AUTOMORPHISMS OF ENDOMORPHISM MONOIDS OF RELATIVELY FREE BANDS

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Abstract   For a set X and a variety $\mathcal{V}$ of bands, let $B_{\mathcal{V}}(X)$ be the relatively free band in $\mathcal{V}$ on X. For an arbitrary band variety $\mathcal{V}$ and an arbitrary set X, we determine the group of automorphisms of $\mathrm{End}(B_{\mathcal{V}}(X))$, the monoid of endomorphisms of $B_{\mathcal{V}}(X)$.

1. Introduction

In general, given a variety $\mathcal{A}$ of algebras, it is of independent interest to determine the group of automorphisms of $\mathrm{End}(A)$, where A is a free algebra in $\mathcal{A}$ and $\mathrm{End}(A)$ is the monoid of endomorphisms of A. Many years ago, Schreier [22] and Mal'cev [15] solved this problem for unstructured sets (that is, algebras without operations) and Gluskĭn [11] solved it for vector spaces. In recent years, this kind of question has gained in importance and attracted wider attention for its links to universal algebraic geometry. (For a detailed explanation of these links, a list of references and related problems, see the excellent paper by Mashevitzky *et al.* [17].) Given the large number of published papers, preprints, lectures and, especially, open problems that have recently appeared on this subject—prompted by the links to universal algebraic geometry—we can anticipate that, for years to come, many new papers will be written describing the group of automorphisms of $\mathrm{End}(A)$ for various varieties $\mathcal{A}$.

Along this line of research, our aim is to describe the automorphisms of $\mathrm{End}(A)$ for any free object A of any variety $\mathcal{V}$ of bands (idempotent semigroups). Analogous problems dealing with varieties of semigroups were solved by Mashevitzky and Schein [16] for the variety of all monoids and by Formanek [8] for the variety of all groups. However, we solve the problem not only for the variety of all bands but also for all subvarieties of this

variety. In fact, with the very particular exceptions of sets and vector spaces, there is no other variety of algebras for which this problem has been solved for all free objects of all subvarieties.

Varieties of bands and their free objects have been studied by numerous authors. The lattice of all varieties of bands was described by Birjukov [**4**], Fennemore [**6**,**7**] and Gerhard [**9**] (see Figure 1 on p. 15). A simplified system of identities for band varieties (one identity for each variety if one excludes $x^2 = x$) and a solution of the world problem in relatively free bands were given by Gerhard and Petrich [**10**].

For a non-empty set X and a variety $\mathcal{V}$ of bands, denote by $B_\mathcal{V}(X)$ the relatively free band in $\mathcal{V}$ on X and by $\mathrm{End}(B_\mathcal{V}(X))$ the monoid of endomorphisms of $B_\mathcal{V}(X)$. To determine the group $\mathrm{Aut}(\mathrm{End}(B_\mathcal{V}(X)))$, we use Theorem 2.1, which states that, for every band B, the automorphisms of $\mathrm{End}(B)$ are induced by permutations on B. This theorem and other preliminary material is presented in § 2. In § 3, we introduce the concept of a d-invariant relatively free band and determine $\mathrm{Aut}(\mathrm{End}(B_\mathcal{V}(X)))$ for every band variety $\mathcal{V}$ that includes the variety $\mathcal{SL}$ of semilattices. This group is either $S(X) \times C_2$ or $S(X)$, depending on whether or not $B_\mathcal{V}(X)$ is d-invariant. Section 4 deals with the varieties of rectangular bands. It turns out that the variety $\mathcal{V}$ of all rectangular bands is exceptional in the sense that $\mathrm{Aut}(\mathrm{End}(B_\mathcal{V}(X)))$ is not encountered in any other band variety. In § 5, we determine which relatively free bands are d-invariant and which are not. This gives us corollaries that determine $\mathrm{Aut}(\mathrm{End}(B_\mathcal{V}(X)))$ given any band variety $\mathcal{V}$ and any set X. Finally, § 6 provides a number of open problems.

2. Preliminaries

Let S be a semigroup. Any bijection $\phi : S \to S$ such that $(ab)\phi = (a\phi)(b\phi)$ for all $a, b \in S$ is called an *automorphism* of S. (We write mappings on the right, that is, $a\phi$ rather than $\phi(a)$.) The group of automorphisms of S will be denoted by $\mathrm{Aut}(S)$. A subset C of S is called *characteristic in S* if $C\phi = C$ for every $\phi \in \mathrm{Aut}(S)$, where $C\phi = \{a\phi : a \in C\}$.

Let X be an arbitrary non-empty set. The full transformation semigroup $T(X)$ consists of all mappings from X to X (full transformations of X) with composition as the semigroup operation. The semigroup $T(X)$ has the symmetric group $S(X)$ of all permutations of X as its group of units. Let S be a subsemigroup of $T(X)$. An automorphism ϕ of S is called *inner* if there exists a $g \in S(X)$ such that $a\phi = g^{-1}ag$ for every $a \in S$. (In such a case, we write $\phi = \phi_S^g$.) The set $\mathrm{Inn}(S)$ of all inner automorphisms of S is a subgroup of $\mathrm{Aut}(S)$. We denote by G_S the normalizer of S in $S(X)$, that is, G_S denotes the subgroup of the symmetric group $S(X)$ consisting of all permutations g of X such that $g^{-1}Sg = S$. Note that every element $g \in G_S$ induces an inner automorphism ϕ_S^g of S.

A *band* is a semigroup in which every element is an idempotent. Let B be a band. Define a ternary relation ρ on B by

$$\rho = \{(x, y, z) : z = xy\}.$$

Since $x = xx$ for every $x \in B$, the relation ρ is reflexive. It is clear that, for every mapping $a : B \to B$, a is an endomorphism of B ($(xy)a = (xa)(ya)$ for all $x, y \in B$) if and only if

a preserves ρ $((x, y, z) \in \rho \implies (xa, ya, za) \in \rho$ for all $x, y, z \in B)$. Thus, $\mathrm{End}(B)$, the semigroup of endomorphisms of B, consists of all transformations of B that preserve the reflexive relation ρ. It is well known (see [**24**, Theorem 1] or [**3**, Lemma 2.1]) that if S is a semigroup of transformations of X that preserve a given reflexive relation on X, then $\mathrm{Aut}(S) \cong G_S$ (where '$\cong$' means 'is isomorphic to'). Hence, we have the following result.

Theorem 2.1. *Let B be any band. Then*

$$\mathrm{Aut}(\mathrm{End}(B)) \cong G_{\mathrm{End}(B)}.$$

We now introduce some standard material about band varieties in order to clearly establish notation and definitions. A *variety* of semigroups is a class of semigroups closed under subsemigroups, homomorphic images and direct products. Let A be a countably infinite set. Denote by A^+ the free semigroup on A. Let $u, v \in A^+$. We say that a semigroup S satisfies the *identity* $u = v$ if $u\hat{\alpha} = v\hat{\alpha}$ for every $\alpha : A \to S$, where $\hat{\alpha}$ is the unique extension of α to a homomorphism from A^+ to S. Let R be a binary relation on A^+. We denote by $[R]$ the class of all semigroups S such that S satisfies the identity $u = v$ for every $(u, v) \in R$. If R is finite, say $R = \{(u_1, v_1), \ldots, (u_n, v_n)\}$, we write $[u_1 = v_1, \ldots, u_n = v_n]$ for $[R]$. Now, for every binary relation R on A^+, the class $[R]$ is a variety of semigroups. Conversely, for every variety $\mathcal{V}$ of semigroups, there is a binary relation R on A^+ such that $\mathcal{V} = [R]$ (see [**13**, p. 111]). For example, $[xy = yx]$ is the variety of commutative semigroups.

Let $\mathcal{V}$ be a non-trivial (containing a semigroup with at least two elements) variety of semigroups, and let X be a non-empty set. We say that a semigroup $F \in \mathcal{V}$ is a $\mathcal{V}$-*free semigroup on X* (or a *relatively free semigroup in $\mathcal{V}$ on X*) if it satisfies the following properties:

(1) X generates F;

(2) for every $S \in \mathcal{V}$ and every mapping $\phi : X \to S$, there is an extension of ϕ to a homomorphism $\bar{\phi} : F \to S$.

Since X generates F, an extension $\bar{\phi}$ is necessarily unique. For the variety $\mathcal{T}$ of trivial semigroups, we define a $\mathcal{T}$-free semigroup on X to be any trivial semigroup. A $\mathcal{V}$-free semigroup on X is isomorphic to $X^+/\rho_{\mathcal{V}}$, where $\rho_{\mathcal{V}}$ is the congruence on X^+ defined by

$$(u, v) \in \rho_{\mathcal{V}} \iff \text{for every } S \in \mathcal{V} \text{ and every } \alpha : X \to S, \ u\hat{\alpha} = v\hat{\alpha} \qquad (2.1)$$

(see [**13**, p. 110]). If $\mathcal{V}$ is a non-trivial variety, then X is embedded in $X^+/\rho_{\mathcal{V}}$ via the injection $x \to x\rho_{\mathcal{V}}$ (where $x \in X$ and $x\rho_{\mathcal{V}}$ is the congruence class of x modulo $\rho_{\mathcal{V}}$) [**19**, Lemma I.8.5]. Note that, for all varieties $\mathcal{V}_1$ and $\mathcal{V}_2$ of semigroups,

$$\mathcal{V}_1 \subseteq \mathcal{V}_2 \iff \rho_{\mathcal{V}_2} \subseteq \rho_{\mathcal{V}_1}. \qquad (2.2)$$

A variety of bands is a variety of semigroups in which every semigroup is a band. The band varieties form a lattice (under inclusion), with $\mathcal{B} = [x^2 = x]$ as the greatest elements and $\mathcal{T} = [x = y]$ as the least element (see Figure 1).

For a variety $\mathcal{V}$ of bands and a non-empty set X, we denote by $B_{\mathcal{V}}(X)$ the $\mathcal{V}$-free semigroup on X. We will refer to $B_{\mathcal{V}}(X)$ as the *$\mathcal{V}$-free band on X*, and assume that $B_{\mathcal{V}}(X) = X^+/\rho_{\mathcal{V}}$, where $\rho_{\mathcal{V}}$ is the congruence on X^+ defined by (2.1). That is,

$$B_{\mathcal{V}}(X) = \{w\rho_{\mathcal{V}} : w \in X^+\}.$$

We will represent the congruence classes modulo $\rho_{\mathcal{V}}$ (the elements of $B_{\mathcal{V}}(X)$) by their representatives, that is, we will write $w \in B_{\mathcal{V}}(X)$ instead of $w\rho_{\mathcal{V}} \in B_{\mathcal{V}}(X)$. Moreover, for $u, v \in X^+$, we shall write $u = v$ both when $u\rho_{\mathcal{V}} = v\rho_{\mathcal{V}}$ (that is, when u and v are equal as elements of $B_{\mathcal{V}}(X)$) and when u and v are equal as words in X^+. It should always be clear from the context which equality is meant.

Let $\mathcal{V}$ be a non-trivial variety of bands. Since the mapping $x \to x\rho_{\mathcal{V}}$ from X to $B_{\mathcal{V}}(X)$ is injective, we have the following:

$$\text{for all } x, y \in X, \quad x = y \text{ in } B_{\mathcal{V}}(X) \quad \Longleftrightarrow \quad x = y \text{ in } X^+. \tag{2.3}$$

Our objective is to determine the group of automorphisms of $\mathrm{End}(B_{\mathcal{V}}(X))$, where $\mathcal{V}$ is an arbitrary variety of bands and X is an arbitrary non-empty set. Almost all varieties of bands contain the variety of semilattices (see Figure 1). We begin our study with these varieties.

3. Band varieties $\mathcal{V}$ with $\mathcal{SL} \subseteq \mathcal{V}$

A *semilattice* is a commutative band. The class $\mathcal{SL}$ of semilattices is a variety. Throughout this section, we assume that X is a non-empty set and that $\mathcal{V}$ is a variety of bands such that $\mathcal{SL} \subseteq \mathcal{V}$.

Our objective is to prove that $\mathrm{Aut}(\mathrm{End}(B_{\mathcal{V}}(X)))$ is isomorphic to either $S(X)$ or $S(X) \times C_2$, where C_2 is the group with two elements. In view of Theorem 2.1, it suffices to prove the statement for $G_{\mathrm{End}(B)}$, where $B = B_{\mathcal{V}}(X)$. Recall that $G_{\mathrm{End}(B)}$ is the group of permutations $g \in S(B)$ such that $g^{-1} \mathrm{End}(B)g = \mathrm{End}(B)$.

For $w \in X^+$, the *content* $C(w)$ of w is defined as the set of all elements of X (letters) that occur in w. For example, if $w = abacabc$, then $C(w) = \{a, b, c\}$. For the $\mathcal{SL}$-free band $B_{\mathcal{SL}}(X)$, we have the following [**13**, Proposition 4.6.5]:

$$\text{for all } u, v \in X^+, \quad u = v \text{ in } B_{\mathcal{SL}}(X) \quad \Longleftrightarrow \quad C(u) = C(v). \tag{3.1}$$

Since $\mathcal{SL} \subseteq \mathcal{V}$, it follows from (2.2) and (3.1) that

$$\text{for all } u, v \in X^+, \quad u = v \text{ in } B_{\mathcal{V}}(X) \quad \Longrightarrow \quad C(u) = C(v). \tag{3.2}$$

Let $x \in X$. It follows from (3.2) that $x\rho_{\mathcal{V}} = \{x^k : k \geqslant 1\}$, which we shall always represent by x.

Let $B = B_{\mathcal{V}}(X)$. We say that a subset J of B is *independent* if, for all $u, v \in J$, $u \neq v$ (in $B_{\mathcal{V}}(X)$) $\Longrightarrow C(u) \cap C(v) = \emptyset$. (Note that this definition makes sense because of (3.2).) If $J \subseteq B$ is not independent, we say that it is *dependent*.

Lemma 3.1. *Let $B = B_{\mathcal{V}}(X)$ and $g \in G_{\mathrm{End}(B)}$. Then $Xg = X$.*

Proof. Suppose X is finite with, say, n elements. Since g is a bijection, $|Xg| = n$. Observe that if $J \subseteq B$ is dependent and $a \in \text{End}(B)$ restricted to J is one-to-one, then Ja is dependent. (Indeed, if $u, v \in J$ with $u \neq v$ and $x \in C(u) \cap C(v)$, then $ua, va \in Ja$ with $ua \neq va$ and $C(xa) \subseteq C(ua) \cap C(va)$.)

Define $a \in \text{End}(B)$ by $xa = xg^{-1}$ for every $x \in X$. (Such an a exists and is unique since B is the $\mathcal{V}$-free band on X.) For every $x \in X$,

$$(xg)(g^{-1}ag) = xag = xg^{-1}g = x.$$

Thus, $(Xg)(g^{-1}ag) = X$. It follows that $g^{-1}ag$ restricted to Xg is one-to-one (since $|Xg| = |X| = n$). Hence, by the observation above, Xg must be independent (since $g^{-1}ag$ is an endomorphism and X is independent). It follows that $Xg = X$ (since X is the only subset of B that is independent and has n elements).

Suppose X is infinite. Let $x \in X$. We claim that $xg \in X$. Let $u = xg$ and suppose, by way of contradiction, that $u \notin X$. Then $u = pu'$ for some $p \in X$ and $u' \in B$.

Let $v = xg^{-1}$ and $w = pg^{-1}$. Then $w \neq x$ (since $w = x$ would imply $u = xg = wg = (pg^{-1})g = p \in X$, but we have that $u \notin X$). Since X is infinite, there exists $z \in X$ such that $z \notin C(v)$. Define $a \in \text{End}(B)$ by $xa = v$ and $ya = z$ for every $y \in X - \{x\}$. Then

$$u(g^{-1}ag) = x(ag) = vg = x.$$

Hence, since $g^{-1}ag \in \text{End}(B)$, we have

$$x = u(g^{-1}ag) = p(g^{-1}ag)u'(g^{-1}ag).$$

Then $C(p(g^{-1}ag)u'(g^{-1}ag)) = C(x) = \{x\}$, and so $p(g^{-1}ag) = u'(g^{-1}ag) = x$. Thus, $x = p(g^{-1}ag) = (pg^{-1})ag = (wa)g$, and so $wa = v$ (since $x = vg$ and g is one-to-one). Since $w \neq x$, there exists $y \in C(w)$ such that $y \neq x$. But then $z = ya$, and so $\{z\} = C(ya) \subseteq C(wa) = C(v)$, which is a contradiction (since $z \notin C(v)$).

The claim has been proved, that is, $Xg \subseteq X$. By the foregoing argument applied to g^{-1}, we also have $Xg^{-1} \subseteq X$. The result follows. $\qquad\square$

For a function $f : A \to B$ and $A_1 \subseteq A$, we denote by $f \mid A_1$ the restriction of f to A_1.

Lemma 3.2. *Let $B = B_\mathcal{V}(X)$ and let $g \in G_{\text{End}(B)}$ be such that $g \mid X = \text{id}_X$. Then for every $w \in B$, $C(wg) = C(w)$.*

Proof. Let $w \in B$. We first show that $C(wg) \subseteq C(w)$. Let $w = x_1 \cdots x_k$ and $wg = y_1 \cdots y_n$, where $x_i, y_j \in X$. Define $a \in \text{End}(B)$ by $xa = x$ for every $x \in C(w)$, and $za = x_1$ for every $z \in X - C(w)$. Note that $C(ua) \subseteq C(w)$ for every $u \in B$. Now,

$$(x_1 \cdots x_k)(gg^{-1}ag) = (x_1 \cdots x_k)ag = ((x_1a)\dots(x_ka))g = (x_1 \cdots x_k)g = wg.$$

On the other hand,

$$
\begin{aligned}
(x_1 \cdots x_k)(gg^{-1}ag) &= (y_1 \cdots y_n)(g^{-1}ag) \\
&= y_1(g^{-1}ag) \cdots y_n(g^{-1}ag) \quad \text{(since } g^{-1}ag \in \operatorname{End}(B)) \\
&= (y_1ag) \ldots (y_nag) \quad \text{(since } g \mid X \text{ is the identity)} \\
&= (y_1a) \ldots (y_na) \quad \text{(since each } y_ja \in X \text{ and } g \mid X \text{ is the identity)} \\
&= (y_1 \cdots y_n)a.
\end{aligned}
$$

Thus, $wg = (y_1 \cdots y_n)a$, and so $C(wg) = C((y_1 \cdots y_n)a) \subseteq C(w)$.

By the foregoing argument applied to g^{-1} and wg, we have $C(w) = C((wg)g^{-1}) \subseteq C(wg)$. Hence, $C(wg) = C(w)$. □

Let S be a semigroup. A bijection $\phi : S \to S$ is called an *anti-automorphism* of S if $(ab)\phi = (b\phi)(a\phi)$ for all $a, b \in S$.

For a word $w \in X^+$, the *dual* $\bar{w}$ of w is the word obtained from w by reversing the order of the letters. For example, if $w = cabacba$, then $\bar{w} = abcabac$. Note that, for all $u, v \in X^+$,

$$
\bar{\bar{u}} = u \quad \text{and} \quad \overline{uv} = \bar{v}\bar{u}. \tag{3.3}
$$

Let $B = B_{\mathcal{V}}(X)$. We say that B is *d-invariant* if for all $u, v \in X^+$,

$$
u = v \text{ in } B \quad \Longrightarrow \quad \bar{u} = \bar{v} \text{ in } B.
$$

Let $B = B_{\mathcal{V}}(X)$ be *d*-invariant. The mapping $d : B \to B$ defined by $wd = \bar{w}$ will be called the *duality mapping* on B. Since B is *d*-invariant, we have, for all $u, v \in B$, that

$$
u = v \quad \Longleftrightarrow \quad \bar{u} = \bar{v} \quad \Longleftrightarrow \quad ud = vd.
$$

Thus, d is a well-defined bijection on B, that is $d \in S(B)$. It follows from the definition of d and (3.3) that d has the following properties:

(1) $d \mid X = \operatorname{id}_X$, that is, $xd = x$ for every $x \in X$;

(2) d is an anti-automorphism of B, that is, d is a bijection and $(uv)d = (vd)(ud)$ for all $u, v \in B$;

(3) $d = d^{-1}$, that is, $w(dd) = w$ for every $w \in B$.

Note that if $B = B_{\mathcal{V}}(X)$ and $d : B \to B$ given by $wd = \bar{w}$ is well defined, then B is *d*-invariant.

We prove that if $B = B_{\mathcal{V}}(X)$ and $g \in G_{\operatorname{End}(B)}$, then g is an automorphism of B or an anti-automorphism of B (Lemmas 3.3–3.7).

Lemma 3.3. *Let $B = B_{\mathcal{V}}(X)$ be d-invariant and let d be the duality mapping on B. Then $d \in G_{\operatorname{End}(B)}$.*

Proof. Since $d^{-1} = d$, it suffices to show that $d \operatorname{End}(B) d \subseteq \operatorname{End}(B)$. Let $a \in \operatorname{End}(B)$ and $u, v \in B$. Then

$$(uv)(dad) = (vdud)(ad) = (v(da)u(da))d = u(dad)v(dad).$$

Thus, $dad \in \operatorname{End}(B)$, and the result follows. $\qquad\square$

In fact, using a similar argument, one can prove a more general statement: if $B = B_{\mathcal{V}}(X)$ and g is an anti-automorphism of B, then $g \in G_{\operatorname{End}(B)}$.

Lemma 3.4. *Let $B = B_{\mathcal{V}}(X)$, let $g \in G_{\operatorname{End}(B)}$ be such that $g \mid X = \operatorname{id}_X$, and let $x, y \in X$ with $x \neq y$. Then,*

(1) *if $(xy)g = xy$, then g is the identity mapping on B,*

(2) *if $(xy)g = yx$, then g is the duality mapping on B.*

Proof. Let u and v be arbitrary elements of B. Let $a \in \operatorname{End}(B)$ be such that $xa = u$ and $ya = v$. Then

$$(xy)(gg^{-1}ag) = (xy)(ag) = (xaya)g = (uv)g.$$

To prove (1), suppose $(xy)g = xy$. Then, since $g^{-1}ag \in \operatorname{End}(B)$ and $g \mid X = \operatorname{id}_X$, we have

$$(xy)(gg^{-1}ag) = (xy)(g^{-1}ag) = x(g^{-1}ag)y(g^{-1}ag) = x(ag)y(ag) = (ug)(vg).$$

Thus, $(uv)g = (ug)(vg)$, and so g is an automorphism of B. But the only automorphism of B that fixes every element of X is id_B. Hence, $g = \operatorname{id}_B$.

To prove (2), suppose that $(xy)g = yx$. Then, since $g^{-1}ag \in \operatorname{End}(B)$ and $g \mid X = \operatorname{id}_X$, we have

$$(xy)(gg^{-1}ag) = (yx)(g^{-1}ag) = y(g^{-1}ag)x(g^{-1}ag) = y(ag)x(ag) = (vg)(ug).$$

Thus, $(uv)g = (vg)(ug)$, and so g is an anti-automorphism of B. But the only anti-automorphism of B that fixes every element of X is d, the duality mapping on B. Hence, $g = d$. $\qquad\square$

Lemma 3.5. *Let $B = B_{\mathcal{V}}(X)$, let $g \in G_{\operatorname{End}(B)}$ be such that $g \mid X = \operatorname{id}_X$, and let $x, y \in X$ with $x \neq y$. Then either $(xy)g = xy$ or $(xy)g = yx$.*

Proof. By Lemma 3.2, $C((xy)g) = C(xy) = \{x, y\}$. Since B is a band, the only elements of B with content $\{x, y\}$ are xy, yx, xyx and yxy.

Suppose $(xy)g = xyx$. Let $a \in \operatorname{End}(B)$ be such that $xa = xy$ and $ya = x$. Then

$$(xy)(gg^{-1}ag) = (xy)(ag) = (xaya)g = (xyx)g.$$

On the other hand, since $(xy)g = xyx$, $g^{-1}ag \in \operatorname{End}(B)$, and $g \mid X = \operatorname{id}_X$, we have

$$\begin{aligned}(xy)(gg^{-1}ag) &= (xyx)(g^{-1}ag) = x(g^{-1}ag)y(g^{-1}ag)x(g^{-1}ag) = x(ag)y(ag)x(ag)\\ &= ((xy)g)(xg)((xy)g) = (xyx)x(xyx) = xyx.\end{aligned}$$

Thus, $(xyx)g = xyx = (xy)g$, implying that $xyx = xy$ (since g is one-to-one). Hence, $(xy)g = xyx = xy$.

Using a similar argument (with $a \in \mathrm{End}(B)$ such that $xa = y$ and $ya = xy$), we can prove that if $(xy)g = yxy$, then $yxy = xy$, and so $(xy)g = yxy = xy$.

Since the only remaining possibilities are $(xy)g = xy$ and $(xy)g = yx$, the result follows. $\qquad\square$

Lemma 3.6. *Let $B = B_{\mathcal{V}}(X)$ and let $g \in G_{\mathrm{End}(B)}$ be such that $g \mid X = \mathrm{id}_X$. Then $g = \mathrm{id}_B$ or $g = d$.*

Proof. If $|X| = 1$, say $X = \{x\}$, then $B = \{x\}$, and so $g = \mathrm{id}_B$. If $|X| \geqslant 2$, then the result follows immediately from Lemmas 3.4 and 3.5. $\qquad\square$

Lemma 3.7. *Let $B = B_{\mathcal{V}}(X)$ and $g \in G_{\mathrm{End}(B)}$. Then we have the following.*

(1) *If B is d-invariant, then g is an automorphism of B or an anti-automorphism of B.*

(2) *If B is not d-invariant, then g is an automorphism of B.*

(3) *If $|X| \geqslant 2$ and g is both an automorphism of B and an anti-automorphism of B, then $\mathcal{V} = \mathcal{SL}$.*

Proof. To prove (1) and (2), define $\theta : G_{\mathrm{End}(B)} \to S(X)$ by $h\theta = h \mid X$. By Lemma 3.1, θ is well defined. Since $(h_1 h_2) \mid X = (h_1 \mid X)(h_2 \mid X)$ for all $h_1, h_2 \in G_{\mathrm{End}(B)}$, we find that θ is a group homomorphism.

Let $\alpha = g\theta$ (that is, $\alpha = g \mid X$). Consider $h \in \mathrm{End}(B)$ defined by $xh = x\alpha$ for every $x \in X$. Then h is an automorphism of B. (Indeed, define $h' \in \mathrm{End}(B)$ by $xh' = x\alpha^{-1}$ for every $x \in X$. Then $x(hh') = x(h'h) = x$ for every $x \in X$, which implies that $hh' = h'h = \mathrm{id}_B$.) Clearly, $h\theta = \alpha = g\theta$.

Suppose that B is d-invariant. Then, by Lemma 3.6, $\mathrm{Ker}(\theta) = \{\mathrm{id}_B, d\}$. Thus, since $h\theta = g\theta$, we have $gh^{-1} \in \mathrm{Ker}(\theta) = \{\mathrm{id}_B, d\}$, that is, $g = \mathrm{id}_B h$ or $g = dh$. In the former case, g is an automorphism of B; and, in the latter case, it is an anti-automorphism of B. This proves (1).

Suppose that B is not d-invariant. Then the duality mapping d is not well defined on B, and so, by Lemma 3.6, $\mathrm{Ker}(\theta) = \{\mathrm{id}_B\}$. Thus, θ is one-to-one, and so $h\theta = g\theta$ implies $h = g$. Since h is an automorphism of B, (2) follows.

To prove (3), suppose $|X| \geqslant 2$ and g is both an automorphism of B and an anti-automorphism of B. Let $x, y \in X$. Since g is onto, there exist $u, v \in B$ such that $x = ug$ and $y = vg$. Since g is both an automorphism and an anti-automorphism, we have

$$ xy = (ug)(vg) = (uv)g = (vg)(ug) = yx. $$

Thus, since B is the $\mathcal{V}$-free band on X and $|X| \geqslant 2$, it follows that, for every $S \in \mathcal{V}$ and for all $a, b \in S$, $ab = ba$. Hence, every band in $\mathcal{V}$ is a semilattice, which implies $\mathcal{V} = \mathcal{SL}$. $\qquad\square$

With the preceding lemmas, we can determine the group of automorphisms of the semigroup $\mathrm{End}(B_{\mathcal{V}}(X))$ for any variety $\mathcal{V}$ of bands that contains the variety $\mathcal{SL}$ of semilattices. We denote by C_2 the cyclic group of order 2, say $C_2 = \{1, t\}$.

Theorem 3.8. *Let X be a set with $|X| \geqslant 2$, let $\mathcal{V}$ be a variety of bands such that $\mathcal{SL} \subseteq \mathcal{V}$, and let $B = B_{\mathcal{V}}(X)$. Then we have the following.*

(1) *If $\mathcal{V} = \mathcal{SL}$ or B is not d-invariant, then $\mathrm{Aut}(\mathrm{End}(B))$ is isomorphic to $S(X)$.*

(2) *If $\mathcal{V} \neq \mathcal{SL}$ and B is d-invariant, then $\mathrm{Aut}(\mathrm{End}(B))$ is isomorphic to $S(X) \times C_2$.*

Proof. Since $\mathrm{Aut}(\mathrm{End}(B))$ is isomorphic to $G_{\mathrm{End}(B)}$ (Theorem 2.1), $\mathrm{Aut}(\mathrm{End}(B))$ in statements (1) and (2) can be replaced with $G_{\mathrm{End}(B)}$.

To prove (1), suppose that $\mathcal{V} = \mathcal{SL}$ or $\mathcal{V}$ is not d-invariant. Define $\theta : G_{\mathrm{End}(B)} \to S(X)$ by $g\theta = g \mid X$. By the proof of Lemma 3.7, θ is a well-defined group homomorphism, $\mathrm{Ker}(\theta) = \{\mathrm{id}_B\}$, and for every $\alpha \in S(X)$ there exists an $h \in G_{\mathrm{End}(B)}$ such that $\alpha = h\theta$. (If $\mathcal{V} = \mathcal{SL}$, then $\mathcal{V}$ is d-invariant, and so, according to the proof of Lemma 3.7, $\mathrm{Ker}(\theta) = \{\mathrm{id}_B, d\}$. But if $\mathcal{V} = \mathcal{SL}$, then $d = \mathrm{id}_B$, and so $\mathrm{Ker}(\theta) = \{\mathrm{id}_B\}$.) Hence, θ is a group isomorphism, which proves (1).

To prove (2), suppose that $\mathcal{V} \neq \mathcal{SL}$ and $\mathcal{V}$ is d-invariant. Define $\phi : G_{\mathrm{End}(B)} \to S(X) \times C_2$ by

$$g\phi = \begin{cases} (g \mid X, 1) & \text{if } g \text{ is an automorphism of } B, \\ (g \mid X, t) & \text{if } g \text{ is an anti-automorphism of } B. \end{cases}$$

Then ϕ is well defined by Lemmas 3.1 and 3.7. It is routine to verify that ϕ is a group homomorphism. For example, suppose that $g_1, g_2 \in G_{\mathrm{End}(B)}$ are anti-automorphisms of B. Then $g_1 g_2$ is an automorphism of B, and so

$$(g_1 g_2)\phi = ((g_1 g_2) \mid X, 1) = ((g_1 \mid X)(g_2 \mid X), 1) = (g_1 \mid X, t)(g_2 \mid X, t) = (g_1 \phi)(g_2 \phi).$$

Suppose $g\phi = (\mathrm{id}_X, 1)$. Then $g \mid X = \mathrm{id}_X$ and g is an automorphism of B. Thus, $g = \mathrm{id}_B$. Hence, $\mathrm{Ker}(\phi) = \{\mathrm{id}_B\}$, and so ϕ is one-to-one.

Let $\alpha \in S(X)$. By the proof of Lemma 3.7, there is an automorphism h of B such that $h \mid X = \alpha$ and $hd \mid X = \alpha$. Hence, $h\phi = (\alpha, 1)$ and $(hd)\phi = (\alpha, t)$ (since d is an anti-automorphism of B, and so hd is an anti-automorphism of B), which shows that ϕ is onto. Thus, ϕ is a group isomorphism, which proves (2). $\qquad\square$

4. Varieties of rectangular bands

A semigroup S is called a *rectangular band* if $aba = a$ for all $a, b \in S$. It is called a *left zero semigroup* if $ab = a$ for all $a, b \in S$, and it is called a *right zero semigroup* if $ab = b$ for all $a, b \in S$. It is clear that the left zero semigroups and right zero semigroups are rectangular bands.

The classes of all rectangular bands, left zero semigroups and right zero semigroups are varieties of bands, which we shall denote, respectively, by $\mathcal{RB}$, $\mathcal{LZ}$ and $\mathcal{RZ}$. These

three varieties and the variety $\mathcal{T}$ of trivial bands are the only varieties of bands that do not contain the variety $\mathcal{SL}$ of semilattices (see Figure 1).

Every left zero semigroup B is relatively free in $\mathcal{LZ}$ on $X = B$. Since $ab = a$ for all $a, b \in B$, every transformation of B is an endomorphism, that is, $\mathrm{End}(B) = T(B)$. It is well known [**15**, **22**] that $\mathrm{Aut}(T(B)) \cong S(B)$. The same argument applies to right zero semigroups. It follows that if $\mathcal{V} = \mathcal{LZ}$ or $\mathcal{V} = \mathcal{RZ}$, then $\mathrm{Aut}(\mathrm{End}(B_{\mathcal{V}}(X)))$ is isomorphic to $S(X)$.

It remains to consider the variety $\mathcal{RB}$ of all rectangular bands. Every rectangular band B is isomorphic to a semigroup $L \times R$, where $L = Bz$ and $R = zB$ (for a fixed $z \in B$), with multiplication $(k_1, p_1)(k_2, p_2) = (k_1, p_2)$ [**13**, Theorem 1.1.3]. The monoid $\mathrm{End}(B)$ is isomorphic to $T(L) \times T(R)$. The following lemma shows that if B is a $\mathcal{V}$-free band on X, then $B = X \times X$.

Lemma 4.1. *Let* $\mathcal{V} = \mathcal{RB}$, $B = B_{\mathcal{V}}(X)$, *and* $z \in B$. *Then*

(1) $Bz = Xz$ *and* $zB = zX$,

(2) $|Xz| = |zX| = |X|$.

Proof. To prove (1), let $wz \in Bz$, where $w = x_1 \cdots x_n \in B$ $(n \geqslant 1, \ x_i \in X)$. Since B is a rectangular band, $uvp = up$ for all $u, v, p \in B$ [**13**, Theorem 1.1.3]. Thus, $wz = x_1 \cdots x_n z = x_1 z \in Xz$. Hence, $Bz = Xz$. By a similar argument, we obtain $zB = zX$.

To prove (2), define $f : X \to Xz$ by $xf = xz$. It is clear that f is onto. Let $x, y \in X$ with $x \neq y$. Consider the rectangular band $L \times R$ with $L = \{1, 2\}$ and $R = \{1\}$. Since B is the $\mathcal{RB}$-free band on X, there exists a homomorphism $\phi : B \to L \times R$ such that $x\phi = (1, 1)$ and $y\phi = (2, 1)$. We have $z\phi = (a, 1)$, where $a = 1$ or $a = 2$, and so

$$(xz)\phi = (x\phi)(z\phi) = (1, 1)(a, 1) = (1, 1) \quad \text{and} \quad (yz)\phi = (y\phi)(z\phi) = (2, 1)(a, 1) = (2, 1).$$

It follows that $xz \neq yz$, and so f is one-to-one. Hence, $|Xz| = |X|$. By a similar argument, we obtain $|zX| = |X|$. $\qquad\qquad\square$

We first consider $B = X \times X$ with $|X| = 2$, say $X = \{1, 2\}$. Here we use the fact that $\mathrm{Aut}(\mathrm{End}(B)) \cong G_{\mathrm{End}(B)}$. Let $a : B \to B$. We say that a *left-splits* $k_0 \in X$ if there exist $p_1, p_2 \in X$ such that $(k_0, p_1)a = (k_1, q_1)$ and $(k_0, p_2)a = (k_2, q_2)$ with $k_1 \neq k_2$. We have a dual definition of what it means that a *right-splits* $p_0 \in X$. It follows from [**13**, Proposition 4.4.2] that $a \in \mathrm{End}(B)$ if and only if there exist $a_1, a_2 \in T(X)$ such that $a = a_1 \times a_2$. (For $f : A \to B$ and $g : C \to D$, we denote by $f \times g$ the mapping from $A \times C$ to $B \times D$ defined by $(x, y)(f \times g) = (xf, yg)$.) The existence of such a_1 and a_2 is clearly equivalent to a not (left- or right-) splitting any element of X.

Let $g \in G_{\mathrm{End}(B)}$. (We continue to assume that $X = \{1, 2\}$.) Suppose that g left-splits 1, that is,

$$(1, 1)g = (k, p) \quad \text{and} \quad (1, 2)g = (l, q)$$

with $k \neq l$. We claim that $p = q$. Suppose $p \neq q$. Then either

(i) $(2,1)g = (k,q)$ and $(2,2)g = (l,p)$ or

(ii) $(2,1)g = (l,p)$ and $(2,2)g = (k,q)$.

Suppose that (i) is true. Consider $a = a_1 \times a_2 \in \mathrm{End}(B)$, where $1a_1 = 2a_1 = 1$, $1a_2 = 1$, and $2a_2 = 2$. Then $(k,p)(g^{-1}ag) = (k,p)$ and $(l,p)(g^{-1}ag) = (l,q)$. Thus, $g^{-1}ag$ right-splits p, which is a contradiction since $g^{-1}ag \in \mathrm{End}(B)$. If (ii) is true, we obtain a contradiction in a similar way. Hence,

$$(1,1)g = (k,p), \quad (1,2)g = (l,p), \quad (2,1)g = (m,q) \quad \text{and} \quad (2,2)g = (n,q),$$

where $k \neq l$, $m \neq n$ and $p \neq q$. It follows that if g left-splits 1, then it left- and right-splits 1 and 2, and $(\{1\} \times X)g = X \times \{p\}$. We can use a similar argument when g right-splits 1, and when g left- or right-splits 2, obtaining the following lemma.

Lemma 4.2. *Let $X = \{1,2\}$, $B = X \times X$, and $g \in G_{\mathrm{End}(B)}$. If g splits any element of X, then, for every $i \in X$,*

$$(\{i\} \times X)g = X \times \{p_i\} \quad \text{and} \quad (X \times \{i\})g = \{k_i\} \times X.$$

Lemma 4.3. *Let $X = \{1,2\}$, $B = X \times X$ and $g : B \to B$. Then $g \in G_{\mathrm{End}(B)}$ if and only if there exist $g_1, g_2 \in S(X)$ such that $g = g_1 \times g_2$ or $g = (g_1 \times g_2)\delta$, where $\delta \in S(B)$ is defined by $(k,p)\delta = (p,k)$.*

Proof. It is clear that if $g = g_1 \times g_2$ or $g = (g_1 \times g_2)\delta$, then $g \in G_{\mathrm{End}(B)}$. To prove the converse, suppose $g \in G_{\mathrm{End}(B)}$. If g does not split any element of X, then $g = g_1 \times g_2$ for some $g_1, g_2 \in S(X)$. Suppose that g splits an element of X. Define $g_1, g_2 \in S(X)$ by $ig_1 = p_i$ and $ig_2 = k_i$, where p_i and k_i are the elements from Lemma 4.2 ($i = 1, 2$). It is routine to check that, for all $k, p \in X$, $(k,p)g = (pg_2, kg_1)$, that is, $g = (g_1 \times g_2)\delta$. □

We now consider $B = X \times X$ with $|X| > 2$. Here we use the fact that $\mathrm{End}(B) \cong T(X) \times T(X)$. Let S be a characteristic subsemigroup of a semigroup T. Then a *lift* of $\phi \in \mathrm{Aut}(S)$ is any $\psi \in \mathrm{Aut}(T)$ such that $\psi \mid S = \phi$.

Lemma 4.4. *Let X be a set with $|X| > 2$, $T = T(X)$, and $\psi : T \times T \to T \times T$. Then $\psi \in \mathrm{Aut}(T \times T)$ if and only if there exist $g_1, g_2 \in S(X)$ such that $\psi = \phi_T^{g_1} \times \phi_T^{g_2}$ or $\psi = (\phi_T^{g_1} \times \phi_T^{g_2})\lambda$, where $\lambda : T \times T \to T \times T$ is defined by $(a,b)\lambda = (b,a)$.*

Proof. Let $S = S(X)$, and let $\phi : S \times S \to S \times S$. It is well known that if $|X| > 2$, then $\phi \in \mathrm{Aut}(S \times S)$ if and only if there exist $\phi_1, \phi_2 \in \mathrm{Aut}(S)$ such that $\phi = \phi_1 \times \phi_2$ or $\phi = (\phi_1 \times \phi_2)\mu$, where $\mu : S \times S \to S \times S$ is defined by $(g,h)\mu = (h,g)$.

Since $S \times S$ is the group of units of $T \times T$, it is characteristic in $T \times T$. Thus, the automorphisms of $T \times T$ are precisely the lifts of the automorphisms of $S \times S$.

We claim that every $\phi \in \mathrm{Aut}(S \times S)$ has at most one lift to $T \times T$, or equivalently, that $\mathrm{id}_{S \times S}$, the identity of $\mathrm{Aut}(S \times S)$, has exactly one lift, namely $\mathrm{id}_{T \times T}$. Indeed, suppose τ is a lift of $\mathrm{id}_{S \times S}$, that is, $\tau \in \mathrm{Aut}(T \times T)$ and $\tau \mid S \times S = \mathrm{id}_{S \times S}$. For $u, v \in T$, let

$(u,v)\tau = (u^*, v^*)$. For $x \in X$, denote by a_x the element of T with range $\{x\}$, and by S_x denote the stabilizer of x in S. Let $u \in T$. Then $ug = u$ for every $g \in S_x$ if and only if $u = a_x$. Let $x, y \in X$. Then, for all $g \in S_x$ and $h \in S_y$,

$$(a_x^*, a_y^*) = (a_x, a_y)\tau = ((a_x, a_y)(g, h))\tau = (a_x, a_y)\tau(g, h)\tau = (a_x, a_y)\tau(g, h) = (a_x^* g, a_y^* h).$$

It follows that, for all $x, y \in X$, $(a_x^*, a_y^*) = (a_x, a_y)$, that is, $(a_x, a_y)\tau = (a_x, a_y)$.

Now let $x, y \in X$ and $u, v \in T$. It is obvious that $(a_x, a_y)(u, v) = (a_{xu}, a_{yv})$. Thus,

$$\begin{aligned}(a_{xu}, a_{yv}) &= (a_{xu}, a_{yv})\tau = ((a_x, a_y)(u, v))\tau = (a_x, a_y)\tau(u, v)\tau \\ &= (a_x, a_y)((u, v)\tau) = (a_x, a_y)(u^*, v^*) = (a_{xu^*}, a_{yv^*}).\end{aligned}$$

Thus, if $(u^*, v^*) = (u, v)\tau$, then $xu^* = xu$ and $yv^* = yv$ for all $x, y \in X$. It follows that $(u, v)\tau = (u, v)$, so $\tau = \mathrm{id}_{T \times T}$. The claim has been proved.

Now, let $\psi \in \mathrm{Aut}(T \times T)$. Then ψ is a lift of $\phi = \psi \mid S \times S \in \mathrm{Aut}(S \times S)$. Since $|X| > 2$, there exist $\phi_1, \phi_2 \in \mathrm{Aut}(S)$ such that $\phi = \phi_1 \times \phi_2$ or $\phi = (\phi_1 \times \phi_2)\mu$. Suppose $|X| \neq 6$. Then every automorphism of $S = S(X)$ is inner, so $\phi_1 = \phi_S^{g_1}$ and $\phi_2 = \phi_S^{g_2}$ for some $g_1, g_2 \in S(X)$. Then $\phi_1 \times \phi_2$ can be lifted to $\phi_T^{g_1} \times \phi_T^{g_2} \in \mathrm{Aut}(T \times T)$. Clearly, μ can be lifted to λ. Since a lift of ϕ is unique, we have $\psi = \phi_T^{g_1} \times \phi_T^{g_2}$ or $\psi = (\phi_T^{g_1} \times \phi_T^{g_2})\lambda$.

If $|X| = 6$, then there are outer automorphisms of $S(X)$. However, we will prove that since ϕ can be lifted to ψ, ϕ_1 and ϕ_2 must be inner. Suppose $\phi = \phi_1 \times \phi_2$. We claim that $(T \times \{1\})\psi = T \times \{1\}$, where $1 = \mathrm{id}_X$. For $x \in X$, let $(a_x, 1)\psi = (u_x, v_x)$. Let $g \in S$ and $x \in X$. Since $(a_x, 1) = (g, 1)(a_x, 1)$, we have

$$(u_x, v_x) = (g, 1)\phi(a_x, 1)\psi = (g\phi_1, 1\phi_2)(u_x, v_x) = (g\phi_1 u_x, v_x).$$

Since g was arbitrary, it follows that $u_x = hu_x$ for every $h \in G$, which implies that u_x is a constant mapping. Thus, for every $x \in X$, $(a_x, 1)\psi = (a_{x_1}, v_x)$, where $x_1 \in X$. In a similar way, we prove that, for every $y \in X$, $(1, a_y)\psi = (w_y, a_{y_2})$, where $w_y \in T$ and $y_2 \in X$.

It is clear that $M = \{(a_x, a_y) : x, y \in X\}$ is the minimal ideal of $T \times T$. Let $x', y' \in X$. Since ψ maps M onto M, there exist $x, y \in X$ such that $(a_x, a_y)\psi = (a_{x'}, a_{y'})$. Since $(a_x, a_y) = (a_x, 1)(1, a_y)$, we have

$$(a_{x'}, a_{y'}) = (a_x, 1)\psi(1, a_y)\psi = (a_{x_1}, v_x)(w_y, a_{y_2}) = (a_{x_1} w_y, v_x a_{y_2}).$$

Thus, $a_{y'} = v_x a_{y_2}$, which implies $y' = y_2$. Using $(a_x, a_y) = (1, a_y)(a_x, 1)$, we obtain $a_{y'} = a_{y_2} v_x = a_{y'} v_x$. This gives $y' v_x = y'$, and so, since y' was arbitrary, $v_x = 1$. Let $z \in X$. Select $g \in S$ such that $xg = z$. Then $(a_z, 1) = (a_x, 1)(g, 1)$, and so

$$(a_{z_1}, v_z) = (a_x, 1)\psi(g, 1)\phi = (a_{x_1}, 1)(g\phi_1, 1) = (a_{x_1} g\phi_1, 1).$$

Thus, $v_z = 1$. We proved that, for every $x \in X$, $(a_x, 1)\psi = (a_{x_1}, 1)$.

Let $a \in T$ and $x \in X$. Let $(a, 1)\psi = (u_a, v_a)$. Since $(a_x, 1) = (a, 1)(a_x, 1)$, we have

$$(a_{x_1}, 1) = (a, 1)\psi(a_x, 1)\psi = (u_a, v_a)(a_{x_1}, 1) = (a_{x_1}, v_a).$$

Thus, $v_a = 1$, and so $(a, 1)\psi = (u_a, 1)$. The claim has been proved.

Define $\psi_1 : T \to T$ by $a\psi_1 = u_a$. It is clear that $\psi_1 \in \mathrm{Aut}(T)$ and $\psi_1 \mid S = \phi_1$. It is well known that all automorphisms of $T(X)$ are inner. Thus, ψ_1 is inner, and so ϕ_1 is also inner. In a similar way, we prove that $(\{1\} \times T)\psi = \{1\} \times T$, which implies that ϕ_2 is inner. A proof that ϕ_1 and ϕ_2 are inner when $\phi = (\phi_1 \times \phi_2)\mu$ is similar. (In this case, we have $(T \times \{1\})\psi = \{1\} \times T$ and $(\{1\} \times T)\psi = T \times \{1\}$.) Now we complete the proof as in the case when $|X| \neq 6$. $\square$

Let $C_2 = \{1, t\}$ be the cyclic group of order 2. Consider the semidirect product $[S(X) \times S(X)] \ltimes C_2$, where C_2 acts on $S(X) \times S(X)$ by $(g_1, g_2)^1 = (g_1, g_2)$ and $(g_1, g_2)^t = (g_2, g_1)$. Note that the multiplication in $[S(X) \times S(X)] \ltimes C_2$ is given by

$$((g_1, g_2), 1)((h_1, h_2), s) = ((g_1, g_2)(h_1, h_2)^1, 1s) = ((g_1 h_1, g_2 h_2), s),$$
$$((g_1, g_2), t)((h_1, h_2), s) = ((g_1, g_2)(h_1, h_2)^t, ts) = ((g_1 h_2, g_2 h_1), ts).$$

Let $M = \{1, 2\}$. We can interpret $S(X) \times S(X)$ as $S(X)^M$ (the set of functions from M to $S(X)$) and C_2 as $S(M)$. With this interpretation, the semidirect product $[S(X) \times S(X)] \ltimes C_2$ becomes the wreath product $S(X) \wr C_2$.

Theorem 4.5. *Let X be a non-empty set, let $\mathcal{V}$ be a variety of rectangular bands and let $B = B_{\mathcal{V}}(X)$. Then we have the following.*

(1) *If $\mathcal{V} = \mathcal{LZ}$ or $\mathcal{V} = \mathcal{RZ}$, then $\mathrm{Aut}(\mathrm{End}(B)) \cong S(X)$.*

(2) *If $\mathcal{V} = \mathcal{RB}$ and $|X| \geqslant 2$, then $\mathrm{Aut}(\mathrm{End}(B)) \cong S(X) \wr C_2$.*

Proof. We established (1) at the beginning of this section. To prove (2), let B be a free rectangular band. By Lemma 4.1, $B = X \times X$.

Suppose that $|X| = 2$ and let $g \in G_{\mathrm{End}(B)}$. By Lemma 4.3, $g = g_1 \times g_2$ or $g = (g_1 \times g_2)\delta$, where $g_1, g_2 \in S(X)$ and $(k, p)\delta = (p, k)$ $(k, p \in X)$. Map $G_{\mathrm{End}(B)}$ to $[S(X) \times S(X)] \ltimes C_2$ by

$$g_1 \times g_2 \to ((g_1, g_2), 1) \quad \text{and} \quad (g_1 \times g_2)\delta \to ((g_1, g_2), t).$$

It is routine to check that this mapping is a group isomorphism. Thus, $\mathrm{Aut}(\mathrm{End}(B)) \cong G_{\mathrm{End}(B)} \cong [S(X) \times S(X)] \ltimes C_2 \cong S(X) \wr C_2$.

Suppose that $|X| > 2$ and let $\psi \in \mathrm{Aut}(T(X) \times T(X))$ and $T = T(X)$. By Lemma 4.4, $\psi = \phi_T^{g_1} \times \phi_T^{g_1}$ or $\psi = (\phi_T^{g_1} \times \phi_T^{g_1})\lambda$, where $g_1, g_2 \in S(X)$ and $(a, b)\lambda = (b, a)$ $(a, b \in T(X))$. Map $\mathrm{Aut}(T(X) \times T(X))$ to $[S(X) \times S(X)] \ltimes C_2$ by

$$\phi_T^{g_1} \times \phi_T^{g_1} \to ((g_1, g_2), 1) \quad \text{and} \quad (\phi_T^{g_1} \times \phi_T^{g_1})\lambda \to ((g_1, g_2), t).$$

Again, it is routine to check that this mapping is an isomorphism. Thus,

$$\mathrm{Aut}(\mathrm{End}(B)) \cong \mathrm{Aut}(T(X) \times T(X)) \cong [S(X) \times S(X)] \ltimes C_2 \cong S(X) \wr C_2.$$

$\square$

5. Relatively free bands that are *d*-invariant

Let $\mathcal{V}$ be a variety of bands with $\mathcal{SL} \subseteq \mathcal{V}$ and $\mathcal{SL} \neq \mathcal{V}$. Then, by Theorem 3.8, the group $\mathrm{Aut}(\mathrm{End}(B_{\mathcal{V}}(X)))$ is isomorphic either to $S(X) \times C_2$ (if $B_{\mathcal{V}}(X)$ is *d*-invariant) or to $S(X)$ (if $B_{\mathcal{V}}(X)$ is not *d*-invariant). Therefore, to have a complete picture, we need to determine which relatively free bands $B_{\mathcal{V}}(X)$ are *d*-invariant and which are not.

Figure 1 shows the lattice of band varieties. At the bottom we have the sublattice generated by the atoms $\mathcal{SL} = [xy = yx]$, $\mathcal{LZ} = [xy = x]$ and $\mathcal{RZ} = [xy = y]$ (see [**13**, p. 137]). In addition to the atoms, the sublattice contains the varieties $\mathcal{T} = [x = y]$ of trivial bands, $\mathcal{RB} = [xyx = x]$ of rectangular bands, $\mathcal{LN} = [xyz = xzy]$ of left normal bands, $\mathcal{RN} = [xyz = yxz]$ of right normal bands and $\mathcal{N} = [xyzx = xzyx]$ of normal bands. (Note that we omit the identity $x^2 = x$ in the system of identities that define band varieties.) At the top we have the variety $\mathcal{B}$ of all bands. The remaining varieties are labelled with the identities given in [**10**]. The words that occur in the identities are defined inductively as follows:

$$G_2 = x_2 x_1, \qquad H_2 = x_2, \qquad I_2 = x_2 x_1 x_2,$$
$$G_n = x_n \bar{G}_{n-1}, \quad H_n = G_n x_n \bar{H}_{n-1}, \quad I_n = G_n x_n \bar{I}_{n-1}, \quad (n \geqslant 3).$$

For example, $G_3 = x_3 x_1 x_2$ and $H_3 = x_3 x_1 x_2 x_3 x_2$. Thus, if $\mathcal{V}$ is the variety labelled $G_3 = H_3$ (second from the bottom in column 1), then $\mathcal{V} = [x_3 x_1 x_2 = x_3 x_1 x_2 x_3 x_2]$.

The varieties $\mathcal{V}$ in column 3 are self-dual, that is, if $\mathcal{V}$ is given by the identity $A = B$, then $\bar{A} = A$ and $\bar{B} = B$. It easily follows that, for all $u, v \in B_{\mathcal{V}}(X)$,

$$u = v \text{ in } B_{\mathcal{V}}(X) \quad \Longleftrightarrow \quad \bar{u} = \bar{v} \text{ in } B_{\mathcal{V}}(X).$$

In other words, for every variety $\mathcal{V}$ in column 3 and every set X, $B_{\mathcal{V}}(X)$ is *d*-invariant.

The varieties in column 5 are dual to the corresponding (same-level) varieties in column 1. That is, if a variety $\mathcal{V}$ in column 1 is given by the identity $A = B$, then the corresponding variety $\mathcal{V}^*$ in column 5 is given by $\bar{A} = \bar{B}$. Similarly, the varieties in column 4 are dual to the corresponding varieties in column 2.

Let $\mathcal{V} = [A = B]$ be a variety in column 1 (column 2) and $\mathcal{V}^* = [\bar{A} = \bar{B}]$ the dual variety in column 5 (column 4). Let X be any set with cardinality at least $\max\{|C(A)|, |C(B)|\}$. Then $\bar{A} \neq \bar{B}$ in $B_{\mathcal{V}}(X)$, since otherwise every semigroup S in $\mathcal{V}$ would satisfy the identity $\bar{A} = \bar{B}$ and we would have $\mathcal{V} \subseteq \mathcal{V}^*$, which is a contradiction. Hence, for every variety $\mathcal{V} = [A = B]$ not in column 3 and every set X with $|X| \geqslant n$, where n is the number of variables (letters) in the identity $A = B$, $B_{\mathcal{V}}(X)$ is not *d*-invariant.

The foregoing discussion and Theorems 3.8 and 4.5 give us the following corollaries.

Corollary 5.1. *Let X be any set with $|X| \geqslant 2$, let $\mathcal{V}$ be any variety of bands in column 3 of Figure 1 (including $\mathcal{B}$), and let $B = B_{\mathcal{V}}(X)$. Then we have the following.*

(1) *If $\mathcal{V} = \mathcal{T}$, then $\mathrm{Aut}(\mathrm{End}(B)) = \{1\}$.*

(2) *If $\mathcal{V} = \mathcal{SL}$, then $\mathrm{Aut}(\mathrm{End}(B)) \cong S(X)$.*

(3) *If $\mathcal{V} = \mathcal{RB}$, then $\mathrm{Aut}(\mathrm{End}(B)) \cong S(X) \wr C_2$.*

(4) *If $\mathcal{V} \neq \mathcal{T}$, $\mathcal{V} \neq \mathcal{SL}$, and $\mathcal{V} \neq \mathcal{RB}$, then $\mathrm{Aut}(\mathrm{End}(B)) \cong S(X) \times C_2$.*

$\mathcal{B}$

Figure 1. The lattice of band varieties.

Corollary 5.2. *Let $\mathcal{V} = [A = B]$ be any variety of bands in Figure 1 that is not in column 3. Then, for every set X with $|X| \geqslant n$, where n is the number of variables in $A = B$,*

$$\mathrm{Aut}(\mathrm{End}(B_{\mathcal{V}}(X))) \cong S(X).$$

It remains to investigate the free bands $B_{\mathcal{V}}(X)$ such that $\mathcal{V} = [A = B]$ is not in column 3 and $|X| < n$, where n is the number of variables in $A = B$. By duality, it suffices to consider the varieties in columns 1 and 2.

We will use the operators on $X^* = X^+ \cup \{1\}$ introduced in [10], which solve the word problem in the relatively free bands (see also [20, 23]). We say that an operator $t : X^* \to X^*$ solves the word problem in the relatively free band $B_{\mathcal{V}}(X)$ if for all $u, v \in X^+$, $u = v$ in $B_{\mathcal{V}}(X)$ if and only if $t(u) = t(v)$ in X^*. (Note that we write the operators on X^* on the left, that is, $t(w)$ rather than wt.) Any such operator t is called an *invariant* for the variety $\mathcal{V}$. (We assume, of course, that the definition of t makes sense for every non-empty set X.)

Following [10], we define operators h_2, i_2, σ, s, ε and e on X^* by $h_2(1) = i_2(1) = \sigma(1) = s(1) = \varepsilon(1) = e(1) = 1$ and, for any $w \in X^+$,

$h_2(w) =$ the first letter occurring in w,

$i_2(w) =$ the word obtained from w by retaining only the first occurrence of each letter,

$\sigma(w) =$ the letter in w that is last to make its first appearance,

$s(w) =$ the subword of w that precedes the first appearance of $\sigma(w)$,

$\varepsilon(w) =$ the letter in w that is first to make its last appearance,

$e(w) =$ the subword of w that follows the last appearance of $\varepsilon(w)$.

For example, $h_2(xzxyz) = x$, $i_2(xzxyz) = xzy$, $\sigma(xzxyz) = y$, $s(xzxyz) = xzx$, $\varepsilon(xzxyz) = x$ and $e(xzxyx) = yz$.

For an operator $t : X^* \to X^*$, define the dual operator $\bar{t}$ by $\bar{t}(w) = \overline{t(\overline{w})}$. For $n \geqslant 3$, define inductively operators h_n and i_n on X^* by

$$h_n(w) = h_n(s(w))\sigma(w)\bar{h}_{n-1}(w),$$

$$i_n(w) = i_n(s(w))\sigma(w)\bar{i}_{n-1}(w).$$

The operators h_n ($n \geqslant 3$) and i_n ($n \geqslant 2$) are invariants for the varieties $[G_n = H_n]$ and $[G_n = I_n]$, respectively [10]. The duals $\bar{h}_n$ and $\bar{i}_n$ are invariants for the corresponding dual varieties. The operators h_2 and $\bar{h}_2$ are invariants for $\mathcal{LZ}$ and $\mathcal{RZ}$, respectively. Recall that, for the variety $\mathcal{SL}$ of semilattices, we have $u = v$ in $B_{\mathcal{SL}}(X)$ if and only if $C(u) = C(v)$. If t_1 and t_2 are invariants for varieties $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively, then the conjunction of t_1 and t_2 is an invariant for the join $\mathcal{V}_1 \vee \mathcal{V}_2$. For example, since $\mathcal{V} = [\bar{G}_4 G_3 = \bar{H}_4 I_3]$ is the join of $[G_3 = H_3]$ and $[\bar{G}_2 = \bar{I}_2]$ (see Figure 1), then, for all $u, v \in X^+$, $u = v$ in $B_{\mathcal{V}}(X)$ if and only if $h_3(u) = h_3(v)$ and $\bar{i}_2(u) = \bar{i}_2(v)$ in X^+.

The above paragraph gives us invariants for all band varieties except the variety $\mathcal{B}$ of all bands. An invariant $b : X^* \to X^*$ for $\mathcal{B}$ is given in [10]. We define $b(w)$ inductively

on the number of letters in a word w as follows:

$$b(1) = 1 \quad \text{and} \quad b(w) = b(s(w))\sigma(w)\varepsilon(w)b(e(w)), \quad w \in X^+.$$

5.1. Column 1

We show that, for all varieties $\mathcal{V} = [A = B]$ in column 1, if $|X| < n$, where n is the number of variables in $A = B$, then $B_{\mathcal{V}}(X)$ is d-invariant. The result will follow from the following lemma.

Lemma 5.3. *Let X be a non-empty set and $n \geqslant 3$. For all words $u, v \in X^+$, if $u = v$ has less than n variables and $h_n(u) = h_n(v)$, then $b(u) = b(v)$.*

Proof. We proceed by induction on n. Let $n = 3$. By [**23**, Proposition 3.7], $h_3(u) = h_3(v)$ implies $h_3(s(u)) = h_3(s(v))$, $\sigma(u) = \sigma(v)$, and $\bar{h}_2(u) = \bar{h}_2(v)$. Since $n = 3$, $u = v$ has one or two variables. If $u = v$ has one variable, the result is trivial since $b(u) = b(v)$ for all words u and v in a single variable x. Suppose $u = v$ has two variables, say x and y. Then $s(u) = s(v)$ has one variable, and $b(s(u)) = b(s(v))$ as above. We have $\bar{h}_2(u) = \bar{h}_2(v)$, that is, u and v have the same last variable, say y. It follows that $\varepsilon(u) = \varepsilon(v) = x$. Thus, $e(u) = e(v)$ has a single variable, y, and so $b(e(u)) = b(e(v))$. Hence, $b(u) = b(v)$.

Let $n \geqslant 4$. By [**23**, Proposition 3.4], for every word w,

$$h_n(w) = h_n(s(w))\sigma(w)\bar{h}_{n-1}(w) = h_n(s(w))\sigma(w)h_{n-2}(w)\varepsilon(w)\bar{h}_{n-1}(e(w)).$$

Again by [**23**, Proposition 3.7], $h_n(u) = h_n(v)$ implies that $h_n(s(u)) = h_n(s(v))$, $\sigma(u) = \sigma(v)$, $\varepsilon(u) = \varepsilon(v)$ and $\bar{h}_{n-1}(e(u)) = \bar{h}_{n-1}(e(v))$. Furthermore, $\bar{h}_{n-1}(e(u)) = \bar{h}_{n-1}(e(v))$ implies that $h_{n-1}(\overline{e(u)}) = h_{n-1}(\overline{e(v)})$. Thus, since $e(u) = e(v)$ has at most $n-2$ variables, $b(\overline{e(u)}) = b(\overline{e(v)})$ by induction. Thus, $b(e(u)) = b(e(v))$ since the variety $\mathcal{B}$ of all bands is self-dual.

To complete the proof that $b(u) = b(v)$, we use the induction on the number of variables in $u = v$. If $u = v$ has one variable, the result is trivial. Suppose $u = v$ has at least two variables. Then $h_n(s(u)) = h_n(s(v))$ implies that $b(s(u)) = b(s(v))$, since $s(u) = s(v)$ has one less variable than $u = v$. We already established $\sigma(u) = \sigma(v)$, $\varepsilon(u) = \varepsilon(v)$ and $b(e(u)) = b(e(v))$. It follows that $b(u) = b(v)$. □

Proposition 5.4. *Let $\mathcal{V} = [A = B]$ be any variety of bands in column 1 or column 5 of Figure 1. Then, for every set X with $|X| < n$, where n is the number of variables in $A = B$, $B_{\mathcal{V}}(X)$ is d-invariant.*

Proof. By duality, we may assume that $\mathcal{V}$ is in column 1. Suppose that $\mathcal{V} = [G_n = H_n]$, $n \geqslant 3$. Let $u, v \in X^+$ be such that $u = v$ in $B_{\mathcal{V}}(X)$, that is, $h_n(u) = h_n(v)$. Since $|X| < n$, Lemma 5.3 gives $b(u) = b(v)$, that is, $u = v$ in $B_{\mathcal{B}}(X)$, where $\mathcal{B}$ is the variety of all bands. Since $B_{\mathcal{B}}(X)$ is d-invariant, $\bar{u} = \bar{v}$ in $B_{\mathcal{B}}(X)$, and so $\bar{u} = \bar{v}$ in $B_{\mathcal{V}}(X)$ since $\mathcal{V} \subset \mathcal{B}$.

Suppose $\mathcal{V} = [G_n = I_n]$, $n \geqslant 2$. If $n = 2$, the result is trivial. Let $n \geqslant 3$. Then $\mathcal{V}_1 = [G_n = H_n]$ is included in $\mathcal{V}$ (see Figure 1). Thus, for all $u, v \in X^+$,

$$
\begin{aligned}
u = v \text{ in } B_{\mathcal{V}}(X) \quad &\implies \quad u = v \text{ in } B_{\mathcal{V}_1}(X) \\
&\implies \quad u = v \text{ in } B_{\mathcal{B}}(X) \\
&\implies \quad \bar{u} = \bar{v} \text{ in } B_{\mathcal{B}}(X) \\
&\implies \quad \bar{u} = \bar{v} \text{ in } B_{\mathcal{V}}(X).
\end{aligned}
$$

This concludes the proof. $\qquad\qquad\square$

Proposition 5.4, together with Theorem 3.8, gives us the following corollary.

Corollary 5.5. *Let $\mathcal{V} = [A = B]$ be any variety of bands in column 1 or column 5 of Figure 1. Then for every set X with $2 \leqslant |X| < n$, where n is the number of variables in $A = B$, $\mathrm{Aut}(\mathrm{End}(B_{\mathcal{V}}(X))) \cong S(X) \times C_2$.*

5.2. Column 2

Here the situation is more complicated than it is in column 1. Proposition 5.4 is true for the varieties $\mathcal{V} = [\bar{G}_n G_n = \bar{I}_n H_n]$, $n \geqslant 3$, and the variety $\mathcal{LZ}$ of left zero semigroups, but not for the varieties $\mathcal{V} = [\bar{G}_n G_{n-1} = \bar{H}_n I_{n-1}]$, $n \geqslant 3$, or the variety $\mathcal{LN}$ of left normal bands.

We first consider the former, for which the result will follow from the following lemma, which is similar to Lemma 5.3.

Lemma 5.6. *Let X be a non-empty set and $n \geqslant 2$. For all words $u, v \in X^+$, if $u = v$ has at most n variables, $i_n(u) = i_n(v)$ and $\bar{h}_n(u) = \bar{h}_n(v)$, then $b(u) = b(v)$.*

Proof. We proceed by induction on n. Let $n = 2$. Then $u = v$ has one or two variables. If $u = v$ has one variable, the result is trivial. Suppose $u = v$ has two variables. Then $i_2(u) = i_2(v)$ implies that $\sigma(u) = \sigma(v)$ and that $s(u) = s(v)$ has one variable. Further, $\bar{h}_2(u) = \bar{h}_2(v)$ implies that $\varepsilon(u) = \varepsilon(v)$ and that $e(u) = e(v)$ has one variable. It follows that $b(u) = b(v)$.

Let $n \geqslant 3$. Since $[G_n = H_n] \subset [G_n = I_n]$ (see Figure 1), $i_n(u) = i_n(v)$ implies $h_n(u) = h_n(v)$. If $u = v$ has less than n variables, then $h_n(u) = h_n(v)$ implies $b(u) = b(v)$ by Lemma 5.3. Suppose $u = v$ has n variables. We have that $i_n(u) = i_n(v)$ implies $i_n(s(u)) = i_n(s(v))$, $\sigma(u) = \sigma(v)$ and $\bar{i}_{n-1}(u) = \bar{i}_{n-1}(v)$, and $\bar{h}_n(u) = \bar{h}_n(v)$ implies that $h_{n-1}(u) = h_{n-1}(v)$, $\varepsilon(u) = \varepsilon(v)$ and $\bar{h}_n(e(u)) = \bar{h}_n(e(v))$ (by [**23**, Proposition 3.7]). Since $s(u) = s(v)$ and $e(u) = e(v)$ have $n-1$ variables each, it follows by Lemma 5.3 that $b(s(u)) = b(s(v))$ and $b(e(u)) = b(e(v))$. Thus, $b(u) = b(v)$. $\qquad\square$

Proposition 5.7. *Let $\mathcal{V}$ be a variety of bands in column 2 or column 4 of Figure 1. If $\mathcal{V} = [\bar{G}_n G_n = \bar{I}_n H_n]$ or $\mathcal{V} = [\bar{G}_n G_n = \bar{H}_n I_n]$, $n \geqslant 3$, then, for every set X with $|X| < n$, $B_{\mathcal{V}}(X)$ is d-invariant.*

Proof. By duality, we may assume that $\mathcal{V} = [\bar{G}_n G_n = \bar{I}_n H_n]$. Then for all $u, v \in X^+$, $u = v$ in $B_{\mathcal{V}}(X)$ if and only if $i_{n-1}(u) = i_{n-1}(v)$ and $\bar{h}_{n-1}(u) = \bar{h}_{n-1}(v)$. Thus, the result follows from Lemma 5.6. $\qquad\square$

For the varieties $\mathcal{V} = [\bar{G}_n G_{n-1} = \bar{H}_n I_{n-1}]$ and their duals, the result is as follows.

Proposition 5.8. *Let $\mathcal{V}$ be a variety of bands in column 2 or column 4 of Figure 1. If $\mathcal{V} = [\bar{G}_n G_{n-1} = \bar{H}_n I_{n-1}]$ or $\mathcal{V} = [\bar{G}_{n-1} G_n = \bar{I}_{n-1} H_n]$, $n \geqslant 4$, then, for every set X,*

(1) *if $|X| < n - 1$, then $B_{\mathcal{V}}(X)$ is d-invariant,*

(2) *if $|X| = n - 1$, then $B_{\mathcal{V}}(X)$ is not d-invariant.*

Proof. By duality, we may assume that $\mathcal{V} = [\bar{G}_n G_{n-1} = \bar{H}_n I_{n-1}]$. To prove (1), suppose $|X| < n-1$, and let $u, v \in X^+$ with $u = v$ in $B_{\mathcal{V}}(X)$. Consider $\mathcal{V}_1 = [G_{n-1} = H_{n-1}]$. Since $\mathcal{V}_1 \subset \mathcal{V}$, $u = v$ in $B_{\mathcal{V}_1}(X)$, so $h_{n-1}(u) = h_{n-1}(v)$. Since $u = v$ has less than $n - 1$ variables, $u = v$ in $B_{\mathcal{B}}(X)$ by Lemma 5.3. Thus, $\bar{u} = \bar{v}$ in $B_{\mathcal{B}}(X)$, and so $\bar{u} = \bar{v}$ in $B_{\mathcal{V}}(X)$.

To prove (2), suppose that $|X| = n-1$. Then $G_{n-1}, I_{n-1} \in X^+$. We have $G_{n-1} = I_{n-1}$ in $B_{\mathcal{V}}(X)$ (since $\mathcal{V} \subset [G_{n-1} = I_{n-1}]$) and $\bar{G}_{n-1} \neq \bar{I}_{n-1}$ in $B_{\mathcal{V}}(X)$ (since $\mathcal{V} \not\subset [\bar{G}_{n-1} = \bar{I}_{n-1}]$) (see Figure 1). Thus, $B_{\mathcal{V}}(X)$ is not d-invariant. $\qquad\square$

We dealt with the variety $\mathcal{LZ} = [xy = x]$ of left zero semigroups and its dual $\mathcal{RZ} = [xy = y]$ in §4. It remains to consider the variety $\mathcal{LN} = [xyz = xzy]$ of left normal bands. (Its dual is the variety $\mathcal{RN} = [xyz = yxz]$ of right normal bands.) The identity $[xyz = xzy]$ defining $\mathcal{LN}$ has three variables. Let X be a set with two variables, say $X = \{x, y\}$. Since $\mathcal{LN}$ is the join of $\mathcal{LZ}$ and $\mathcal{SL}$, for all $u, v \in X^+$, $u = v$ in $B_{\mathcal{LN}}(X)$ if and only if $h_2(u) = h_2(v)$ and $C(u) = C(v)$. It follows that $B_{\mathcal{LN}}(X)$ is not d-invariant, since $xy = xyx$ in $B_{\mathcal{LN}}(X)$ but $yx \neq xyx$ in $B_{\mathcal{LN}}(X)$.

The foregoing discussion, Propositions 5.7 and 5.8 and Theorem 3.8 give us the following corollary.

Corollary 5.9. *Let $\mathcal{V}$ be a variety of bands in column 2 or column 4 of Figure 1. Then, for every set X with $|X| \geqslant 2$,*

(1) *if $\mathcal{V} = [\bar{G}_n G_n = \bar{I}_n H_n]$ or $\mathcal{V} = [\bar{G}_n G_n = \bar{H}_n I_n]$, $n \geqslant 3$, and $|X| < n$, then $\mathrm{Aut}(\mathrm{End}(B_{\mathcal{V}}(X))) \cong S(X) \times C_2$,*

(2) *if $\mathcal{V} = [\bar{G}_n G_{n-1} = \bar{H}_n I_{n-1}]$ or $\mathcal{V} = [\bar{G}_{n-1} G_n = \bar{I}_{n-1} H_n]$, $n \geqslant 4$, and $|X| < n - 1$, then $\mathrm{Aut}(\mathrm{End}(B_{\mathcal{V}}(X))) \cong S(X) \times C_2$,*

(3) *if $\mathcal{V} = [\bar{G}_n G_{n-1} = \bar{H}_n I_{n-1}]$ or $\mathcal{V} = [\bar{G}_{n-1} G_n = \bar{I}_{n-1} H_n]$, $n \geqslant 4$, and $|X| = n - 1$, then $\mathrm{Aut}(\mathrm{End}(B_{\mathcal{V}}(X))) \cong S(X)$,*

(4) *if $\mathcal{V} = \mathcal{LN}$ or $\mathcal{V} = \mathcal{RN}$, and $|X| = 2$, then $\mathrm{Aut}(\mathrm{End}(B_{\mathcal{V}}(X))) \cong S(X)$.*

If $|X| = 1$, say $X = \{x\}$, then for, every variety $\mathcal{V}$ of bands $B_{\mathcal{V}}(X) = \{x\}$, and so $\mathrm{Aut}(\mathrm{End}(B_{\mathcal{V}}(X))) = \{1\}$. Hence, we have covered all possible varieties of bands $\mathcal{V}$ and all possible sets X, and our determination of $\mathrm{Aut}(\mathrm{End}(B_{\mathcal{V}}(X)))$ is complete.

6. Problems

In connection with the general problem that motivated this paper and in parallel with many similar open problems in various branches of algebra, we suggest the following problems from semigroup theory.

(1) Describe $\mathrm{Aut}(\mathrm{Aut}(F))$ and $\mathrm{Aut}(\mathrm{End}(F))$, where F is a free inverse semigroup, that is, a free algebra in the variety of all inverse semigroups. (This is the most important free semigroup apart from the free group (see [**18**, Chapter VIII]).)

(2) Solve analogous problems for other varieties of inverse semigroups (see [**18**, Chapter XII]).

(3) Describe $\mathrm{Aut}(\mathrm{Aut}(F))$ and $\mathrm{Aut}(\mathrm{End}(F))$, where F is a free completely regular semigroup or a free $*$-regular semigroup (see [**14**, **21**]).

(4) For a pseudovariety $\mathcal{V}$ of semigroups, describe $\mathrm{Aut}(\mathrm{Aut}(F))$ and $\mathrm{Aut}(\mathrm{End}(F))$, where F is the free pro-$\mathcal{V}$ semigroup over a finite alphabet (see [**1**, **2**]).

(5) Describe $\mathrm{Aut}(\mathrm{Aut}(F))$ and $\mathrm{Aut}(\mathrm{End}(F))$, where F is an independence algebra (see [**5**, **12**]).

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