Canad Math. Bull. Vol. **17** (5), 1975

HAUSDORFF COMPACTIFICATIONS AS EPIREFLECTIONS

BY

W. N. HUNSAKER AND S. A. NAIMPALLY

ABSTRACT. We answer the following problem posed by Herrlich in the affirmative: "Can the Freudenthal compactification be regarded as a reflection in a sensible way?" This is accomplished by exploiting the one-to-one correspondence between proximities compatible with a given Tihonov space and compactifications of that space. We give topological characterizations of proximally continuous functions for the proximities associated with the Freudenthal and Fan-Gottesman compactifications.

The purpose of this note is to answer Problem (9) of Herrlich [4]: "Can the Freudenthal-compactification be regarded as a reflection in a sensible way?" In fact, we show that Smirnov [7] solved a general problem of this type in his theory of compactifications of proximity spaces.

We briefly recall the relevant facts (for details see [6]). Every separated proximity space (X, δ) has a unique Smirnov compactification X^*_δ and the map $\delta \to X^*_\delta$ is an order isomorphism. Every Hausdorff compactification X^* of a Tihonov space X is the Smirnov compactification of X corrsponding to the proximity δ which is induced by the unique proximity on X^* viz. $A\delta B$ if and only if $\mathrm{Cl}_{X^*} A \cap \mathrm{Cl}_{X^*} B \neq \varnothing$; further if Y is compact Hausdorff, then a function f from X to Y has a continuous extension $\bar{f}: X^* \to Y$ if and only if f is proximally continuous.

The categorical implication of the above result is as follows. Let $\mathscr{F}$ denote any class of proximity spaces. For each object X in $\mathscr{F}$, X^* will denote its Smirnov compactification. From Smirnov's theory we get the following.

THEOREM 1. *The category of compact Hausdorff spaces and continuous functions is an epireflective subcategory of the category of proximity spaces and proximally continuous functions. The epireflection of X in $\mathscr{F}$ is X^*.*

We examine the special cases of the Freudenthal and Fan-Gottesman compactifications in more detail.

Freudenthal compactification [3]. Let $\mathscr{F}$ denote the class of all semi-compact (or rim compact) Tihonov spaces and for each X in $\mathscr{F}$, let X^* denote the Freudenthal compactification of X.

Received by the editors June 4, 1973.

AMS 1970 *Subject Classification.* Primary 54C10, 54D35, 54E05, 54E10, 18A40, 18A99; Secondary 54D30.

Key words and phrases. Proximity space, proximally continuous, Freudenthal compactification, epireflection, semi-compact, Fan-Gottesman compactification.

Morita [5] (see also [1]) has characterized X^* as the unique Hausdorff compactification satisfying (1) and (2) below:

(1) For any x in X^* and for any open set U of X^* with $x \in U$, there exists an open set V of X^* with $x \in V$, $V \subset U$, and $\mathrm{Fr}(V) \subset X$.

(2) Any two disjoint closed subsets of X with compact boundaries have disjoint closures in X^*.

THEOREM 2. *Let X be a semi-compact Tihonov space and let δ be the proximity on X induced by X^*. Then $A \not\delta B$ if and only if A and B are contained in two disjoint closed sets with compact boundaries.*

Proof. Let δ be the proximity induced by X^* on X. Clearly, if A and B are contained in two disjoint closed sets with compact boundaries, then $A \not\delta B$. Conversely, suppose $A \not\delta B$, then $\mathrm{Cl}_{X^*} A \cap \mathrm{Cl}_{X^*} B = \varnothing$. By (1) above, there exists a finite open cover $\mathscr{U} = \{U_i : 1 \leq i \leq n\}$ of X^* such that each $\mathrm{Fr}(U_i)$ is compact, $\mathrm{Fr}(U_i) \subset X$, and $\mathrm{St}(\bar{A}, \mathscr{U}) \cap \bar{B} = \varnothing$. Put $V_i = X \cap U_i$, then $\{V_i : 1 \leq i \leq n\}$ is an open cover of X, and each finite union of V_i's has compact boundary. Let

$$W_A = X - \bigcup \{V_i : V_i \cap A = \varnothing\}$$

and

$$W_B = X - \bigcup \{V_i : V_i \cap B = \varnothing\},$$

then $A \subset W_A$, $B \subset W_B$, $W_A \cap W_B = \varnothing$, and W_A and W_B are each closed sets with compact boundary.

COROLLARY. *For X and Y in $\mathscr{F}$, a function f from X to Y is proximally continuous if and only if, for each pair of disjoint closed sets F_1, F_2 in Y with compact boundaries, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are contained in two disjoint closed sets in X with compact boundaries.*

We call a map satisfying in the conditions of the above corollary an F-map.

THEOREM 3. *The category of compact Hausdorff spaces and continuous maps is an epireflective subcategory of the category of semi-compact spaces and F-maps. The epireflection of a semi-compact space X is its Freudenthal compactification X^*.*

Fan and Gottesman [2] have generalized the Freudenthal procedure by considering a normal base $\mathscr{B}_X$ associated with a Tihonov space X. The proof of the following Theorem is similar to the proof of Theorem 2. It uses Lemmas 4, 6, and 9 of [2] together with the definition of a normal base. Let $\mathscr{D}$ denote the class of all Tihonov spaces with normal bases, and for each X in $\mathscr{D}$, let $\hat{X}$ denote the Fan-Gottesman compactification of X.

THEOREM 4. *Let X be a Tihonov space and let δ be the proximity induced on X by $\hat{X}$, then $A \not\delta B$ if and only if there exists $G_i \in \mathscr{B}_X$ such that $A \subset G_1$, $B \subset G_2$, and $\bar{G}_1 \cap \bar{G}_2 = \varnothing$.*

https://doi.org/10.4153/CMB-1974-122-0 Published online by Cambridge University Press

COROLLARY. *For X and Y in $\mathscr{D}$ a function f from X to Y is proximally continuous if and only if for each $F_1, F_2 \in \mathscr{B}_Y$ with $\bar{F}_1 \cap \bar{F}_2 = \varnothing$, there exist G_1, G_2 in $\mathscr{B}_X$ such that $f^{-1}(F_i) \subset \bar{G}_i$ $i=1, 2$, and $\bar{G}_1 \cap \bar{G}_2 = \varnothing$.*

We call a map satisfying the above conditions an *FG*-map.

THEOREM 5. *The category of compact Hausdorff spaces and continuous maps is an epireflective subcategory of the category whose objects are Tihonov spaces with normal bases, and whose morphisms are the FG-maps.*

REFERENCES

1. R. F. Dickman, Jr., *Some characterizations of the Freudenthal compactification of a semi-compact space*, Proc. Amer. Math. Soc. **19** (1968), 631–633.

2. K. Fan and N. Gottesman, *On compactifications of Freudenthal and Wallman*, Indag. Math. **14** (1952), 504–510.

3. H. Freudenthal, *Kompaktisierungen und Bikompaktisierungen*, Indag. Math. **13** (1951) 184–192.

4. H. Herrlich, *Categorical Topology*, General Topology and Applications **1** (1971), 1–15.

5. K. Morita, *On bicompactifications of semibicompact spaces*, Sci. Rep. Tokyo Bunrika Daigaku Sect. A **4** (1952), 222–229.

6. S. A. Naimpally and B. D. Warrack, *Proximity Spaces*, Cambridge Tract in Mathematics and Mathematical Physics No. 59, Cambridge University Press (1970).

7. Y. M. Smirnov, *On proximity spaces*, Math. Sb. **31** (73) (1952), 543–574; English translation in Amer. Math. Soc. Transl. (2) **38** (1964), 5–36.

DEPARTMENT OF MATHEMATICS,
SOUTHERN ILLINOIS UNIVERSITY,
CARBONDALE, ILLINOIS 62901, U.S.A.

DEPARTMENT OF MATHEMATICS,
LAKEHEAD UNIVERSITY,
THUNDER BAY, ONTARIO,
CANADA P7B 5E1