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EXPONENTIAL DOMINANCE AND UNCERTAINTY FOR WEIGHTED RESIDUAL LIFE MEASURES

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In this article notions of exponential dominance and uncertainty for weighted and unweighted distributions are explored and used to compare values of the informational energy function and the differential entropy. Stochastic inequalities and bounds for cross-discrimination and uncertainty measures in weighted and unweighted residual life distribution functions and related reliability measures are presented. Moment-type inequalities for the comparisons of weighted and unweighted residual life distributions are also presented.

1. INTRODUCTION

In many situations, the usual random sample from a population of interest might not be available, due in part to the data having unequal probabilities of entering the sample. The class of weighted distributions can be used to model this bias by adjusting the probabilities of actual occurrence of events. Weighted distributions are of tremendous practical importance in various aspects of reliability, biometry, survival analysis, and renewal theory, to mention a few areas. In renewal theory, the residual lifetime has a limiting distribution that is a weighted distribution with the weight function equal to the reciprocal of the of the hazard or failure rate function. When observations are selected with probability proportional to their "length," the resulting distribution is referred to as a length-biased distribution. Zelen and Feinleib [10] showed that cases of chronic diseases identified by early detection screening programs constitute a length-biased sampling because individuals with a long preclinical disease phase have greater probability of being identified. Length-biased distributions occur naturally in a wide

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variety of settings and are discussed by several authors, including but not limited to Gupta and Keating [3], Nanda and Jain [5], Oluyede [6], and Patil and Rao [7].

The main objective of this article is to explore the notions of exponential dominance and uncertainty as well as obtain and compare cross-discrimination and uncertainty measures for weighted residual life distributions, in general, and for length-biased distributions, in particular. This article is organized as follows. Section 2 contains some basic definitions and utility notions. In Section 3 we present results on exponential dominance and compare cross-entropy or discrimination information and uncertainty measures for weighted and unweighted residual life distributions. Some results on the differential entropy and informational energy function involving weighted distributions as well as weighted residual life distributions and related reliability measures are presented. Section 4 is concerned with comparisons, bounds, and moment-type inequalities for weighted residual life reliability functions and related distributions.

2. SOME DEFINITIONS, UTILITY NOTIONS, AND COMPARISONS

In this section we present some basic definitions and useful notions. Let $\mathcal{F}$ be the set of absolutely continuous distribution function satisfying

$$F(0) = 0, \qquad \lim_{x\to\infty} F(x) = 1, \qquad \mathrm{Sup}\{x : F(x) < 1\} = \infty. \tag{1}$$

Note that if the mean of a random variable with distribution function in $\mathcal{F}$ is finite, it is positive.

Let X be a nonnegative random variable with reliability or survival function $\overline{F}$ and probability density function (p.d.f.) f, where $\overline{F}(x) = \int_x^\infty f(t)\,dt$. The weighted random variable X_W has a p.d.f. given by

$$f_W(x) = \frac{W(x)f(x)}{\delta^*}, \tag{2}$$

where δ^* is a normalizing constant. The weighted reliability function is given by

$$\overline{F}_W(x) = \delta^{*-1}\int_x^\infty W(y)f(y)\,dy, \tag{3}$$

where $W(x)$ is a positive real function and $0 < \delta^* = E(W(X)) < \infty$. See Patil and Rao [7] for details.

The weighted survival function $\overline{F}_W(x)$ can be rewritten as

$$\frac{\overline{F}_W(x)}{\overline{F}(x)} = \frac{E[W(X)|X > x]}{\delta^*} \tag{4}$$

and the corresponding ratio of hazard functions is given by

$$\frac{\lambda_{F_W}(x)}{\lambda_F(x)} = \frac{W(x)}{E[W(X)|X > x]} = \frac{W(x)}{V_{F_W}(x)}. \tag{5}$$

Under length-biased distribution, $W(x) = x$ and $V_{F_l}(x) = E_{F_l}(X|X > x)$ is the vitality function. The vitality function can also be written as

$$V_{F_l}(x) = E_{F_l}(X - x|X > x) + x = \delta_{F_l}(x) + x, \tag{6}$$

where

$$\delta_{F_l}(x) = \int_x^\infty \frac{\overline{F}_l(y)\,dy}{\overline{F}_l(x)} \tag{7}$$

is the mean residual life function (MRLF) of the length-biased distribution function F_l.

It is clear that if $V_{F_l}(x)$ is increasing in x, then

$$\overline{F_l}(x) = \mu^{-1} \int_x^\infty y\,dF(y) \geq c^{-1}\overline{F}(x) \geq \overline{F}(x),$$

where $c = \overline{F}(0), \overline{F}_W(0) = \overline{F}_l(0) = 1$, and $c^{-1} = \overline{F}_l(0)/\overline{F}(0)$. Indeed, if the function $E[W(X)|X > x]$ is increasing in x, then $\overline{F}_W(x)/\overline{F}(x)$ is increasing in $x \geq 0$.

For the ease of reference, we give here succinct definitions of stochastic ordering, increasing (decreasing) hazard rate, monotone likelihood ratio ordering, as well as information energy, differential entropy and exponential dominance that are useful for the results presented in subsequent sections. See Szekli [8] and references therein.

DEFINITION 2.1: *Let X and X_W be two nonnegative random variables with distribution functions F and F_W, respectively. We say $F <_{\mathrm{st}} F_W$ if $\overline{F}(x) \leq \overline{F}_W(x)$ for $x \geq 0$.*

DEFINITION 2.2:

1. *A distribution function F is said to have increasing (decreasing) hazard rate on $[0, \infty)$, denoted by IHR (DHR), if $F(0-) = 0$, $F(0) < 1$, and $P(X > x + t|X > t) = \overline{F}(x + t)/\overline{F}(t)$ is decreasing (increasing) in $t \geq 0$ for each $x > 0$.*
2. *A distribution function F is called new better than used in expectation (NBUE) if $\int_x^\infty \overline{F}(y)\,dy \leq \mu_F \overline{F}(x)$, where $\mu_F = \int_0^\infty \overline{F}(x)\,dx$. The inequality is reversed for new worst than used in expectation (NWUE).*

Note that if F has decreasing hazard rate (DHR) and $\mu_F = \int_0^\infty \overline{F}(x)\,dx < \infty$, then F has increasing mean residual life (IMRL).

DEFINITION 2.3: *Let X and X_W be two nonnegative random variables with probability density functions f and f_W, respectively. The random variable X is said to be larger than X_W in monotone likelihood ratio ordering ($X \geq_{\mathrm{lr}} X_W$) if $f(x)/f_W(x)$ is nondecreasing in $x \geq 0$.*

Note that if the weight function $W(x)$ is increasing on $[0, \infty)$, then $X_W \geq_{\mathrm{lr}} X$ and $\lambda_{\overline{F}_W}(x) \leq \lambda_F(x)$ for all $x \geq 0$, so that $\overline{F}_W(x) \geq \overline{F}(x)$ for all $x \geq 0$, where $\lambda_F(x) = f(x)/\overline{F}(x)$.

DEFINITION 2.4: *The energy associated with a probability density function f in $\mathcal{F}$ is given by*

$$e(f) = \int f^2(x)\,dx, \tag{8}$$

where $f(x) = dF(x)/dx$ and F is the corresponding distribution function.

Example 2.5: Normal distribution. The energy associated with the normal p.d.f. f is

$$\begin{aligned} e(f(\mu,\sigma)) &= \int_{\mathbf{R}} f^2(x;\mu,\sigma)\,dx \\ &= \pi^{-1/2}(2\sigma)^{-1}. \end{aligned} \tag{9}$$

Let U and V be sets. A function $h : U \to V$ that is one-to-one and onto is called a bijection or bijective function from U to V; that is, a function $h : U \to V$ is called a bijection if and only if for each y in V, there exists exactly one x in U such that $h(x) = y$. Note that $e(f(\mu,\sigma))$ is a bijective function of σ. Consequently, if σ_f and σ_{f_W} are the standard deviations of the distribution functions F and F_W, then

$$e(f(\mu,\sigma)) \geq e(f_W(\mu,\sigma)) \qquad \text{if and only if } \sigma_{f_W} \geq \sigma_f. \tag{10}$$

DEFINITION 2.6: *Let X be a nonnegative random variable with finite variance and differentiable p.d.f f. The uncertainty measure associated with a distribution function F in $\mathcal{F}$ is the differential entropy given by*

$$H(f(X)) = -E_f(\log f(X)) = -\int f(x)\log f(x)\,dx, \tag{11}$$

where $f(x) = dF(x)/dx$.

DEFINITION 2.7: *Let f and g be two bounded functions on $(0,\infty)$. Then f is exponentially dominated by g if for each ϵ, $0 < \epsilon < 1$, there exist $A(\epsilon) < \infty$, such that*

$$f(x) \leq A(\epsilon)(g(x))^{1-\epsilon} \qquad \textit{for all } x > 0. \tag{12}$$

If f and g are exponentially dominated by each other, then they are said to be exponentially equivalent. The definition of exponential dominance is useful in the comparison of small values of bounded nonnegative functions.

Consider two renewal processes with life distribution $F(x)$ and weighted distribution function $F_W(x)$, with weight function $W(x) > 0$. Let X_t denote the residual life of the unit functioning at time t. Then, as $t \longrightarrow \infty$, X_t has the limiting reliability

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function given by

$$\overline{F}_e(x) = \mu_F^{-1} \int_x^\infty \overline{F}(y)dy, \qquad \text{for all } x \geq 0. \tag{13}$$

The corresponding limiting p.d.f. is

$$f_e(x) = \overline{F}(x)/\mu_F \qquad \text{for all } x \geq 0. \tag{14}$$

The weighted equilibrium reliability or survival function is

$$\overline{F}_{W_e}(x) = \mu_{F_W}^{-1} \int_x^\infty \overline{F}_W(y)dy, \tag{15}$$

where

$$\overline{F}_{W_e}(x) = \mu_{F_W}^{-1} \overline{F}_W(x)\delta_{F_W}(x) \qquad \text{for all } x \geq 0. \tag{16}$$

Additionally, note that if F_W is NBUE, then $\mu_{F_W} \geq \delta_{F_W}(x) = \int_x^\infty \overline{F}_W(y)\,dy/\overline{F}_W(x)$ and

$$\frac{\overline{F}_{W_e}(x)}{\overline{F}_W(x)} = \frac{\delta_{F_W}(x)}{\mu_{F_W}} \leq 1 \tag{17}$$

for all $x \geq 0$. In fact, if F_W is NBUE, then $\overline{F}_{W_e}(x)$ and $\overline{F}_W(x)$ are stochastically ordered. The corresponding weighted equilibrium hazard function is

$$\lambda_{F_{W_e}}(x) = f_{W_e}(x)/\overline{F}_{W_e}(x) = (\delta_{F_{W_e}}(x))^{-1} \qquad \text{for all } x \geq 0, \tag{18}$$

where

$$\delta_{F_{W_e}}(x) = (\delta_F(x)\overline{F}(x))^{-1} \int_x^\infty \overline{F}(y)\delta_F(y)\,dy \qquad \text{for all } x \geq 0. \tag{19}$$

3. DISCRIMINATION AND UNCERTAINTY

In this section, we present and compare cross-entropy or discrimination information and uncertainty measures for weighted and unweighted residual life distributions. For any such comparisons, interest should be in comparisons that are compatible and practically possible. In this light, one might be inclined to investigate and compare the possibility of sampling or selection of experiment from weighted distribution as opposed to the parent or original distribution. In a similar setting, comparison might be restricted to a class of distributions, including possibly distributions with monotone likelihood ratios, comparisons via some informational measures such as informational energy function, Fisher, Shannon [9], or Kullback–Leibler information. We also present results on the uncertainty of the weighted residual life distribution of a component at time t, denoted by $H(X_{W_e}; t)$. The quantity $H(X_{W_e}; t)$ is the entropy of the residual life variable $(X_{W_e} - t | X_{W_e} > t)$ and is the expected uncertainty in

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the conditional distribution of $X_{W_e} - t$, given $X_{W_e} > t$ about the predictability of the remaining lifetime of the component that has survived for time t.

Now, consider the following weighted cross-entropy or discrimination information measures:

$$H^*(f, f_{W_e}; t) = \int_t^\infty f(x)\, \log(f(x)/f_{W_e}(x))\, dx \tag{20}$$

and

$$H^*(f_e, f_{W_e}; t) = \int_t^\infty f_e(x)\, \log(f_e(x)/f_{W_e}(x))\, dx. \tag{21}$$

Similarly, let

$$H^*(f_{W_e}, f_e; t) = \int_t^\infty f_{W_e}(x)\, \log(f_{W_e}(x)/f_e(x))\, dx. \tag{22}$$

Additionally, define

$$H(f, f_{W_e}; t) = \int_t^\infty \frac{f(x)}{\overline{F}(t)} \log\left\{\frac{f(x)/\overline{F}(t)}{f_{W_e}(x)/\overline{F}_{W_e}(t)}\right\} dx \tag{23}$$

and

$$H(f_e, f_{W_e}; t) = \int_t^\infty \frac{f_e(x)}{\overline{F}_e(t)} \log\left\{\frac{f_e(x)/\overline{F}_e(t)}{f_{W_e}(x)/\overline{F}_{W_e}(t)}\right\} dx \tag{24}$$

and the variation measures as

$$I_{V_e}(f_e, f_{W_e}; t) = \int_t^\infty |f_e(x) - f_{W_e}(x)|\, dx. \tag{25}$$

The following definition is due to Ebrahimi and Pellerey [2].

DEFINITION 3.1: *The uncertainty of residual life distribution $H(X;t)$ of a component at time t is the entropy of the residual life random variable $(X - t|X > t)$ and is given by*

$$\begin{aligned} H(X;t) &= -\int_t^\infty \frac{f(x)}{\overline{F}(t)} \log \frac{f(x)}{\overline{F}(t)}\, dx \\ &= \log \overline{F}(t) - \{\overline{F}(t)\}^{-1} \int_t^\infty f(x)\, \log f(x)\, dx \\ &= 1 - \{\overline{F}(t)\}^{-1} \int_t^\infty f(x)\, \log\{\lambda_F(x)\}\, dx, \end{aligned} \tag{26}$$

where $\lambda_F(x) = f(x)/\overline{F}(x)$ is the hazard or failure rate function.

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The entropy of the weighted residual life random variable $(X_W - t|X_W > t)$ is given by

$$H(X_W;t) = -\int_t^\infty \frac{f_W(x)}{\overline{F}_W(t)} \log \frac{f_W(x)}{\overline{F}_W(t)}\, dx$$
$$= \log \overline{F}_W(t) - \{\overline{F}_W(t)\}^{-1} \int_t^\infty f_W(x) \log f_W(x)\, dx. \qquad \textbf{(27)}$$

Under length-biased sampling, $H(X_W;t)$ reduces to

$$H(X_l;t) = 1 - \{\overline{F}(t)V_F(t)\}^{-1} \int_t^\infty xf(x) \log(x\lambda_F(x)/V_F(x))\, dx, \qquad \textbf{(28)}$$

where $V_F(t) = E(X - t|X > t) + t$. Note that $H(f;t) = H(X;t)$ is the expected uncertainty in the conditional distribution of $X - t$, given $X > t$ about the predictability of the remaining lifetime of the component that has survived for time t.

THEOREM 3.2: *Suppose f_w is exponentially dominated by g_w. If $\{f_{W_n}\}_{n\geq 1}$ and $\{g_{W_n}\}_{n\geq 1}$ are sequences of bounded functions, where $\{f_{W_n}\}_{n\geq 1}$ is exponentially dominated by $\{g_{W_n}\}_{n\geq 1}$, then the following hold:*

1. *$e(f_w) = \int f_w^2(x)\, dx$ is exponentially dominated by $e(g_w)$.*
2. *$H(f_w) = -\int f_w(x) \log f_w(x)\, dx$ is exponentially dominated by $H(g_w)$.*
3. *$\lim_{k\to\infty} \mathrm{Sup}\{e(g_{W_n})^{1/k} - (e(f_{W_n})^{1/k}\} \leq 0$.*
4. *$\lim_{k\to\infty} \mathrm{Sup}\{H(g_{W_n})^{1/k} - (H(f_{W_n})^{1/k}\} \geq 0$.*

PROOF:

1. Let $f_w^*(x) = f_w^2(x)$ and $g_w^*(x) = g_w^2(x)$ and apply Jensen's inequality to the concave function $y \mapsto y^{1-\epsilon}$ to obtain

$$e(f_w) = \int f_w^*(x)\, dx \qquad \textbf{(29)}$$
$$= \int f_w^2(x)\, dx$$
$$\leq \int A^2(\epsilon)(g_w^2(x))^{1-\epsilon}\, dx$$
$$= \int C(\epsilon)(g_w^*(x))^{1-\epsilon}\, dx$$
$$\leq C(\epsilon)\left(\int g_w^*(x)\, dx\right)^{1-\epsilon}$$
$$= C(\epsilon)(e(g_w))^{1-\epsilon}, \qquad \textbf{(30)}$$

where $C(\epsilon) = A^2(\epsilon) < \infty$.

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2. Let $f_{_W}^{**}(x) = -f_{_W}(x)\,\log(f_{_W}(x))$ and $g_{_W}^{**}(x) = -g_{_W}(x)\,\log(g_{_W}(x))$, $\forall x > 0$. Since the function $y\,\log(y)$ is convex, apply Jensen's inequality to the concave function $y \mapsto (-y\,\log(y))^{1-\epsilon}$ to obtain

$$\begin{aligned} H(f_W) &= \int f_W^{**}(x)\,dx && (31)\\ &= -\int f_W(x)\,\log(f_W(x))\,dx \\ &\le \int D(\epsilon)(g_W^{**}(x))^{1-\epsilon}\,dx \\ &\le D(\epsilon)\left(\int g_W^{**}(x)\,dx\right)^{1-\epsilon} \\ &= D(\epsilon)\left(-\int g_W(x)\,\log(g_W(x)\,dx\right)^{1-\epsilon} \\ &= D(\epsilon)(H(g_W))^{1-\epsilon}, && (32) \end{aligned}$$

where $D(\epsilon) < \infty$.

3. Note that since $\{f_{_{W_n}}^{*}\}_{n\ge1}$ and $\{g_{_{W_n}}^{*}\}_{n\ge1}$ are sequences of bounded functions, there exists convergent subsequences such that $e(f) = \lim_{j\to\infty} e(f_{_{W_{n_j}}})$ and $e(g) = \lim_{j\to\infty} e(g_{_{W_{n_j}}})$.
Let $e(g_{_{W_n}})^{1/k} = L_k$ and $e(f_{_{W_n}})^{1/k} = M_k$. Then

$$\lim_{k\to\infty} e(g_{_{W_n}})^{1/k} = \lim_{k\to\infty} L_k = L$$

and

$$\lim_{k\to\infty} e(f_{_{W_n}})^{1/k} = \lim_{k\to\infty} M_k = M.$$

Consequently, $L \le A(\epsilon)M^{1-\epsilon}$ for every $\epsilon \in (0,1)$, and the result now follows.

4. This follows by considering the bounded sequences of functions $\{f_{W_n}^{**}\}_{n\ge1}$ and $\{g_{W_n}^{**}\}_{n\ge1}$, as in the proof of part 3.

■

THEOREM 3.3: *Let $W(x) > 0$ be a nondecreasing weight function with $W(x) > 0$ for all $x \ge 0$. If F_W is NBUE, then*

$$H(f_{_W}, f_{_{W_e}}; t) \ge H^*(f_{_W}, f_{_{W_e}}; t) \tag{33}$$

for all $t \ge 0$.

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PROOF: Note that since F_W is NBUE, we have $\delta_{F_W}(x)/\mu_{F_W} \leq 1$ and $\overline{F}_{W_e}(t) \leq \overline{F}_W(t)$, so that $\overline{F}_{W_e}(t)/\overline{F}_W(t) \geq 1$ and $\log(\overline{F}_{W_e}(t)/\overline{F}_W(t)) \geq 0$, $\forall t \geq 0$. Now,

$$\begin{aligned} H(f_w, f_{W_e}; t) &= \log \frac{\overline{F}_{W_e}(t)}{\overline{F}_W(t)} + (\overline{F}_W(t))^{-1} \int_t^\infty f_w(x) \log(f_w(x)/f_{w_e})(x)\, dx \\ &= \log \frac{\overline{F}_{W_e}(t)}{\overline{F}_W(t)} + (\overline{F}_W(t))^{-1} H^*(f_W, f_{W_e}; t) \\ &\geq (\overline{F}_W(t))^{-1} H^*(f_W, f_{W_e}; t) \\ &\geq H^*(f_W, f_{W_e}; t), \end{aligned} \quad \mathbf{(34)}$$

$\forall t \geq 0$. ■

The next result follows from Theorem 2.1 of Ebrahimi and Pellerey [2].

THEOREM 3.4: *Let $W(x) > 0$ be a nondecreasing weight function with $W(x) > 0$ for all $x \geq 0$. If F_e or F_{W_e} are DFR distributions, then*

$$H(f_{W_e}; t) \geq H(f_e; t) \quad \mathbf{(35)}$$

for all $t \geq 0$.

PROOF: Note that $W(x) > 0$ nondecreasing implies $\lambda_{F_{W_e}}(x) \geq \lambda_{F_e}(x)$, so that $(\overline{F}_{W_e}(x))^{-1} \geq (\overline{F}_e(x))^{-1}$ for all $x \geq 0$. Now, using the fact that F_e or F_{W_e} are DFR distributions leads to

$$(\overline{F}_{W_e}(x))^{-1} \int_t^\infty f_{W_e}(x) \log(\lambda_{F_{W_e}}(x))\, dx \geq (\overline{F}_e(x))^{-1} \int_t^\infty f_e(x) \log(\lambda_{F_e}(x))\, dx, \quad \mathbf{(36)}$$

so that

$$-(\overline{F}_{W_e}(x))^{-1} \int_t^\infty f_{W_e}(x) \log(\lambda_{F_{W_e}}(x))\, dx \leq -(\overline{F}_e(x))^{-1} \int_t^\infty f_e(x) \log(\lambda_{F_e}(x))\, dx. \quad \mathbf{(37)}$$

Consequently,

$$H(f_{W_e}; t) \geq H(f_e; t) \qquad \text{for all } t \geq 0. \quad \mathbf{(38)}$$

■

Example 3.5: Let $\overline{F}_l(x) = (1 + x/\theta)\exp(-x/\theta)$ be the length-biased exponential reliability function corresponding to $\overline{F}(x) = \exp(-x/\theta)$. Then the corresponding

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length-biased equilibrium reliability function is given by

$$\overline{F}_{l_e}(x) = \left(1 + \frac{x}{2\theta}\right) \exp\left(-\frac{x}{\theta}\right) \qquad \text{for all } x \geq 0 \text{ and } \theta > 0. \tag{39}$$

The length-biased equilibrium hazard and mean residual life functions are

$$\lambda_{F_{l_e}}(x) = \frac{x+1}{2\theta + x} \qquad \text{and} \qquad \delta_{F_{l_e}}(x) = \frac{x + 2\theta}{x+1}, \qquad \text{respectively.} \tag{40}$$

Clearly, $\lambda_{F_{l_e}}(x)$ is decreasing in $x \geq 0$; it follows therefore that

$$H(f_{W_e}; x) \geq H(f_e; x) \qquad \text{for all } x \geq 0. \tag{41}$$

THEOREM 3.6: *Let $W(x) \geq 0$ be a nondecreasing weight function. Then*

$$H^*(f_e, f_{W_e}; 0) = C^* - E_{F_e}[\log(W(X))], \tag{42}$$

where $C^* = \log(\delta^*)$ and

$$I_{V_e}(f_e, f_{W_e}; t) \geq H^*(f_{W_e}, f_e; t) \tag{43}$$

for all $t \geq 0$.

PROOF:

$$\begin{aligned} H^*(f_e, f_{W_e}; 0) &= \int_0^\infty \frac{f_e(x)}{\overline{F}_e(0)} \log\left(\frac{f_e(x)/\overline{F}_e(0)}{f_{W_e}(x)/\overline{F}_{W_e}(0)}\right) dx \\ &= \int_0^\infty f_e(x) \log\left(\frac{f_e(x)}{f_{W_e}(x)}\right) dx \\ &= \int_0^\infty f_e(x) \log\left(\frac{f_e(x)}{W(x) f_e(x)/\delta^*}\right) dx \\ &= -\int_0^\infty f_e(x) \log(W(x))\, dx + \log(\delta^*) \\ &= C^* - E_{F_e}(\log W(X)), \end{aligned} \tag{44}$$

where $C^* = \log(\delta^*)$. Now to show that $I_{V_e}(f_e, f_{W_e}; t) \geq H^*(f_{W_e}, f_e; t)$, note that

$$\begin{aligned} I_{V_e}(f_e, f_{W_e}; t) &= \int_t^\infty |f_e(x) - f_{W_e}(x)|\, dx \\ &= -\int_t^\infty (f_{W_e}(x) - f_e(x))\, dx \\ &= -\int_t^\infty f_{W_e}(x) \left(1 - \frac{f_e(x)}{f_{W_e}(x)}\right) dx \end{aligned}$$

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$$\geq -\int_t^\infty f_{W_e}(x)\,\log\left(\frac{f_e(x)}{f_{W_e}(x)}\right)\,dx$$

$$= H^*(f_{W_e}, f_e; t). \tag{45}$$

Consequently, $I_{V_e}(f_e, f_{W_e}; t) \geq H^*(f_{W_e}, f_e; t), \forall t \geq 0.$ ■

4. COMPARISONS AND INEQUALITIES FOR WEIGHTED RELIABILITY MEASURES

In this section we present some moment-type inequalities for the comparisons of weighted equilibrium life distributions. We assume that the weighted distribution function F_W and the weighted equilibrium life distribution function F_{W_e} are in $\mathcal{F}$, the set of absolutely continuous distribution functions given by (1).

The following results are due in part to the application of the lemma given by Brown [1].

THEOREM 4.1: *Assume that the weighted residual life function $\delta_{F_W}(x)$ is nondecreasing in $x \geq 0$ or that F_W has IMRL, $0 < \delta^* = E_F[W(X)] < \infty$; then*

$$\mathrm{Sup}_x|\overline{F}_{W_e}(x) - \overline{F}_W(x)| \leq 1 - \delta^*/\mu_{F_{W_e}} \tag{46}$$

whenever $\mu_{F_{W_e}} = \int_0^\infty \overline{F}_{W_e}(x)\,dx \geq \delta^*$.

PROOF: Since $\delta_{F_W}(x) = \mu_{F_W}\overline{F}_{W_e}(x)/\overline{F}_W(x)$ is nondecreasing, we obtain

$$\mathrm{Sup}_x|\overline{F}_{W_e}(x) - \overline{F}_W(x)| \leq 1 - (\delta^*)^{-1}\int_0^\infty (\overline{F}_W(x)/\overline{F}_{W_e}(x))W(x)f(x)\,dx$$

$$\leq 1 - (\delta^*)^{-1}\mu_{F_{W_e}}\left(\int_0^\infty W(x)f(x)\,dx/\mu_{F_{W_e}}\right)^2$$

$$= 1 - \delta^*/\mu_{F_{W_e}}. \tag{47}$$

■

THEOREM 4.2: *Assume that the weighted residual life function $\delta_F(x)$ is nondecreasing in $x \geq 0$ or that F has IMRL, and $0 < \mu_F = \int_0^\infty \overline{F}(x)\,dx < \infty$; then*

$$\mathrm{Sup}_x|\overline{F}_{W_e}(x) - \overline{F}(x)| \leq 1 - \mu_F/\mu_{F_{W_e}}, \tag{48}$$

where $\mu_{F_{W_e}} = \int_0^\infty \overline{F}_{W_e}(x)\,dx \geq \mu_F$.

PROOF: The result follow directly from Theorem 4.1. ■

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THEOREM 4.3: *Under residual length or size-biased sampling,*

$$\text{Sup}_x|\overline{F}_{l_e}(x) - \overline{F}_e(x)| \leq 1 - \mu^2/\mu_2$$
$$= \sigma_F^2/(\sigma_F^2 + \mu_F^2), \tag{49}$$

where $\overline{F}_{l_e}(x) = (\mu_{F_l})^{-1}\int_x^\infty \overline{F}_l(y)\,dy$, $\overline{F}_e(x)$ *is given by (13),* $\mu_2 = E(X^2)$, *and* $\sigma_F^2 =$ $\text{Var}(X)$ *is the variance of* X.

PROOF: This follows from the fact that

$$E_{F_e}(X^r) = E_F(X^{r+1})/(r+1)\mu_F, \tag{50}$$

$r \geq 1$, and $\mu_F = \int_0^\infty \overline{F}(x)\,dx$. ■

Example 4.4: Keilson [4] defines two first passage times, T_V (the ergodic sojourn time) and T_E (the ergodic exit time). The stationary renewal distribution for the finite-state ergodic Markov process in continuous time is

$$\overline{F}_E(x) = \mu_V^{-1}\int_x^\infty \overline{F}_V(t)\,dt, \tag{51}$$

where $\mu_V = E(T_V)$. Consider a repairable three-component parallel system with component failure rate 0.01 and component repair rate 1. The time to first system failure is the first passage to a subset B of the state space, where $B = \{(0,0,0)\}$. Brown [1] showed that $E_{F_E}(T_E) = 345,181.85$ and $E_{F_V}(T_V) = 343,333.33$, so that

$$\text{Sup}_x|\overline{F}_{l_e}(x) - \overline{F}(x)| \leq 1 - 343,333.33/345,181.85 = 0.005355, \tag{52}$$

where $\overline{F}_{l_e}(x) = (\mu_{F_l})^{-1}\int_x^\infty \overline{F}_l(y)\,dy$ and $\overline{F}_V = \overline{F}$. Consequently, F_{l_e} is within 0.005355 of F.

5. CONCLUDING REMARKS

In this article exponential dominance, uncertainty, and cross-discrimination information measures as well as inequalities and bounds are obtained for weighted equilibrium life distributions. Some results on the entropy of weighted equilibrium life distributions are given and comparisons made with the original or parent equilibrium life distribution. Moment-type inequalities and comparisons are also presented.

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