Proceedings of the Edinburgh Mathematical Society (2022) **65**, 244–262
doi:10.1017/S0013091522000050

IRRATIONAL-SLOPE VERSIONS OF THOMPSON'S GROUPS T AND V

JOSÉ BURILLO[1], BRITA NUCINKIS[2] AND LAWRENCE REEVES[3]

[1] *Departament de Matemàtiques, Universitat Politècnica de Catalunya, Jordi Girona 1-3, 08034 Barcelona, Spain* (pep.burillo@upc.edu)
[2] *Department of Mathematics, Royal Holloway, University of London, Egham TW20 0EX, UK* (Brita.Nucinkis@rhul.ac.uk)
[3] *School of Mathematics and Statistics, University of Melbourne, Parkville, VIC 3010, Australia* (lreeves@unimelb.edu.au)

(Received 5 November 2020; first published online 11 February 2022)

Abstract In this paper, we consider the T- and V-versions, T_τ and V_τ, of the irrational slope Thompson group F_τ considered in J. Burillo, B. Nucinkis and L. Reeves [An irrational-slope Thompson's group, Publ. Mat. 65 (2021), 809–839]. We give infinite presentations for these groups and show how they can be represented by tree-pair diagrams similar to those for T and V. We also show that T_τ and V_τ have index-2 normal subgroups, unlike their original Thompson counterparts T and V. These index-2 subgroups are shown to be simple.

Keywords: Thompson group; irrational slope; simple group

2020 *Mathematics subject classification:* Primary 20E32; 20F05; 20F65;
Secondary 20F12; 20F38

Introduction

This paper is a continuation of the authors' previous paper [4], in which they studied in detail the irrational-slope Thompson's group F_τ, introduced by Cleary in [7], a group of piecewise-linear maps of the interval with irrational slopes that are powers of the small golden ratio $\tau = (\sqrt{5}-1)/2$ and having breakpoints in the ring $\mathbb{Z}[\tau]$. Note that in this context, by breakpoint, we mean a point at which the slope is discontinuous. In the standard fashion within the family of Thompson's groups, that study is extended here to the T and V versions, analogously defined via maps on the circle or left-continuous maps of the interval. The traditional Thompson's groups T and V are simple; however, the irrational-slope versions have subgroups of index two. This phenomenon stems from a special relation in the group, which introduces 2-torsion into the abelianization of F_τ.

Finally, we mention the possibility of taking a V_3-version with irrational slopes to construct Thompson-like groups with normal subgroups of index 4.

The paper is organised as follows. We begin by defining the groups and indicating how to work with their elements. Then we go into the specifics for each group, beginning with T_τ where we give a presentation and a proof of the simplicity of the index-two subgroup. Then follows a section for V_τ which also contains a presentation and a proof of the simplicity of the index-two subgroup, although the proof of simplicity of that subgroup is substantially different than that for T_τ. Finally, we also explore the possibility of the index-four subgroups for the three-caret version.

Note that these groups are of type F_∞. The result for F_τ was proven in [7], see [6] for detail. The extension of the result to T_τ and V_τ follows by directly applying the methods of Stein [12]. These methods are by now standard, see, for example, [8, 10, 11], and we will not present them here.

This paper is the natural continuation of [4], and many results and definitions are taken directly from that paper. Many of the arguments used are similar to those in [3].

1. The groups T_τ and V_τ

Let τ be the small golden ratio, namely $\tau = (\sqrt{5}-1)/2 = 0.618\ldots$, which satisfies $1 = \tau + \tau^2$. In [7], Cleary introduced the group F_τ, the irrational Thompson group, with breaks in $\mathbb{Z}[\tau]$ and slopes powers of τ. Given the equality $1 = \tau + \tau^2$, we can subdivide an interval of length τ^n into one of length τ^{n+1} and one of length τ^{n+2}. The group F_τ is the group of piecewise-linear maps on the interval with these breaks and slopes, see [4] for details.

As is standard in the Thompson family, we construct the irrational slope versions corresponding to the groups T and V. For the group T_τ, one can consider maps on the circle instead of the interval. The circle will be obtained by identifying the two endpoints of the interval $[0, 1]$, so we can consider maps of the interval such that the images of 0 and 1 are equal. So the group T_τ is the group of piecewise-linear, orientation-preserving homeomorphisms of the circle such that the breakpoints are in $\mathbb{Z}[\tau]$ and the slopes of the linear parts are powers of τ. The V-version V_τ consists of the left-continuous, piecewise-linear maps of $(0, 1]$, also with breaks in $\mathbb{Z}[\tau]$ and slopes that are powers of τ. See [5] for details of this construction for V, which is analogous to this one. See Figure 1 for examples of elements in T_τ and V_τ.

Thompson's group V is also commonly seen as a group of homeomorphisms of the Cantor set; however, this interpretation is not possible here. The fact that we have two types of carets and we have subdivisions which can be obtained in more than one way makes this interpretation unclear. The boundary is not well defined the traditional way, because there are different tree configurations that give the same subdivisions. Hence we will stick with left-continuous maps, but we will favour throughout the paper the expression of the elements in tree-permutation form, as is explained in what follows. Similarly, it is not clear how the arguments used to show that F is a diagram group could be adapted to the irrational slope setting.

It was shown in [7] that all elements of F_τ can be obtained by pairs of iterated subdivisions of the unit interval as above. When subdividing an interval of length τ^n into two intervals of lengths τ^{n+1} and τ^{n+2}, we have two options, either put the short interval, or

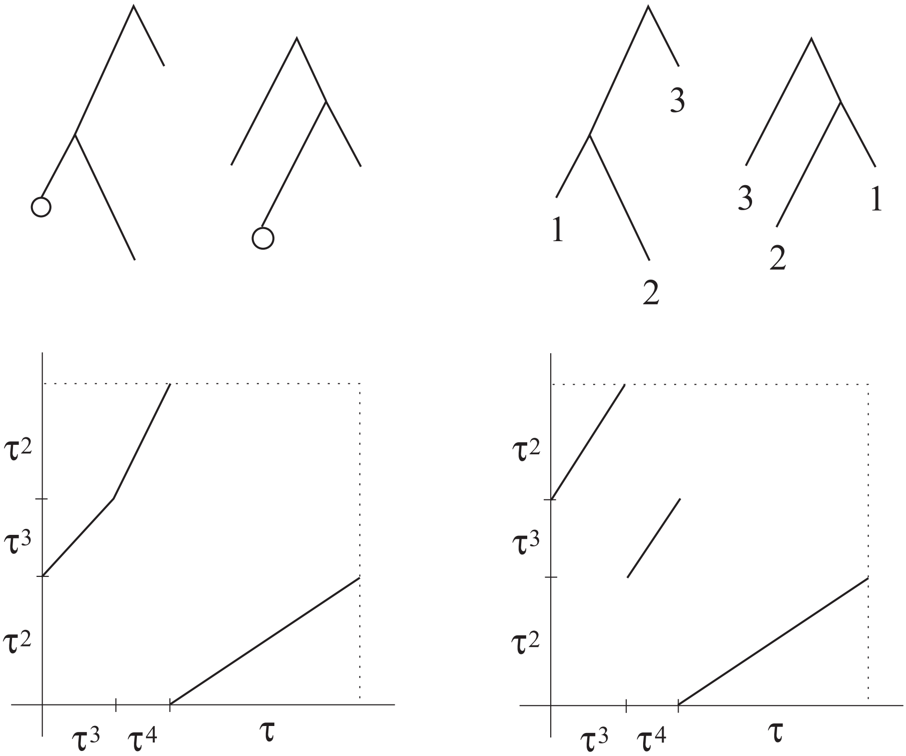

Figure 1. Elements in T_τ and V_τ, based on the same subdivisions. In the element in T_τ the circles indicate that the first leaf is mapped to the second leaf (and the rest in cyclic order), while for the element in V_τ the numbers indicate the permutation and the way leaves are mapped.

the long interval first. When representing iterated subdivisions of the interval by a tree, we will use two different types of carets. Those we call x-carets have a long leg on the left (representing a short first interval) and those we call y-carets have their left leg short (representing a long first interval). See [4] for details for the representation of elements of F_τ by tree-pair diagrams.

We now represent the elements of T_τ and V_τ by the same tree-pair diagrams, but with an added permutation of the leaves of the trees. See [5] for the analogous description for T and V. For the group T_τ these permutations preserve the cyclic order of the leaves, hence giving maps of the circle. For the group V_τ any permutation of the leaves is allowed. These permutations are represented by labels on the leaves indicating the permutation, although the cyclic permutations in T_τ are determined by the image of the first leaf, so we may abbreviate the permutation with just a circle in the image of the first leaf.

The diagrams are usually denoted by (T_1, π, T_2), where T_1 and T_2 are trees and π is the permutation of the leaves.

Results from [4] can be used to compose elements. The target tree of the first element has to be matched with the source tree of the second element, as is standard in the Thompson family. To do this, one may need to change the type of some carets, but this is

https://doi.org/10.1017/S0013091522000050 Published online by Cambridge University Press

independent of the two permutations of the elements and so can be done exactly as in F_τ, using Proposition 2.1 and Lemma 2.2 from [4]. The only difference is that when a caret is added, the counterpart caret added in the other tree has to go into the corresponding leaf determined by the permutation.

A reader familiar with F_τ and with Thompson's groups T and V will have no problem with all these definitions as they are very similar.

Finally, we can use the methods developed in [4] to construct a particularly appropriate diagram for elements of the groups T_τ and V_τ. As T_τ is a subgroup of V_τ, the lemma is stated for elements of V_τ.

Lemma 1.1. *Let $v \in V_\tau$. Then there exists a triple (T_1, π, T_2) representing v, where:*

(1) *T_1 and T_2 are trees with the same number of leaves, k say.*

(2) *$\pi \in \mathcal{S}_k$.*

(3) *T_2 is a tree consisting entirely of x-carets.*

(4) *The y-carets in T_1 have no left children.*

Proof. The proof is entirely analogous to that of [4, Lemma 6.1], with the exception of including a pair of spines in the middle in which the leaves are permuted. When adding carets to T_1 and T_2 simultaneously, one needs to take account of the permutation of leaves, but this does not change the outcome of the procedure. □

Corollary 1.2. *Every element in $v \in V_\tau$ has an expression*

$$v = x_0^{a_0} y_0^{\varepsilon_0} \ldots . x_n^{a_n} y_n^{\varepsilon_n} \pi x_m^{-b_m} \ldots . x_0^{-b_0},$$

where a_i, b_j, n, $m \in \mathbb{Z}_{\geq 0}$ $(i = 0, ..., n;\ j = 0, ...m)$, $\pi \in \mathcal{S}_k$ for some k, and $\varepsilon_i \in \{0, 1\}$.

This expression is called, in short, the $p\pi q^{-1}$ form of an element. The only particularity satisfied by elements of T_τ is that the permutation obtained will be a cyclic permutation c of the leaves, and in this case we call it the pcq^{-1} form, to keep with tradition. The $p\pi q^{-1}$ form can be used to solve the word problem by using the following lemma.

Lemma 1.3. *Suppose the identity element is given by an expression $p\pi q^{-1}$, where p and q are positive words in the x_i, y_i, $i \geq 0$, and $\pi \in \mathcal{S}_n$ for some n. Then $\pi = id_{\mathcal{S}_n}$.*

Proof. We represent id by a reduced tree-pair diagram (T_1, T_2), where the leaves of T_1 are labelled in order, and the labelling of the leaves of T_2 is given by π. We show that the leaves of T_2 are also labelled in order, therefore $\pi = 1$.

Since $id(0) = 0$, leaf 1 of T_1 is mapped to the left-most leaf of T_2. Furthermore, the slope of id is 1, and hence the depths of leaves 1 in T_1 and T_2 are the same, and hence the next leaves in each tree represent the same break-point $x \in \mathbb{Z}[\tau] \cap [0, 1]$. Now $id(x) = x$, and hence the second leaf in T_2 is also labelled by 2, and, since the slope is 1, also of the same level as leaf 2 in T_1. We continue this argument to see that $\pi = 1$. □

From this lemma, it is straightforward to solve the word problem for the groups T_τ and V_τ. Given an element, find its diagram and its $p\pi q^{-1}$ form. If π is not the identity,

https://doi.org/10.1017/S0013091522000050 Published online by Cambridge University Press

the element is not the identity. And if the permutation is the identity, then the element is actually in F_τ, and that word problem was solved in [4].

Finally, we observe that the method we will follow to obtain presentations for T_τ and V_τ is to see that the $p\pi q^{-1}$ form can be obtained algebraically with the set of generators stated. Hence, we proceed with the details for both groups.

2. A presentation for T_τ

2.1. Generators for T_τ

As happens with the standard dyadic groups, the only thing one needs to do to obtain generating sets for T is to add cycles. Recall that in F_τ we gave preference to x-type carets, where the left leg was long and the right one was short. Generators were constructed with spines made out of x-carets. The y_n generators had one y-type caret in the only non-spine position. So it only makes sense here to consider cyclic permutations of a spine as generators, following the methods of [5]. Hence, the generator c_n, for $n \geq 1$, will be defined as an element whose two trees are spines with $n+1$ carets, and where the first leaf is mapped to the last leaf. This is an element of order $n+2$ (or one of its divisors), since it is a cycle on its $n+2$ leaves. See Figure 2 for the diagrams of the c-generators.

From here we can already find a set of generators.

Proposition 2.1. *The set* $\{x_n, y_n, c_n \mid n \geq 1\}$ *is a generating set for the group* T_τ.

Proof. To see this, one only needs to decompose a tree diagram into three parts: a diagram for a positive element from F_τ, a cycle (i.e., a power of a c_n) and a negative element of F_τ. This can be done easily by introducing two spines in the middle of the diagram. The positive piece is given by the first tree and a spine (with the same number of carets), then a cycle with this same spine (which gives the permutation part of the element), and finally the spine with the target tree. Since the F_τ elements can be generated by the x_n and y_n, and the central cycle is a power of a c_n, we have a generating set as claimed. □

We need to evaluate the interactions of the generators with each other in order to find the needed relations. Clearly, all relations from F_τ are satisfied, so we need to see how the x_n and y_n generators interact with the c_n. Observe, to start with, that the generators x_n and c_n by themselves generate a copy of T inside T_τ, because all elements have only x-type carets and the combinatorics are exactly the same as those in T. Hence, the relations on T between x_n and c_n are satisfied the same way. From the simplicity of T, we conclude that T embeds in T_τ. So we already have the following relations:

(1) The relators from F_τ:
   (1.1) $x_j x_i = x_i x_{j+1}$, if $i < j$.
   (1.2) $x_j y_i = y_i x_{j+1}$, if $i < j$.
   (1.3) $y_j x_i = x_i y_{j+1}$, if $i < j$.
   (1.4) $y_j y_i = y_i y_{j+1}$, if $i < j$.
   (1.5) $y_n^2 = x_n x_{n+1}$.

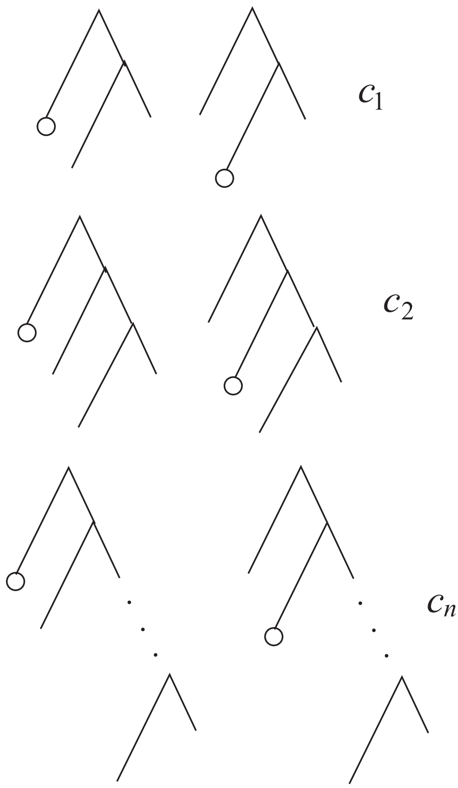

Figure 2. The c generators in T_τ.

(2) The relations involving the generators c_n and x_n, as in T:

(2.1) $x_k c_{n+1} = c_n x_{k+1}$, if $k < n$.

(2.2) $c_n x_0 = c_{n+1}^2$.

(2.3) $c_n = x_n c_{n+1}$.

(2.4) $c_n^{n+2} = 1$.

We only need to find analogs for the relations (2.1), (2.2) and (2.3) between the y_n and c_n by checking how these generators interact. It is straightforward to check that the relations $y_k c_{n+1} = c_n y_{k+1}$ are satisfied for all $k < n$. Because the y-caret does not affect the cycle, the relations work in the exact same way as with the x-carets. To find an analog of (2.2) for the y_n, we observe that the product $c_n y_0$ acquires a y-caret in the last leaf, continuing the spine, and this must be eliminated to rewrite it in terms of the generators. After adding a caret and performing a basic move, we obtain the product $y_{n+1}^{-1} c_{n+1}^2$. Finally, since normal forms for the elements of F_τ do not have any appearances of y_n^{-1}, a relator analogous to (2.3) will not be needed.

https://doi.org/10.1017/S0013091522000050 Published online by Cambridge University Press

So, to the above families of relators, we can add the following two:

(3.1) $y_k c_{n+1} = c_n y_{k+1}$, if $k < n$.

(3.2) $c_n y_0 = y_{n+1}^{-1} c_{n+1}^2$.

It is easy to check by direct inspection that these relations are all true in T_τ. The goal of the following section will be to prove that this gives a presentation for the group.

2.2. Relators for T_τ

The goal of this section is to exhibit a presentation of T_τ. The theorem we will prove is the following.

Theorem 2.2. *A presentation for the group T_τ is given by the generators x_n, y_n and c_n, for $n \geq 1$ with the following relators:*

(1) *The relators from F_τ:*
  (1.1) $x_j x_i = x_i x_{j+1}$, *if $i < j$.*
  (1.2) $x_j y_i = y_i x_{j+1}$, *if $i < j$.*
  (1.3) $y_j x_i = x_i y_{j+1}$, *if $i < j$.*
  (1.4) $y_j y_i = y_i y_{j+1}$, *if $i < j$.*
  (1.5) $y_n^2 = x_n x_{n+1}$.

(2) *The relations involving the generators c_n and x_n, same as in T:*
  (2.1) $x_k c_{n+1} = c_n x_{k+1}$, *if $k < n$.*
  (2.2) $c_n x_0 = c_{n+1}^2$.
  (2.3) $c_n = x_n c_{n+1}$.
  (2.4) *The finite order for the c_n, namely, $c_n^{n+2} = 1$.*

(3) *The relations involving the generators c_n and y_n :*
  (3.1) $y_k c_{n+1} = c_n y_{k+1}$, *if $k < n$.*
  (3.2) $c_n y_0 = y_{n+1}^{-1} c_{n+1}^2$.

The main tool to prove this theorem is the method to write any element in pcq^{-1} form. This is the algebraic analog of splitting a diagram into three pieces as used at the beginning of the previous section to show that these generators suffice.

Proposition 2.3. *Every element of T_τ can be written in pcq^{-1} form, where p and q are positive elements of F_τ and c is short for c_n^m for some m, n.*

Proof. We need the following well-known lemma, which is known to hold in T: □

https://doi.org/10.1017/S0013091522000050 Published online by Cambridge University Press

Lemma 2.4. *Let n be a positive integer, The generators x_n and c_n satisfy the following equalities*:

(i) $c_n^m = x_{n+1-m} c_{n+1}^m$, *for* $m \leq n+1$

(ii) $c_n^m = c_{n+1}^{m+1} x_{m-1}^{-1}$

This is contained in Lemma 5.6 in [5].

We continue with the proof of our proposition. Given a word in the x_n, y_n, c_n generators, we want to transform it into pcq^{-1} form. First of all, in this word, between two consecutive c_n generators, there is a word only in x_n and y_n, representing an element in F_τ. According to [4], we can put this element of F_τ in seminormal form

$$x_0^{a_0} y_0^{\epsilon_0} x_1^{a_1} y_1^{\epsilon_1} \dots x_n^{a_n} y_n^{\epsilon_n} x_m^{-b_m} x_{m-1}^{-b_{m-1}} \dots x_1^{-b_1} x_0^{-b_0}$$

where a_i, $b_i \geq 0$ and $\epsilon_i \in \{0, 1\}$. The proof of Proposition 2.3 will follow from a repeated application of the following process: transform

$$c_n^m x_0^{a_0} y_0^{\epsilon_0} x_1^{a_1} y_1^{\epsilon_1} \dots x_n^{a_n} y_n^{\epsilon_n} x_m^{-b_m} x_{m-1}^{-b_{m-1}} \dots x_1^{-b_1} x_0^{-b_0} c_k^l$$

into $w(x_n, y_n) c_N^M$, where $w(x_n, y_n)$ is a word in the x_n and y_n only, and N, M are appropriately chosen positive integers. Applying this process repeatedly from left to right will yield the result.

To move the c_n^m past the x_0 use the standard T relations. Observe that the index of a given c_n^m can be raised as much as we need, at the price of adding some instances of x_n to its left (which is fine) by repeated uses of property (i) of Lemma 2.4. To move an element c_n^m past y_0, we apply:

$$c_n^m y_0 = c_n^{m-1} c_n y_0 = c_n^{m-1} y_{n+1}^{-1} c_{n+1}^2$$

after using relation (3.2). To move the y_{n+1}^{-1} to the left past the c_n^{m-1}, we will take inverses. If we take inverses directly as $y_{n+1} c_n^{1-m}$, to apply (3.1), we need to have raised the index of c_n beforehand, because on the left-hand side of (3.1) the term $y_k c_{n+1}$ requires that the index of c is at least two higher than that of y. Since there are several instances of (3.1) to be taken, and each one increases the index of y by 1, we will need to raise the index several times, using (i) from the lemma. Hence:

$$\begin{aligned} c_n^{m-1} y_{n+1}^{-1} &= x_{n-m+2} c_{n+1}^{m-1} y_{n+1}^{-1} = x_{n-m+2} x_{n-m+3} c_{n+2}^{m-1} y_{n+1}^{-1} \\ &= x_{n-m+2} x_{n-m+3} \dots x_N c_N^{m-1} y_{n+1}^{-1} \end{aligned}$$

where N is an index large enough, much larger than $n+1$, to be able to apply the following:

$$c_N^{m-1} y_{n+1}^{-1} = [y_{n+1} c_N^{N-m+3}]^{-1}$$

Now use repeated applications of (3.1) to obtain

$$[y_{n+1} c_N^{N-m+3}]^{-1} = [c_{N-1}^{N-m+3} y_{n+N-m+2}]^{-1}.$$

This yields the equality

$$c_n^m y_0 = x_{n-m+2} \dots x_N y_{n+N-m+2}^{-1} c_{N-1}^{N-m+3}$$

which is what we desired to get the c_n^m past the y_0.

https://doi.org/10.1017/S0013091522000050 Published online by Cambridge University Press

To get the c_n^m past x_k, use T again. And to go past y_k, note that now we can use relation (3.1) directly. Raise the index of the c_n^m if necessary using (i) in the lemma and when the index is high enough, reverse them using (3.1). If the index of y_k becomes zero, use the algorithm above.

Ultimately, this produces an element with a word on x_n and y_n, followed by a product $c_n^m c_k^l$. To transform this into a power of a single c-generator, use Lemma 2.4 to raise the smaller index of n and k, using (i) if n needs to be raised, and (ii) if k is smaller. The only drawback of doing this that it introduces x-generators to the left or right of the pair of c-generators, which in fact works well for our purposes. This finishes the process of getting c_n^m past an element of F_τ and merging it into the following c_k^l. Continue this process until only one power of a c_n is left.

This produces an element of the type $pq^{-1}crs^{-1}$, where p, q, r, s are positive words and we can assume (using the properties of F_τ) that q and s contain only x-generators. To finalize the process, move the word r to the left, using the method above, leaving only negative elements to the right of c giving a word wcs in which w clearly can have negative elements. Put w in F_τ-form and use T to move the negative x-generators to the right of c. This finishes the proof of Proposition 2.3.

This process constructs an easy expression for an element which can be used to solve the word problem and to show that these relators suffice to have a presentation. To prove that the relators in Theorem 2.2 are enough to give a presentation, given a word in the x_n, y_n, c_n which represents the identity in T_τ, rewrite it in pcq^{-1} form. Since the word represents the identity, apply Lemma 1.3 to see that c is actually the identity. This then means that the original element was in F_τ and is thus a product of conjugates of the relators for this group, which are in the presentation as well.

This process finishes the proof of Theorem 2.2.

3. The group T_{xz} is simple

The appearance of the relator (1.5), present already in F_τ, was the reason for the 2-torsion observed in the abelianization of F_τ. Observe that, as happens in F_τ, the parity of the y-generators is preserved in all relations of T_τ (see Theorem 2.2). This shows that there is a homomorphism from T_τ onto $\mathbb{Z}/2\mathbb{Z}$, which sends x_n and c_n to 0 and y_n to 1.

Definition 3.1. The kernel of the map $\varphi : T_\tau \to \mathbb{Z}/2\mathbb{Z}$ will be denoted by T_{xz}.

The group T_{xz} is generated by the x_n, c_n and by the new generators $z_n = y_{2n}y_{2n+2}$. Hence, the group T_τ has no chance of being simple like T, but this kernel is.

Theorem 3.2. *The group T_{xz} is simple.*

Proof. Let N be a normal subgroup of T_{xz} and let $a \neq 1$ be an element in N. We want to show that $N = T_{xz}$. Let

$$\theta : T_{xz} \longrightarrow T_{xz}/N$$

https://doi.org/10.1017/S0013091522000050 Published online by Cambridge University Press

be the canonical quotient map. Since a is non-trivial, we know it can be written in the form pcq^{-1}, and we only need that p, $q \in F_\tau$. The parity of y is irrelevant for this argument. Since $\theta(a) = 1$, we have that $\theta(p^{-1}q) = \theta(c)$, and hence, since c is of finite order, we have that $\theta((p^{-1}q)^n) = 1$ for some n. We now have two possibilities:

(1) $p^{-1}q \neq 1$. If this is the case, and since F_τ is torsion-free, we have that $(p^{-1}q)^n$ is an element which is in F_τ, is not the identity but lies in the kernel of θ.

(2) $p^{-1}q = 1$. Then $\theta(c) = 1$. Suppose $c = c_n^m$. Use relation (2.3) to see that $c_n^m = c_n^{m-1}x_n c_{n+1}$ and applying relator (2.1) $m-1$ times, we get $c_n^m = x_{n-(m-1)}c_{n+1}^m$. Hence, since $\theta(c_n^m) = 1$, we have that $\theta(x_{n-(m-1)}) = \theta(c_{n+1}^{-m})$. But this means that $\theta(x_{n-(m-1)}^{n+3}) = 1$, and again we have found a non-trivial element in F_τ and in the kernel of θ.

Restrict θ to F_τ and recall that a quotient of F_τ is either isomorphic to F_τ, or else is abelian. From the two cases above, we deduce that $\theta|_{F_\tau}$ is not an isomorphism, so we conclude that $\theta(F_\tau)$ is an abelian group.

From here we have that the images by θ of all x_n and y_n commute. As was done in [5], this means that:

(1) From relation (2.1) with $n = 2$, $k = 1$, namely, $x_1c_3 = c_2x_2$, that is, $x_1(x_0^{-2}c_1x_1^2) = (x_0^{-1}c_1x_1)(x_0^{-1}x_1x_0)$ we deduce that $\theta(x_1) = \theta(x_0)$.

(2) From relation (2.3) with $n = 1$, namely, $c_1 = x_1(x_0^{-1}c_1x_1)$, we deduce that $\theta(x_0) = \theta(x_1) = 1$.

(3) From relation (2.2) with $n = 1$, namely, $c_1x_0 = (x_0^{-1}c_1x_1)^2$, we deduce that $\theta(c_1) = 1$.

Hence $\theta(x_n) = \theta(c_n) = 1$ for all n. Finally, from relation (3.1), we deduce that $y_ny_{n+1}^{-1}$ also maps to 1, and from here that $\theta(y_iy_j^{-1}) = 1$ for all i, j. From relation (3.2) with $n = 1$ we deduce that $z_0 = y_0y_2$ is also in the kernel, and by induction for any other z_n using $z_{n-1} = y_{2n-2}y_{2n}$ and $y_{2n-2}^{-1}y_{2n+2}$. So θ is the trivial map and $T_{xz} = N$. □

4. A presentation for V_τ

The construction of a generating set for V_τ is straightforward. Following [5] and in the same way that we have done for T_τ, we need to add a series of generators π_n, for $n \geq 0$, which introduce noncyclic permutations to the leaves of a spine. The generator π_n has $n+2$ x-carets, hence it has $n+3$ leaves, and the permutation just transposes leaves $n+1$ and $n+2$. This is analogous to the generators for V in [5]. See the generators π_n in Figure 3. It is clear that the series x_n, y_n, c_n and π_n generate V_τ. Observe that we write the element in $p\pi q^{-1}$ form and then the permutation π is in some $\mathcal{S}_k$, which is generated by a k-cycle and a transposition. We suppose $k \geq 3$, the case $k = 2$ being straightforward and is actually in T_τ. Then the k-cycle is c_{k-2} and the transposition is π_{k-3}.

Hence, we need to find relations involving these generators. As before, the relations will be obtained by seeing how the generators of different types interact with each other. First of all, from Theorem 2.2, involving x_n, y_n and c_n, we have all relators from T_τ:

https://doi.org/10.1017/S0013091522000050 Published online by Cambridge University Press

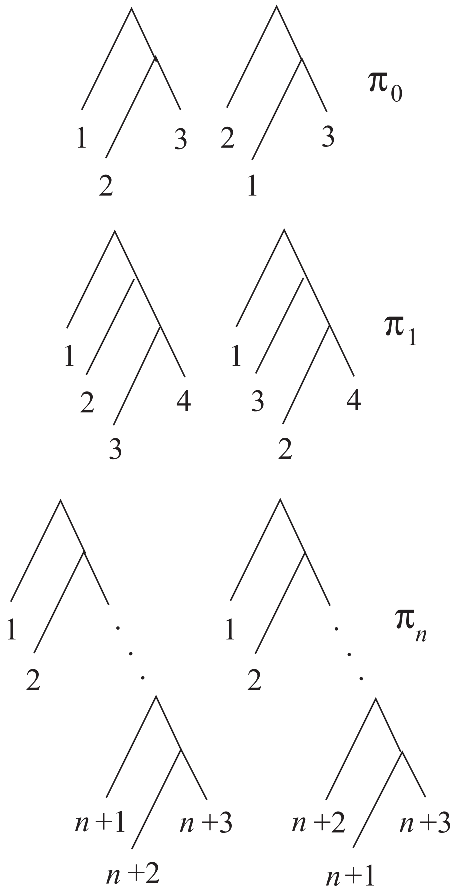

Figure 3. The π generators in V_τ.

(1) The relations involving x_n and y_n:
 (1.1) $x_jx_i = x_ix_{j+1}$, if $i < j$.

 (1.2) $x_jy_i = y_ix_{j+1}$, if $i < j$.

 (1.3) $y_jx_i = x_iy_{j+1}$, if $i < j$.

 (1.4) $y_jy_i = y_iy_{j+1}$, if $i < j$.

 (1.5) $y_n^2 = x_nx_{n+1}$.

(2) The relations involving x_n and c_n:
 (2.1) $x_kc_{n+1} = c_nx_{k+1}$, if $k < n$.

 (2.2) $c_nx_0 = c_{n+1}^2$.

 (2.3) $c_n = x_nc_{n+1}$.

 (2.4) $c_n^{n+2} = 1$.

https://doi.org/10.1017/S0013091522000050 Published online by Cambridge University Press

(3) The relations involving c_n and y_n:
(3.1) $y_k c_{n+1} = c_n y_{k+1}$, if $k < n$.

(3.2) $c_n y_0 = y_{n+1}^{-1} c_{n+1}^2$.

Observe that the combinatorics involving x_n, c_n and π_n are exactly the same as in V, the only difference is that a binary caret is replaced by an x-caret; however, the methods are exactly the same. Hence, we have all relators from V (as before when we had all relators for T in T_τ).

(4) The relations involving x_n and π_n:
(4.1) $\pi_i x_j = x_j \pi_i$ if $j \geq i+2$.

(4.2) $\pi_i x_{i+1} = x_i \pi_{i+1} \pi_i$.

(4.3) $\pi_i x_i = x_{i+1} \pi_i \pi_{i+1}$.

(4.4) $\pi_i x_j = x_j \pi_{i+1}$ if $0 \leq j < i$.

(4.5) $\pi_i^2 = 1$.

(4.6) $(\pi_{i+1}\pi_i)^3 = 1$.

(4.7) $\pi_i \pi_j = \pi_j \pi_i$ if $j \geq i+2$.

(5) The relations involving c_n and π_n:
(5.1) $c_n \pi_k = \pi_{k-1} c_n$.

(5.2) $c_n \pi_0 = \pi_0 \cdots \pi_{n-1} c_n^2$.

(5.3) $c_n^2 \pi_0 = \pi_{n-1} \cdots \pi_0 c_n$.

(5.4) $c_n^3 \pi_0 = \pi_{n-1} c_n^3$.

As was the case in T_τ, the only relators we need to add are those which involve y_n and π_n. But the case here differs from what we have found in T_τ, because the transposition in π_n never involves the last leaf of the caret. In T_τ when we combined y_n with c_n, there were some cases where the y-caret ends up being on the spine, the last caret of the tree, and since we are taking the spine with only x-carets, another y-caret had to be added to switch. This is the reason why relations (3.1) and (3.2) have an extra y-generator than their counterparts with x_n, which are (2.1) and (2.2). But this does not happen here, the relations involving y_n and π_n look exactly the same as for x_n, so they are the same as (4.1)–(4.4) with x replaced by y.

(6) The relations involving y_n and π_n:
(6.1) $\pi_i y_j = y_j \pi_i$ if $j \geq i+2$.

(6.2) $\pi_i y_{i+1} = y_i \pi_{i+1} \pi_i$.

https://doi.org/10.1017/S0013091522000050 Published online by Cambridge University Press

(6.3) $\pi_i y_i = y_{i+1}\pi_i\pi_{i+1}$.

(6.4) $\pi_i y_j = y_j \pi_{i+1}$ if $0 \leq j < i$.

It is for this reason that the proof of the fact that these are all the relations we need is straightforward. For the interactions between x_n, y_n and c_n we already have the π_n-relators, and the way the π_n interact with the other three series of generators is exactly the same as in V. Hence, all results for V in [5] apply here, and clearly an element can be put into $p\pi q^{-1}$ form using all these relations. Once it is in this form, if it was the identity, it is in F_τ and it is, therefore, a product of the relators (1.1)–(1.5). This finishes the construction of the presentation for V_τ.

5. The group V_{xz} is simple

Inspecting the presentation for V_τ we observe the same phenomenon as in F_τ and T_τ, namely, that the y-generators only appear in the relators an even number of times. Hence there is a well-defined epimorphism

$$\varphi : V_\tau \longrightarrow \mathbb{Z}/2\mathbb{Z},$$

given by $\varphi(x_n) = \varphi(c_n) = \varphi(\pi_n) = 0$, and $\varphi(y_n) = 1$. The kernel of this epimorphism is generated by those elements in V_τ where y-generators appear an even number of times.

In particular, V_τ is not simple. We will consider the kernel of the homomorphism φ:

Definition 5.1. We denote by V_{xz} the kernel of the map

$$\varphi : V_\tau \longrightarrow \mathbb{Z}/2\mathbb{Z}.$$

Theorem 5.2. *The group V_{xz} is a simple group.*

Here we follow an argument described by Brin in [1], and attributed to M. Rubin.

Lemma 5.3. *V_{xz} is generated by permutations.*

Proof. Let $v \in V_{xz}$. Then, by Lemma 1.1 and the parity of y, it can be represented by a triple (T, π, S) such that: T and S have k leaves, S has no y-carets, there are an even number of y-carets in T all with no left children, and $\pi \in \mathcal{S}_k$.

To begin the process, we need an exposed y-caret in the tree T, that is, a y-caret with no children. Observe that we only know that the y-carets have no left children, but they could have right ones. Take any y-caret in T and make it exposed in the following way. See Figure 4 for reference. Add two x-carets to its left leaf (and the corresponding ones in S). Perform a basic move on these two x-carets to transform them to y-carets, which produces three consecutive y-carets. Now switch the two top carets back to x-type, again using a basic move. This makes the bottom y-caret exposed. The number of y-carets has not changed. Note that this process is the reverse of a *hidden cancellation* described in [4, Section 6].

Now we have ensured that in T, there is at least one exposed caret. Pick a pair of y-carets c_1 and c_2 in T and assume c_1 is exposed. Then add a y-caret to the left leaf of c_2;

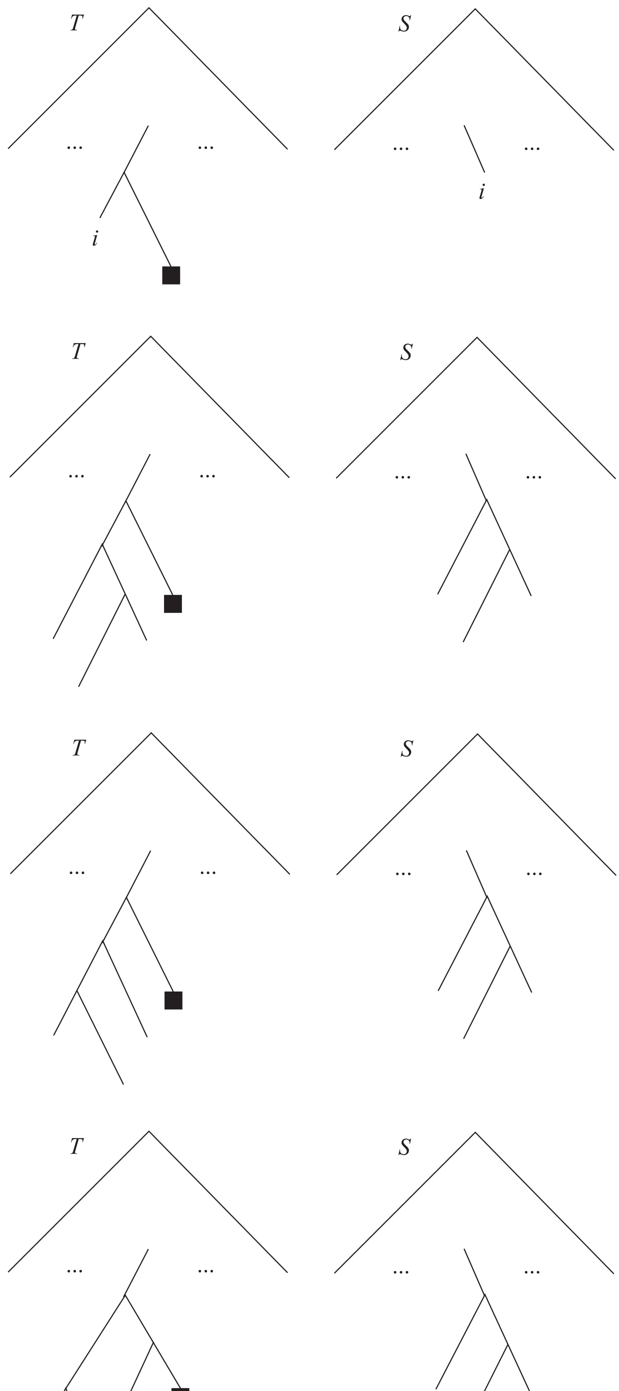

Figure 4. The process of creating an exposed y-caret in the proof of Lemma 5.3. The black box indicates that a subtree is present in that leaf.

https://doi.org/10.1017/S0013091522000050 Published online by Cambridge University Press

this means that we have to add a y-caret d_1 to the corresponding leaf in S. Switch c_2 and the recently added caret to type x using a basic move. We have now a representation of v by a triple (T', π', S'), where T' and S' are now trees with $k+1$ carets, and $\pi' \in \mathcal{S}_{k+1}$, and the carets c_1 and d_1 are both exposed, one in each tree.

We now precompose with a permutation σ of the leaves of T', so that c_1 in the left-most tree has the same labelling as d_1 in S'. We now have obtained an element $v_1 = \sigma v$ represented by

$$(T', \sigma, T')(T', \pi', S'),$$

where the y-carets c_1 and d_1 map to each other, so they have become redundant and can be removed. Hence v_1 is now represented by a triple as above with $k+2$ leaves, but with two fewer y-carets.

We repeat this process until we have $u = \rho v$, where ρ is a product of permutations and u is an element represented by a tree-pair with only x-carets.

We can now follow the procedure set out by Brin as if we were in V: find a pair of exposed carets, one in each tree, premultiply with a permutation making this pair of carets redundant, then remove the carets to obtain an element represented by a tree with fewer carets. Then continue this process until no tree is left. Our original element is now a product of permutations. □

Definition 5.4. We say an element $\pi \in V_\tau$ is a proper transposition if π is a transposition which can be represented by a triple (T, π, T), where T is a tree with at least three carets.

Corollary 5.5. *V_{xz} is generated by proper transpositions.*

Proof. Suppose we have a transposition represented by a tree-pair with two or fewer carets, then adding redundant carets simultaneously on both sides gives a permutation with the right number of carets, which, in turn, is a product of proper transpositions. □

Lemma 5.6. *Any two proper transpositions are conjugate in V_{xz}.*

Proof. Let π_1 and π_2 be two proper transpositions represented by (T, π_1, T) and (S, π_2, S), respectively. Since both transpositions are proper, we have "uninvolved" leaves in both trees, so we can add redundant carets to ensure that T and S have the same number of leaves. Now the element $v = (T, \sigma, S)$, where σ maps the leaves involved in π_1 to the leaves involved in π_2, conjugates π_2 into π_1. A priori this is an element of V_τ. Suppose that both S and T have the same parity of y-leaves. Then, automatically, $v \in V_{xz}$. Should the parity of y-leaves in T and S differ, we can always add a redundant y-caret to T and a corresponding redundant x-caret to S to obtain a tree-pair (T', σ', S') representing an element of V_{xz} conjugating π_2 into π_1. □

Lemma 5.7. *Any normal subgroup N of V_{xz} contains a proper transposition.*

Proof. This proof is exactly the proof of Brin [1], so we only provide an outline here. Let $v = (T, \pi, S) \in N$. By expanding, we can always represent v by a tree-pair with at least three carets, and so an interval gets completely moved off itself. We now consider all

https://doi.org/10.1017/S0013091522000050 Published online by Cambridge University Press

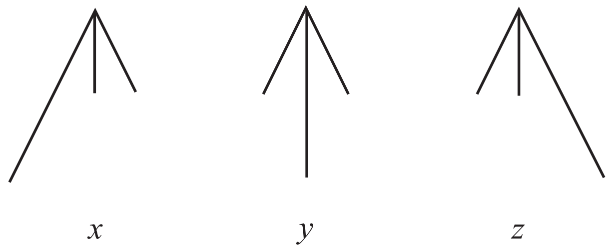

Figure 5. The three different caret types in V_β.

elements as moving nodes in an infinite tree. In particular, there is a node l mapping to a node k so that neither is in a subtree containing the other. We expand l and k with a caret of the same type, an x-caret say. Let g be the element switching $(l_o,\, l_1)$, the leaves of the caret starting at l. Then $u = [g,\, v] \in N$ and is a product of two transpositions. Now, set h to be the transposition switching l_0 with k_0, the left-hand leaf of the caret starting at k. Then $w = [h,\, u] \in N$ is a permutation swapping l_0 with k_0 and l_1 with k_1, hence w is a transposition switching l with k. □

This sequence of results, 5.3 through to 5.7, now yield Theorem 5.2.

6. A V-type Thompson's group with a normal subgroup of index 4

As mentioned above, Higman [9] showed that some of the V-type groups have index-2 normal subgroups. This, however, occurs only when the n-ary carets in the trees representing the elements have n odd. Hence, this is a very different phenomenon than the one we observe for V_τ. Let $\beta = \sqrt{2} - 1$, the positive root of $X^2 + 2X - 1$. In [6], Cleary also showed that the group $G(I,\, \mathbb{Z}[\beta],\, \langle\beta\rangle) = F_\beta$ can be seen as a group of rearrangements of β-regular subdivisions of the unit interval. In his thesis, J. Brown [2], shows that elements of F_β can also be represented by tree-pair diagrams, this time having three caret-types with three legs each, see Figure 5.

He also produced an infinite presentation involving x-carets and y-carets only:

Proposition 6.1 (Brown [2, 6.1.1.]). *A presentation for F_β is*:

$$\left\langle x_n, y_n,\ \text{for } n \geq 0 \;\middle|\; \begin{array}{l} a_i b_j = b_j a_{i+2},\ \text{for all } i > j,\ \text{and } a, b \in \{x, y\}, \\ y_k^2 = x_k x_{k+1},\ \text{for } k \geq 0 \end{array} \right\rangle.$$

The generators are represented by the following tree-pairs $(T_{a_i},\, S_k)$, where T_{a_i} are as in Figure 6 ($a \in \{x,\, y\}$), and S_k is a spine with $\lfloor \frac{i}{2} \rfloor + 2$ x-carets.

Analogously to V_τ, we can now define the group V_β, whose elements are represented by triples $(T,\, \pi,\, S)$, where T and S are trees with the same number of leaves, and π is a permutation of leaves. We now follow the arguments of § 5.

Lemma 6.2. *We have the following*:

(1) *Every element $v \in V_\beta$ has an expression as $p\sigma q^{-1}$, where p and q are words in positive powers of x_i and y_i ($i \geq 0$), and σ is a permutation of the leaves of a spine with only x-carets.*

https://doi.org/10.1017/S0013091522000050 Published online by Cambridge University Press

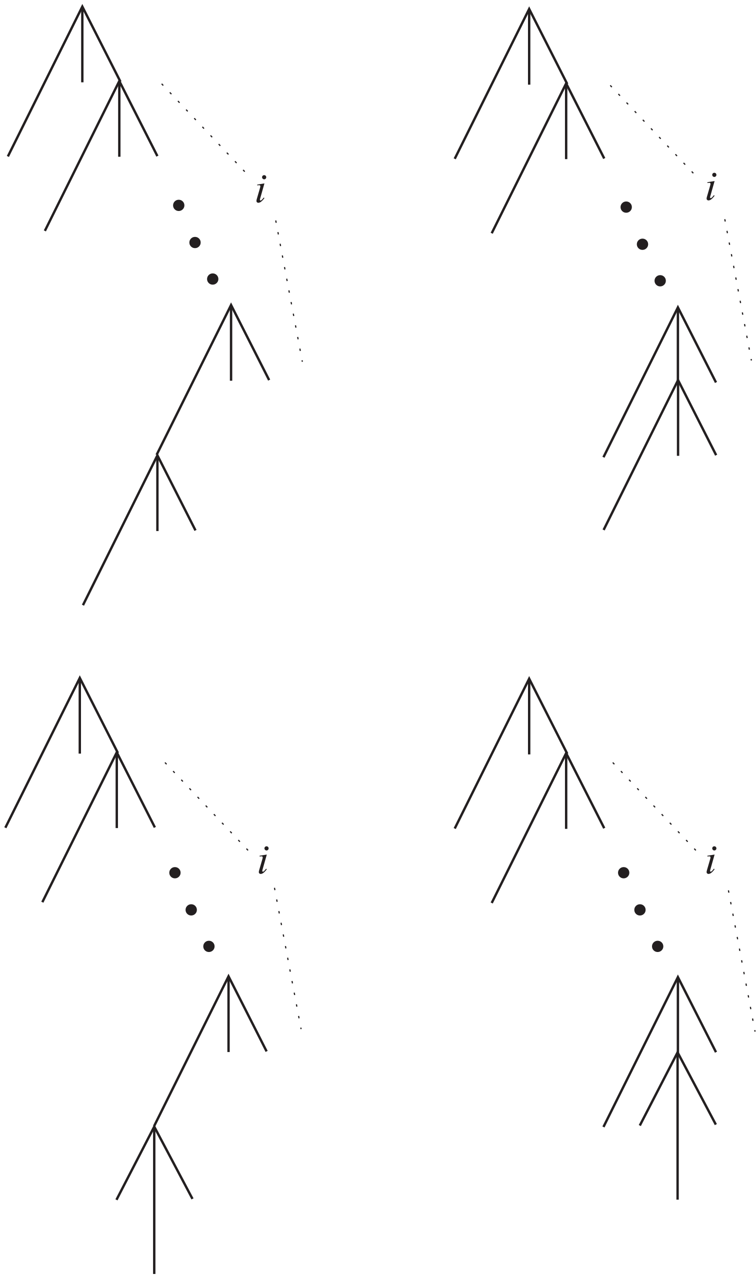

Figure 6. The left-hand trees for the generators. Top left $T_{x_{2i}}$, top right $T_{x_{2i+1}}$, bottom left $T_{y_{2i}}$, bottom right $T_{y_{2i+1}}$.

https://doi.org/10.1017/S0013091522000050 Published online by Cambridge University Press

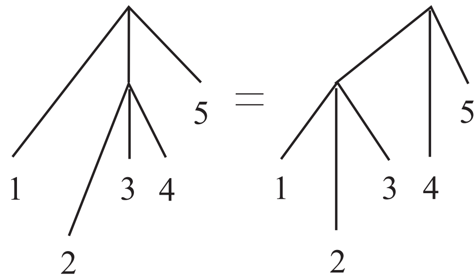

Figure 7. The order of leaves does not change when we perform a basic move replacing two carets by another two giving the same subdivision.

(2) *In every such expression of $v \in V_\beta$ as a word in x_i, y_j, π_k, $(i, j, k \geq 0)$ the parity of the sum of the exponents of the y_j is constant.*

Proof. The first statement is immediate by the usual method of introducing a spine in the middle and permuting the leaves there. The second statement is a check similar to that for V_τ. □

Lemma 6.3. *There is a well-defined surjective homomorphism*

$$\varphi : V_\beta \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

given by $\varphi(x_i) = \varphi(\sigma) = 0$ and $\varphi(y_i) = 1$ $(i \geq 0)$, where σ is a permutation of leaves.

Analogously to Higman's original argument [9, Section 5], we can also find a subgroup of index-2 in V_β given by the sign of the permutation in (T, π, S). This sign is unchanged by the expansion of the trees, see [9, Section 5], and also by the relations between the different carets, which follows from the fact that the relations do not change the order of the leaves, see Figure 7.

Lemma 6.4. *There is a well-defined subgroup ${V_\beta}^+$ of index 2 in V_β given by the elements of even parity. Hence there is a canonical projection*:

$$\rho : V_\beta \longrightarrow V_\beta / {V_\beta}^+ \cong \mathbb{Z}/2\mathbb{Z}.$$

Theorem 6.5. *V_β has an index-4 normal subgroup.*

Proof. We have a surjection $V_\beta \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ given by $v \mapsto (\rho(v), \varphi(v))$. □

The kernel K of this map is the subgroup given by all even elements, which have an even number of y_i in each expression $v = p\sigma q^{-1}$ where p and q are words in positive powers of x_i and y_i $(i \geq 0)$. We suspect that K is simple, but the methods above for V_τ do not work, and one would have to follow Higman's original proof of the simplicity of $G^+_{n,r}$. This is the subject of N. Winstone's thesis [13], where Thompson groups for more general algebraic numbers are considered.

https://doi.org/10.1017/S0013091522000050 Published online by Cambridge University Press

References

1. M. G. Brin, Higher dimensional Thompson groups, *Geom. Dedicata* **108** (2004), 163–192.
2. J. Brown, A class of PL-homeomorphism groups with irrational slopes. M.Sc Thesis (University of Melbourne, 2018).
3. J. Burillo, S. Cleary, M. Stein and J. Taback, Combinatorial and metric properties of Thompson's group T, *Trans. Amer. Math. Soc.* **361**(2) (2009), 631–652.
4. J. Burillo, B. Nucinkis and L. Reeves, An irrational-slope Thompson's group, *Publ. Mat.* **65** (2021), 809–839.
5. J. W. Cannon, W. J. Floyd and W. R. Parry, Introductory notes on Richard Thompson's groups, *Enseign. Math. (2)* **42**(3-4) (1996), 215–256.
6. S. Cleary, Groups of piecewise-linear homeomorphisms with irrational slopes, *Rocky Mountain J. Math.* **25**(3) (1995), 935–955.
7. S. Cleary, Regular subdivision in $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$, *Illinois J. Math.* **44**(3) (2000), 453–464.
8. M. G. Fluch, M. Marschler, S. Witzel and M. C. B. Zaremsky, The Brin-Thompson groups sV are of type F_∞, *Pacific J. Math.* **266**(2) (2013), 283–295.
9. G. Higman, *Finitely presented infinite simple groups*. Department of Pure Mathematics, Department of Mathematics, I.A.S. Australian National University, Canberra, 1974. Notes on Pure Mathematics, No. 8 (1974)
10. D. H. Kochloukova, C. Martínez-Pérez and B. E. A. Nucinkis, Cohomological finiteness properties of the Brin-Thompson-Higman groups $2V$ and $3V$, *Proc. Edinb. Math. Soc. (2)* **56**(3) (2013), 777–804.
11. C. Martínez-Pérez, F. Matucci and B. E. A. Nucinkis, Cohomological finiteness conditions and centralisers in generalisations of Thompson's group V, *Forum Math.* **28**(5) (2016), 909–921.
12. M. Stein, Groups of piecewise linear homeomorphisms, *Trans. Amer. Math. Soc.* **332**(2) (1992), 477–514.
13. N. Winstone, Bieri–Strebel groups with irrational slopes. Ph.D thesis (Royal Holloway, University of London, 2021).