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FINITENESS PROPERTIES OF SOME GROUPS OF LOCAL SIMILARITIES

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Abstract Hughes has defined a class of groups that we call finite similarity structure (FSS) groups. Each FSS group acts on a compact ultrametric space by local similarities. The best-known example is Thompson's group V. Guided by previous work on Thompson's group, we show that many FSS groups are of type F_∞. This generalizes work of Ken Brown from the 1980s.

Keywords: finite similarity structure; Nekrashevych–Röver groups; Thompson's group; type F_∞; ultrametric; zipper action

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1. Introduction

In [**6**], Hughes defined a class of groups that act by homeomorphisms on compact ultrametric spaces. Let X be a compact ultrametric space. A finite similarity structure (FSS) Sim_X on X assigns to each pair of balls $B_1, B_2 \subseteq X$ a finite set $\mathrm{Sim}_X(B_1, B_2)$ of surjective similarities from B_1 to B_2. The sets $\mathrm{Sim}_X(B_1, B_2)$ are required to have certain additional properties, such as closure under compositions and under restrictions to subballs. (A complete list of the required properties appears in Definition 2.5.) Given an FSS, one defines an associated group $\Gamma(\mathrm{Sim}_X)$: it is the group of homeomorphisms of X that locally resemble elements of Sim_X. We will call the groups $\Gamma(\mathrm{Sim}_X)$ FSS groups. Perhaps the best-known example of an FSS group is Thompson's group V. Section 2 contains a review of FSS groups.

Hughes [**6**] proved that all FSS groups have the Haagerup property. His argument even established the stronger conclusion that all FSS groups act properly by isometries on CAT(0) cubical complexes. This greatly extended earlier results [**3**] that showed that V has the Haagerup property.

The results of [**6**] left many open questions about the new class of FSS groups. In this paper, guided by previous work on Thompson's group V and related groups, we will establish a finiteness property for some FSS groups. Brown [**2**] proved that V has type F_∞. It seems natural to expect some more general class of FSS groups to have type F_∞ as well. Our main theorem states a fairly general sufficient condition for an FSS group to have type F_∞. Recall that a group has *type* F_∞ if there is a $K(\Gamma, 1)$-complex having a finite n-skeleton for each $n \geqslant 0$.

Theorem 1.1 (Main Theorem). *Let X be a compact ultrametric space together with an FSS Sim_X that is rich in simple contractions and has at most finitely many Sim_X-equivalence classes of balls of X. If Γ is the FSS group associated with Sim_X, then Γ is of type F_∞.*

This theorem is proved as Theorem 6.5 below. Thompson's group V is covered by the theorem above, and our method of proof can be considered a generalization of Brown's original argument. The strategy can be briefly sketched as follows. We show that every FSS group Γ acts on a certain simplicial complex K, which we call its similarity complex. Under the hypothesis that there are finitely many Sim_X-equivalence classes of balls (see Definition 3.2), we show that the complex K will be filtered by Γ-finite subcomplexes. If the FSS Sim_X is also rich in simple contractions (see Definition 5.11), then one can argue that the connectivity of the Γ-finite subcomplexes tends to infinity. The fact that Γ has type F_∞ then follows from well-established principles. The proof of Theorem 1.1 occupies §§ 3–6.

Section 6 also contains a proof that for an arbitrary FSS group Γ, the similarity complex K is a model for $E_{\mathrm{Fin}} = \underline{E}\Gamma$, the classifying space for proper Γ actions.

2. Groups defined by FSSs

2.1. Review of FSSs

We begin with a review of FSSs on compact ultrametric spaces, as defined in [**6**].

Definition 2.1. An *ultrametric space* is a metric space (X, d) such that $d(x, y) \leqslant \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$.

If (X, d) is a metric space, $x \in X$, and $r > 0$, then $B(x, r) = \{y \in X \mid d(x, y) \leqslant r\}$ denotes the closed ball about x of radius r. In an ultrametric space, closed balls are open sets. In a compact ultrametric space, closed balls are also open balls (perhaps with a different radius). Moreover, in an ultrametric space, if two balls intersect, then one must contain the other.

Throughout this paper, a *ball* in X means a closed ball in X.

Definition 2.2. If $\lambda > 0$, then a map $g\colon X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is a λ-*similarity* provided that $d_Y(gx, gy) = \lambda d_X(x, y)$ for all $x, y \in X$.

Definition 2.3. A homeomorphism $g\colon X \to Y$ between metric spaces is a *local similarity* if, for every $x \in X$, there exists $r, \lambda > 0$ such that g restricts to a surjective λ-similarity $g|\colon B(x, r) \to B(gx, \lambda r)$. In this case, λ is the *similarity modulus of g at x* and we write $\mathrm{sim}(g, x) = \lambda$. A *local similarity embedding* is a local similarity onto its image.

Convention 2.4. For a local similarity g, the similarity modulus $\mathrm{sim}(g,x)$ is uniquely determined by g and x, except for the case in which x is an isolated point of X. In that case, we will always take $\mathrm{sim}(g,x) = 1$. Likewise, if $g\colon X \to Y$ is a map between metric spaces and $X = \{x\}$ is a singleton, then g will only be referred to as a λ-similarity for $\lambda = 1$.

The *group of all local similarities* of a metric space X onto X is denoted by $\mathrm{LS}(X)$ and is a subgroup of the group of self-homeomorphisms on X.

Let (X,d) be a compact ultrametric space. Usually, the metric will not be mentioned explicitly.

Definition 2.5. An *FSS for* X is a function Sim_X that assigns to each ordered pair B_1, B_2 of balls in X a (possibly empty) set $\mathrm{Sim}_X(B_1, B_2)$ of surjective similarities $B_1 \to B_2$ such that, whenever B_1, B_2, B_3 are balls in X, the following properties hold.

(1) (Finiteness) $\mathrm{Sim}_X(B_1, B_2)$ is a finite set.

(2) (Identities) $\mathrm{id}_{B_1} \in \mathrm{Sim}_X(B_1, B_1)$.

(3) (Inverses) If $h \in \mathrm{Sim}_X(B_1, B_2)$, then $h^{-1} \in \mathrm{Sim}_X(B_2, B_1)$.

(4) (Compositions) If $h_1 \in \mathrm{Sim}_X(B_1, B_2)$ and $h_2 \in \mathrm{Sim}_X(B_2, B_3)$, then we have that $h_2 h_1 \in \mathrm{Sim}_X(B_1, B_3)$.

(5) (Restrictions) If $h \in \mathrm{Sim}_X(B_1, B_2)$ and $B_3 \subseteq B_1$, then

$$h \mid B_3 \in \mathrm{Sim}_X(B_3, h(B_3)).$$

In other words, Sim_X is a category whose objects are the balls of X and whose morphisms are finite sets of surjective similarities together with a restriction operation.

Definition 2.6. If B is a ball in X, then an embedding $h\colon B \to X$ is *locally determined by* Sim_X provided that, for every $x \in B$, there exists a ball B' in X such that $x \in B' \subseteq B$, $h(B')$ is a ball in X, and $h \mid B' \in \mathrm{Sim}_X(B', h(B'))$.

Definition 2.7. The *FSS group* $\mathit{\Gamma} = \mathit{\Gamma}(\mathrm{Sim}_X)$ *associated with* Sim_X is the set of all homeomorphisms $h\colon X \to X$ such that h is locally determined by Sim_X.

Properties (2)–(5) of Definition 2.5 imply that $\mathit{\Gamma}(\mathrm{Sim}_X)$ is indeed a group. In fact, it is the maximal subgroup of the homeomorphism group of X consisting of homeomorphisms locally determined by Sim_X. Moreover, $\mathit{\Gamma}(\mathrm{Sim}_X)$ is a subgroup of the group $\mathrm{LS}(X)$.

Definition 2.8. A subgroup of $\mathit{\Gamma}(\mathrm{Sim}_X)$ is said to be a *group locally determined by* Sim_X.

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2.2. Examples of FSS groups

We recall standard alphabet language and notation. An *alphabet* is a non-empty finite set A. Finite (perhaps empty) n-tuples of A are *words*. We typically write a word as a string of letters from A. The set of all words is denoted by A^* and the set of *infinite words* is denoted by A^ω; that is,

$$A^* = \coprod_{n=0}^{\infty} A^n \quad \text{and} \quad A^\omega = \prod_{1}^{\infty} A.$$

The set of non-empty words is denoted by A^+; that is, $A^+ = \coprod_{n=1}^{\infty} A^n$. If $u \in A^*$, then $|u| = n$ means $u \in A^n$. If $u \in A^*$ with $u \neq \emptyset$ and n is a non-negative integer, then $u^n := uu \cdots u$ (n times) $\in A^*$ and $\bar{u} := uuu \cdots \in A^\omega$.

Let T_A be the tree associated with A. The vertex set of T_A is A^*. Two words v, w are connected by an edge if and only if there exists $x \in A$ such that $v = wx$ or $vx = w$. The root of T_A is $\emptyset$. Thus, $A^\omega = \mathrm{Ends}(T_A, \emptyset)$, the end space of the tree T_A with root $\emptyset$, and so comes with a natural ultrametric d making A^ω compact. That is, if $x = x_1x_2x_3 \cdots$ and $y = y_1y_2y_3 \cdots$ are in A^ω, then

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \mathrm{e}^{1-n} & \text{if } n = \min\{k \mid x_k \neq y_k\}. \end{cases}$$

Remark 2.9. The metric balls in A^ω are of the form wA^ω, where $w \in A^*$.

We may assume that A is totally ordered. There is then an induced total order on A^ω, namely, the lexicographic order.

Let $A = \{a_1, a_2, \dots, a_d\}$ and let Σ_d be the symmetric group on A. There is an action of Σ_d on A^* given by $\sigma(x_1 \cdots x_n) = \sigma(x_1) \cdots \sigma(x_n)$; this action induces an action of Σ_d on the tree T_A. Indeed, there is an action of Σ_d on A^ω given by

$$\sigma(x_1x_2x_3\cdots) = \sigma(x_1)\sigma(x_2)\sigma(x_3)\cdots.$$

Notation 2.10. Let H be a subgroup of Σ_d.

Definition 2.11. If $w_1, w_2 \in A^*$, then let $\mathrm{Sim}(w_1A^\omega, w_2A^\omega)$ consist of all homeomorphisms $h\colon w_1A^\omega \to w_2A^\omega$ for which there exists a $\sigma \in H$ such that $h(w_1x) = w_2\sigma(x)$ for all $x \in A^\omega$. Then Sim is the *FSS for A^ω determined by H.*

Remark 2.12. The following are some observations related to Definition 2.11.

(1) Sim is an FSS for A^ω.

(2) The element $\sigma \in H$ is uniquely determined by $h \in \mathrm{Sim}(w_1A^\omega, w_2A^\omega)$.

(3) Even though w_1 and w_2 are not uniquely determined by h, the integer $|w_2| - |w_1|$ is the natural logarithm of the similarity modulus of h at each point of w_1A^ω. Hence, $|w_2| - |w_1|$ is uniquely determined by h. Moreover, h, together with either w_1 or w_2, uniquely determines the other.

(4) If $p, q \in A^*$ are such that $h| \in \mathrm{Sim}(w_1 pA^\omega, w_2 qA^\omega)$, then $h|$ is given by $w_1 px \mapsto w_2 q\sigma(x)$ for all $x \in A^\omega$ and $|p| = |q|$.

(5) $\mathrm{Sim}(w_1 A^\omega, w_2 A^\omega)$ contains the unique order-preserving similarity, which is given by $w_1 x \mapsto w_2 x$ for all $x \in A^\omega$.

Remark 2.13. If $\Gamma = \Gamma(\mathrm{Sim})$ is the FSS group associated with Sim, then Γ is isomorphic to the Nekrashevych–Röver groups $V_d(H)$. See [**6**] for comments about the groups of Nekrashevych [**7**] and Röver [**8**]. For example, note that, in the special case $H = \{1\}$, the group $V_d(H)$ is $G_{d,1}$, which is a Higman–Thompson group.

3. The similarity complex associated with an FSS

Throughout this section, X will denote a non-empty, compact, ultrametric space with an FSS $\mathrm{Sim} = \mathrm{Sim}_X$ on X.

Note that the image of a local similarity embedding $f\colon B \to X$, where B is a ball in X, is a finite union of mutually disjoint balls in X (see [**6**, Lemma 2.4]).

We begin by recalling the zipper as defined in [**6**]. Consider the set

$$\mathcal{S} := \{(f, B) \mid B \text{ is a ball in } X \text{ and } f\colon B \to X \text{ is an embedding locally determined by Sim}\}.$$

Define an equivalence relation on $\mathcal{S}$ by declaring that (f_1, B_1) and (f_2, B_2) are *equivalent* provided that there exists $h \in \mathrm{Sim}(B_1, B_2)$ such that $f_2 h = f_1$ (in particular, $f_1(B_1) = f_2(B_2)$). The verification that this is an equivalence relation requires the Identities, Compositions, and Inverses properties of the similarity structure. Equivalence classes are denoted by $[f, B]$. Let $\mathcal{E}$ be the set of equivalence classes of pairs $(f, B) \in \mathcal{S}$. Thus,

$$\mathcal{E} := \{[f, B] \mid (f, B) \in \mathcal{S}\}.$$

The *zipper* is

$$Z := \{[f, B] \in \mathcal{E} \mid f(B) \text{ is a ball in } X \text{ and } f \in \mathrm{Sim}(B, f(B))\}.$$

Note that an element $[f, B] \in Z$ is uniquely determined by the ball $f(B)$. In fact, $[f, B] = [\mathrm{incl}_{f(B)}, f(B)]$, where $\mathrm{incl}_Y\colon Y \to X$ denotes the inclusion map. Thus,

$$Z = \{[\mathrm{incl}_B, B] \in \mathcal{E} \mid B \text{ is a ball in } X\}.$$

In particular, Z can be identified with the collection of all balls in X.

We now begin the construction of a complex on which Γ acts.

Definition 3.1. Let k be a positive integer. A *pseudo-vertex v of height k* is a set

$$v = \{[f_i, B_i] \mid 1 \leqslant i \leqslant k\},$$

where $[f_i, B_i] \in \mathcal{E}$ for each $i = 1, \ldots, k$ and such that $\{f_i(B_i)\}_{i=1}^k$ is a collection of disjoint subsets of X. The height of v is denoted by $\|v\| = k$. The *image of v* is $\mathrm{im}(v) := \bigcup_{i=1}^k f_i(B_i) \subseteq X$.

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Note that the image of a pseudo-vertex v is well defined. Note also that the set of pseudo-vertices of height 1 is $\{\{[f,B]\} \mid [f,B] \in \mathcal{E}\}$. That is, with a slight abuse of notation, $\mathcal{E}$ is the set of pseudo-vertices of height 1.

Definition 3.2. The Sim-*equivalence class* of a ball B in X is

$$[B] := \{A \subseteq X \mid A \text{ is a ball and } \mathrm{Sim}(A,B) \neq \emptyset\}.$$

The Identities, Inverses, and Compositions properties imply that Sim-equivalence is an equivalence relation on the set of all balls in X.

Definition 3.3. The *second coordinate* of a pseudo-vertex $v = \{[f,B]\}$ of height 1 is the Sim-equivalence class $[B]$. The *set of second coordinates* of a pseudo-vertex $v = \{[f_i,B_i] \mid 1 \leqslant i \leqslant k\}$ of height k is the set $\{[B_i] \mid 1 \leqslant i \leqslant k\}$.

Note that this is well defined; that is, if $[f,B] = [f',B']$, then $[B] = [B']$.

Definition 3.4. A *vertex v of height k* is a pseudo-vertex

$$v = \{[f_i,B_i] \mid 1 \leqslant i \leqslant k\}$$

of height k such that $X = \coprod_{i=1}^{k} f_i(B_i)$, where $\coprod$ denotes disjoint union. The set of all vertices of all heights is denoted by K^0.

Note that a pseudo-vertex v is a vertex if and only if $\mathrm{im}(v) = X$. Note also that every homeomorphism $\gamma\colon X \to X$ locally determined by Sim represents a vertex $[\gamma, X]$ of height 1.

Definition 3.5. A pseudo-vertex v is *positive* if each element of v is in the zipper Z.

Remark 3.6. As noted above, there is a bijection from the zipper Z to the set of balls in X. That bijection induces a bijection from the set of positive vertices to the set of partitions of X into balls. This bijection sends a positive vertex $v = \{[f_i,B_i]\}_{i=1}^{k}$ to the partition $\{f_i(B_i)\}_{i=1}^{k}$. The inverse of this bijection sends a partition $\{B_i\}_{i=1}^{k}$ of X into balls to the positive vertex $\{[\mathrm{incl}_{B_i}, B_i]\}_{i=1}^{k}$.

Definition 3.7. If v is a pseudo-vertex and $[f,B] \in v$ with B containing more than one point, then the *simple expansion of v at* $[f,B]$ is the pseudo-vertex

$$w = \{[g,A] \in v \mid [g,A] \neq [f,B]\} \cup \{[f|A, A] \mid A \text{ is a maximal proper sub-ball of } B\}.$$

Moreover, v is the *simple contraction of w at*

$$\{[f|A, A] \mid A \text{ is a maximal proper sub-ball of } B\}.$$

In this situation, we write $v \nearrow w$ and $w \searrow v$.

If v is a pseudo-vertex and $[f,B] \in v$ with B containing exactly one point (which is to say, B does not contain a proper sub-ball), then the expansion of v at $[f,B]$ is not defined.

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Remark 3.8. If v and w are pseudo-vertices such that $v \nearrow w$, then the following hold.

(1) $\|v\| < \|w\|$.

(2) v is a vertex if and only if w is a vertex.

(3) If v is positive, then w is positive.

Remark 3.9. Simple expansions are well defined in the following sense. If $[f_1, B_1] = [f_2, B_2] \in v$, then

$$\{[f_1|A_1, A_1] \mid A_1 \text{ is a maximal proper sub-ball of } B_1\} \\ = \{[f_2|A_2, A_2] \mid A_2 \text{ is a maximal proper sub-ball of } B_2\}.$$

(This follows from the fact that a surjective similarity $B_1 \to B_2$ carries maximal proper sub-balls of B_1 to maximal proper sub-balls of B_2 and from the Restrictions property of Sim.) The converse need not be true. That is, if w is a pseudo-vertex and $u \subseteq w$, then it might be the case that there is more than one pseudo-vertex that is a simple contraction of w at u. However, if v is a simple contraction of w at u, then u is uniquely determined: if v is also a simple contraction of w at u', then $u = u'$.

Remark 3.10. Let v and w be pseudo-vertices such that $\operatorname{im}(v) \cap \operatorname{im}(w) = \emptyset$. The following observations are immediate.

(1) $v \cup w$ is a pseudo-vertex and $\|v \cup w\| = \|v\| + \|w\|$.

(2) If $v \nearrow v'$, then $\operatorname{im}(v') \cap \operatorname{im}(w) = \emptyset$ and $v \cup w \nearrow v' \cup w$.

(3) If v and w are positive, then so is $v \cup w$.

Definition 3.11. If v and w are pseudo-vertices, then write $v \leqslant w$ if and only if there is a finite sequence of simple expansions $v = v_1 \nearrow v_2 \nearrow \cdots \nearrow v_n = w$. The pseudo-vertex w is an *expansion* of v, and v *expands to* w.

Lemma 3.12. *The set of pseudo-vertices is partially ordered by* $\leqslant$.

Proof. The relation is clearly reflexive. It is antisymmetric because if w is an expansion of v, then $\|v\| < \|w\|$. The relation is transitive because it is defined to be the transitive closure of a reflexive, antisymmetric relation. □

The following remark is an immediate consequence of Remark 3.10 (2) and the definitions.

Remark 3.13. If v, w, v', w' are pseudo-vertices such that $\operatorname{im}(v) \cap \operatorname{im}(w) = \emptyset$, $v \leqslant v'$ and $w \leqslant w'$, then $\operatorname{im}(v') \cap \operatorname{im}(w') = \emptyset$ and $v \cup w \leqslant v' \cup w'$.

Remark 3.14. The only pseudo-vertices that are maximal with respect to $\leqslant$ are those of the form $\{[f_i, B_i] \mid 1 \leqslant i \leqslant k\}$, where B_i is a singleton for each $i = 1, \ldots, k$. In particular, if X has no isolated points, then there are no maximal pseudo-vertices.

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Remark 3.15. If v, w are pseudo-vertices, $v \leqslant w$ and $[f, B] \in w$, then there exists a unique $[g, A] \in v$ such that $f(B) \subseteq g(A)$.

Definition 3.16. If $v = \{[f_i, B_i] \mid 1 \leqslant i \leqslant k\}$ is a pseudo-vertex, then the *complete expansion of* v is the pseudo-vertex

$$\text{expansion}(v) := \{[f_i|A, A] \mid 1 \leqslant i \leqslant k \text{ and } A \text{ is a maximal, proper sub-ball of } B_i, \\ \text{or } A = B \text{ if } B_i \text{ is a singleton}\}.$$

Remark 3.17. If v is a pseudo-vertex of height 1, then $v \nearrow \text{expansion}(v)$. It follows from Remark 3.10 (2) that if $v = \{[f_i, B_i] \mid 1 \leqslant i \leqslant k\}$ is a pseudo-vertex of height k, then

$$v \nearrow \text{expansion}\{[f_1, B_1]\} \cup \{[f_i, B_i] \mid 2 \leqslant i \leqslant k\} \nearrow \cdots \\ \nearrow \bigcup_{i=1}^{k} \text{expansion}\{[f_i, B_i]\} = \text{expansion}(v).$$

In particular, $v \leqslant \text{expansion}(v)$.

Definition 3.18. Let B be a ball in X. Inductively define a sequence $\{\mathcal{B}_i\}_{i=0}^{\infty}$ of partitions of B into sub-balls as follows. First, $\mathcal{B}_0 = \{B\}$. Assuming $i > 0$ and $\mathcal{B}_i$ has been defined, a sub-ball A of B is in $\mathcal{B}_{i+1}$ if and only if there exists a ball $C \in \mathcal{B}_i$ such that A is a maximal proper sub-ball of C, or C is a singleton and $A = C$. The sequence $\{\mathcal{B}_i\}_{i=0}^{\infty}$ is the *ball hierarchy of* B.

Suppose that $(f, B) \in \mathcal{S}$ and let $\{\mathcal{B}_i\}_{i=0}^{\infty}$ be the ball hierarchy of B. Observe that if $i \geqslant 1$ and $A \in \mathcal{B}_i$, then the Restrictions property implies that $(f|A, A) \in \mathcal{S}$. For each $x \in B$, let $D((f, B), x)$ denote the smallest non-negative integer i such that there exists $A \in \mathcal{B}_i$ with $x \in A$, $f(A)$ a ball, and $f|A \in \text{Sim}(A, f(A))$. The integer $D((f, B), x)$ is called the *depth of* (f, B) *at* x. Note that if $y \in A$, then $D((f, B), y) = D((f, B), x)$ (since any two balls are either disjoint or one contains the other). Thus, $D((f, B), \cdot)$ is a locally constant function on X.

Definition 3.19. If $(f, B) \in \mathcal{S}$, then the *depth of* $[f, B] \in \mathcal{E}$ is

$$D[f, B] := \max\{D((f, B), x) \mid x \in B\}.$$

Note that $D[f, B]$ is well defined; that is, it is independent of the representative of $[f, B] \in \mathcal{E}$ in $\mathcal{S}$.

Definition 3.20. If $v = \{[f_i, B_i] \mid 1 \leqslant i \leqslant k\}$ is a pseudo-vertex, then the *depth of* v is

$$\text{depth}(v) := \max\{D[f_i, B_i] \mid 1 \leqslant i \leqslant k\}.$$

Remark 3.21. If v is a pseudo-vertex, then the following hold.

(1) $\text{depth}(v) = 0$ if and only if v is positive.

(2) $\text{depth}(\text{expansion}(v)) \leqslant \text{depth}(v)$, with equality if and only if $\text{depth}(v) = 0$.

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Lemma 3.22. *Every pseudo-vertex expands to a positive pseudo-vertex. In particular, for every vertex $v \in K^0$, there exists a positive vertex w such that $v \leqslant w$.*

Proof. If $k = \operatorname{depth}(v)$, then it follows from Remarks 3.17 and 3.21 that $v := v_0 \leqslant v_1 \leqslant \cdots \leqslant v_k$, where $v_i := \operatorname{expansion}(v_{i-1})$ for $1 \leqslant i \leqslant k$, and that $\operatorname{depth}(v_k) = 0$. Thus, $w := v_k$ is the desired positive pseudo-vertex.

The second statement of the lemma follows from the first, together with the observation that the expansion of a vertex is a vertex. □

Lemma 3.23. *If B is a ball in X and $\mathcal{P}$ is a partition of B into sub-balls, then the positive pseudo-vertex $\{[\operatorname{incl}_B, B]\}$ expands to the positive pseudo-vertex $\{[\operatorname{incl}_A, A] \mid A \in \mathcal{P}\}$.*

Proof. Observe first that if B' is a sub-ball of B, and $B'' \in \mathcal{P}$ is a sub-ball of B', then there is $\mathcal{P}' \subseteq \mathcal{P}$ partitioning B'. The proof of the lemma is by induction on the cardinality of $\mathcal{P}$. If $|\mathcal{P}| = 1$, then $\mathcal{P} = \{B\}$ and there is nothing to prove. Assume $|\mathcal{P}| > 1$ and that the statement is true for partitions of smaller cardinality. Let $\{\mathcal{B}_i\}_{i=0}^{\infty}$ be the ball hierarchy of B and let $N = \max\{i > 0 \mid \mathcal{P} \cap \mathcal{B}_i \neq \emptyset\}$ and choose $C \in \mathcal{P} \cap \mathcal{B}_N$. Note that $C \neq B$. Let D be the smallest sub-ball of B such that $C \neq D$ and $C \subseteq D$. Note that C is a maximal proper sub-ball of D. By the observation above, $\mathcal{P}$ contains a partition $\mathcal{P}_D$ of D. By the definition of N, $\mathcal{P}_D$ is the partition of D into maximal proper sub-balls. Clearly, $C \in \mathcal{P}_D$ and $|\mathcal{P}_D| > 1$. Let $\mathcal{P}' = \mathcal{P} \setminus \mathcal{P}_D \cup \{D\}$. Since $\mathcal{P}'$ is a partition of B by balls and $|\mathcal{P}'| < |\mathcal{P}|$, the inductive assumption implies that $\{[\operatorname{incl}_B, B]\}$ expands to the pseudo-vertex $w = \{[\operatorname{incl}_A, A] \mid A \in \mathcal{P}'\}$. The proof is now complete upon observing that the simple expansion of w at $[\operatorname{incl}_D, D]$ is the pseudo-vertex $\{[\operatorname{incl}_A, A] \mid A \in \mathcal{P}\}$. □

Definition 3.24. The *similarity complex associated with* Sim is the simplicial complex $K = K_{\mathrm{Sim}}$ obtained from $(K^0, \leqslant)$. Thus, an n-simplex of K is an ascending chain $(v_0, v_1, \ldots, v_n)$ of distinct vertices $v_0 < v_1 < \cdots < v_n$.

Note that the vertices of an n-simplex of K are totally ordered by $\leqslant$. Note also that $K \neq \emptyset$ because it contains the positive vertex $\{[\operatorname{id}_X, X]\}$ of height 1.

Proposition 3.25. *The partially ordered set $(K^0, \leqslant)$ is a directed set. Hence, K is contractible.*

Proof. By Lemma 3.22, $(K^0, \leqslant)$ is a directed set if any two positive vertices have an upper bound. If v_1 and v_2 are positive vertices, then there are partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of X into balls such that $v_i = \{[\operatorname{incl}_B, B] \mid B \in \mathcal{P}_i\}$ for $i = 1, 2$. Let $\mathcal{P} = \{B_1 \cap B_2 \mid B_1 \in \mathcal{P}_1, B_2 \in \mathcal{P}_2, \text{ and } B_1 \cap B_2 \neq \emptyset\}$. Thus, $\mathcal{P}$ is a common refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$, and $\mathcal{P}$ is a partition of X into balls. Moreover, $\mathcal{P}$ contains a partition of any ball in $\mathcal{P}_1$ or in $\mathcal{P}_2$. Lemma 3.23 implies that if $i = 1$ or 2 and $B \in \mathcal{P}_i$, then the pseudo-vertex $\{[\operatorname{incl}_B, B]\}$ expands to the pseudo-vertex $\{[\operatorname{incl}_A, A] \mid A \in \mathcal{P} \text{ and } A \subseteq B\}$ for $i = 1, 2$. Remark 3.10 implies that both v_1 and v_2 expand to the vertex $\{[\operatorname{incl}_A, A] \mid A \in \mathcal{P}\}$. This completes the proof of the first statement of the proposition. The second statement follows from the well known fact that the complex obtained from a directed, partially ordered set is contractible (see [**5**, Proposition 9.3.14, pp. 210]). □

Example 3.26. Let $X = \{x_1, \ldots, x_n\}$ be a finite ultrametric space in which the distance between any two distinct points is 1. Note that $\{x_i\}$ (for $i \in \{1, \ldots, n\}$) and X itself are the only balls in X. For a pair of balls $B_1, B_2 \subseteq X$, we define $\mathrm{Sim}_X(B_1, B_2)$ as follows:

(1) $\mathrm{Sim}_X(\{x_i\}, \{x_j\}) = \{\phi_{ij}\}$, where ϕ_{ij} is the only possible map $\phi_{ij} \colon \{x_i\} \to \{x_j\}$;

(2) $\mathrm{Sim}_X(X, X) = \{\mathrm{id}_X\}$.

It is straightforward to check that Sim_X is an FSS, and that $\Gamma(\mathrm{Sim}_X) = \Sigma_X$, the symmetric group on X. There are exactly $n! + 1$ vertices:

- if $\phi \in \Sigma_X$, then $\{[\phi, X]\}$ is a vertex of height 1; since $\mathrm{Sim}_X(X, X) = \{\mathrm{id}_X\}$, $\{[\phi_1, X]\} \neq \{[\phi_2, X]\}$ if $\phi_1 \neq \phi_2$, and so there are $n!$ vertices of this type;
- the remaining vertex is $\{[\phi_{ii}, \{x_i\}] \mid 1 \leqslant i \leqslant n\}$.

Every vertex of the form $\{[\phi, X]\}$ expands to $\{[\phi_{ii}, \{x_i\}]\}$. It follows that K may be identified with the cone on Σ_X; that is,

$$K = (\Sigma_X \times I)/\sim,$$

where I denotes the unit interval and $(\phi_1, t_1) \sim (\phi_2, t_2)$ if $t_1 = t_2 = 0$. The action of $\Gamma(\mathrm{Sim}_X)$ on K under this identification is the same as the natural action of Σ_X on its cone.

On the other hand, we might set $\mathrm{Sim}_X(X, X) = \Sigma_X$ (in place of (2) above). The result is still an FSS. In this case, there are just two vertices, $\{[\mathrm{id}_X, X]\}$ and $\{[\phi_{ii}, \{x_i\}] \mid i \in \{1, \ldots, n\}\}$, and K may be identified with the unit interval. We still have $\Gamma(\mathrm{Sim}_X) = \Sigma_X$, but the action of $\Gamma(\mathrm{Sim}_X)$ on K is now trivial.

Various intermediate constructions are possible, depending on the size of the group $\mathrm{Sim}_X(X, X)$.

Note that up to this point we have not used the Finiteness property of the Sim structure.

4. Local finiteness of the sublevel complexes

We continue to use the same notation as in the previous section. In particular, X denotes a non-empty, compact ultrametric space with an FSS Sim. Moreover, K denotes the similarity complex associated with Sim.

The goal of this section is to filter K by subcomplexes that are locally finite if the set of Sim-equivalence classes of balls in X is assumed to be finite (see Proposition 4.6).

Definition 4.1. For $n \in \mathbb{N}$, the *sublevel complex* $K_{\leqslant n}$ is the subcomplex of K spanned by all vertices of height less than or equal to n.

Lemma 4.2. *Suppose that B is a ball in X, w is a pseudo-vertex, and $P_{w,B}$ denotes the set of all pseudo-vertices v of height 1 such that the second coordinate of v is $[B]$ and such that $v \nearrow w$. Then $P_{w,B}$ is finite.*

Proof. Write $w = \{[f_i, B_i] \mid 1 \leqslant i \leqslant k\}$. We may assume that $P_{w,B}$ is not empty so that there is an element in $P_{w,B}$ of the form $[f, B]$. The fact that $[f, B] \nearrow w$ implies that there are exactly k maximal proper sub-balls of B, say $\hat{B}_1, \ldots, \hat{B}_k$, indexed so that if $\hat{f}_i = f|\hat{B}_i$, then $[\hat{f}_i, \hat{B}_i] = [f_i, B_i]$ for $i = 1, \ldots, k$. Let $\mathcal{S}_{w,B} = \{(g, B) \in \mathcal{S} \mid [g, B] \in P_{w,B}\}$. Since the function $\mathcal{S}_{w,B} \to P_{w,B}$, defined by $(g, B) \mapsto [g, B]$, is surjective, it suffices to show that $\mathcal{S}_{w,B}$ is finite. Let Σ_k be the set of all permutations of $\{1, \ldots, k\}$. The proof will be completed by defining an injection

$$\Psi \colon \mathcal{S}_{w,B} \to \coprod_{\sigma \in \Sigma_k} \prod_{i=1}^{k} \mathrm{Sim}(\hat{B}_i, B_{\sigma(i)}).$$

Given $(g, B) \in \mathcal{S}_{w,B}$, we know that

$$\{[g_i, \hat{B}_i] \mid 1 \leqslant i \leqslant k\} = \{[f_i, B_i] \mid 1 \leqslant i \leqslant k\},$$

where $g_i = g|\hat{B}_i$. It follows that there exists a unique $\sigma \in \Sigma_k$ such that $[g_i, \hat{B}_i] = [f_{\sigma(i)}, B_{\sigma(i)}]$ for $i = 1, \ldots, k$. Thus, $f_{\sigma(i)}^{-1} g_i \in \mathrm{Sim}(\hat{B}_i, B_{\sigma(i)})$ and we can define

$$\Psi(g, B) = (f_{\sigma(1)}^{-1} g_1, \ldots, f_{\sigma(k)}^{-1} g_k) \in \prod_{i=1}^{k} \mathrm{Sim}(\hat{B}_i, B_{\sigma(i)}).$$

To see that Ψ is injective, suppose we have another element $(h, B) \in \mathcal{S}_{w,B}$ and $\Psi(h, B) = \Psi(g, B)$. It follows that $f_{\sigma(i)}^{-1} g_i = f_{\sigma(i)}^{-1} h_i$ for each $i = 1, \ldots, k$, where $h_i = h|\hat{B}_i$. Thus, $g = h$ and $(g, B) = (h, B)$. □

Remark 4.3. Note that the previous argument relied on the Finiteness property of the similarity structure.

Lemma 4.4. *If v is a pseudo-vertex, then v has only finitely many immediate successors.*

Proof. This is clear because v contains only finitely many elements at which a simple expansion may be performed. □

In the next result, we will begin using the assumption that the set of Sim-equivalence classes of balls in X is finite. This assumption will be required for the main result, Theorem 6.5.

Lemma 4.5. *If w is a pseudo-vertex and the set of* Sim*-equivalence classes of balls in X is finite, then w has only finitely many immediate predecessors.*

Proof. An immediate predecessor of w is a pseudo-vertex v such that there is an elementary expansion $v \nearrow w$. Thus, there is a subset $w' \subseteq w$ and a pseudo-vertex $v' \subseteq v$ of height 1 such that $v' \nearrow w'$ and $w \backslash w' = v \backslash v'$. There are only finitely many possibilities for w' (since w has only finitely many subsets). Once w' is fixed, there are only finitely many possibilities for the second coordinate of v' (by the assumption of the finiteness of the set of Sim-equivalence classes of balls). Finally, once w' and the second coordinate of v' are fixed, there are only finitely many possibilities for v' by Lemma 4.2. □

Proposition 4.6. *If the set of* Sim*-equivalence classes of balls in* X *is finite and* $n \in \mathbb{N}$, *then the sublevel complex* $K_{\leqslant n}$ *is locally finite.*

Proof. It follows from Lemmas 4.4 and 4.5 that any vertex v of $K_{\leqslant n}$ is contained in at most finitely many ascending chains of vertices in $K_{\leqslant n}$. That is to say, v is in only finitely many simplices of $K_{\leqslant n}$. □

Remark 4.7. The complex K is usually not locally finite. In fact, the following are equivalent:

(1) K is finite;

(2) K is locally finite;

(3) X is finite.

Proof. If X is not finite, then, since X is compact, there exists a sequence of balls $X = B_1 \supsetneq B_2 \supsetneq B_3 \supsetneq \cdots$ such that B_{i+1} is a maximal proper sub-ball of B_i for each $i \in \mathbb{N}$. Define vertices $v_1 < v_2 < v_3 < \cdots$ inductively as follows. Let $v_1 = \{[\mathrm{incl}_{B_1}, B_1]\}$. If $i > 1$ and v_i has been defined so that $[\mathrm{incl}_{B_i}, B_i] \in v_i$, let v_{i+1} be obtained from v_i by a simple expansion at $[\mathrm{incl}_{B_i}, B_i]$. Then v_1 is a vertex of the simplex spanned by $\{v_1, \ldots, v_n\}$ for every $n \in \mathbb{N}$, showing that K is not locally finite.

On the other hand, if X is finite, then it is rather obvious that K is finite: if X has cardinality n, then there are only finitely many partitions of X and each has cardinality less than or equal to n, there are only finitely many collections of at most n balls, and only a finite number of functions between any two subsets of X. This shows that there are only finitely many vertices of K. □

5. Connectivity of the descending links

We continue to use the same notation as in the previous two sections. In particular, X denotes a non-empty, compact, ultrametric space with an FSS Sim. Moreover, K denotes the similarity complex associated with Sim.

The goal of this section is to prove, under the assumptions in the Main Theorem 1.1, that the descending link of a vertex in K is highly connected depending on the height of the vertex (see Corollary 5.22). The main technical result is Theorem 5.20.

Definition 5.1. A pseudo-vertex v is *contracting* if there exists $[f, B] \in \mathcal{E}$ such that $v = \{[f|A, A] \mid A \text{ is a maximal proper sub-ball of } B\}$.

Note that v is contracting if and only if there exists $[f, B] \in \mathcal{E}$ such that B is not a singleton and $v = \mathrm{expansion}\{[f, B]\}$. Note also that every simple contraction of a vertex v takes place at a subset w of v, where w is a contracting pseudo-vertex.

Definition 5.2. For $1 \leqslant i \leqslant k$, let v_i be pseudo-vertices each obtained by simple contractions of a pseudo-vertex v at contracting pseudo-vertices $w_i \subseteq v$. Then $v_1, \ldots, v_k$ are *obtained from* v *by pairwise disjoint simple contractions* if $w_i \cap w_j = \emptyset$ whenever $i \neq j$.

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We note that, by the final line of Remark 3.9, the property of being obtained from v by pairwise disjoint simple contractions is well defined.

Lemma 5.3. *Suppose v, w and y are pseudo-vertices, $v \leqslant w$, $[f, B] \in v$, and $[f, B] \notin w$. If $\{[f, B]\} \nearrow y$, then $z := (v \setminus \{[f, B]\}) \cup y$ is a pseudo-vertex and $z \leqslant w$.*

Proof. The fact that $f(B) = \mathrm{im}(y)$ implies that z is a pseudo-vertex. Now choose a sequence of simple expansions $v = v_1 \nearrow v_2 \nearrow \cdots \nearrow v_n = w$ and let m be the greatest integer such that $[f, B] \in v_m$. It follows that $v_{m+1} = (v_m \setminus \{[f, B]\}) \cup y$ and $v \setminus \{[f, B]\} \leqslant v_m \setminus \{[f, B]\}$. Thus, $z \leqslant v_{m+1} \leqslant w$. □

A pseudo-vertex $\hat{q}$ is a *maximal lower bound for* $v_1, \ldots, v_k$ if $\hat{q}$ is a lower bound for $v_1, \ldots, v_k$, and if $\hat{q} < q$, then q is not a lower bound for $v_1, \ldots, v_k$. By contrast, $\hat{q}$ is the *greatest lower bound for* $v_1, \ldots, v_k$ if $\hat{q}$ is a lower bound for $v_1, \ldots, v_k$, and if q is another lower bound for $v_1, \ldots, v_k$, then $q \leqslant \hat{q}$. A greatest lower bound is maximal, but the converse need not hold in arbitrary partially ordered sets.

Lemma 5.4. *Let $\hat{q}$ be a maximal lower bound for $v_1, \ldots, v_k$. If $[g, A] \in \bigcap_{i=1}^{k} v_i$, then $[g, A] \in \hat{q}$.*

Proof. Remark 3.15 implies that there exists a unique $[\hat{g}, \hat{A}] \in \hat{q}$ such that $g(A) \subseteq \hat{g}(\hat{A})$. We note that, since $\hat{g}(\hat{A}) \cap g(A) \neq \emptyset$ and $v_1, \ldots, v_k, \hat{q}$ are pseudo-vertices, either $[\hat{g}, \hat{A}] = [g, A]$ or $[\hat{g}, \hat{A}] \notin v_i$ for all $i = 1, \ldots, k$. Let y be such that $[\hat{g}, \hat{A}] \nearrow y$. If $[\hat{g}, \hat{A}] \notin v_i$ for all $i = 1, \ldots, k$, then Lemma 5.3 implies that $q' := y \cup (\hat{q} \setminus \{[\hat{g}, \hat{A}]\}) \leqslant v_i$, for all $i = 1, \ldots, k$. Since $\hat{q} \nearrow q'$, this contradicts maximality of $\hat{q}$. Therefore, $[\hat{g}, \hat{A}] = [g, A]$ and $[g, A] \in \hat{q}$. □

Lemma 5.5. *Let v be a pseudo-vertex containing distinct contracting pseudo-vertices $w_1, \ldots, w_k$ and let v_i be a pseudo-vertex obtained from a simple contraction of v at w_i for $1 \leqslant i \leqslant k$.*

(1) *The pseudo-vertices $v_1, \ldots, v_k$ have a lower bound if and only if $v_1, \ldots, v_k$ are obtained from v by pairwise disjoint simple contractions.*

(2) *If the pseudo-vertices $v_1, \ldots, v_k$ have a lower bound, then they have a greatest lower bound.*

Proof. For notation that will be used throughout the proof, choose $[f_i, B_i] \in \mathcal{E}$ such that $[f_i, B_i] \in v_i$, and if $u_i := \{[f_i, B_i]\}$, then $u_i \subseteq v_i$ and $u_i \nearrow w_i$ for $1 \leqslant i \leqslant k$. Note that $v \setminus w_i = v_i \setminus u_i$ for $1 \leqslant i \leqslant k$.

To prove the 'if' part of the first statement, the assumption is that $w_i \cap w_j = \emptyset$ whenever $i \neq j$. Define a pseudo-vertex

$$\hat{v} = \left[v \setminus \bigcup_{i=1}^{k} w_i \right] \cup \bigcup_{i=1}^{k} u_i.$$

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It follows that $\hat{v} \leqslant v_j$ for $1 \leqslant j \leqslant k$ as is amply illustrated for the case $j = k$:

$$\hat{v} \nearrow \left[v \setminus \bigcup_{i=2}^{k} w_i\right] \cup \bigcup_{i=2}^{k} u_i \nearrow \left[v \setminus \bigcup_{i=3}^{k} w_i\right] \cup \bigcup_{i=3}^{k} u_i \nearrow \cdots \nearrow [v \setminus w_k] \cup u_k = v_k,$$

where the ℓth simple expansion in the sequence above uses $u_\ell \nearrow w_\ell$.

To prove the 'only if' part of the first statement, it suffices to consider the case $k = 2$. Suppose z is a lower bound of v_1 and v_2. The goal is to show $w_1 \cap w_2 = \emptyset$. Suppose on the contrary that there exists $[f, B] \in w_1 \cap w_2$. Since $u_1 \nearrow w_1$ and $u_2 \nearrow w_2$, it follows that there exist maximal proper sub-balls, $\hat{B}_1 \subseteq B_1$ and $\hat{B}_2 \subseteq B_2$, such that $[f_1|\hat{B}_1, \hat{B}_1] = [f, B] = [f_2|\hat{B}_2, \hat{B}_2]$. Since $z \leqslant v$ and $[f, B] \in v$, it follows from Remark 3.15 that there exists a unique $[h, C] \in z$ such that $f(B) \subseteq h(C)$. Now, since $z \leqslant v_i$ and $[f_i, B_i] \in v_i$ $(i = 1, 2)$, it follows from Remark 3.15 that there are unique $[h_i, D_i] \in z$ $(i = 1, 2)$ such that $f_i(B_i) \subseteq h_i(D_i)$ $(i = 1, 2)$. Since $[h_1, D_1], [h_2, D_2], [h, C] \in z$ and z is a pseudo-vertex, we must have that any two of $h_1(D_1)$, $h_2(D_2)$, $h(C)$ are either identical or disjoint. We have $h_i(D_i) \cap h(C) \neq \emptyset$ for $i = 1, 2$, however (since $f(B)$ is a subset of both). It follows that $h_1(D_1) = h_2(D_2) = h(C)$, and so $f_i(B_i) \subseteq h(C)$ for $i = 1, 2$. Since z expands to v_i $(i = 1, 2)$ and $f_i(B_i) \subseteq h(C)$, there exist sub-balls $C_1, C_2 \subseteq C$ such that $[h|_{C_i}, C_i] = [f_i, B_i]$ $(i = 1, 2)$. Since v_i $(i = 1, 2)$ expands to v, there exist sub-balls $\hat{C}_1 \subseteq C_1$ and $\hat{C}_2 \subseteq C_2$ such that $[h|\hat{C}_1, \hat{C}_1] = [f, B] = [h|\hat{C}_2, \hat{C}_2]$. In particular, $h(\hat{C}_1) = f(B) = h(\hat{C}_2)$, from which it follows that $\hat{C}_1 = \hat{C}_2$.

We will now show that $\hat{C}_1$ is a maximal proper sub-ball of C_1. There exists $g \in \mathrm{Sim}(B, \hat{B}_1)$ such that $f_1 g = f$. There exists $\hat{h} \in \mathrm{Sim}(B_1, C_1)$ such that $h\hat{h} = f_1$. Since $\hat{B}_1$ is a maximal proper sub-ball of B_1, $\hat{h}(\hat{B}_1)$ is a maximal proper sub-ball of C_1. Now, $h\hat{h}(\hat{B}_1) = f_1(\hat{B}_1) = fg^{-1}(\hat{B}_1) = f(B) = h(\hat{C}_1)$. Thus, $\hat{h}(\hat{B}_1) = \hat{C}_1$ and $\hat{C}_1$ is a maximal proper sub-ball of C_1 as claimed. Likewise, $\hat{C}_2$ is a maximal proper sub-ball of C_2. Since $\hat{C}_1 = \hat{C}_2$, it follows that $C_1 = C_2$ (in an ultrametric space a ball is a maximal proper sub-ball of at most one ball). Therefore, $[f_1, B_1] = [f_2, B_2]$; that is, $u_1 = u_2$ and $w_1 = w_2$, contradicting the assumption that w_1 and w_2 are distinct.

To prove the second statement, assuming $v_1, \ldots, v_k$ have a lower bound (equivalently, they are obtained from v by pairwise disjoint simple contractions), we will show that the pseudo-vertex $\hat{v}$ defined above is the greatest lower bound of $v_1, \ldots, v_k$. Let $\hat{q}$ be a maximal lower bound for $v_1, \ldots, v_k$. Let $i \in \{1, \ldots, k\}$ be arbitrary. We claim that $u_i \subseteq \hat{q}$. Since $\hat{q} \leqslant v_i$ and $u_i = \{[f_i, B_i]\} \subseteq v_i$, Remark 3.15 implies that there is a unique $[\hat{f}_i, \hat{B}_i] \in \hat{q}$ such that $f_i(B_i) \subseteq \hat{f}_i(\hat{B}_i)$. Suppose, for a contradiction, that $i \neq j$, but $[\hat{f}_i, \hat{B}_i] \in v_j$. Since $w_i \in v_j$, we have

$$\text{expansion}\{u_i\} = \{[f_i|_{B'_l}, B'_l] \mid B'_l \text{ is a maximal proper sub-ball of } B_i\} \subseteq v_j.$$

Clearly, each $f_i(B'_l)$ is a proper subset of $\hat{f}_i(\hat{B}_i)$. Since v_j is a pseudo-vertex and $[f_i|_{B'_l}, B'_l]$, $[\hat{f}_i, \hat{B}_i] \in v_j$, we have a contradiction. Thus, $[\hat{f}_i, \hat{B}_i] \notin v_j$ if $i \neq j$. Now, if $[\hat{f}_i, \hat{B}_i] \notin v_i$, then, by Lemma 5.3, the simple expansion $\tilde{q}_i$ of $\hat{q}$ at $[\hat{f}_i, \hat{B}_i]$ satisfies $\tilde{q}_i \leqslant v_j$ for all $j \in \{1, \ldots, k\}$, violating maximality of $\hat{q}$. Thus, $u_i \subseteq \hat{q}$. It follows that $[f_i, B_i] \in \hat{q}$ for $i = 1, \ldots, k$.

Lemma 5.4 implies that $z \subseteq \hat{q}$. Since $\mathrm{im}(u_1 \cup \cdots \cup u_k \cup z) = \mathrm{im}(\hat{v})$, we must have $\hat{q} = \bigcup_{i=1}^{k} u_i \cup z = \hat{v}$. □

Definition 5.6. Let v be a vertex.

(1) The *descending link of* v, denoted by $\mathrm{lk}_{\downarrow}(v)$, is the subcomplex of K spanned by $\{v' \in K^0 \mid v' < v\}$.

(2) The *complex below* v, denoted by $B(v)$, is the subcomplex of K spanned by $\{v' \in K^0 \mid v' \leqslant v\}$.

Note that the set of vertices of $B(v)$ is a directed set; in fact, it has a greatest element v. Thus, $B(v)$ is contractible.

Definition 5.7. The *nerve complex associated with a pseudo-vertex* v, denoted by $\mathcal{N}_v$, is the abstract simplicial complex of which a vertex is a pseudo-vertex obtained from v by a simple contraction and a k-simplex is a set of the form $\{v_0, \dots, v_k\}$, where $v_0, \dots, v_k$ are pseudo-vertices obtained from v by pairwise disjoint simple contractions.

Remark 5.8. The reason for the nerve terminology is the following alternative interpretation of $\mathcal{N}_v$ for the case in which v is a vertex. Recall that in general, if $\mathcal{U}$ is a cover of a space, then the *nerve of* $\mathcal{U}$ is the simplicial complex, denoted by $N(\mathcal{U})$, whose vertices are the elements of $\mathcal{U}$ and such that a collection $\{U_0, \dots, U_n\}$ of vertices spans an n-simplex of $N(\mathcal{U})$ if and only if $\bigcap_{i=0}^{n} U_i \neq \emptyset$. Let $v_1, \dots, v_n$ be the complete list of distinct vertices that can be obtained from v by simple contractions; that is, $v_i \nearrow v$ for $1 \leqslant i \leqslant n$. Note that $\mathcal{U} = \{B(v_1), \dots, B(v_n)\}$ is a cover of $\mathrm{lk}_{\downarrow}(v)$ by subcomplexes. Moreover, a k-element subset $\{B(v_{i_1}), \dots, B(v_{i_k})\}$ of $\mathcal{U}$ has a non-empty intersection if and only if $v_{i_1}, \dots, v_{i_k}$ have a lower bound. By Lemma 5.5, this means that $\{B(v_{i_1}), \dots, B(v_{i_k})\}$ has a non-empty intersection if and only if $v_{i_1}, \dots, v_{i_k}$ are obtained from v by pairwise disjoint simple contractions. Therefore, $\mathcal{N}_v$ is the nerve of the cover $\mathcal{U}$.

Proposition 5.9. *If v is a vertex, then $\mathrm{lk}_{\downarrow}(v)$ is homotopy equivalent to $\mathcal{N}_v$.*

Proof. Let $v_1, \dots, v_n$ be the complete list of distinct vertices that can be obtained from v by simple contractions. Using the alternative interpretation of $\mathcal{N}_v$ in Remark 5.8 and a standard fact about nerves of covers (which may be found in [**5**, Proposition 9.3.20]), it suffices to show that $\bigcap_{j=1}^{k} B(v_{i_j})$ is contractible whenever it is non-empty. The intersection is non-empty precisely when the vertices $v_{i_1}, \dots, v_{i_k}$ have a lower bound. In that case, Lemma 5.5 implies that the vertices have a greatest lower bound. That is to say, $\bigcap_{j=1}^{k} B(v_{i_j})^0$ has a greatest element, and in particular, it is a directed set. Therefore, the intersection $\bigcap_{j=1}^{k} B(v_{i_j})$ is contractible. □

Recall that a simplicial complex M is a *flag complex* if every finite subset of vertices of M that is pairwise joined by edges spans a simplex.

Lemma 5.10. *If v is a pseudo-vertex, then $\mathcal{N}_v$ is a flag complex.*

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Proof. Let $v_0, \ldots, v_k$ be vertices of $\mathcal{N}_v$ such that any pair spans a 1-simplex of $\mathcal{N}_v$. Thus, $v_0, \ldots, v_k$ are pseudo-vertices obtained from v by pairwise disjoint simple contractions. That is to say, $\{v_0, \ldots, v_k\}$ is a k-simplex of $\mathcal{N}_v$. □

We will need to assume the following property in order to establish our main finiteness result, Theorem 1.1.

Definition 5.11. The space X together with Sim is *rich in simple contractions* if there exists a constant $C_0 > 0$ such that, if $k \geqslant C_0$ and v is a pseudo-vertex of height k, there exists a pseudo-vertex $w \subseteq v$ with $\|w\| > 1$ and a simple contraction of v at w.

Note that the condition $\|w\| > 1$ in the definition above is redundant because it is implied by the definition of a simple contraction.

The property of Definition 5.11 is the one that we will need in our proof; however, the following property, which is a bit more cumbersome to state, is easier to verify and implies rich in simple contractions.

Definition 5.12. The space X together with Sim is *rich in ball contractions* if there exists a constant $C_0 > 0$ such that if $k \geqslant C_0$ and $(B_1, \ldots, B_k)$ is a k-tuple of balls of X, then there exists a ball $B \subseteq X$ such that if $\mathcal{M}_B := \{A \mid A \text{ is a maximal, proper sub-ball of } B\}$, then $|\mathcal{M}_B| > 1$ and there is an injection $\sigma\colon \mathcal{M}_B \to \{(B_i, i) \mid 1 \leqslant i \leqslant k\}$ such that $[A] = [B_i]$ whenever $\sigma(A) = (B_i, i)$.

Proposition 5.13. *If X together with* Sim *is rich in ball contractions, then it is rich in simple contractions.*

Proof. Let C_0 be the constant given in Definition 5.12; we will show that Definition 5.11 is satisfied with the same constant. Let $v = \{[f_i, B_i] \mid 1 \leqslant i \leqslant k\}$ be a pseudo-vertex of height $k \geqslant C_0$. Let $B \subseteq X$ be a ball such that $|\mathcal{M}_B| > 1$ and there exists an injection $\sigma\colon \mathcal{M}_B \to \{(B_i, i) \mid 1 \leqslant i \leqslant k\}$. Let σ_1 and σ_2 denote the first and second coordinates of σ, respectively; that is, if $\sigma(A) = (B_i, i)$, then $\sigma_1(A) = B_i$ and $\sigma_2(A) = i$. For each $A \in \mathcal{M}_B$, choose $h_A \in \mathrm{Sim}(A, \sigma_1(A))$. Define $f\colon B \to X$ by setting $f|A = f_{\sigma_2(A)} \circ h_A\colon A \to X$ for each $A \in \mathcal{M}_B$. Let $w = \{[f_i, B_i] \mid i \in \mathrm{im}(\sigma_2)\}$. Then $w \subseteq v$ is a pseudo-vertex and $\|w\| > 1$. Define $u = \{[f, B]\} \cup v \setminus w$. Clearly, u is obtained from a simple contraction at w. □

Example 5.14. We let $A = \{a_1, \ldots, a_d\}$ be a finite alphabet, and consider, for arbitrary $H \leqslant \Sigma_d$, the FSS for A^ω from Definition 2.11.

We claim that A^ω, with the given Sim structure, is rich in ball contractions with $C_0 = d$. Suppose $k \geqslant d$ and $(B_1, \ldots, B_k)$ is a k-tuple of balls in A^ω. We can write $(B_1, \ldots, B_k) = (u_1 A^\omega, \ldots, u_k A^\omega)$ for appropriate words $u_1, \ldots, u_k \in A^*$. We consider $\mathcal{M}_{A^\omega} = \{a_i A^\omega \mid a_i \in A\}$. Let $\sigma\colon \mathcal{M}_{A^\omega} \to \{(u_i A^\omega, i) \mid 1 \leqslant i \leqslant k\}$ be defined by $\sigma(a_i A^\omega) = (u_i A^\omega, i)$. This map is injective, and clearly $\mathrm{Sim}(a_i A^\omega, u_i A^\omega) \neq \emptyset$, so $[a_i A^\omega] = [u_i A^\omega]$.

Lemma 5.15. *If the set of* Sim*-equivalence classes of balls in X is finite, then there exists a constant C_1 such that $\|v\| \leqslant C_1$ whenever v is a contracting pseudo-vertex.*

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Proof. Let $[B_1], \dots, [B_n]$ be the set of Sim-equivalence classes of balls in X. Let N_i be the number of maximal, proper sub-balls of B_i for $1 \leqslant i \leqslant n$. Define $C_1 := \max\{N_i \mid 1 \leqslant i \leqslant n\}$. If v is a contracting pseudo-vertex, then there exist $i \in \{1, \dots, n\}$ and $[f, B_i] \in \mathcal{E}$ such that $v = \text{expansion}\{[f, B_i]\}$. Thus, $\|v\| \leqslant N_i$. □

Hypothesis 5.16. The following two conditions are satisfied.

(1) There exist at most finitely many Sim-equivalence classes of balls of X, and $C_1 > 0$ is the constant given by Lemma 5.15.

(2) The space X together with Sim is rich in simple contractions and $C_0 > 0$ is the constant in Definition 5.11.

For the proof of Theorem 5.20 we need the following three results concerning connectivity in simplicial complexes.

Recall that the *star* of a vertex v in a simplicial complex M, denoted by $\mathrm{st}(v, M)$, or $\mathrm{st}(v)$ if M is understood, is the subcomplex of M consisting of all the simplices containing v, together with the faces of these simplices. The *link* of a vertex v in a simplicial complex M, denoted by $\mathrm{l}(v, M)$ or $\mathrm{l}(v)$, consists of all simplices in $\mathrm{st}(v, M)$ that do not contain v.

A reference for the following well known result is [**1**, Theorem 10.6, pp. 1850].

Theorem 5.17 (Nerve Theorem). *Let M be a simplicial complex and let $\{M_i\}_{i\in I}$ be a family of subcomplexes such that $M = \bigcup_{i\in I} M_i$. If every non-empty intersection $M_{i_1} \cap \cdots \cap M_{i_t}$ is $(k - t + 1)$-connected, then M is k-connected if and only if the nerve of the cover $\{M_i\}_{i\in I}$ is k-connected.*

Lemma 5.18. *Suppose $v_1, \dots, v_n$ are vertices in a flag complex M. If*

$$\bigcap_{i=1}^{n} \mathrm{st}(v_i, M) \neq \emptyset \quad \textit{but} \quad \bigcap_{i=1}^{n} \mathrm{l}(v_i, M) = \emptyset,$$

then $\bigcap_{i=1}^{n} \mathrm{st}(v_i, M)$ is a simplex.

Proof. By the flag property, it suffices to show that any two vertices of $\bigcap_{i=1}^{n} \mathrm{st}(v_i, M)$ are adjacent. If u, w are vertices of $\bigcap_{i=1}^{n} \mathrm{st}(v_i, M)$, then, since the intersection of the links is empty, $u, w \in \{v_1, \dots, v_n\}$. It follows that $w \in \mathrm{st}(u, M)$, which is to say, u and w are adjacent. □

The following result is due to Farley [**4**, Lemma 6]. We only require the second item; however, we state both parts in order to clarify the statement in [**4**].

Lemma 5.19 (Farley). *Let M be a non-empty finite flag complex.*

(1) *Assume that $k \geqslant 0$ and for any collection S of vertices of M such that $|S| \geqslant 2$,*

$$\bigcap_{v\in S} \mathrm{l}(v) \textit{ is } (k - |S| + 1)\textit{-connected.}$$

Then M is k-connected.

(2) *Assume that $n \geqslant -1$. If S is any collection of vertices of M and $\bigcap_{v\in S} \mathrm{l}(v)$ is n-connected, then so is $\bigcap_{v\in S} \mathrm{st}(v)$.*

We are now ready for the main technical result of this section.

Theorem 5.20. *If Hypothesis 5.16 is satisfied, v is a pseudo-vertex, $k \geqslant -1$ is an integer, and*

$$\|v\| \geqslant (2k+2)C_1 + C_0,$$

then $\mathcal{N}_v$ is k-connected.

Proof. The proof is by induction on k. We begin with the case $k = -1$. Then $\|v\| \geqslant C_0$. Thus, there exists a pseudo-vertex $w \subseteq v$ and a simple contraction of v at w. Let v_1 be a pseudo-vertex resulting from such a simple contraction. Hence, v_1 is a vertex of $\mathcal{N}_v$; that is, $\mathcal{N}_v \neq \emptyset$, which is to say, $\mathcal{N}_v$ is (-1)-connected.

Now consider the case $k = 0$; then $\|v\| \geqslant 2C_1 + C_0$. To show that $\mathcal{N}_v$ is 0-connected, let v_1, v_2 be vertices of $\mathcal{N}_v$. Thus, there exist pseudo-vertices $w_1, w_2 \subseteq v$ such that v_i is obtained from a simple contraction of v at w_i for $i = 1, 2$. Thus, w_1, w_2 are contracting pseudo-vertices and $\|w_i\| \leqslant C_1$ for $i = 1, 2$. Hence, $\|v \setminus (w_1 \cup w_2)\| \geqslant C_0$ and so there is a pseudo-vertex $w \subseteq v \setminus (w_1 \cup w_2)$ and a pseudo-vertex v_3' resulting from a simple contraction of $v \setminus (w_1 \cup w_2)$ at w. It follows that $v_3 := v_3' \cup w_1 \cup w_2$ is a pseudo-vertex such that $v_3 \nearrow v$. Therefore, since $w_1 \cap w = \emptyset = w_2 \cap w$, $\{v_1, v_3\}$ and $\{v_2, v_3\}$ are 1-simplices of $\mathcal{N}_v$ showing that v_1 and v_2 are in the same component.

Now suppose that $k > 0$ and that the nerve complex $\mathcal{N}_w$ is ℓ-connected whenever w is a pseudo-vertex, $-1 \leqslant \ell < k$ and $\|w\| \geqslant (2\ell + 2)C_1 + C_0$. We continue to let v be a pseudo-vertex with $\|v\| \geqslant (2k+2)C_1 + C_0$. We will show that $\mathcal{N}_v$ is k-connected by appealing to the Nerve Theorem 5.17. Let $v_1, \ldots, v_n$ be the distinct pseudo-vertices obtained from v by simple contractions (since $\|v\| \geqslant C_0$, $n \geqslant 1$). Thus, $v_1, \ldots, v_n$ are the vertices of $\mathcal{N}_v$ and $\mathcal{N}_v = \bigcup_{i=1}^n \mathrm{st}(v_i, \mathcal{N}_v)$. To apply the Nerve Theorem 5.17, we must verify the following two items.

(1) If $\emptyset \neq \{i_1, \ldots, i_t\} \subseteq \{1, \ldots, n\}$ and $S := \mathrm{st}(v_{i_1}, \mathcal{N}_v) \cap \cdots \cap \mathrm{st}(v_{i_t}, \mathcal{N}_v) \neq \emptyset$, then S is $(k - t + 1)$-connected.

(2) The nerve of the cover $\{\mathrm{st}(v_i, \mathcal{N}_v)\}_{i=1}^n$ is k-connected.

We begin by introducing some notation. For $1 \leqslant i \leqslant n$ there is $w_i \subseteq v$ such that v_i is obtained from v by a simple contraction at w_i. By the choice of constants, $\|w_i\| \leqslant C_1$ for $1 \leqslant i \leqslant n$.

We now begin the verification of item (1). If $t = 1$, then S is a star, which is contractible. Now assume that $t \geqslant 2$. If $\mathrm{l}(v_{i_1}, \mathcal{N}_v) \cap \cdots \cap \mathrm{l}(v_{i_t}, \mathcal{N}_v) = \emptyset$, then Lemma 5.18 implies that S is a simplex. Hence, we may assume that $\mathrm{l}(v_{i_1}, \mathcal{N}_v) \cap \cdots \cap \mathrm{l}(v_{i_t}, \mathcal{N}_v) \neq \emptyset$. Lemma 5.19 (2) implies that it suffices to show that $\mathrm{l}(v_{i_1}, \mathcal{N}_v) \cap \cdots \cap \mathrm{l}(v_{i_t}, \mathcal{N}_v)$ is $(k-t+1)$-connected. If $t \geqslant k+2$, then $-1 \geqslant k-t+1$, and there is nothing to prove (since $\mathrm{l}(v_{i_1}, \mathcal{N}_v) \cap \cdots \cap \mathrm{l}(v_{i_t}, \mathcal{N}_v) \neq \emptyset$ by hypothesis). Thus, we may assume that $t \leqslant k+1$. Define $u := v \setminus (w_{i_1} \cup \cdots \cup w_{i_t})$ and estimate the height of u:

$$\|u\| \geqslant (2k+2)C_1 + C_0 - tC_1 = (2k - t + 2)C_1 + C_0.$$

Since $t \leqslant k + 1$, $\|u\| \geqslant C_0$ and $\mathcal{N}_u \neq \emptyset$.

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We now show that $\mathrm{l}(v_{i_1}, \mathcal{N}_v) \cap \cdots \cap \mathrm{l}(v_{i_t}, \mathcal{N}_v)$ is isomorphic to $\mathcal{N}_u$. We begin by showing that $\mathcal{N}_u$ is isomorphic to a subcomplex $\mathcal{N}'_u$ of $\mathcal{N}_v$. If y is a pseudo-vertex obtained from u by a simple contraction at $w \subseteq u$, then let $y' := y \cup (v \setminus u)$. Thus, $y' \nearrow v$, and so y' is a vertex of $\mathcal{N}_v$. Define a simplical map $\mathcal{N}_u \to \mathcal{N}_v$ by $y \mapsto y'$. This induces an isomorphism of $\mathcal{N}_u$ onto its image $\mathcal{N}'_u$. We now show that $\mathrm{l}(v_{i_1}, \mathcal{N}_v) \cap \cdots \cap \mathrm{l}(v_{i_t}, \mathcal{N}_v) = \mathcal{N}'_u$. Let v_m be obtained from v by a simple contraction at w_m. For $1 \leqslant j \leqslant t$, the vertex v_m of $\mathcal{N}_v$ is in $\mathrm{l}(v_{i_j}, \mathcal{N}_v)$ if and only if $\{v_{i_j}, v_m\}$ is a 1-simplex of $\mathcal{N}_v$. Thus, v_m is in $\mathrm{l}(v_{i_j}, \mathcal{N}_v)$ if and only if v_{i_j} and v_m are obtained from v by disjoint simple contractions. Therefore, $v_m \in \mathrm{l}(v_{i_1}, \mathcal{N}_v) \cap \cdots \cap \mathrm{l}(v_{i_t}, \mathcal{N}_v)$ if and only if $w_m \subseteq u$. It follows that $\mathcal{N}'_u$ and $\mathrm{l}(v_{i_1}, \mathcal{N}_v) \cap \cdots \cap \mathrm{l}(v_{i_t}, \mathcal{N}_v)$ are both subcomplexes of $\mathcal{N}_v$ with the same sets of vertices. Since they are both full subcomplexes, they are equal. (Recall that a subcomplex A of a complex B is *full* if A is the largest subcomplex of B having A^0 as its 0-skeleton. It is obvious that $\mathcal{N}'_u$ is a full subcomplex of $\mathcal{N}_v$. In general, the link of a vertex in a flag complex is a full subcomplex. Moreover, intersections of full subcomplexes are full.)

Define $\ell := k - t + 1$. To finish the verification of item (1), we need to show that $\mathcal{N}_u$ is ℓ-connected. Since $t \geqslant 2$, we have $\ell < k$. Therefore, we will be able to invoke the inductive hypothesis to conclude that $\mathcal{N}_u$ is ℓ-connected if it is true that $\|u\| \geqslant (2\ell + 2)C_1 + C_0$. We continue from the estimate above:

$$\begin{aligned}\|u\| &\geqslant (2k - t + 2)C_1 + C_0\\ &= (2(\ell + t - 1) - t + 2)C_1 + C_0\\ &= (2\ell + t)C_1 + C_0\\ &\geqslant (2\ell + 2)C_1 + C_0,\end{aligned}$$

which completes the verification of item (1).

For the verification of item (2), let M denote the nerve of the cover $\{\mathrm{st}(v_i, \mathcal{N}_v)\}_{i=1}^n$ of $\mathcal{N}_v$. To show that M is k-connected, it suffices to prove that the $(k+1)$-skeleton of M is isomorphic to the $(k+1)$-skeleton of the n-simplex. Thus, we need to show that if $1 \leqslant t \leqslant k+2$, then any collection of t vertices of M spans a $(t-1)$-simplex in M. To this end, let $\emptyset \neq \{i_1, \ldots, i_t\} \subseteq \{1, \ldots, n\}$, $1 \leqslant t \leqslant k+2$, and show that $S := \mathrm{st}(v_{i_1}, \mathcal{N}_v) \cap \cdots \cap \mathrm{st}(v_{i_t}, \mathcal{N}_v)$ is non-empty. As above, let $u := v \setminus (w_{i_1} \cup \cdots \cup w_{i_t})$. The estimate of the height if u (using $1 \leqslant t \leqslant k+2$) is

$$\|u\| \geqslant (2k+2)C_1 + C_0 - tC_1 = (2k - t + 2)C_1 + C_0 \geqslant (k+1)C_1 + C_0 \geqslant C_0.$$

This implies that there exists a pseudo-vertex $w \subseteq u$ and a simple contraction of u at w. Let y be the resulting pseudo-vertex. Thus, $y \nearrow u$ and $y \cup (w_{i_1} \cup \cdots \cup w_{i_t}) \nearrow v$. Hence, $\hat{y} := y \cup (w_{i_1} \cup \cdots \cup w_{i_t})$ is a vertex of $\mathcal{N}_v$. Since w is disjoint from $w_{i_1} \cup \cdots \cup w_{i_t}$, it follows that, for $1 \leqslant j \leqslant t$, $\{v_{i_j}, \hat{y}\}$ is a 1-simplex of $\mathcal{N}_v$ and, in particular, $\hat{y} \in \mathrm{st}(v_{i_j}, \mathcal{N}_v)$. Thus, $\hat{y} \in S$ and $S \neq \emptyset$, as desired. □

Corollary 5.21. *Suppose Hypothesis 5.16 is satisfied. If v is a vertex of K such that*

$$\|v\| \geqslant (2k+2)C_1 + C_0,$$

then $\mathrm{l}_{\downarrow} v$ is k-connected.

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Corollary 5.22. *Suppose Hypothesis 5.16 is satisfied. There is a function* $f\colon \mathbb{N} \to \mathbb{N} \cup \{0, -1\}$ *such that* $\lim_{n\to\infty} f(n) = \infty$, *and if* v *is a vertex, then* $\mathrm{l}_{\downarrow} v$ *is* $f(\|v\|)$*-connected.*

6. The zipper action of an FSS group on the similarity complex

Throughout this section, X will denote a compact ultrametric space with an FSS $\mathrm{Sim} = \mathrm{Sim}_X$ and $\Gamma = \Gamma(\mathrm{Sim})$ will be the FSS group associated with Sim.

The goal of this section is to define an action of Γ on the similarity complex and use this action, together with Brown's finiteness criterion [**2**], to prove the Main Theorem 1.1 (see Theorem 6.5). We also show that the similarity complex K is a model for $\underline{E}\Gamma$, the classifying space for proper Γ actions (see Proposition 6.11).

We begin by recalling the action of Γ on $\mathcal{E}$ as defined in [**6**]. The *zipper action* is the left action $\Gamma \curvearrowright \mathcal{E}$ defined by $\gamma[f, B] = [\gamma f, B]$. The fact that $[\gamma f, B] \in \mathcal{E}$ follows from the Compositions and Restrictions properties of the similarity structure.

Remark 6.1. The zipper action $\Gamma \curvearrowright \mathcal{E}$ extends to an action of Γ on the set of all pseudo-vertices as follows. If $\gamma \in \Gamma$ and $v = \{[f_i, B_i] \mid 1 \leqslant i \leqslant k\}$ is a pseudo-vertex, then $\gamma v := \{[\gamma f_i, B_i] \mid 1 \leqslant i \leqslant k\}$. The following facts are easily verified.

(1) Height is Γ-invariant; that is, if $\gamma \in \Gamma$ and v is a pseudo-vertex, then $\|\gamma v\| = \|v\|$.

(2) K^0 is Γ-invariant; that is, if $g \in \Gamma$ and v is a vertex, then gv is a vertex.

(3) If $v \nearrow w$, where v and w are pseudo-vertices and $\gamma \in \Gamma$, then $\gamma v \nearrow \gamma w$.

(4) If $\gamma \in \Gamma$ permutes the vertices of an n-simplex Δ of K, then γ fixes each vertex of Δ.

It follows that the partial order on pseudo-vertices is preserved by the Γ-action. Hence, there is an induced simplicial action $\Gamma \curvearrowright K$. Each of the actions of Γ on pseudo-vertices, on vertices and on K are called the *zipper action*.

The next task is to characterize orbits under the zipper action.

Lemma 6.2. *Let* $v = \{[f_i, B_i] \mid 1 \leqslant i \leqslant k\}$ *be a pseudo-vertex. If the pseudo-vertex* $w = \{[\hat{f}_i, \hat{B}_i] \mid 1 \leqslant i \leqslant \ell\}$ *is in the* Γ*-orbit of* v*, then* $k = \ell$ *and there exists a permutation* σ *of* $\{1, \ldots, k\}$ *such that* $\mathrm{Sim}(B_i, \hat{B}_{\sigma(i)}) \neq \emptyset$ *for* $i = 1, \ldots, k$*. If* v *is a vertex, then the converse holds.*

Proof. Assume first that $w = \gamma v$ for some $\gamma \in \Gamma$. The fact that height is Γ-invariant (see Remark 6.1) implies that $k = \ell$. Since the sets $\gamma v = \{[\gamma f_i, B_i] \mid 1 \leqslant i \leqslant k\}$ and $w = \{[\hat{f}_i, \hat{B}_i] \mid 1 \leqslant i \leqslant k\}$ are equal, there exists a permutation σ such that $[\gamma f_i, B_i] = [\hat{f}_{\sigma(i)}, \hat{B}_{\sigma(i)}]$ for each $i = 1, \ldots, k$. The definition of the equivalence relation immediately implies that $\mathrm{Sim}(B_i, \hat{B}_{\sigma(i)}) \neq \emptyset$ for $i = 1, \ldots, k$.

Conversely, if v is a vertex, choose $h_i \in \mathrm{Sim}(B_i, \hat{B}_{\sigma(i)})$ for each $i = 1, \ldots, k$. Define $\gamma \in \Gamma$ by $\gamma|f_i(B_i) = \hat{f}_{\sigma(i)} h_i f_i^{-1}\colon f_i(B_i) \to \hat{f}_{\sigma(i)}(B_{\sigma(i)})$. Since v and w are vertices, $X = \coprod_{i=1}^{k} f_i(B_i) = \coprod_{i=1}^{k} \hat{f}_{\sigma(i)}(\hat{B}_{\sigma(i)})$ and so γ is a homeomorphism on X. Since Γ is the maximal group of homeomorphisms locally determined by Sim, it follows that $\gamma \in \Gamma$. Clearly, $\gamma v = w$. □

We next show that the zipper action has finite vertex stabilizers.

Lemma 6.3. *The isotropy group of any vertex of K under the zipper action is a finite subgroup of Γ.*

Proof. Let $v = \{[f_i, B_i] \mid 1 \leqslant i \leqslant k\}$ be a vertex of height k, where representatives $(f_i, B_i) \in \mathcal{S}$, $1 \leqslant i \leqslant k$, have been chosen for each member of v. Let Γ_v be the isotropy subgroup of Γ fixing v. Let Σ_k be the set of permutations of $\{1, \ldots, k\}$. The proof will be completed by defining an injection

$$\Psi\colon \Gamma_v \to \coprod_{\sigma \in \Sigma_k} \prod_{i=1}^{k} \mathrm{Sim}(B_i, B_{\sigma(i)}).$$

Given that $\gamma \in \Gamma_v$, $v = \gamma v = \{[\gamma f_i, B_i] \mid 1 \leqslant i \leqslant k\}$ implies that there exists a unique $\sigma \in \Sigma_k$ such that $[\gamma f_i, B_i] = [f_{\sigma(i)}, B_{\sigma(i)}]$ for $1 \leqslant i \leqslant k$. It follows that $f_{\sigma(i)}^{-1} \gamma f_i \in \mathrm{Sim}(B_i, B_{\sigma(i)})$ for $1 \leqslant i \leqslant k$. Define

$$\Psi(\gamma) = (f_{\sigma(1)}^{-1} \gamma f_1, \ldots, f_{\sigma(k)}^{-1} \gamma f_k) \in \prod_{i=1}^{k} \mathrm{Sim}(B_i, B_{\sigma(i)}).$$

To see that Ψ is injective, suppose that we are given another element $\beta \in \Gamma_k$ and $\Psi(\gamma) = \Psi(\beta)$. It follows that $\gamma | f_i(B_i) = \beta | f_i(B_i)$ for $1 \leqslant i \leqslant k$. Thus, $\gamma = \beta$ since $X = \coprod_{i=1}^{k} f_i(B_i)$. □

We next show that the zipper action restricted to sublevel sets is cocompact if the set of Sim-equivalence classes of balls in X is finite.

Proposition 6.4. *If the set of* Sim*-equivalence classes of balls in X is finite and $n \in \mathbb{N}$, then the sublevel set $K_{\leqslant n}$ is Γ-finite; that is, $\Gamma \setminus K_{\leqslant n}$ is a finite complex.*

Proof. By Proposition 4.6, it suffices to show that $\Gamma \setminus K^0_{\leqslant k}$ is finite for each $k = 1, 2, 3, \ldots$. Let $[B_1], \ldots, [B_l]$ be the distinct Sim-equivalence classes of balls in X. For a vertex $v \in K^0$, define

$$n_{[B_i]}(v) = |\{[\hat{f}, \hat{B}] \in v \mid [\hat{B}] = [B_i]\}|.$$

Let $\tilde{\phi}\colon K^0 \to \prod_{i=1}^{l} (\mathbb{N} \cup \{0\})$ be defined by $\tilde{\phi} = n_{[B_1]} \times \cdots \times n_{[B_l]}$. By Lemma 6.2, $\tilde{\phi}$ descends to a well-defined injection ϕ on the quotient; that is, $\phi\colon \Gamma \setminus K^0 \to \prod_{i=1}^{l} (\mathbb{N} \cup \{0\})$. Fixing a height k, we get an injection

$$\phi_k\colon \Gamma \setminus K^0_{\leqslant k} \to \prod_{i=1}^{l} (\mathbb{N} \cup \{0\}),$$

where the entries of an element in the image of ϕ_k must add up to k. There are $\binom{k+l-1}{k}$ distinct ordered l-tuples of non-negative integers which add up to k, so

$$|\Gamma \setminus K^0_{\leqslant k}| \leqslant \binom{k+l-1}{k}.$$

□

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We can now prove the Main Theorem 1.1, which is restated here.

Theorem 6.5 (Main Theorem). *If Hypothesis 5.16 is satisfied, then Γ is of type F_∞.*

Proof. This is a standard application of Brown's criterion for finiteness. See [**5**, § 7.4] for an exposition, and [**5**, Exercise, pp. 179] for the result we need.

We refer to the original statement from [**2**, Corollary 3.3(a)]. Note that the similarity complex K is a contractible Γ-complex (Proposition 3.25; Remark 6.1), it is filtered by the Γ-finite Γ-complexes $K_{\leqslant n}$ (Proposition 6.4; Remark 6.1 (1)), and the stabilizer of each vertex is finite (Lemma 6.3). The final point to check is that the connectivity of the pair $(K_{\leqslant n+1}, K_{\leqslant n})$ tends to infinity as n tends to infinity. We may assume that there are vertices of height $n+1$, so $K_{\leqslant n+1} \neq K_{\leqslant n}$. The complex $K_{\leqslant n+1}$, up to homotopy, is $K_{\leqslant n}$ with a collection $\{C_i\}_{i\in\mathcal{I}}$ of cones attached along their bases, each of which is homotopy equivalent to the descending link of a vertex of height $n+1$. By Corollary 5.22, the connectivity of such descending links tends to infinity with n, and it follows (from elementary Mayer–Vietoris and van Kampen arguments) that the connectivity of $(K_{\leqslant n+1}, K_{\leqslant n})$ tends to infinity as well. Therefore, Γ has type F_∞.

See [**4**] for an illustration of how to put these ingredients together in a related context. □

Corollary 6.6. *The groups $V_d(H)$ have type F_∞ for all $d \in \mathbb{N}$ and $H \leqslant \Sigma_d$.*

Proof. We fix $A = \{a_1, \dots, a_d\}$. Recall that Σ_d denotes the symmetric group on A. We choose $H \leqslant \Sigma_d$. We equip the space A^ω with the FSS Sim from Definition 2.11. By Remark 2.13, $V_d(H)$ is the FSS group associated with Sim. By Example 5.14, A^ω, with the given FSS, is rich in ball contractions with constant $C_0 = d$. Since there is only one Sim-equivalence class $[B]$ of balls, and B has d maximal proper sub-balls, Hypothesis 5.16 (1) is satisfied with $C_1 = d$. Thus, $V_d(H)$ has type F_∞ by Theorem 6.5. □

Example 6.7. The action $\Gamma \curvearrowright K$ is usually not free. In fact, a vertex $v = \{[f_i, B_i] \mid 1 \leqslant i \leqslant m\}$ will have a non-trivial stabilizer in either of the following cases:

(1) if $[B_i] = [B_j]$ for some $[f_i, B_i] \neq [f_j, B_j]$ with $i, j \in \{1, \dots, m\}$, or

(2) if the group $\mathrm{Sim}_X(B_i, B_i) \neq \{\mathrm{id}_{B_i}\}$ for some $[f_i, B_i] \in v$.

For (1), suppose that $[B_i] = [B_j]$ and $[f_i, B_i] \neq [f_j, B_j]$. We choose $h \in \mathrm{Sim}_X(B_i, B_j)$ and define a homeomorphism $g\colon X \to X$ as follows. If $k \neq i, j$, then $g|_{f_k(B_k)} = \mathrm{id}_{f_k(B_k)}$. We set $g|_{f_i(B_i)} = f_j h f_i^{-1}$ and $g|_{f_j(B_j)} = f_i h^{-1} f_j^{-1}$. These assignments completely determine g on all of X, since $\{f_1(B_1), \dots, f_m(B_m)\}$ is a partition of X. The map $g\colon X \to X$ is continuous since the partition $\{f_1(B_1), \dots, f_m(B_m)\}$ is made up of open (and also, therefore, closed) sets, and g is continuous on each piece. The map g is bijective since it induces a bijection on the partition $\{f_1(B_1), \dots, f_m(B_m)\}$, and g also maps any element of the partition bijectively to another such element. Lastly, g is locally determined by Sim_X since it is locally determined by Sim_X on each piece $f_i(B_i)$, $i = 1, \dots, m$. It follows that $g \in \Gamma(\mathrm{Sim}_X)$. One easily checks that $[gf_k, B_k] = [f_k, B_k]$ for $k \neq i, j$,

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$[gf_i, B_i] = [f_j, B_j]$ and $[gf_j, B_j] = [f_i, B_i]$. Thus $g \cdot v = v$. On the other hand, g is not the identity, since $g(f_i(B_i)) \cap f_i(B_i) = f_j(B_j) \cap f_i(B_i) = \emptyset$.

For (2), suppose that $\psi \in \mathrm{Sim}_X(B_i, B_i)$, where $\psi \neq \mathrm{id}_{B_i}$. We define $g \in \Gamma(\mathrm{Sim}_X)$ such that $g|_{f_k(B_k)} = \mathrm{id}_{f_k(B_k)}$ when $k \neq i$, and $g|_{f_i(B_i)} = f_i \psi f_i^{-1}$. By reasoning similar to that from (1), g is a non-trivial element of $\Gamma(\mathrm{Sim}_X)$, $g \cdot v = v$, and $g \neq \mathrm{id}_X$ since $g|_{f_i(B_i)} = f_i \psi f_i^{-1} \neq \mathrm{id}_{f_i(B_i)}$.

Example 6.8. The quotient $\Gamma \setminus K$ is usually not locally finite. In fact, the following are equivalent:

(1) $\Gamma \setminus K$ is finite.

(2) $\Gamma \setminus K$ is locally finite.

(3) X is finite.

Proof. It is clear that (1) implies (2). If X is finite, then K is finite by Remark 4.7, so $\Gamma \setminus K$ will also be finite. If X is infinite, then the argument from Remark 4.7 shows that there is an infinite chain of vertices $v_0 < v_1 < v_2 < \cdots$. Any two of these vertices are adjacent in K, and at different heights. Since the action of Γ preserves height by Remark 6.1, the vertex v_0 is adjacent to infinitely many vertices in the quotient $\Gamma \setminus K$. Thus $\Gamma \setminus K$ is not locally finite. □

6.1. The similarity complex as a classifying space

We now show that the similarity complex K is a classifying space with finite isotropy; that is, K is a model for $E_{\mathrm{Fin}}\Gamma$, where Γ is the FSS group associated with the given FSS and Fin denotes the family of finite subgroups of Γ.

Definition 6.9. If Γ is any group, then a *family* $\mathcal{F}$ of subgroups of Γ is a non-empty collection of subgroups that is closed under conjugation by elements of Γ and passage to subgroups. If Γ is any group, then we let Fin denote the family of finite groups.

Definition 6.10. Let X be a Γ-CW complex (where 'CW' indicates 'closure-finite weak topology'). Suppose that if $c \subseteq X$ is a cell of X, then $\gamma \cdot c = c$ if and only if γ fixes c pointwise. Let $\mathcal{F}$ be a family of subgroups of Γ. We say that X is an *$E_{\mathcal{F}}\Gamma$-complex* if

(1) X is contractible;

(2) whenever $H \in \mathcal{F}$, the fixed set $\mathrm{Fix}(H) = \{x \in X \mid \gamma \cdot x = x \text{ for all } \gamma \in H\}$ is contractible;

(3) whenever $H \notin \mathcal{F}$, $\mathrm{Fix}(H)$ is empty.

Proposition 6.11. *K is a model for $E_{\mathrm{Fin}} = \underline{E}\Gamma$; that is, the fixed set by the action on K of a subgroup G of Γ is empty if G is infinite, and contractible if G is finite.*

Proof. It follows from Lemma 6.3 that the fixed set of an infinite subgroup of Γ is empty. Assume that G is a finite subgroup of Γ. We first claim that there is a positive

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vertex $\hat{v}$ such that the orbit $G \cdot \hat{v}$ contains only positive vertices. For a vertex v, we let $\text{expansion}^k(v)$ denote the result of applying the expansion function (Definition 3.16) to v k times. By Lemma 3.22, there is, for each vertex $v_g = \{[g, X]\}$ $(g \in G)$, a positive integer n_g such that $\text{expansion}^{n_g}(v_g)$ is positive. Since G is finite, it follows that there is an $N \in \mathbb{N}$ such that $\text{expansion}^N(v_g)$ is positive for all $g \in G$. This immediately implies that the orbit $G \cdot \text{expansion}^N(v_{\text{id}_X})$ consists of positive vertices, proving the claim with $\hat{v} = \text{expansion}^N(v_{\text{id}_X})$.

The usual partial order $\leqslant$ on vertices has the property that any two positive vertices v_1, v_2 have a least upper bound; that is, there is $\tilde{v} \in K^0$ such that $\tilde{v} \geqslant v_1, v_2$, and if $v' \in K^0$ is such that $v' \geqslant v_1, v_2$, then $v' \geqslant \tilde{v}$. (In fact, if $v_1 = \{[\text{incl}_{B_i}, B_i] \mid 1 \leqslant i \leqslant m\}$ and $v_2 = \{[\text{incl}_{\hat{B}_j}, \hat{B}_j] \mid 1 \leqslant j \leqslant n\}$, then $\tilde{v} = \{[\text{incl}_{B_i \cap \hat{B}_j}, B_i \cap \hat{B}_j] \mid B_i \cap \hat{B}_j \neq \emptyset,\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$ is the required vertex.) The least upper bound is necessarily unique.

It follows from an entirely straightforward argument that any finite collection of positive vertices has a (unique) least upper bound in K^0. Now, since $G \cdot \hat{v}$ consists of positive vertices, G must fix the least upper bound of $G \cdot \hat{v}$ by the uniqueness of the least upper bound. Therefore, the fixed set of G is non-empty.

We now show that the fixed set of G is contractible. It is enough to show that the set of fixed vertices is directed. Note that if v, w are vertices, $g \in \Gamma$, $gv = v$, and $\text{expansion}(v) = w$, then $gw = w$. Thus, given vertices v_1, v_2 such that $gv_i = v_i$ $(i = 1, 2,$ $g \in G)$, we can use Lemma 3.22 to find positive vertices v_1', v_2' such that $gv_i' = v_i'$ and $v_i \leqslant v_i'$ $(i = 1, 2,\ g \in G)$. We let $\tilde{v}$ be the least upper bound of $\{v_1', v_2'\}$. Since v_1' and v_2' are fixed by G, $\tilde{v}$ must also be fixed by G due to the uniqueness of the least upper bound. Thus $v_1, v_2 \leqslant \tilde{v}$, and all three vertices are fixed by G, so the set of fixed vertices is directed. □

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