theorem for $n = 2$. This domain is on the boundary between rupturing and non-rupturing domains.

Remark 2.9 This analysis of asymptotic shapes is similar to the analysis of the shapes of the necks of bubbles during break-off which is done in [7].

3 Points of partial contraction

3.1 Definition and properties of points of partial contraction

Let $B_0(t) \subset B(t)$ be a connected component of the air domain (i.e. a single bubble), which exists on the interval of time (τ_f, τ_c). For example, we assume that τ_f is the time

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press

of formation of the bubble B_0, and τ_c is the time of its disappearance (contraction). A point contained in $B_0(t)$ for all $t \in (\tau_f, \tau_c)$ will be called a *point of partial contraction*. In Figure 1, P is a point of complete contraction and Q is a point of partial contraction. A point of either complete or partial contraction will be called a contraction point. To every contraction point, there corresponds a time of contraction τ_c.

Theorem 3.1 *(i) Every component B_0 of the air domain which contracts without breakup contains a unique contraction point.*

(ii) A contraction point at a time t is an (isolated) global minimum point of the potential $\Pi_{B(0)\setminus B(t)}$, and vice versa. *In particular, the number of contraction points is finite.*

Proof By formula (2.6),

$$\Pi_{B(0)\setminus B(t)}(\xi,\eta) = K_t - \int_0^t \Phi(\xi,\eta,\tau)\,\mathrm{d}\tau,$$

where K_t is a constant. Since $\Phi(\xi,\eta,\tau) \leqslant 0$, and $\Phi(\xi,\eta,\tau) = 0$ if and only if (ξ,η) is contained in the closure of $B(t)$, we have $\Pi_{B(0)\setminus B(t)} \geqslant K_t$, and $\Pi_{B(0)\setminus B(t)} = K_t$ if and only if $(\xi,\eta) \in \cap_{\tau<t} B(\tau)$. Thus, if (ξ,η) is a contraction point at a time t, then $\Pi_{B(0)\setminus B(t)}(\xi,\eta) = K_t$, i.e. the potential achieves its minimal value at (ξ,η), and *vice versa.*

On the other hand, the contraction points of the component B_0 are contained in the domain $B(0) \setminus B(t)$, where the potential $\Pi_{B(0)\setminus B(t)}$ satisfies the Poisson equation $\Delta\Pi = 1$, i.e. is a real analytic function. Thus, the set of contraction points of B_0 is analytic (as it is a connected component of the set of solutions of $\Pi_{B(0)\setminus B(t)} = K_t$). Also, it is compact and simply connected. This implies that it consists of one point (see [1]). This point is thus an isolated point of global minimum of the potential. On the other hand, it is clear that any global minimum point of $\Pi_{B(0)\setminus B(t)}$ is a contraction point. The theorem is proved. □

3.2 Finding points of partial contraction

If the bubble breaks up into two parts, which contract without further breakup, then Theorem 3.1 allows us to find the partial contraction point explicitly, provided that the derivative of the conformal map $f_0(\zeta)$ from the unit disk to the complement of the initial domain $B(0)$ is a rational function. Indeed, let τ be the moment of contraction of the bubble that contracts sooner, and z_0 be its contraction point. The domain $B(\tau)$ is connected, so there exists a conformal map $f_\tau(\zeta)$ of the unit disk into the complement of $B(\tau)$, which also has rational derivative. It can be found as described in [10]. Assume that the map f_τ is known. Then, by a direct computation one finds the potential $\Pi_{B(\tau)}$ on the whole plane. Next, the point z_0 is found from the condition

$$\nabla(\Pi_{B(0)} - \Pi_{B(\tau)})(z_0) = 0.$$

Moreover, the value of the potential $\Pi_{B(0)} - \Pi_{B(\tau)}$ at z_0 must coincide with the value of this potential inside the domain $B(\tau)$ (this value is constant inside $B(\tau)$ by Theorem 2.2). This is a condition on the unknown time τ of contraction.

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press

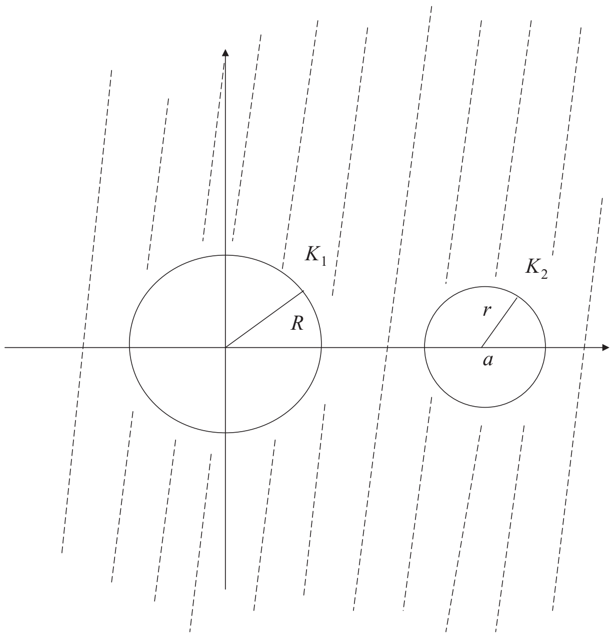

FIGURE 2. Simultaneous contraction of two circular bubbles.

A similar method allows one to find the contraction point of the smaller bubble in the problem of simultaneous contraction of two circular bubbles, $K_1 = \{z\,;|z| < R\}$ and $K_2 = \{z\,;|z-a| < r\}$, $a, R, r > 0$, $R > r$, $a > R + r$ (Figure 2).

In this case, as shown by Kufarev [6], the conformal map of the unit disk to the complement of the domain $B(\tau)$ at the time τ of partial contraction has the form

$$f_\tau(\zeta) = \frac{\beta\alpha^{-1}}{1-\alpha\zeta} + \frac{\gamma}{\zeta}, \qquad \gamma = \frac{1}{2}\left(a\alpha + \frac{2r^2 + R^2 - \frac{qt}{\pi}}{a\alpha}\right), \tag{3.1}$$

$$\beta = \frac{1-\alpha^2}{2}\left(a\alpha - \frac{2r^2 + R^2 - \frac{qt}{\pi}}{a\alpha}\right)$$

where $\alpha > 0$ and α^2 is the middle root of the cubic equation

$$2a^4x^3 - (2a^2(R^2 - qt/\pi) + a^4)x^2 + (2r^2 + R^2 - qt/\pi)^2 = 0, \tag{3.2}$$

and from the equations

$$\nabla \Pi_{B(0)\setminus B(\tau)}(z_0) = 0, \qquad \Pi_{B(0)\setminus B(\tau)}(z_0) = \Pi_{B(0)\setminus B(\tau)}|_{B(\tau)}, \tag{3.3}$$

one finds the contraction point z_0 of the smaller bubble and the time τ of partial contraction.

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press

3.3 Asymptotics of partial contraction

As in the case of complete contraction, if the contraction point is a non-degenerate minimum of the potential, then at times close to the time τ of partial contraction, the boundary of the bubble has an approximate shape of an ellipse, whose half-axes are directed along the eigenvectors of the Hessian of $\Pi_{B(0)\setminus B(\tau)}$ at the contraction point, and their lengths are inverse proportional to its eigenvalues. This fact is proved analogously to the case of complete contraction (see [10]).

It is interesting to study the rate of partial contraction, in the case when the bubbles contract at different times. For example, assume we have two bubbles B_1 and B_2, and B_2 contracts to a point P_2 at a time t', at which B_1 assumes the shape of a domain E. We assume that P_2 is a non-degenerate minimum of the potential $\Pi_{B(0)\setminus B(t')}$. Let us conformally map the outside of E onto the unit disk, so that ∞ maps to 0, and the point P_2 to a point $b \in (0,1)$. Obviously, such b and the map are unique. Let ζ be the complex coordinate in the disk; then the potential Φ at a time $t < t'$ close to t' has the form

$$\Phi = -\frac{q}{2\pi}\log|\zeta| + \frac{Q}{2\pi}\log\left|\frac{\zeta - b}{1 - \zeta b}\right| + O(t' - t),$$

where $Q = Q(t)$ is the rate of contraction of the bubble B_2, i.e. $Q(t) = -A'(t)$, where $A(t)$ is the area of $B_2(t)$. It is clear that $Q(t) \to 0$ as $t \to t'$.

Let $\Gamma(t)$ be the image of the boundary of $B_2(t)$ in the disk. Since the boundary of $B_2(t)$ for t close to t' is almost elliptic, the distance from the points of Γ to the point b is sandwiched between $c_1 A(t)^{1/2}$ and $c_2 A(t)^{1/2}$, where c_1, c_2 are some constants. On the other hand, Φ must vanish on $\Gamma(t)$. This yields

$$\frac{Q}{2}\log A \to q\log b, \qquad t \to t'.$$

Thus, $Q(t)$ is equivalent to $2q\frac{\log b}{\log A(t)}$ as $t \to t'$.

This allows us to determine the asymptotic behaviour of $A(t)$ as $t \to t'$. To do so, let us introduce the variable $\tau = t' - t$. Then, asymptotically A behaves as the solution of the differential equation

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} = 2q\frac{\log b}{\log A},$$

with the initial condition $A = 0$ as $\tau = 0$. Solving this equation, we get the solution which is implicitly defined by the equation

$$A\log A - A = 2q\tau\log b.$$

Therefore, we obtain the following result.

Theorem 3.2 *With the above assumptions, we have*

$$A \sim \frac{2q\tau\log b}{\log\tau}, \qquad Q \sim \frac{2q\log b}{\log\tau}, \qquad \tau \to 0.$$

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press

This result can be generalized to the case when the bubble B_2 contracts to a degenerate minimum. For example, assume that the contraction point is a critical point of degree n. Then, conducting a similar asymptotic analysis and using the results of Section 2.5, we obtain the following theorem.

Theorem 3.3 *We have*

$$A \sim \frac{2nq\tau \log b}{\log \tau}, \qquad Q \sim \frac{2nq \log b}{\log \tau}, \qquad \tau \to 0.$$

Thus, we see that when t is close to t' then the contraction of the bubble B_2 is logarithmically slow, and almost all air is extracted from B_1.

4 Sufficient conditions for breakup of symmetric bubbles

Theorem 4.1 *Assume that the gravity potential of a simply connected domain $B(0)$ symmetric with respect to a point P (respectively, a line ℓ) achieves a global minimum at a point $Q \neq P$ (respectively, $Q \notin \ell$). Then, $B(0)$ breaks up in the process of contraction.*

Proof The symmetric point $Q' \neq Q$ is also the global minimum point for the gravity potential. By Theorem 2.3, the points Q and Q' are points of complete contraction of the domain $B(0)$. Therefore, $B(0)$ must break up. □

Corollary 4.2 *(see [1]) Let $B(0) = \{(x, y); |x| < b, y^2 < f(x)\}$, where $f(x)$ is an even piecewise smooth function, non-negative on $[-b, b]$, such that $f(b) = 0$ (Figure 3).*

Then $B(0)$ breaks up in the process of contraction if

$$\int_0^b \frac{\mathrm{d}(xf(x)^{1/2})}{x^2 + f(x)} > \pi/2. \tag{4.1}$$

Proof

We first need the following formula.

Lemma 4.3 *One has*

$$\frac{\partial^2 \Pi_{B(0)}}{\partial x^2}(0) = \frac{1}{2} - \frac{1}{\pi}\int_0^b \frac{\mathrm{d}(xf(x)^{1/2})}{x^2 + f(x)}.$$

Proof Let $z = x + iy$ and $\Pi = \Pi_{B(0)}$. We have

$$\partial_z \Pi = \frac{1}{4}(\overline{z} - h_B(z)),$$

where h_B is the Cauchy transform of B. Therefore

$$\partial_z^2 \Pi = -\frac{1}{4}h'_B(z).$$

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press

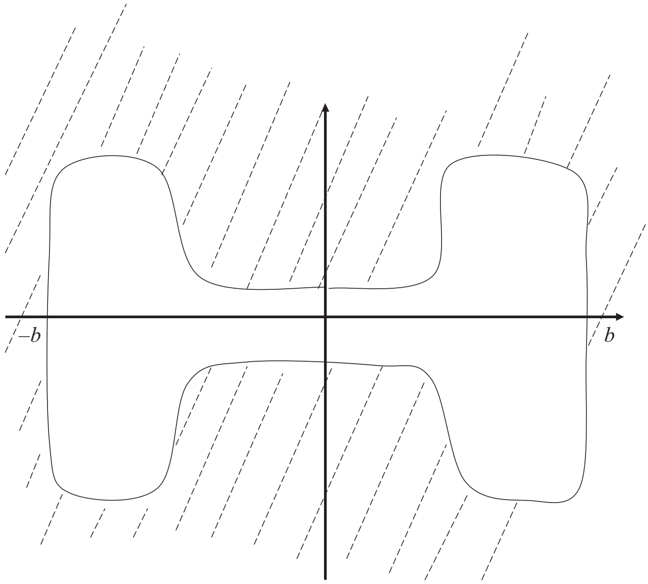

FIGURE 3. Contraction of an axially symmetric bubble.

Thus, for $y = 0$, where $\partial_y \Pi = 0$, we have

$$(\partial_x^2 - \partial_y^2)\Pi = -h'_B.$$

On the other hand, $\Delta\Pi = 1$, therefore

$$\partial_x^2 \Pi(0) = \frac{1 - h'_B(0)}{2}.$$

On the other hand, we have

$$h_B(w) = \frac{1}{2\pi i}\int_{\partial B} \frac{z\,\mathrm{d}\bar{z}}{z - w},$$

(where the boundary is oriented counterclockwise), therefore

$$h'_B(0) = \frac{1}{2\pi i}\int_{\partial B} \frac{\mathrm{d}\bar{z}}{z} = -\frac{1}{\pi}\,\mathrm{Im}\int_{-b}^{b} \frac{\mathrm{d}(x - if(x)^{1/2})}{x + if(x)^{1/2}}.$$

Thus

$$h'_B(0) = \frac{1}{\pi}\int_{-b}^{b} \frac{\mathrm{d}(xf(x)^{1/2})}{x^2 + f(x)} = \frac{2}{\pi}\int_{0}^{b} \frac{\mathrm{d}(xf(x)^{1/2})}{x^2 + f(x)},$$

which implies the statement. □

Now we proceed to prove the corollary. By the lemma, if (4.1) holds, then

$$\frac{\partial^2 \Pi_{B(0)}}{\partial x^2}(0) < 0,$$

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press

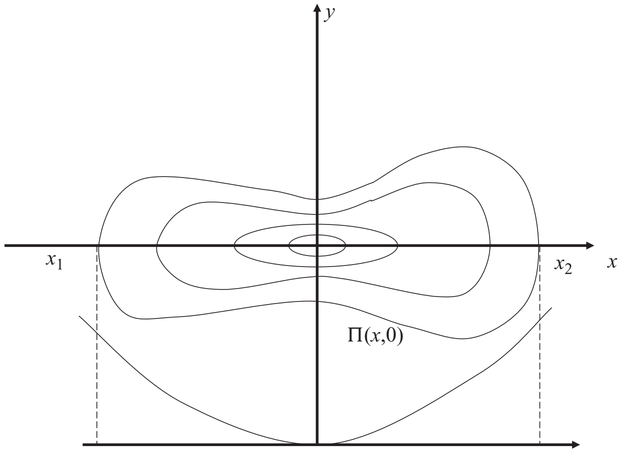

FIGURE 4. The potential of an axially symmetric bubble.

so the origin is not a global minimum point of the potential. Hence, the global minimum is attained at another point. Because of the central symmetry, the domain $B(0)$ must break up. □

Theorem 4.4 *(see* [10], *problem 2 on p. 32) Let* $B(0)$ *be a simply connected domain, symmetric with respect to the horizontal axis, and the function* $\pi(x) := \Pi_{B(0)}(x, 0)$ *have more than one local extremum on* $(-\infty, \infty)$. *Then the domain* $B(0)$ *breaks up in the process of contraction.*

Proof Assume the contrary, i.e. the domain does not break up. Let $x_1(t) < x_2(t)$ be the intersection points of the boundary $\partial B(t)$ with the horizontal axis (Figure 4).

By the monotonicity property of contraction, the function $x_1(t)$ is increasing, and the function $x_2(t)$ is decreasing on $[0, t^*)$, and $\lim_{t\to t^*} x_1(t) = \lim_{t\to t^*} x_2(t) = x_0$. The rays $[x_2(t), +\infty)$ and $(-\infty, x_1(t)]$ are flowlines of the flow, and the flow is directed from infinity, so the potential $\Phi(x, 0, t)$ increases from $-\infty$ to 0 on the interval $(-\infty, x_1(t)]$, equals zero on $(x_1(t), x_2(t))$, and decreases from 0 to $-\infty$ on $[x_2(t), +\infty)$. Thus, if $\xi_1 \geqslant \xi_2 \geqslant x_0$ or $\xi_1 \leqslant \xi_2 \leqslant x_0$, then for any $t \in (0, t^*)$, one has $\Phi(\xi_1, 0, t) \leqslant \Phi(\xi_2, 0, t)$. Moreover, if this inequality turns into an equality for all t, then $\xi_1 = \xi_2$. Since $\Pi_{B(0}(\xi, \eta) = K - \int_0^{t^*} \Phi(\xi, \eta, t)\,\mathrm{d}t$, these arguments imply that the function $\Pi_{B(0)}(x, 0)$ is strictly increasing on $(x_0, +\infty)$ and strictly decreasing on $(-\infty, x_0)$, i.e. its unique local extremum is a minimum at the point x_0. This is a contradiction. □

5 Regulated contraction

5.1 Definition of regulated contraction

Consider contraction of a domain which breaks up into two parts at a time $\tau \in [0, t^*)$. After the time τ, the process of contraction may be controlled, creating different pressures inside the two components of the air domain by regulating the amount of air which is pumped out of each component. This is a generalization of the problem from Section 2. In particular, it is interesting whether one can regulate contraction in such a way that

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press

both bubbles contract at the same time. In this case, as we will see, the contraction points have the convenient property that they are critical points of the potential of the initial bubble, and thus can be easily found. Let us consider this generalized problem in more detail.

Consider the process of contraction of a system of two bubbles; we do not assume that they were obtained as a result of breakup of a single bubble. Assume that at a time $t \in [0, t^*)$, the air domain $B(t)$ consists of the components $B_1(t)$ and $B_2(t)$, and the air is pumped from $B_1(t)$ at the rate $q_1(t)$, and from $B_2(t)$ at a rate $q_2(t)$. This means that the velocity potential $\Phi(x, y, t)$ is a solution of the boundary value problem

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0, \qquad (x, y) \in B(t)^c; \qquad \Phi|_{\partial B_i(t)} = \Phi_i(t), \qquad i = 1, 2,$$

$$\Phi(x, y) = -\frac{q_1(t) + q_2(t)}{2\pi} \log(r) + O(1), \quad r \to \infty,$$

and the constants $\Phi_1(t)$ and $\Phi_2(t)$ are chosen in such a way that

$$\int_{\partial B_i} \frac{\partial \Phi}{\partial n} \, \mathrm{d}\ell = q_i(t), \quad i = 1, 2. \tag{5.1}$$

It is useful to extend Φ to the interior of $B(t)$: $\Phi = \Phi_i(t)$ in $B_i(t)$ for $i = 1, 2$. Then $\Phi(x, y, t)$ is an everywhere continuous function. The velocity of motion of the boundaries ∂B_1 and ∂B_2 is $\frac{\partial \Phi}{\partial n}$. The motion can be considered up to the time of disappearance of one of the bubbles (we assume that there is no topological transformations of the first kind, i.e. formations of new bubbles). The contraction process in this situation will be called *regulated contraction* (as opposed to free contraction, i.e. with equal pressures in the bubbles). The vector function $(q_1(t), q_2(t))$ will be called *the strategy of air extraction.*

Theorem 5.1 *The gravity potential inside every component of the contracting domain changes by a constant in the process of regulated contraction (the constants may be different for different components).*

This theorem is proved analogously to part 1 of Theorem 2.2.

Theorem 5.2 *Let B_1^s, B_2^s be two smooth families of simply connected domains, $s = (s_1, s_2)$, $B_1^s \cap B_2^s = \emptyset$. If the potential of the system $\{B_1^s, B_2^s\}$ changes by a constant inside B_1^s and inside B_2^s under the change of s, and the areas of B_1^s and B_2^s are independent of s, then B_1^s and B_2^s do not change under the change of s.*

Thus, a pair of simply connected domains is locally uniquely determined by the potential inside them, given up to additive constants, and by the areas of the domains.

Proof Let $v_s : \partial B_1^s \cup \partial B_2^s \to \mathbb{R}$ be the velocity of the boundary under the change of s. For any continuous function $u(x, y)$,

$$\frac{\mathrm{d}}{\mathrm{d}s} \int_{B_1^s \cup B_2^s} u \, \mathrm{d}x \, \mathrm{d}y = \int_{\partial B_1^s \cup \partial B_2^s} u v_s \, \mathrm{d}\ell.$$

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press

Thus, the conditions of the theorem imply that the function v_s is orthogonal on $\partial B_1^s \cup \partial B_2^s$ to the following functions:

(i) the characteristic functions $\xi_j(z)$ of the boundary components ∂B_j^s (because the areas of B_j^s are constant in s); and

(ii) $\eta_1^w(z) = \mathrm{Re}(w-z)^{-1}$, $\eta_2^w(z) = \mathrm{Im}(w-z)^{-1}$, where $w \in B_1^s \cup B_2^s$ (because of the fact that the gradient of the gravity potential is constant in s). These functions are dense in $L^2(\partial B_1^s \cup \partial B_2^s)$. Hence, v_s is identically zero. The theorem is proved. □

Theorem 5.3 *Let $\bar{q}_s(t)$, $t \in (\tau, \theta)$, $\bar{q} = (q_1, q_2)$, $s \in [0,1]$, be a smooth family of strategies of extraction such that $\int_\tau^\theta \bar{q}_s(t)dt = \bar{Q}$ is independent of s. Let $B^s(\theta) = B_1^s(\theta) \cup B_2^s(\theta)$ be the result of extraction according to the strategy $\bar{q}_s$ from the same initial domain $B(\tau) = B_1(\tau) \cup B_2(\tau)$. Then $B^s(\theta)$ is independent of s.*

Proof By Theorem 5.1, the gravity potential $B^s(\theta)$ is independent of s (and equals the potential of $B(\tau)$) up to an additive constant (in each connected component). The areas of the domains $B_1^s(\theta)$ and $B_2^s(\theta)$ are also independent of s and equal $S_1 - Q_1$, $S_2 - Q_2$, respectively, where S_j are the areas of the components $B_j(\tau)$, and Q_j are the coordinates of the vector $\bar{Q}$, i.e. the volumes of the air extracted from the first and second bubbles. By Theorem 5.2, $B_j^s(\theta)$ do not change under the change of s, as desired. □

Thus, the result of contraction depends only on the total quantities of air extracted from the bubbles for a given period of time and does not depend on other parameters of the strategy. In other words, the transformations in the space of domains defined by extraction of air from the first and second bubbles, respectively, commute with each other. This is an analogue of Richardson's result [9] on the commutativity of injection operations at different points (see also [10]).

5.2 The phase rectangle and the accessibility region

Theorem 5.3 shows that the domains which can be obtained from $B(0)$ under regulated contraction can be visualized by points of the 'phase' rectangle $0 \leqslant X \leqslant S_1$, $0 \leqslant Y \leqslant S_2$, where S_1 and S_2 are areas of $B_1(\tau)$ and $B_2(\tau)$, respectively: the point (X, Y) corresponds to the domain which is obtained by extraction of the volumes $S_1 - X, S_2 - Y$ from the first and second bubbles, respectively. The strategy of extraction is shown by a path inside the rectangle, which emanates from the corner (S_1, S_2) (Figure 5).

It is important to note that in the process of regulated contraction, pieces of the boundary may move towards the fluid region. If this happens for a particular strategy, the problem of regulated contraction is ill-posed in the vicinity of this strategy. Thus, in general, one has neither the monotonicity property nor the existence of a weak solution up to the time of complete contraction. In other words, the solution of the problem of regulated contraction may not exist for some strategies of extraction. More specifically, in a generic situation, solutions develop singularities in the following way: at some point of the boundary, one of the two bubbles develops a semicubic cusp directed towards the fluid region (Figure 6), and after this time the solution cannot be continued.

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press

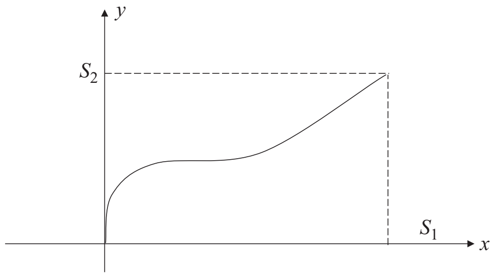

FIGURE 5. A strategy of extraction in the phase rectangle.

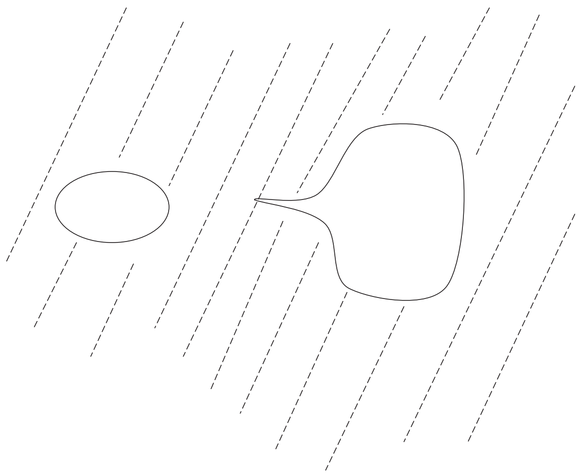

FIGURE 6. Cusp formation on one of the two contracting bubbles.

The development of such a cusp was first discovered by Polubarinova-Kochina [8] for the problem of contraction of the boundary of the oil region. The fact that the singularity generically takes the shape of a semicubic cusp is related to the fact that while the solution exists, the boundary of the bubble is an analytic curve, and a semicubic cusp is the simplest singularity of a non-self-intersecting analytic curve.

It follows from the above discussion that not all paths in the phase rectangle correspond to actual solutions, but only those that lie in some region Ω, which is the set of all points of the rectangle that can be accessed by a strategy of extraction in which both bubbles exist all the way up to the last moment. We will call Ω *the accessibility region*. The boundary of the accessibility region consists (in the generic situation) of parts of the boundary of the phase rectangle, and curves, whose points correspond to pairs of bubbles, one of which has a cusp (Figure 7).

The path γ corresponding to free contraction divides the accessibility region into two parts. Namely, if the pressure in the first bubble is kept higher than in the second bubble, then the corresponding path lies below γ, and if it is kept lower, then the path lies above γ. Similarly, one can construct the trajectory γ_P of free contraction, starting from a

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press

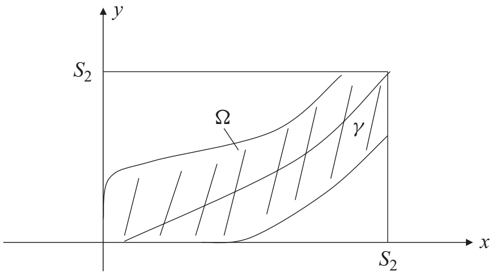

FIGURE 7. The accessibility region.

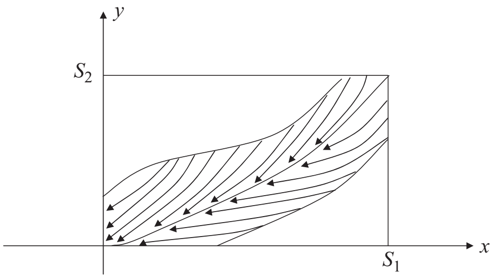

FIGURE 8. Foliation of the accessibility region by trajectories of free contraction.

domain corresponding to any point $P \in \Omega$. The accessibility region Ω is foliated by such trajectories (Figure 8), which implies that Ω is contractible (i.e. simply connected).

If the initial domain consists of two symmetric bubbles, then the accessibility region is symmetric with respect to the diagonal $Y = X$ of the phase square, and the path γ of the free contraction is diagonal.

Remark 5.4 Theorem 3.2 implies that the trajectories γ_P of free contraction which do not end in the origin are tangent to the boundary of the phase rectangle at the endpoint, and the tangency is of the type $Y - a = cX \log(1/X)$ if the endpoint is $(0, a)$, and $X - a = cY \log(1/Y)$ if the endpoint is $(a, 0)$ (for some $c > 0$).

6 Synchronizing strategies of extraction

Let us say that a strategy $\bar{q}(t)$ is synchronizing for the system of bubbles B_1, B_2, if the extraction according to this strategy leads to simultaneous contraction of the bubbles B_1 and B_2 to a point. The path in the accessibility region which corresponds to a synchronizing strategy ends in the origin. Obviously, a synchronizing strategy exists iff the accessibility region contains the origin. A domain $B(0)$, which breaks up under contraction into two bubbles which have this property, will be called *synchronizable.*

Under extraction of air according to synchronizing strategy $\bar{q}(t)$, the bubbles B_1 and B_2 simultaneously contract to points P_1 and P_2. This means that the points P_1 and P_2 are limiting positions of the boundaries $\partial B_1(t)$ and $\partial B_2(t)$ when the time t tends to the time t^* of contraction. These points are easily found from the shape of the initial domain.

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press

Theorem 6.1 *The contraction points P_1 and P_2 are critical points of the potential $\Pi_{B(0)}$ (if they both belong to $B(0)$).*

Proof We first prove the following.

Lemma 6.2 *Let G be a bounded domain of area S. Then*

$$|\nabla \Pi_G(x, y)| \leqslant \sqrt{S/\pi}, x, y \in \mathbb{R}. \tag{6.1}$$

Proof Let $z = x + iy, w = u + iv$. Let K be the disk of area S centred at (x, y). Then we have

$$\begin{aligned} |\nabla \Pi_G(x, y)| &= \frac{1}{2\pi}\Big|\int_G \frac{(z-w)\,\mathrm{d}u\,\mathrm{d}v}{|z-w|^2}\Big| \leqslant \frac{1}{2\pi}\int_G \frac{\mathrm{d}u\mathrm{d}v}{|z-w|} \\ &\leqslant \frac{1}{2\pi}\int_K \frac{\mathrm{d}u\,\mathrm{d}v}{|z-w|} = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\rho\int_0^{\sqrt{S/\pi}}\mathrm{d}\lambda = \sqrt{S/\pi}, \end{aligned}$$

as required. □

Now we prove the theorem. Let t_n be a sequence of times which tends from below to t^*. Let $P_1^{(n)}, P_2^{(n)}$ be sequences of points in $B_1(t_n)$ and $B_2(t_n)$, which converge to P_1 and P_2 as $n \to \infty$. From Theorem 5.1, we have

$$\nabla \Pi_{B(0)}\big(P_j^{(n)}\big) = \nabla \Pi_{B(t_n)}\big(P_j^{(n)}\big).$$

From Lemma 6.2, $|\nabla \Pi_{B(t_n)}(P_j^{(n)})| \to 0$ as $n \to \infty$, as the area of $B(t_n)$ tends to 0 for $n \to \infty$. Hence, $\nabla \Pi_{B(0)}(P_j^{(n)}) \to 0$, $n \to \infty$, $j = 1, 2$. Since the gravity potential is a C^1-function, this implies that $\nabla \Pi_{B(0)}(P_j) = 0$, as desired. □

Clearly, one of the contraction points P_1, P_2 coincides with the point P of complete contraction in the sense of Section 1 (namely, the contraction point for the bubble that contracts later under free contraction). This point, as we mentioned, is the global minimum point of the gravity potential. Regarding the second point, it is shown below that it is either a local minimum point or a degenerate critical point, and the degenerate critical point arises for initial domains which lie on the boundary between synchronizable and non-synchronizable domains in the space of domains.

Let us call an initial domain $B(0)$ strictly synchronizable if the corresponding accessibility region contains a sector

$$\{(X, Y)|X \geqslant 0, Y \geqslant 0, X^2 + Y^2 < \varepsilon\}$$

for sufficiently small ε. Clearly, strict synchronizability is an open condition, i.e. this is a property which is stable under small deformations. Non-strictly synchronizable domains form the boundary between synchronizable and non-synchronizable domains. To illustrate

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press

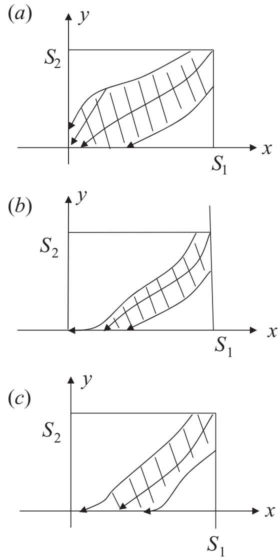

FIGURE 9. The accessibility regions of a strictly synchronizable, non-strictly synchronizable, and non-synchronizable domain.

this, we show in Figures 9a–c what the accessibility region looks like for a strictly synchronizable, non-strictly synchronizable and non-synchronizable domain.

Theorem 6.3 *If a domain $B(0)$ is strictly synchronizable, then the contraction points P_1, P_2 for a synchronizing strategy are local minima of the potential $B(0)$ (if they lie in $B(0)$).*

Proof If the domain $B(0)$ is strictly synchronizable, then one of the trajectories of free contraction ends in the origin (Figure 9).

Therefore, there exists a synchronizing strategy which corresponds to free contraction on the interval $(t^* - \varepsilon, t^*)$ for some $\varepsilon > 0$. Then, by Theorem 3.1, the points P_1, P_2 are points of global minimum of the potential $\Pi_{B(t^*-\varepsilon)}$. Since inside $B_i(t^* - \varepsilon)$, $i = 1, 2$, the potential $\Pi_{t^*-\varepsilon}$ coincides with $\Pi_{B(0)}$ up to constants, we see that P_1 and P_2 are points of local minimum of $\Pi_{B(0)}$. □

Remark 6.4 If the point P_1 or P_2 is outside the domain $B(0)$ (a priori, such a situation cannot be ruled out because of the failure of the monotonicity property), then it is a local minimum point of the analytic continuation of the potential of the domain $B(0)$ from its inside to its outside along the track of the corresponding bubble.

7 Asymptotics of regulated contraction under a synchronizing strategy

Theorem 7.1 *Let $B(t) = (B_1(t), B_2(t))$ be the evolution of the air domain after its breakup, under extraction of air according to a synchronizing strategy $\bar{q}(t)$, $P_1, P_2 \in B(0)$ be the contraction points and A_1, A_2 be the Hessian matrices of the potential $\Pi_{B(0)}$ at these points. Let $\widehat{B_j(t)}$ be the domains obtained from $B_j(t)$ by dilation $\lambda_j(t)$ times, where $\lambda_j(t)$ is chosen in*

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press

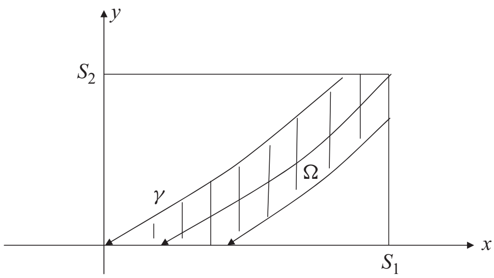

FIGURE 10. Contraction along the boundary of the accessibility region.

such a way that $\widehat{B_j(t)}$ has a fixed diameter d. Then if the matrices A_1, A_2 are non-degenerate, then the boundaries $\partial\widehat{B_i(t)}$ tend to ellipses, whose axes are directed along eigenvectors of A_1, A_2, and the lengths of the half-axes are inverse proportional to the eigenvalues of these matrices.

Proof We prove the theorem for B_1; the proof for B_2 is the same. Assume that the point P_1 is the origin. Obviously, for t close to t^*, the potential $\Pi_{B_1(t)}$ has a non-degenerate local minimum at some point $a(t)$ inside $B_1(t)$. Let $E(t) = B_1(t) - a(t)$ be the domain obtained by translating $B_1(t)$ by the vector $-a(t)$. It can be seen, along the lines of [10], that the boundary $\partial E(t)$ converges to some curve Γ.

Let $A_D(z)$ be the matrix of second derivatives of the gravity potential of a domain D. It is easy to see that $A_{E(t)}(z) \to A_{B(0)}(0) = A_1$, $t \to t^*$, if $z \in E(t)$ for t close to t^*. This implies that the potential of the domain E bounded by the curve Γ is a quadratic function, whose Hessian matrix is A_1, and which has a minimum at zero, i.e. it is $(A_1 z, z) + C$. By Sakai's theorem, this implies (see e.g. [10]) that E is an ellipse, whose axes are directed along the eigenvectors of A_1, and the lengths of half-axes are inverse proportional to its eigenvalues. The centre of the ellipse is situated at the origin. The theorem is proved. □

8 Properties of the potential of non-strictly synchronizable domains

Suppose that $B(0)$ is a synchronizable domain, and P_1 and P_2 are points of its contraction under a synchronizing strategy. Assume that P_1 is the point of free contraction. If $B(0)$ is strictly synchronizable, then by Theorem 6.3 the points P_1 and P_2 must be local minima of the potential $\Pi_{B(0)}$. However, if $B(0)$ is synchronizable, but not strictly synchronizable, this does not have to be the case.

For example, in this case there is a synchronizing strategy γ that goes along the boundary of the accessibility region Ω (see Figure 10). Let $B(t)$ be the evolution of the bubble under this strategy. Then, for t close to the time t^* of full contraction, $B(t)$ is a union of two bubbles $B_1(t)$ and $B_2(t)$, and the boundary of $B_2(t)$ has a persistent singularity (typically, a cusp).

In this situation, we expect that similar to Section 2, the limiting shape of $B_2(t)$ as $t \to t^*$ is not an ellipse, but rather a line segment, which means that the Hessian matrix of $\Pi_{B(0)}$ at P_2 is degenerate. However, unlike Section 2, P_2 does not have to be (and typically, won't be) a local minimum of $\Pi_{B(0)}$. Indeed, in the generic situation, P_2 is a critical point

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press

of $\Pi_{B(0)}$ of type saddle-node, i.e. in some Cartesian coordinates

$$\Pi_{B(0)}(x, y) = \frac{y^2}{2} + \beta \frac{x^3}{3} + \cdots ,$$

where the dots stand for the terms inside the Newton polygon.

It is interesting to study finer asymptotics of contraction in this situation. Consider the generic case when the contraction point of the singular bubble $B_2(t)$ is 0, and the potential at this point is as above:

$$\Pi_{B(0)}(z) = -\frac{(z - \bar{z})^2}{8} + \frac{\beta \operatorname{Re}(z^3)}{3} + \cdots .$$

Let $B_*(t)$ be the image of the bubble $B_2(t)$ at the time t under the renormalization $x \to cx$, $y \to c^2 y$, where $c = c(t)$ is chosen in such a way that the diameter of $B_*(t)$ is 2.

Theorem 8.1 (*i*) *The boundary of $B_*(t)$ tends to the curve*

$$y^2 = \beta^2 \left(x + \frac{1}{2}\right)^3 \left(x - \frac{3}{2}\right).$$

(*ii*) *The bubble $B_1(t)$ contracts at some point $a < 0$ of the real axis. Thus, the cusp of B_2 is always directed precisely towards B_1 at the time of contraction.*

Proof The method of proof is the same as that we used for the asymptotic analysis of a bubble contracting to a degenerate minimum. For example, we have

$$h_{B_2(t)}(z) = K(t) + z - 2\beta z^2 + \ldots,$$

where the dots are negligible terms as $t \to t^*$. (Note that $h_{B_2(t)}(z)$ depends on t because of the presence of the second bubble $B_1(t)$.) Thus, the conformal map of the unit disk into the outside of $B_1(t)$ which maps 0 to ∞ looks like

$$f(\zeta) = A(\zeta + \zeta^{-1}) + B\zeta^2 + C\zeta + D + \ldots .$$

Since $f^*(\zeta^{-1}) - h_D(f(\zeta))$ is holomorphic and vanishes at infinity, we get

$$B = -2\beta A^2, \quad C = -4\beta AD,$$

modulo negligible terms. Also, we have a cusp in the 'saddle' direction of the singularity, i.e. in the negative direction in our case. Thus, $f_t'(-1) = 0$, which gives $2B - C = 0$ modulo negligible terms. This means that modulo negligible terms we have

$$D = A, \quad C = -4\beta A^2.$$

This implies that

$$x = \operatorname{Re} f = 2A\left(\cos\theta + \frac{1}{2}\right) + o(A),$$

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press

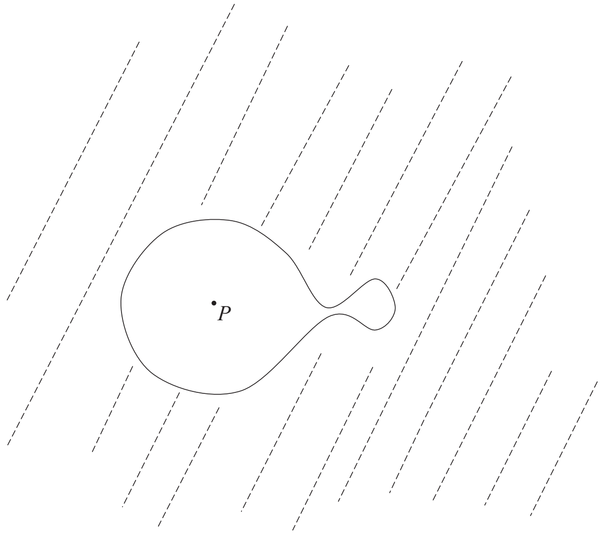

FIGURE 11. Breakup of a bubble with a unique critical point of the potential.

and

$$y = -2\beta A^2(\sin 2\theta + 2\sin\theta) + o(A^2).$$

Thus, after the rescaling $x \to x/2A$, $y = y/4A^2$ and sending A to zero, we get the limiting curve

$$y^2 = \beta^2\left(x+\frac{1}{2}\right)^3\left(x-\frac{3}{2}\right),$$

and part (i) of the theorem follows. We also get $K = 6\beta A^2 + o(A^2) > 0$ for small A, which implies that the second bubble disappears at some point $a < 0$, hence (ii). □

We see from the proof of this theorem that when A goes to zero, the area of B_1 goes down as c_1A^2 and the area of B_2 as c_2A^3. This shows that the trajectory corresponding to our strategy (i.e. the upper boundary of the accessibility region) behaves near the origin as a semicubic parabola $Y = cX^{3/2}$ (Figure 9b).

9 The boundary between rupturing and non-rupturing domains

Having multiple local minima of the gravity potential is not a necessary condition for the breakup of a bubble. Indeed, it is obvious that a domain in Figure 11 breaks up in the process of contraction, although its potential has a unique critical point (so this domain is not synchronizable).

In this connection, it is interesting to study domains which lie on the boundary between rupturing and non-rupturing domains.

Theorem 9.1 *Suppose that B is a generic domain on the boundary between rupturing and non-rupturing domains. Then, B does not break up under contraction, but at some time t,*

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press

the domain $B(t)$ has a cusp on the boundary, locally equivalent to $y = x^{5/2}$. This singularity disappears immediately under further contraction.

In other words, on the boundary between the sets of rupturing and non-rupturing domains in the space of (C^k-smooth) domains, a dense open set is formed by domains that have the property stated in the theorem.

Proof (sketch). Consider a smooth family of simply connected bounded domain of generic position, B^s, $s \in [0,1]$. Assume that for $s \leqslant \sigma$ the domain B^s does not break up, while for $s > \sigma$ it does. Let $\tau(s)$ be the time of breakup of the domain B^s for $s > \sigma$. Let $\tau = \lim_{s \to \sigma} \tau(s)$. One can show that in the situation of general position this limit exists and is not equal to zero. Consider the family of curves $\Gamma(s)$, $s \in (\sigma, 1]$, which are obtained from B^s by contraction during the time $\tau(s)$ (where $\tau(\sigma) := \tau$).

The curves $\Gamma(s)$ for $s > \sigma$ have a simple self-tangency at some point. It is easy to see that typical degenerations of such curves into simple closed curves have the structure described in the theorem: for $s = \sigma$ at the point of disappearance of the loop, there forms a cusp of degree 5/2. For example, a typical such family is

$$y^2 = x^4(x + \varepsilon), \varepsilon = \varepsilon(s),$$

where $\varepsilon(\sigma) = 0$, and $\varepsilon(s) > 0$ for $s > \sigma$. □

Remark 9.2 The appearance of the singularity in the process of contraction is, at first sight, a strange phenomenon, as the contraction problem has good properties of correctness and stability. Nevertheless, solutions with such type of instantaneous singularities do exist. They were first discovered by Howison in [5]. More precisely, he showed that in the process of contraction, there can appear instantaneous cusps of degrees $(4n+1)/2$, $n \geqslant 1$, while cusps of degree $(4n-1)/2$ cannot appear. For $n = 1$, these are the instantaneous cusps of degree 5/2 that we have just considered. Solutions with these properties form a subset of codimension 1 in the space of all solutions.

If we restrict ourselves to polynomial domains of degree $\leqslant n$, then the boundary between rupturing and non-rupturing domains, as follows from Theorem 9.1, is a piece of an algebraic hypersurface in the space of coefficients. For small degrees n, this equation is not difficult to write down. If two domains can be connected by a curve not intersecting this surface, then either they both break up under contraction or they both do not.

10 Corrections to [1, 10]

We use the opportunity to correct some errors in [1, 10].

(1) The formula for the gravity potential of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ given in [1, 10] ([1], p. 517, and [10], (4.14)) is incorrect. The correct formula is

$$\Pi = \frac{1}{2}\left(\frac{b}{a+b}x^2 + \frac{a}{a+b}y^2\right) + C(a,b),$$

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press

(i.e. a and b need to be switched). The same correction needs to be made in formulae (24) and (25) of [1], and the two sentences after formula (25). Hence, the correct condition of division of the bubble for $a = 2b$ is $c < \sqrt{3}b$. The same corrections should be made in [10] in the example on page 52 and in the answer to problem 4 on page 68.

(2) As a result of point (1), Theorem 4.10 in [10] should say that the lengths of the half-axes of the limiting ellipse are *inverse* proportional to the eigenvalues of the Hessian matrix. The same correction is to be made on page 535 of [1] (the power −2 of the Hessian matrix should be replaced with +2).

(3) The left-hand side of formula (19) of [1] should read $2\frac{\partial^2 \Pi_B}{\partial x^2}(0,0)$ (the factor of 2 is missing). The same correction should be made in the first formula on page 52 of [10]. The right-hand side of the formula in Theorem 6.4 in [10] should be π, not $\pi/2$. In the first formula on p. 52 of [10], the factor $2/\pi$ should be $1/\pi$.

11 Conclusion

In this paper, we have made a detailed study of the phenomenon of breakup of a bubble in a Hele-Shaw cell under suction of air, and of the process of subsequent contraction of the resulting system of smaller bubbles. In particular, we have studied the asymptotics of contraction of bubbles to degenerate minima of the potential, which includes symmetric bubbles that are on the boundary between breaking and non-breaking bubbles.

We have also studied the phenomenon of non-simultaneous contraction of a system of bubbles. Typically, under free contraction, i.e. when the pressures in the bubbles are kept equal, the bubbles contract at different times and the points of contraction of all of them but the one contracting last are hard to compute. One can, however, control the contraction by regulating the pressures inside the bubbles, which sometimes allows one to make the bubbles contract simultaneously, to critical points of the potential (a synchronizing strategy). In this case, the system of bubbles is called synchronizable. We have discussed various properties of synchronizable systems of bubbles and asymptotics of contraction under a synchronizing strategy. In particular, we have explained that if a system of bubbles is on the border between synchronizable and non-synchronizable systems, then typically the contraction point is not a minimum but rather a saddle-node of the potential (i.e. the simplest degenerate critical point of a function of two variables), and studied the asymptotics of such contraction. Finally, we have discussed typical bubbles which are on the border between breaking and non-breaking bubbles and explained that they develop an instantaneous cusp (of order 5/2) during contraction.

There are many questions about breakup and contraction of systems of bubbles which deserved to be studied further.

For example, an interesting question is the following. For a domain B, a point P in the plane, and $s > 0$, let $B(P,s)$ be the domain obtained by rescaling the size of B by s, and shifting it so that its centre of mass is at P. Let B_1, B_2 be bounded convex domains in the plane with regular analytic boundaries (say, disks). Suppose P_1, P_2 are two distinct points in the plane. Is the system of bubbles $B_1(P_1,s), B_2(P_2,s)$ synchronizable for small enough s?

Another interesting problem is to generalize the asymptotic results of this paper to three dimensions.

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press

Acknowledgements

The work of the second author was partially supported by the NSF grant DMS-0504847. He thanks B. Gustafsson and D. Jerison for useful discussions, and the referee for useful comments.

Note

Sadly, the first author of this paper, Vladimir M. Entov, passed away on 10 April 2008.

References

[1] Entov, V. M. & Etingof, P. I. (1991) Bubble contraction in Hele-Shaw cells. *Q. J. Mech. Appl. Math.* **44**(4), 507–535.

[2] Entov, V. M. & Etingof, P. I. (1997) Viscous flows with time-dependent free boundaries in a non-planar Hele-Shaw cell. *Eur. J. Appl. Math.* **8**(1), 23–35.

[3] Gustafsson, B. (1983) Quadrature identities and the Schottky double. *Acta Appl. Math.* **1**(3), 209–240.

[4] Gustafsson, B. & Vasiliev, A. (2006) *Conformal and Potential Analysis in Hele-Shaw Cells*. Advances in Mathematical Fluid Mechanics, Birkhäuser, Basel.

[5] Howison, S. D. (1986) Cusp development in Hele-Shaw flow with a free surface. *SIAM J. Appl. Math.* **46**(1), 20–26.

[6] Kufarev, P. P. (1950) The problem of the contour of the oil-bearing region for a circle with an arbitrary number of gaps (in Russian). *Dokl. Akad. Nauk SSSR (NS)* **75**, 507–510.

[7] Lee, S.-Y., Bettelheim, E. & Wiegmann, P. (2006) Bubble break-off in Hele-Shaw flows: singularities and integrable structures. *Physica* D **219**(1), 22–34.

[8] Polubarinova-Kochina, P. Y. (1945) On the displacement of the oil-bearing contour. *C. R. (Dokl.) Acad. Sci. URSS (NS)* **47**, 250–254.

[9] Richardson, S. (1981) Some Hele-Shaw flows with time-dependent free boundaries. *J. Fluid Mech.* **102**, 263–278.

[10] Varchenko, A. N. & Etingof, P. I. (1992) *Why the Boundary of a Round Drop Becomes a Curve of Order Four*. American Mathematical Society, Providence, RI.

https://doi.org/10.1017/S095679251000032X Published online by Cambridge University Press