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ON KLOOSTERMAN SUMS WITH OSCILLATING COEFFICIENTS

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ABSTRACT. An estimate for Kloosterman sums with oscillating coefficients is presented. Precisely we show: for any $\epsilon > 0$ and a, b positive integers with $(a, b) = 1$ we have,

$$\sum_{\substack{k \leqq n \\ (k,b)=1, k\bar{k} = 1 (\mathrm{mod}\, b)}} \mu(k) e\left(\frac{\bar{k}a}{b}\right) \ll_\epsilon nb^\epsilon \left(\frac{(\log n)^{5/2}}{b^{1/2}} + \frac{(\log n)^{11/5} b^{3/10}}{n^{1/5}}\right)$$

Similar techniques may be used to estimate other Kloosterman sums with oscillating coefficients which are not smooth.

1. **Introduction.** In this note, we obtain some bounds on Kloosterman sums with oscillating coefficients. Precisely, we obtain an estimate for

$$(1.1) \qquad \sum_{\substack{k \leqq n \\ k\bar{k} = 1 (\mathrm{mod}\, b), (k,b)=1}} \mu(k) e\left(\frac{a\bar{k}}{b}\right)$$

where $(a, b) = 1$. The Theorem we prove about such sums is:

THEOREM. *For any $\epsilon > 0$ and a, b, positive integers with $(a, b) = 1$ we have,*

$$\sum_{k \leqq n} \mu(k) \chi_b(k) e\left(\frac{\bar{k}a}{b}\right) \ll_\epsilon nb^\epsilon \left(\frac{(\log n)^{5/2}}{b^{1/2}} + \frac{(\log n)^{11/5} b^{3/10}}{n^{1/5}}\right)$$

where $\chi_b(k) = 1$ for $(k, b) = 1$ and 0 else, and $k\bar{k} = 1 \pmod{b}$.

The interest in estimating Kloosterman sums of this type stems from applications to additive problems when estimating similar types of Kloosterman sums, but with smooth coefficients. We refer to [2], [5] for various examples. The technique that is used for proving the above estimate is an application of Vaughan's identity [6] along with an estimate for incomplete Kloosterman sums due to Hooley [4] which follows from Weil's estimate for Kloosterman sums. The estimate of Hooley [4] that we shall need is

$$(1.2) \qquad \sum_{k \leqq n} \chi_b(k) e\left(\frac{\bar{k}a}{b}\right) \ll_\epsilon b^{1/2+\epsilon}(a, b)^{1/2}$$

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for $n \leqq b$ (henceforth we shall write $\chi(k)$ instead of $\chi_b(k)$ when there is no confusion). It is readily seen that the above technique adapts to many other Kloosterman sums with non-smooth coefficients. Therefore, we have restricted ourselves to the above theorem. The notation is standard and is as in [3].

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2. **Proof of the Result.** We prove the theorem mentioned in the introduction.

Proof. Let

$$\lambda(x, y) = \mu(y)\chi(x)\chi(y)e\left(\frac{a\overline{xy}}{b}\right).$$

By Vaughan's identity,

$$\sum_{y \leqq N} \lambda(1, y) = S_0 + S_1 - S_2 - S_3$$

where

$$S_0 = \sum_{y \leqq W} \lambda(1, y)$$

$$S_1 = \sum_{d \leqq W} \sum_{y \leqq N/d} \sum_{z \leqq N/yd} \mu(d)\lambda(dz, y)$$

$$S_2 = \sum_{d \leqq W} \sum_{y \leqq W} \sum_{z \leqq N/yd} \mu(d)\lambda(dz, y)$$

$$S_3 = \sum_{x > W} \sum_{\substack{y > W \\ xy \leqq N}} \tau_x \lambda(x, y)$$

$$\tau_x = \sum_{d|x, d \leqq W} \mu(d)$$

Here W is a parameter chosen later. Clearly $S_0 \ll W$. To estimate S_1,

$$\begin{aligned} S_1 &= \sum_{d \leqq W} \mu(d)\chi(d) \sum_{k \leqq N/d} \left(\sum_{yz = k} \chi(z)\chi(y)\mu(y) \right) e\left(\frac{\overline{kd}a}{b}\right) \\ &= \sum_{d \leqq W} \mu(d)\chi(d) \sum_{k \leqq N/d} \chi(k) \left(\sum_{y|k} \mu(y) \right) e\left(\frac{\overline{kd}a}{b}\right) \\ &\ll \left| \sum_{d \leqq W} \mu(d)\chi(d) e\left(\frac{\overline{d}a}{b}\right) \right| \ll W \end{aligned}$$

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To estimate S_2, we have used the estimate

$$\sum_{k \leqq n} \frac{d(k)}{k} \ll (\log n)^2,$$

$$S_2 = \sum_{k \leqq W^2} \sum_{z \leqq N/k} \chi(z) \Bigg(\sum_{\substack{yd=k \\ y \leqq W, d \leqq W}} \mu(d)\mu(y)\chi(k) \Bigg) e\left(\frac{a\bar{k}\bar{z}}{b}\right)$$

$$\ll \sum_{\substack{k \leqq W^2 \\ (k,b)=1}} d(k) \left| \sum_{z \leqq N/k} \chi(z) e\left(\frac{a\bar{k}\bar{z}}{b}\right) \right|$$

$$\ll_\epsilon \sum_{k \leqq W^2} d(k) \left(\frac{N}{kb} + b^{1/2+\epsilon} \right)$$

(by (1.2) and evaluating Ramanujan sums)

$$\ll_\epsilon \frac{N}{b} (\log N)^2 + b^{1/2+\epsilon} W^2 \log W$$

To estimate S_3, we let

$$A = \{2^j W | 0 \leqq j \leqq k,\ 2^k W^2 < N \leqq 2^{k+1} W^2\}$$

$$S(Y) = \sum_{Y < x \leqq 2Y} \sum_{W < y \leqq N/x} \tau_x \chi(x)\chi(y)\mu(y) e\left(\frac{a\bar{x}\bar{y}}{b}\right)$$

where $Y \in A$. Thus

$$S_3 = \sum_{Y \in A} S(Y).$$

Since $\tau_x \ll d(x)$, we have by the Cauchy-Schwartz inequality,

$$|S(Y)|^2 \ll \left(\sum_{x \leqq 2Y} d^2(x) \right) \sum_{Y < x \leqq 2Y} \left| \sum_{W < y \leqq N/x} \chi(y)\mu(y) e\left(\frac{a\bar{x}\bar{y}}{b}\right) \right|^2$$

Upon applying Hooley's estimate [4] (see (1.2) of this paper) and the evaluation of Ramanujan sums in the third line of the estimate that follows,

$$\sum_{Y < x \leqq 2Y} \left| \sum_{W < y \leqq N/x} \chi(y)\mu(y) e\left(\frac{a\bar{x}\bar{y}}{b}\right) \right|^2$$

$$\ll \sum_{y \leqq N/Y} \sum_{z \leqq N/Y} \left| \sum_{Y < x \leqq 2Y} e\left(\frac{a\bar{x}(\bar{y} - \bar{z})}{b}\right) \right|$$

$$\ll_\epsilon \sum_{y \leqq N/Y} \sum_{z \leqq N/Y} \frac{Y(b, \bar{y} - \bar{z})}{b} + b^{1/2+\epsilon}(b, \bar{y} - \bar{z})^{1/2}$$

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$$\ll_\epsilon \sum_{y \leqq N/Y} \sum_{k|b} \left(\frac{N}{Yk} + 1\right)\left(\frac{Yk}{b} + b^{1/2+\epsilon}k^{1/2}\right)$$

$$\ll_\epsilon \sum_{y \leqq N/Y} \sum_{k|b} \frac{N}{b} + \frac{N}{Yk^{1/2}} b^{1/2+\epsilon} + \frac{Yk}{b} + b^{1/2+\epsilon} k^{1/2}$$

$$\ll_\epsilon \sum_{y \leqq N/Y} \frac{N}{b} d(b) + \frac{N}{Y} b^{1/2+\epsilon} + Yb^\epsilon + b^{1+\epsilon}$$

$$\ll_\epsilon \frac{N^2}{Yb^{1-\epsilon}} + \frac{N^2}{Y^2} b^{1/2+\epsilon} + Nb^\epsilon + \frac{N}{Y} b^{1+\epsilon}$$

$$= \frac{N^2}{Y} b^\epsilon \left(\frac{1}{b} + \frac{b^{1/2}}{Y} + \frac{Y}{N} + \frac{b}{N}\right)$$

Above, we have used the elementary estimate [3], that for $\alpha \geqq 0$,

$$\sum_{k|b} k^\alpha \ll_\epsilon b^{\alpha+\epsilon}$$

Next applying the estimate [1], page 140,

$$\sum_{x \leqq 2Y} d^2(x) \ll Y(\log Y)^3$$

we have,

$$|S(Y)|^2 \ll_\epsilon N^2 (\log N)^3 b^\epsilon \left(\frac{1}{b} + \frac{b^{1/2}}{Y} + \frac{Y}{N} + \frac{b}{N}\right) \tag{2.1}$$

Hence, we have the estimate for S_3,

$$S_3 \leqq \sum_{Y \in A} |S(Y)|$$

$$\ll_\epsilon N(\log N)^{3/2} b^\epsilon \sum_{Y \in A} \frac{1}{b^{1/2}} + \frac{b^{1/4}}{Y^{1/2}} + \frac{Y^{1/2}}{N^{1/2}} + \frac{b^{1/2}}{N^{1/2}}$$

$$\ll_\epsilon N(\log N)^{3/2} b^\epsilon \left(\frac{\log N}{b^{1/2}} + \frac{b^{1/4}}{W^{1/2}} + \frac{1}{W^{1/2}} + \frac{b^{1/2}}{N^{1/2}} \log N\right)$$

$$\ll_\epsilon N(\log N)^{5/2} b^\epsilon \left(\frac{1}{b^{1/2}} + \frac{b^{1/4}}{W^{1/2}} + \frac{b^{1/2}}{N^{1/2}}\right)$$

Putting the estimates on S_0, S_1, S_2 and S_3 together we have,

$$\sum_{k \leqq N} \mu(k)\chi(k) e\left(\bar{k}\frac{a}{b}\right) \tag{2.2}$$

$$\ll_\epsilon N(\log N)^{5/2} b^\epsilon \left(\frac{1}{b^{1/2}} + \frac{b^{1/4}}{W^{1/2}} + \frac{b^{1/2}}{N^{1/2}}\right) + b^{1/2+\epsilon} W^2 \log W$$

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Let $W = [N^{2/5}(\log N)^{3/5}b^{-1/10}]$. Then,

$$\frac{N(\log N)^{5/2}b^{\epsilon+1/4}}{W^{1/2}} + b^{1/2+\epsilon}W^2 \log W \ll N^{4/5}(\log N)^{11/5}b^{3/10+\epsilon}$$

Thus (2.2) becomes,

$$\sum_{k \leqq N} \mu(k)\chi(k)e\left(\frac{\bar{k}a}{b}\right) \tag{2.3}$$

$$\ll_\epsilon N(\log N)^{5/2}b^{\epsilon-1/2} + N^{1/2}(\log N)^{5/2}b^{1/2+\epsilon}$$

$$+ N^{4/5}(\log N)^{11/5}b^{3/10+\epsilon}$$

We may assume that $b \leqq N^{2/3}$, otherwise the estimate in (2.3) is trivial. Then,

$$N^{1/2}(\log N)^{5/2}b^{1/2+\epsilon} \leqq N^{4/5}(\log N)^{11/5}b^{3/10+\epsilon}$$

and using this in (2.3) completes the proof.

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