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On the commuting probability for subgroups of a finite group

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Let K be a subgroup of a finite group G. The probability that an element of G commutes with an element of K is denoted by $Pr(K, G)$. Assume that $Pr(K, G) \geqslant \epsilon$ for some fixed $\epsilon > 0$. We show that there is a normal subgroup $T \leqslant G$ and a subgroup $B \leqslant K$ such that the indices $[G : T]$ and $[K : B]$ and the order of the commutator subgroup $[T, B]$ are ϵ-bounded. This extends the well-known theorem, due to P. M. Neumann, that covers the case where $K = G$. We deduce a number of corollaries of this result. A typical application is that if K is the generalized Fitting subgroup $F^*(G)$ then G has a class-2-nilpotent normal subgroup R such that both the index $[G : R]$ and the order of the commutator subgroup $[R, R]$ are ϵ-bounded. In the same spirit we consider the cases where K is a term of the lower central series of G, or a Sylow subgroup, etc.

Keywords: Commutativity degree; conjugacy classes; nilpotent subgroups

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1. Introduction

The probability that two randomly chosen elements of a finite group G commute is given by

$$Pr(G) = \frac{|\{(x, y) \in G \times G \ : \ xy = yx\}|}{|G|^2}.$$

The above number is called the *commuting probability* (or the *commutativity degree*) of G. This is a well-studied concept. In the literature one can find publications dealing with problems on the set of possible values of $Pr(G)$ and the influence of $Pr(G)$ over the structure of G (see [**9**, **15**, **17**, **22**, **23**] and references therein). The reader can consult [**25**, **32**] and references therein for related developments concerning probabilistic identities in groups.

P. M. Neumann [**29**] proved the following theorem (see also [**9**]).

THEOREM 1.1. *Let G be a finite group and let ϵ be a positive number such that $Pr(G) \geqslant \epsilon$. Then G has a nilpotent normal subgroup R of nilpotency class at most 2 such that both the index $[G : R]$ and the order of the commutator subgroup $[R, R]$ are ϵ-bounded.*

Throughout the article we use the expression '$(a, b, \dots)$-bounded' to mean that a quantity is bounded from above by a number depending only on the parameters $a, b, \dots$.

If K is a subgroup of G, write

$$Pr(K, G) = \frac{|\{(x, y) \in K \times G \ : \ xy = yx\}|}{|K||G|}.$$

This is the probability that an element of G commutes with an element of K (the relative commutativity degree of K in G).

This notion has been studied in several recent papers (see in particular [**10**, **26**]). Here we will prove the following proposition.

PROPOSITION 1.2. *Let K be a subgroup of a finite group G and let ϵ be a positive number such that $Pr(K, G) \geqslant \epsilon$. Then there is a normal subgroup $T \leqslant G$ and a subgroup $B \leqslant K$ such that the indices $[G : T]$ and $[K : B]$, and the order of the commutator subgroup $[T, B]$ are ϵ-bounded.*

Theorem 1.1 can be easily obtained from the above result taking $K = G$.

Proposition 1.2 has some interesting consequences. In particular, we will establish the following results.

Recall that the generalized Fitting subgroup $F^*(G)$ of a finite group G is the product of the Fitting subgroup $F(G)$ and all subnormal quasisimple subgroups; here a group is quasisimple if it is perfect and its quotient by the centre is a non-abelian simple group. Throughout, by a class-c-nilpotent group we mean a nilpotent group whose nilpotency class is at most c.

THEOREM 1.3. *Let G be a finite group such that $Pr(F^*(G), G) \geqslant \epsilon$, where ϵ is a positive number. Then G has a class-2-nilpotent normal subgroup R such that both the index $[G : R]$ and the order of the commutator subgroup $[R, R]$ are ϵ-bounded.*

A somewhat surprising aspect of the above theorem is that information on the commuting probability of a subgroup (in this case $F^*(G)$) enables one to draw a conclusion about G as strong as in P. M. Neumann's theorem. Yet, several other results with the same conclusion will be established in this paper.

Our next theorem deals with the case where K is a subgroup containing $\gamma_i(G)$ for some $i \geqslant 1$. Here and throughout the paper $\gamma_i(G)$ denotes the ith term of the lower central series of G.

THEOREM 1.4. *Let K be a subgroup of a finite group G containing $\gamma_i(G)$ for some $i \geqslant 1$. Suppose that $Pr(K, G) \geqslant \epsilon$, where ϵ is a positive number. Then G has a*

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nilpotent normal subgroup R of nilpotency class at most $i+1$ such that both the index $[G:R]$ and the order of $\gamma_{i+1}(R)$ are ϵ-bounded.

P. M. Neumann's theorem is a particular case of the above result (take $i = 1$).

In the same spirit, we conclude that G has a nilpotent subgroup of ϵ-bounded index if K is a verbal subgroup corresponding to a word implying virtual nilpotency such that $Pr(K,G) \geqslant \epsilon$. Given a group-word w, we write $w(G)$ for the corresponding verbal subgroup of a group G, that is the subgroup generated by the values of w in G. Recall that a group-word w is said to imply virtual nilpotency if every finitely generated metabelian group G where w is a law, that is $w(G) = 1$, has a nilpotent subgroup of finite index. Such words admit several important characterizations (see [**2**, **4**, **12**]). In particular, by a result of Gruenberg [**13**], the j-Engel word $[x, y, \ldots, y]$, where y appears $j \geqslant 1$ times, implies virtual nilpotency. Burns and Medvedev proved that for any word w implying virtual nilpotency there exist integers e and c depending only on w such that every finite group G, in which w is a law, has a class-c-nilpotent normal subgroup N such that $G^e \leqslant N$ [**4**]. Here G^e denotes the subgroup generated by all eth powers of elements of G. Our next theorem provides a probabilistic variation of this result.

THEOREM 1.5. *Let w be a group-word implying virtual nilpotency. Suppose that K is a subgroup of a finite group G such that $w(G) \leqslant K$ and $Pr(K,G) \geqslant \epsilon$, where ϵ is a positive number. There is an (ϵ, w)-bounded integer e and a w-bounded integer c such that G^e is nilpotent of class at most c.*

We also consider finite groups with a given value of $Pr(P,G)$, where P is a Sylow p-subgroup of G.

THEOREM 1.6. *Let P be a Sylow p-subgroup of a finite group G such that $Pr(P,G) \geqslant \epsilon$, where ϵ is a positive number. Then G has a class-2-nilpotent normal p-subgroup L such that both the index $[P:L]$ and the order of $[L,L]$ are ϵ-bounded.*

Once we have information on the commuting probability of all Sylow subgroups of G, the result is as strong as in P. M. Neumann's theorem.

THEOREM 1.7. *Let $\epsilon > 0$, and let G be a finite group such that $Pr(P,G) \geqslant \epsilon$ whenever P is a Sylow subgroup. Then G has a nilpotent normal subgroup R of nilpotency class at most 2 such that both the index $[G:R]$ and the order of the commutator subgroup $[R,R]$ are ϵ-bounded.*

If ϕ is an automorphism of a group G, the centralizer $C_G(\phi)$ is the subgroup formed by the elements $x \in G$ such that $x^\phi = x$. In the case where $C_G(\phi) = 1$ the automorphism ϕ is called fixed-point-free. A famous result of Thompson [**33**] says that a finite group admitting a fixed-point-free automorphism of prime order is nilpotent. Higman proved that for each prime p there exists a number $h = h(p)$ depending only on p such that whenever a nilpotent group G admits a fixed-point-free automorphism of order p, it follows that G has nilpotency class at most h [**19**]. Therefore a finite group admitting a fixed-point-free automorphism of order p is nilpotent of class at most h. Khukhro obtained the following 'almost fixed-point-free' generalization of this fact [**21**]: if a finite group G admits an automorphism ϕ

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of prime order p such that $C_G(\phi)$ has order m, then G has a nilpotent subgroup of p-bounded nilpotency class and (m, p)-bounded index. We will establish a probabilistic variation of the above results. Recall that an automorphism ϕ of a finite group G is called coprime if $(|G|, |\phi|) = 1$.

THEOREM 1.8. *Let G be a finite group admitting a coprime automorphism ϕ of prime order p such that $Pr(C_G(\phi), G) \geqslant \epsilon$ where ϵ is a positive number. Then G has a nilpotent subgroup of p-bounded nilpotency class and (ϵ, p)-bounded index.*

An even stronger conclusion will be derived about groups admitting an elementary abelian group of automorphisms of rank at least 2.

THEOREM 1.9. *Let $\epsilon > 0$, and let G be a finite group admitting an elementary abelian coprime group of automorphisms A of order p^2 such that $Pr(C_G(\phi), G) \geqslant \epsilon$ for each nontrivial $\phi \in A$. Then G has a class-2-nilpotent normal subgroup R such that both the index $[G : R]$ and the order of $[R, R]$ are (ϵ, p)-bounded.*

Proposition 1.2, which is a key result of this paper, will be proved in the next section. The other results will be established in § 3–5.

2. The key result

A group is said to be a BFC-group if its conjugacy classes are finite and of bounded size. A famous theorem of B. H. Neumann says that in a BFC-group the commutator subgroup G' is finite [**27**]. It follows that if $|x^G| \leqslant m$ for each $x \in G$, then the order of G' is bounded by a number depending only on m. A first explicit bound for the order of G' was found by J. Wiegold [**34**], and the best known was obtained in [**16**] (see also [**28**] and [**31**]). The main technical tools employed in this paper are provided by the recent results [**1**, **6**–**8**] strengthening B. H. Neumann's theorem.

A well-known lemma due to Baer says that if A, B are normal subgroups of a group G such that $[A : C_A(B)] \leqslant m$ and $[B : C_B(A)] \leqslant m$ for some integer $m \geqslant 1$, then $[A, B]$ has finite m-bounded order (see [**30**, 14.5.2]).

We will require a stronger result. Here and in the rest of the paper, given an element $x \in G$ and a subgroup $H \leqslant G$, we write x^H for the set of conjugates of x by elements from H.

LEMMA 2.1. *Let $m \geqslant 1$, and let G be a group containing normal subgroups A, B such that $[A : C_A(y)] \leqslant m$ and $[B : C_B(x)] \leqslant m$ for all $x \in A$, $y \in B$. Then $[A, B]$ has finite m-bounded order.*

Proof. We first prove that, given $x \in A$ and $y \in B$, the order of $[x, y]$ is m-bounded. Let $H = \langle x, y\rangle$. By assumptions, $[A : C_A(y)] \leqslant m$ and $[B : C_B(x)] \leqslant m$. Hence there exists an m-bounded number l such that x^l and y^l are contained in $Z(H)$ (e.g. we can take $l = m!$). Let $D = A \cap B \cap H$ and $N = \langle D, x^l, y^l\rangle$. Then H/N is abelian of order at most l^2. Both x and y have centralizers of index at most m in N. Moreover every element of N has centralizer of index at most m in N. Indeed $|d^N| \leqslant |d^A| \leqslant m$ for every $d \in D \leqslant A \cap B$. So, every element of H is a product of at most $l^2 + 1$ elements each of which has centralizer of index at most m in N. Therefore each

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element of H has centralizer of m-bounded index in H. We conclude that H is a BFC-group in which the sizes of conjugacy classes are m-bounded. Hence $|H'|$ is m-bounded and so the order of $[x, y]$ is m-bounded, too.

Now we claim that for every $x \in A$, the subgroup $[x, B]$ has finite m-bounded order. Indeed, x has at most m conjugates $\{x^{b_1}, \dots, x^{b_m}\}$ in B, where $b_1, \dots, b_m \in B$, so $[x, B]$ is generated by at most m elements. Let C be a maximal normal subgroup of B contained in $C_B(x)$. Clearly C has m-bounded index in B and centralizes $[x, B]$. Thus, the centre of $[x, B]$ has m-bounded index in $[x, B]$. It follows from Schur's theorem [**30**, 10.1.4] that the derived subgroup of $[x, B]$ has finite m-bounded order. Since $[x, B]$ is generated by at most m elements of m-bounded order, we deduce that the order of $[x, B]$ is finite and m-bounded.

Choose $a \in A$ such that $[B : C_B(a)] = \max_{x \in A}[B : C_B(x)]$ and set $n = [B : C_B(a)]$, where $n \leqslant m$. Let $b_1, \dots, b_n$ be elements of B such that $a^B = \{a^{b_1}, \dots, a^{b_n}\}$ is the set of (distinct) conjugates of a by elements of B. Set $U = C_A(b_1, \dots, b_n)$ and note that U has m-bounded index in A. Given $u \in U$, the elements $(ua)^{b_1}, \dots, (ua)^{b_n}$ form the conjugacy class $(ua)^B$ because they are all different and their number is the allowed maximum. So, for an arbitrary element $y \in B$ there exists i such that $(ua)^y = (ua)^{b_i} = ua^{b_i}$. It follows that $u^{-1}u^y = a^{b_i}a^{-y}$, hence

$$[u, y] = a^{b_i}a^{-y} = [a, b_i^{a^{-1}}][y^{a^{-1}}, a] \in [a, B].$$

Therefore $[U, B] \leqslant [a, B]$. Let $a_1, \dots, a_s$ be coset representatives of U in A and note that s is m-bounded. As each $[x, B]$ is normal in B and $[U, B] \leqslant [a, B]$, we deduce that $[A, B] = [a, B]\prod[a_i, B]$. So $[A, B]$ is a product of m-boundedly many subgroups of m-bounded order. These subgroups are normal in B and therefore their product has finite m-bounded order. □

In the next lemma the subgroup B is not necessarily normal. Instead, we require that B is contained in an abelian normal subgroup. Throughout, $\langle H^G \rangle$ denotes the normal closure of a subgroup H in G.

Lemma 2.2. *Let $m \geqslant 1$, and let G be a group containing a normal subgroup A and a subgroup B such that $[A : C_A(y)] \leqslant m$ and $[B : C_B(x)] \leqslant m$ for all $x \in A$, $y \in B$. Assume further that $\langle B^G \rangle$ is abelian. Then $[A, B]$ has finite m-bounded order.*

Proof. Without loss of generality we can assume that $G = AB$. Set $L = \langle B^G \rangle = \langle B^A \rangle$.

Let $x \in A$. There is an m-bounded number l such that x centralizes y^l for every $y \in B$. Since L is abelian, $[x, y]^i = [x, y^i]$ for each i and therefore the order of $[x, y]$ is at most l. Thus $[x, B]$ is an abelian subgroup generated by at most m elements of m-bounded order, whence $[x, B]$ has finite m-bounded order.

Now we choose $a \in A$ such that $[B : C_B(a)]$ is as big as possible. Let $b_1, \dots, b_m$ be elements of B such that $a^B = \{a^{b_1}, \dots, a^{b_m}\}$. Set $U = C_A(b_1, \dots, b_m)$ and note that U has m-bounded index in A. Arguing as in the previous lemma, we see that for arbitrary $u \in U$ and $y \in B$, the conjugate $(ua)^y$ belongs to the set $\{(ua)^{b_1}, \dots, (ua)^{b_m}\}$. Let $(ua)^y = (ua)^{b_i}$. Then $u^{-1}u^y = a^{b_i}a^{-y}$ and hence $[u, y] = a^{b_i}a^{-y} \in [a, B]$. Therefore $[U, B] \leqslant [a, B]$.

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Let $V = \cap_{x \in A} U^x$ be the maximal normal subgroup of A contained in U. We know that $[V, B]$ has m-bounded order, since $[V, B] \leqslant [a, B]$. Denote the index $[A : V]$ by s. Evidently, s is m-bounded. Let $a_1, \ldots, a_s$ be a transversal of V in A. As $[V, B] \leqslant L = \langle B^A \rangle$ is abelian, we have

$$\langle [V, B]^G \rangle = \langle [V, B]^A \rangle = \prod_{i=1}^{s} [V, B]^{a_i}.$$

Thus $[V, L] = [V, B^A] = \langle [V, B]^A \rangle$ is a product of m-boundedly many subgroups of m-bounded order, and hence it has m-bounded order. Write

$$L = \langle B^A \rangle \leqslant \langle B^{V a_i} \mid i = 1, \ldots s \rangle \leqslant [V, L] \prod_{i=1}^{s} B^{a_i}.$$

Thus, it becomes clear that L is a product of m-boundedly many conjugates of B. Say L is a product of t conjugates of B. Then, every $y \in L$ can be written as a product of at most t conjugates of elements of B and consequently $[A : C_A(y)] \leqslant m^t$. Moreover, as A is normal in G and $|a^B| \leqslant m$ for every $a \in A$, the conjugacy class x^L of an element $x \in A$ has size at most m^t. Now lemma 2.1 shows that $[A, B] \leqslant [A, L]$ has finite m-bounded order. □

We will now show that if K is a subgroup of a finite group G and N is a normal subgroup of G, then $Pr(KN/N, G/N) \geqslant Pr(K, G)$. More precisely, we will establish the following lemma.

Lemma 2.3. *Let N be a normal subgroup of a finite group G, and let $K \leqslant G$. Then $Pr(K, G) \leqslant Pr(KN/N, G/N) Pr(N \cap K, N)$.*

This is an improvement over [**10**, theorem 3.9] where the result was obtained under the additional hypothesis that $N \leqslant K$.

Proof. In what follows $\bar{G} = G/N$ and $\bar{K} = KN/N$. Write $\bar{K}_0$ for the set of cosets $(N \cap K)h$ with $h \in K$. If $S_0 = (N \cap K)h \in \bar{K}_0$, write S for the coset $Nh \in \bar{K}$. Of course, we have a natural one-to-one correspondence between $\bar{K}_0$ and $\bar{K}$.

Write

$$\begin{aligned}
|K||G|Pr(K, G) &= \sum_{x \in K} |C_G(x)| = \sum_{S_0 \in \bar{K}_0} \sum_{x \in S_0} \frac{|C_G(x)N|}{|N|} |C_N(x)| \\
&\leqslant \sum_{S_0 \in \bar{K}_0} \sum_{x \in S_0} |C_{\bar{G}}(xN)||C_N(x)| = \sum_{S \in \bar{K}} |C_{\bar{G}}(S)| \sum_{x \in S_0} |C_N(x)| \\
&= \sum_{S \in \bar{K}} |C_{\bar{G}}(S)| \sum_{y \in N} |C_{S_0}(y)|.
\end{aligned}$$

If $C_{S_0}(y) \neq \emptyset$, then there is $y_0 \in C_{S_0}(y)$ and so $S_0 = (N \cap K)y_0$. Therefore

$$C_{S_0}(y) = (N \cap K)y_0 \cap C_G(y) = C_{N \cap K}(y)y_0, \quad \text{whence } |C_{S_0}(y)| = |C_{N \cap K}(y)|.$$

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Conclude that

$$|K||G|Pr(K,G) \leqslant \sum_{S\in\bar{K}} |C_{\bar{G}}(S)| \sum_{y\in N} |C_{N\cap K}(y)|.$$

Observe that

$$\sum_{S\in\bar{K}} |C_{\bar{G}}(S)| = \frac{|K|}{|N\cap K|}\frac{|G|}{|N|}Pr(\bar{K},\bar{G})$$

and

$$\sum_{y\in N} |C_{N\cap K}(y)| = |N\cap K||N|Pr(N\cap K,N).$$

It follows that $Pr(K,G) \leqslant Pr(\bar{K},\bar{G})Pr(N\cap K,N)$, as required. □

The following theorem is taken from [**1**]. It plays a crucial role in the proof of proposition 1.2.

THEOREM 2.4. *Let m be a positive integer, G a group having a subgroup K such that $|x^G| \leqslant m$ for each $x\in K$, and let $H = \langle K^G\rangle$. Then the order of the commutator subgroup $[H,H]$ is finite and m-bounded.*

A proof of the next lemma can be found in Eberhard [**9**, lemma 2.1].

LEMMA 2.5. *Let G be a finite group and X a symmetric subset of G containing the identity. Then $\langle X\rangle = X^{3r}$ provided $(r+1)|X| > |G|$.*

We are now ready to prove proposition 1.2 which we restate here for the reader's convenience:

Let $\epsilon > 0$, and let G be a finite group having a subgroup K such that $Pr(K,G) \geqslant \epsilon$. Then there is a normal subgroup $T \leqslant G$ and a subgroup $B \leqslant K$ such that the indices $[G:T]$ and $[K:B]$ and the order of $[T,B]$ are ϵ-bounded.

Proof of proposition 1.2. Set

$$X = \{x\in K \mid |x^G| \leqslant 2/\epsilon\} \quad \text{and} \quad B = \langle X\rangle.$$

Note that $K\setminus X = \{x\in K \mid |C_G(x)| \leqslant (\epsilon/2)|G|\}$, whence

$$\begin{aligned}
\epsilon|K||G| &\leqslant |\{(x,y)\in K\times G \mid xy=yx\}| = \sum_{x\in K}|C_G(x)| \\
&\leqslant \sum_{x\in X}|G| + \sum_{x\in K\setminus X}\frac{\epsilon}{2}|G| \\
&\leqslant |X||G| + (|K|-|X|)\frac{\epsilon}{2}|G|.
\end{aligned}$$

Therefore $\epsilon|K| \leqslant |X| + (\epsilon/2)(|K|-|X|)$, whence $(\epsilon/2)|K| < |X|$. Clearly, $|B| \geqslant |X| > (\epsilon/2)|K|$ and so the index of B in K is at most $2/\epsilon$. As X is symmetric

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and $(2/\epsilon)|X| > |K|$, it follows from lemma 2.5 that every element of B is a product of at most $6/\epsilon$ elements of X. Therefore $|b^G| \leqslant (2/\epsilon)^{6/\epsilon}$ for every $b \in B$.

Let $L = \langle B^G \rangle$. Theorem 2.4 tells us that the commutator subgroup $[L, L]$ has ϵ-bounded order. Let us use the bar notation for the images of the subgroups of G in $G/[L, L]$. By lemma 2.3,

$$Pr(\bar{K}, \bar{G}) \geqslant Pr(K, G) \geqslant \epsilon.$$

Moreover, $[\bar{K} : \bar{B}] \leqslant [K : B] \leqslant \epsilon/2$ and $|\bar{b}^{\bar{G}}| \leqslant |b^G| \leqslant (2/\epsilon)^{6/\epsilon}$. Thus we can pass to the quotient over $[L, L]$ and assume that L is abelian.

Now we set

$$Y = \{y \in G \mid |y^K| \leqslant 2/\epsilon\} = \{y \in G \mid |C_K(y)| \geqslant (\epsilon/2)|K|\}.$$

Note that

$$\begin{aligned} \epsilon|K||G| &\leqslant |\{(x, y) \in K \times G \mid xy = yx\}| \\ &\leqslant \sum_{y \in Y} |K| + \sum_{y \in G \setminus Y} \frac{\epsilon}{2}|K| \\ &\leqslant |Y||K| + (|G| - |Y|)\frac{\epsilon}{2}|K| \leqslant |Y||K| + \frac{\epsilon}{2}|G||K|. \end{aligned}$$

Therefore $(\epsilon/2)|G| < |Y|$.

Set $E = \langle Y \rangle$. Thus $|E| \geqslant |Y| > (\epsilon/2)|G|$, and so the index of E in G is at most $2/\epsilon$. As Y is symmetric and $(2/\epsilon)|Y| > |G|$, it follows from lemma 2.5 that every element of E is a product of at most $6/\epsilon$ elements of Y. Since $|y^K| \leqslant 2/\epsilon$ for every $y \in Y$, we conclude that $|e^K| \leqslant (2/\epsilon)^{6/\epsilon}$ for every $e \in E$. Let T be the maximal normal subgroup of G contained in E. Clearly, the index $[G : T]$ is ϵ-bounded.

So, now $|b^G| \leqslant (2/\epsilon)^{6/\epsilon}$ for every $b \in B$ and $|e^B| \leqslant (2/\epsilon)^{6/\epsilon}$ for every $e \in T$. As L is abelian, we can apply lemma 2.2 to conclude that $[T, B]$ has ϵ-bounded order and the result follows. □

Remark 2.6. Under the hypotheses of proposition 1.2 the subgroup $N = \langle [T, B]^G \rangle$ has ϵ-bounded order.

Proof. Since $[T, B]$ is normal in T, it follows that there are only boundedly many conjugates of $[T, B]$ in G and they normalize each other. Since N is the product of those conjugates, N has ϵ-bounded order. □

As usual, $Z_i(G)$ stands for the ith term of the upper central series of a group G.

Remark 2.7. Assume the hypotheses of proposition 1.2. If K is normal, then the subgroup T can be chosen in such a way that $K \cap T \leqslant Z_3(T)$.

Proof. According to remark 2.6, $N = \langle [T, B]^G \rangle$ has ϵ-bounded order. Let $B_0 = \langle B^G \rangle$ and note that $B_0 \leqslant K$ and $[T, B_0] \leqslant N$. Since the index $[K : B_0]$ and the

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order of N are ϵ-bounded, the stabilizer in T of the series

$$1 \leqslant N \leqslant B_0 \leqslant K,$$

that is, the subgroup

$$H = \{g \in T \mid [N, g] = 1 \ \& \ [K, g] \leqslant B_0\}$$

has ϵ-bounded index in G. Note that $K \cap H \leqslant Z_3(H)$, whence the result. □

3. Probabilistic almost nilpotency of finite groups

Our first goal in this section is to furnish a proof of theorem 1.3. We restate it here.

Let G be a finite group such that $Pr(F^*(G), G) \geqslant \epsilon$. Then G has a class-2-nilpotent normal subgroup R such that both the index $[G : R]$ and the order of the commutator subgroup $[R, R]$ are ϵ-bounded.

As mentioned in the introduction, the above result yields a conclusion about G which is as strong as in P. M. Neumann's theorem.

Proof of theorem 1.3. Set $K = F^*(G)$. In view of proposition 1.2 there is a normal subgroup $T \leqslant G$ and a subgroup $B \leqslant K$ such that the indices $[G : T]$ and $[K : B]$, and the order of the commutator subgroup $[T, B]$ are ϵ-bounded. As K is normal in G, according to remark 2.7 the subgroup T can be chosen in such a way that $K \cap T \leqslant Z_3(T)$. By [**20**, corollary X.13.11(c)] we have $K \cap T = F^*(T)$. Therefore $F^*(T) \leqslant Z_3(T)$ and in view of [**20**, theorem X.13.6] we conclude that $T = F^*(T)$ and so $T \leqslant K$. It follows that the index of K in G is ϵ-bounded. By remark 2.6 the subgroup $N = \langle [T, B]^G \rangle$ has ϵ-bounded order. Conclude that $R = \langle B^G \rangle \cap C_G(N)$ has ϵ-bounded index in G. Moreover R is nilpotent of class at most 2 and $[R, R]$ has ϵ-bounded order. This completes the proof. □

Now focus on theorem 1.4, which deals with the case where $\gamma_i(G) \leqslant K$. Start with a couple of remarks on the result. Let G and R be as in theorem 1.4. The fact that both the index $[G : R]$ and the order of $\gamma_{i+1}(R)$ are ϵ-bounded implies that for any $x_1, \ldots, x_i \in R$ the centralizer of the long commutator $[x_1, \ldots, x_i]$ has ϵ-bounded index in G. Therefore there is an ϵ-bounded number e such that G^e centralizes all commutators $[x_1, \ldots, x_i]$ where $x_1, \ldots, x_i \in R$. Then $G_0 = G^e \cap R$ is a nilpotent normal subgroup of nilpotency class at most i with G/G_0 of ϵ-bounded exponent (recall that a positive integer e is the exponent of a finite group G if e is the minimal number for which $G^e = 1$).

If G is additionally assumed to be m-generated for some $m \geqslant 1$, then G has a nilpotent normal subgroup of nilpotency class at most i and (ϵ, m)-bounded index. Indeed, we know that for any $x_1, \ldots, x_i \in R$ the centralizer of the long commutator $[x_1, \ldots, x_i]$ has ϵ-bounded index in G. An m-generated group has only (j, m)-boundedly many subgroups of any given index j [**18**, theorem 7.2.9]. Therefore G has a subgroup J of (ϵ, m)-bounded index that centralizes all commutators $[x_1, \ldots, x_i]$ with $x_1, \ldots, x_i \in R$. Then $J \cap R$ is a nilpotent normal subgroup of nilpotency class at most i and (ϵ, m)-bounded index in G.

These observations are in parallel with Shalev's results on probabilistically nilpotent groups [**32**].

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Our proof of theorem 1.4 requires the following result from [7].

THEOREM 3.1. *Let G be a group such that $|x^{\gamma_k(G)}| \leqslant n$ for any $x \in G$. Then $\gamma_{k+1}(G)$ has finite (k, n)-bounded order.*

We can now prove theorem 1.4.

Proof of theorem 1.4. Recall that K is a subgroup of the finite group G such that $\gamma_k(G) \leqslant K$ and $Pr(K, G) \geqslant \epsilon$. In view of [10, theorem 3.7] observe that $Pr(\gamma_k(G), G) \geqslant \epsilon$. Therefore without loss of generality we can assume that $K = \gamma_k(G)$.

Proposition 1.2 tells us that there is a normal subgroup $T \leqslant G$ and a subgroup $B \leqslant K$ such that the indices $[G : T]$ and $[K : B]$ and the order of $[T, B]$ are ϵ-bounded. In particular, $|x^B|$ is ϵ-bounded for every $x \in T$. Since B has ϵ-bounded index in K, we deduce that $|x^{\gamma_k(G)}|$ is ϵ-bounded for every $x \in T$. Now theorem 3.1 implies that $\gamma_{k+1}(T)$ has ϵ-bounded order. Set $R = C_T(\gamma_{k+1}(T))$. It follows that R is as required. □

Our next goal is a proof of theorem 1.5. As mentioned in the introduction, a group-word w implies virtual nilpotency if every finitely generated metabelian group G where w is a law, that is $w(G) = 1$, has a nilpotent subgroup of finite index. A theorem, due to Burns and Medvedev, states that for any word w implying virtual nilpotency there exist integers e and c depending only on w such that every finite group G, in which w is a law, has a nilpotent of class at most c normal subgroup N with $G^e \leqslant N$ [4].

Proof of theorem 1.5. Recall that w is a group-word implying virtual nilpotency while K is a subgroup of a finite group G such that $w(G) \leqslant K$ and $Pr(K, G) \geqslant \epsilon$. We need to show that there is an (ϵ, w)-bounded integer e and a w-bounded integer c such that G^e is nilpotent of class at most c.

As in the proof of theorem 1.4 without loss of generality we can assume that $K = w(G)$. Proposition 1.2 tells us that there is a normal subgroup $T \leqslant G$ and a subgroup $B \leqslant K$ such that the indices $[G : T]$ and $[K : B]$ and the order of the commutator subgroup $[T, B]$ are ϵ-bounded. According to remark 2.7 the subgroup T can be chosen in such a way that $K \cap T \leqslant Z_3(T)$. In particular $w(T) \leqslant Z_3(T)$. Taking into account that the word w implies virtual nilpotency, we deduce from the Burns–Medvedev theorem that there are w-bounded numbers i and c such that the subgroup generated by the ith powers of elements of T is nilpotent of class at most c. Recall that the index of T in G is ϵ-bounded. Hence there is an ϵ-bounded integer e such that every eth power in G is an ith power of an element of T. The result follows. □

If $[x^i, y_1, \ldots, y_j]$ is a law in a finite group G, then $\gamma_{j+1}(G)$ has $\{i, j\}$-bounded exponent (the case $j = 1$ is a well-known result, due to Mann [24]; see [5, lemma 2.2] for the case $j \geqslant 2$). If the j-Engel word $[x, y, \ldots, y]$, where y is repeated j times, is a law in a finite group G, then G has a normal subgroup N such that the exponent of N is j-bounded while G/N is nilpotent with j-bounded class [3]. Note that both words $[x^i, y_1, \ldots, y_j]$ and $[x, y, \ldots, y]$ imply virtual nilpotency.

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Therefore, in addition to theorem 1.5, we deduce

THEOREM 3.2. *Assume the hypotheses of theorem 1.5.*

- *If $w = [x^n, y_1, \ldots, y_k]$, then G has a normal subgroup T such that the index $[G : T]$ is ϵ-bounded and the exponent of $\gamma_{k+4}(T)$ is w-bounded.*
- *There are k-bounded numbers e_1 and c_1 with the property that if w is the k-Engel word, then G has a normal subgroup T such that the index $[G : T]$ is ϵ-bounded and the exponent of $\gamma_{c_1}(T)$ divides e_1.*

Proof. By [**10**, theorem 3.7], without loss of generality we can assume that $K = w(G)$. Proposition 1.2 tells us that there is a normal subgroup $T \leqslant G$ and a subgroup $B \leqslant w(G)$ such that the indices $[G : T]$ and $[w(G) : B]$ and the order of $[T, B]$ are ϵ-bounded. Since K is normal in G, according to remark 2.7 the subgroup T can be chosen in such a way that $w(G) \cap T \leqslant Z_3(T)$. If $w = [x^n, y_1, \ldots, y_k]$, then $[x^n, y_1, \ldots, y_{k+3}]$ is a law in T, whence the exponent of $\gamma_{k+4}(T)$ is w-bounded. If w is the k-Engel word, then the $(k + 3)$-Engel word is a law in T and the theorem follows from the Burns–Medvedev theorem [**3**]. □

4. Sylow subgroups

As usual, $O_p(G)$ denotes the maximal normal p-subgroup of a finite group G. For the reader's convenience we restate theorem 1.6:

Let P be a Sylow p-subgroup of a finite group G such that $Pr(P, G) \geqslant \epsilon$. Then G has a class-2-nilpotent normal p-subgroup L such that both the index $[P : L]$ and the order of the commutator subgroup $[L, L]$ are ϵ-bounded.

Proof of theorem 1.6. Proposition 1.2 tells us that there is a normal subgroup $T \leqslant G$ and a subgroup $B \leqslant P$ such that the indices $[G : T]$ and $[P : B]$ and the order of the commutator subgroup $[T, B]$ are ϵ-bounded. In view of remark 2.6 the subgroup $N = \langle [T, B]^G \rangle$ has ϵ-bounded order. Therefore $C = C_T(N)$ has ϵ-bounded index in G. Set $B_0 = B \cap C$ and note that $[C, B_0] \leqslant Z(C)$. It follows that $B_0 \leqslant Z_2(C)$ and we conclude that $B_0 \leqslant O_p(G)$. Let $L = \langle {B_0}^G \rangle$. As $B_0 \leqslant L \leqslant O_p(G)$, it is clear that L is contained in P as a subgroup of ϵ-bounded index. Moreover $[L, L] \leqslant N$ and so the order of $[L, L]$ is ϵ-bounded. Hence the result. □

We will now prove theorem 1.7.

Proof of theorem 1.7. Recall that G is a finite group such that $Pr(P, G) \geqslant \epsilon$ whenever P is a Sylow subgroup. We wish to show that G has a nilpotent normal subgroup R of nilpotency class at most 2 such that both the index $[G : R]$ and the order of the commutator subgroup $[R, R]$ are ϵ-bounded.

For each prime $p \in \pi(G)$ choose a Sylow p-subgroup S_p in G. Theorem 1.6 shows that G has a normal p-subgroup L_p of class at most 2 such that both $[S_p : L_p]$ and $|[L_p, L_p]|$ are ϵ-bounded. Since the bounds on $[S_p : L_p]$ and $|[L_p, L_p]|$ do not depend on p, it follows that there is an ϵ-bounded constant C such that $S_p = L_p$ and $[L_p, L_p] = 1$ whenever $p \geqslant C$. Set $R = \prod_{p \in \pi(G)} L_p$. Then all Sylow subgroups

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of G/R have ϵ-bounded order and therefore the index of R in G is ϵ-bounded. Moreover, R is of class at most 2 and $|[R, R]|$ is ϵ-bounded, as required. □

5. Coprime automorphisms and their fixed points

If A is a group of automorphisms of a group G, we write $C_G(A)$ for the centralizer of A in G. The symbol $A^{\#}$ stands for the set of nontrivial elements of the group A.

The next lemma is well-known (see e.g. [**11**, theorem 6.2.2 (iv)]). In the sequel we use it without explicit references.

LEMMA 5.1. *Let A be a group of automorphisms of a finite group G such that $(|G|, |A|) = 1$. Then $C_{G/N}(A) = NC_G(A)/N$ for any A-invariant normal subgroup N of G.*

Proof of theorem 1.8. Recall that G is a finite group admitting a coprime automorphism ϕ of prime order p such that $Pr(K, G) \geqslant \epsilon$, where $K = C_G(\phi)$. We need to show that G has a nilpotent subgroup of p-bounded nilpotency class and (ϵ, p)-bounded index.

By proposition 1.2 there is a normal subgroup $T \leqslant G$ and a subgroup $B \leqslant K$ such that the indices $[G : T]$ and $[K : B]$ and the order of the commutator subgroup $[T, B]$ are ϵ-bounded. Let T_0 be the maximal ϕ-invariant subgroup of T. Evidently, T_0 is normal and the index $[G : T_0]$ is (ϵ, p)-bounded. Since $\langle [T_0, B]^G \rangle \leqslant \langle [T, B]^G \rangle$, remark 2.6 implies that $M = \langle [T_0, B]^G \rangle$ has ϵ-bounded order. Moreover, M is ϕ-invariant. Set $D = C_G(M) \cap T_0$ and $\bar{D} = D/Z_2(D)$, and note that D is ϕ-invariant.

In a natural way ϕ induces an automorphism of $\bar{D}$ which we will denote by the same symbol ϕ. We note that $C_{\bar{D}}(\phi) = C_D(\phi)Z_2(D)/Z_2(D)$, so its order is ϵ-bounded because $B \cap D \leqslant Z_2(D)$. The Khukhro theorem [**21**] now implies that $\bar{D}$ has a nilpotent subgroup of p-bounded class and (ϵ, p)-bounded index. Since $\bar{D} = D/Z_2(D)$ and since the index of D in G is (ϵ, p)-bounded, we deduce that G has a nilpotent subgroup of p-bounded class and (ϵ, p)-bounded index. The proof is complete. □

A proof of the next lemma can be found in [**14**].

LEMMA 5.2. *If A is a noncyclic elementary abelian p-group acting on a finite p'-group G in such a way that $|C_G(a)| \leqslant m$ for each $a \in A^{\#}$, then the order of G is at most m^{p+1}.*

We will now prove theorem 1.9.

Proof of theorem 1.9. By hypotheses, G is a finite group admitting an elementary abelian coprime group of automorphisms A of order p^2 such that $Pr(C_G(\phi), G) \geqslant \epsilon$ for each $\phi \in A^{\#}$. We need to show that G has a nilpotent normal subgroup R of nilpotency class at most 2 such that both the index $[G : R]$ and the order of the commutator subgroup $[R, R]$ are (ϵ, p)-bounded.

Let $A_1, \ldots, A_{p+1}$ be the subgroups of order p of A and set $G_i = C_G(A_i)$ for $i = 1, \ldots, p + 1$. According to proposition 1.2 for each $i = 1, \ldots, p + 1$ there is a normal subgroup $T_i \leqslant G$ and a subgroup $B_i \leqslant G_i$ such that the indices $[G : T_i]$ and

$[G_i : B_i]$ and the order of the commutator subgroup $[T_i, B_i]$ are ϵ-bounded. We let U_i denote the maximal A-invariant subgroup of T_i so that each U_i is a normal subgroup of (ϵ, p)-bounded index. The intersection of all U_i will be denoted by U. Further, we let D_i denote the maximal A-invariant subgroup of B_i so that each D_i has (ϵ, p)-bounded index in G_i. Note that a modification of remark 2.6 implies that $N_i = \langle [U_i, D_i]^G \rangle$ is A-invariant and has ϵ-bounded order. It follows that the order of $N = \prod_i N_i$ is (ϵ, p)-bounded. Let V denote the minimal (A-invariant) normal subgroup of G containing all D_i for $i = 1, \ldots, p+1$. It is easy to see that $[U, V] \leqslant N$.

Obviously, U has (ϵ, p)-bounded index in G. Let us check that this also holds with respect to V. Let $\bar{G} = G/V$. Since V contains D_i for each $i = 1, \ldots, p+1$ and since D_i has (ϵ, p)-bounded index in G_i, we conclude that the image of G_i in $\bar{G}$ has (ϵ, p)-bounded order. Now lemma 5.2 tells us that the order of $\bar{G}$ is (ϵ, p)-bounded and we conclude that indeed V has (ϵ, p)-bounded index in G. Also note that since N has (ϵ, p)-bounded order, $C_G(N)$ has (ϵ, p)-bounded index in G. Let

$$R = U \cap V \cap C_G(N).$$

Then R is as required since the subgroups $U, V, C_G(N)$ have (ϵ, p)-bounded index in G while $[R, R] \leqslant N \leqslant C_G(R)$. The proof is complete. □

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