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CLASSIFICATION OF MAXIMAL FUCHSIAN SUBSGROUPS OF SOME BIANCHI GROUPS

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ABSTRACT. Let $d = 1, 2,$ or p, prime $p \equiv 3 \pmod 4$. Let O_d be the ring of integers of an imaginary quadratic field $\mathbf{Q}(\sqrt{-d})$. A complete classification of conjugacy classes of maximal non-elementary Fuchsian subgroups of $\mathrm{PSL}(2, O_d)$ in $\mathrm{PGL}(2, O_d)$ is given.

1. **Introduction.** Let d be a positive square free integer. Let O_d be the ring of integers in $\mathbf{Q}(\sqrt{-d})$. The groups $\mathrm{PSL}(2, O_d)$ are called collectively the *Bianchi groups*. The group $\mathrm{PSL}(2, O_d)$ acts by linear fractional transformations on the complex plane $\mathbf{C}$. A Fuchsian subgroup of $\mathrm{PSL}(2, O_d)$ fixes a circle or straight line C and the two components of $\mathbf{C}/\,C$. It is *non-elementary* if its limit set on C has more than two points. A Fuchsian subgroup of $\mathrm{PSL}(2, O_d)$ is *maximal* if it is not a subgroup of any other Fuchsian subgroup of $\mathrm{PSL}(2, O_d)$. Fuchsian subgroups has been investigated (see e.g. Fine [2,3], Maclachlan [4], Maclachlan and Reid [5,6]). In [4], for all values of d, C. Maclachlan showed that Fuchsian subgroups of $\mathrm{PSL}(2, O_d)$ exist, that maximal non-elementary ones are all arithmetic Fuchsian groups, and that for each d they are distributed in infinitely many commensurability classes. A maximal non-elementary Fuchsian subgroup of $\mathrm{PSL}(2, O_d)$ can be treated as the $\mathrm{PSL}(2, O_d)$-unit group of an indefinite rational binary Hermitian form in $\mathrm{PSL}(2, O_d)$. Here, for

$$(1) \qquad d = 1, 2, \text{ or } p, \text{ prime } p \equiv 3 \pmod 4,$$

complete classification of the conjugacy classes of maximal non-elementary Fuchsian subgroups of $\mathrm{PSL}(2, O_d)$ in $\mathrm{PGL}(2, O_d)$ is given. This is a simple consequence of the classification of rational indefinite binary Hermitian forms obtained in [10].

For $d = 1$, C. Maclachlan and A. W. Reid [6] also solved the problem of classification of maximal non-elementary Fuchsian subgroups of the Picard group. Their results are more detailed. They found the covolumes and indicated how to find the signatures of these subgroups. It is shown in [12] that, for any square-free d as positive as negative, the approach of the present paper can be applied to classify maximal arithmetic Fuchsian subgroups of $\mathrm{PSL}(2, \mathfrak{o})$ where $\mathfrak{o}$ is an order in $\mathbf{Q}(\sqrt{d})$.

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2. **Hermitian forms.** Let $a, c \in \mathbf{R}$, $b \in \mathbf{C}$, $F = \begin{vmatrix} a & b \\ \bar{b} & c \end{vmatrix}$, $x = (z, w) \in \mathbf{C}^2$, $x^* = (\bar{z}, \bar{w})^T$. The binary form

$$f(x) = f(z, w) = xFx^* \tag{2}$$

is called a *Hermitian form*. Here we shall only be concerned with indefinite Hermitian forms, that is with forms whose determinants

$$\det(F) = -\Delta = ac - |b|^2 < 0. \tag{3}$$

Any indefinite Hermitian form f defines the unique circle or straight line C with equation

$$f(z, 1) = a|z|^2 + b\bar{z} + \bar{b}z + c = 0. \tag{4}$$

It is a circle if and only if $a \neq 0$, in which case the radius of C

$$r(C) = |a|^{-1}|\det F|^{1/2} = |a|^{-1}\Delta^{1/2}. \tag{5}$$

Below, straight lines are considered as circles of infinite radius.

Conversely, given a circle C with equation $g(z) = 0$. Let $w \neq 0$. A form $f(z, w) = k|w|^2 g(z/w)$ is an indefinite Hermitian form for any nonzero real k. Thus, we have obtained a one-to-one correspondence between the set of circles on the complex plane $\mathbf{C}$ and the set of all nonzero real multiples of indefinite Hermitian forms.

Two Hermitian forms are said to be *equivalent* ($f \sim f'$) if there is a matrix $T \in \mathrm{PGL}(2, O_d)$ such that $f'(x) = f(xT)$. In that case, if $f(z, 1) = 0$ and $f'(z, 1) = 0$ are the equations of C' and C correspondingly then $C' = (T^t)^{-1}(C)$, with T defined as above. On the other hand, if $C' = (T^t)^{-1}(C)$, $T \in \mathrm{PGL}(2, O_d)$, and the equations of C' and C are $f'(z, 1) = 0$ and $f(z, 1) = 0$, then there is a nonzero real k such that $f'(x) = kf(xT)$, hence $f' \sim kf$.

A Hermitian form is said to be *rational* if $f(z, w) \in \mathbf{Q}$ for all $z, w \in O_d$. It can be easily shown that f is rational if and only if

$$a, c \in \mathbf{Q}, \quad b \in \mathbf{Q}(\sqrt{-d}). \tag{6}$$

We shall call a circle *rational* if its equation can be written in the form $f(z, 1) = 0$ where f is a rational Hermitian form. One can verify that, for a rational circle C, $C' = T(C)$ is rational for any $T \in \mathrm{PGL}(2, O_d)$. Therefore, there is a one-to-one correspondence between the $\mathrm{PGL}(2, O_d)$-orbits of rational circles and $\mathrm{PGL}(2, O_d)$-equivalency classes of nonzero real multiples of rational indefinite Hermitian forms.

Let f be a Hermitian form. We denote

$$\mu(f) = \inf |f(z, w)|, \tag{7}$$

where the infimum is taken over all $z, w \in O_d$ such that $f(z, w) \neq 0$. It was shown by Margulis [7,8] that for any $\epsilon > 0$ and any indefinite quadratic form Q in $n > 2$ variables, which is not a multiple of an integral form, there is a nonzero vector $x \in \mathbf{Z}^n$ such that $0 < |Q(x)| < \epsilon$.

Let $\{1, \omega\}$ be the standard basis of O_d. The quaternary form $Q(x) = f(x_1 + \omega x_2, x_3 + \omega x_4)$ is indefinite if and only if $f(z, w)$ is an indefinite Hermitian form. The theorem of Margulis implies the following.

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LEMMA 1. *Let f be a binary indefinite Hermitian form. Then $\mu(f) > 0$ if and only if f is a nonzero multiple of a rational Hermitian form.*

For a Hermitian form f, the number

(8) $$\nu(f) = \mu(f)|\Delta|^{-1/2}$$

is said to be the *normalized nonzero minimum* of f or, simply, *nonzero minimum* of f. Since the sets of values of two equivalent forms f and f' coincide, for any nonzero real k,

(9) $$\nu(f') = \nu(kf).$$

Denote the set of all binary indefinite Hermitian forms by H_d. The set $S_d = \{\nu(f) \mid f \in H_d\}$ is called the *spectrum of minima* of binary indefinite Hermitian forms over O_d. As Lemma 1 shows, if $\nu(f) \neq 0$ and $\nu(f) \in S_d$, then f is a multiple of a rational form. For any d, the spectrum S_d is discrete [11] (i.e., for any given $\delta > 0$, there is only a finite number of $\nu(f) \in S_d$ such that $\nu(f) > \delta$). The spectrum S_d was completely described in [10] for $d = 1, 2$, or p, prime $p \equiv 3 \pmod 4$. The author has also obtained a complete description of S_d for any d (as positive, as negative). The results will be published elsewhere.

3. **Fuchsian subgroups.** Upper half-3-space $H^3 = \{(z,t), z \in \mathbf{C}, t > 0\}$ with metric $t^{-2}(|dz|^2 + dt^2)$ can be used as a model for hyperbolic 3-space. The group of all orientation-preserving isometries of H^3 can be identified with $\mathrm{PSL}(2,\mathbf{C})$ (see e.g. [4]). The groups $\mathrm{PSL}(2,O_d)$ are discrete subgroups of $\mathrm{PSL}(2,\mathbf{C})$. A Fuchsian subgroup of $G = \mathrm{PSL}(2,O_d)$ stabilizing the circle C with equation $f(z,1) = 0$ in $\mathbf{C}$ stabilizes the hemisphere S_f in H^3 with equation $f(z,1)+at^2 = 0$. Here $f(z,w)$ is an indefinite Hermitian form. The group $\Gamma = \mathrm{Stab}(C,G)$ can be identified with the group

(10) $$\mathrm{PSU}(f,O_d) = \{T \in G, \mathrm{TFT}^* = \pm F\}$$

where F is the matrix of the Hermitian form f. Let

(11) $$\rho(\Gamma) = \sup r\big(T(C)\big)$$

where the supremum is taken over all $T \in G$ such that $r\big(T(C)\big) < \infty$. It is clear that $\rho(\Gamma)$ is constant on each conjugacy class of Fuchsian subgroups of G in $\mathrm{PGL}(2,O_d)$.

LEMMA 2 (MACLACHLAN [4], P. 306–307). *A circle C in the complex plane $\mathbf{C}$ is rational if and only if its stabilizer Γ is a non-elementary Fuchsian subgroup of G.*

As follows from (5), (7), and (11),

(12) $$\rho(\Gamma) = 1/\nu(f),$$

provided there are $z, w \in O_d$, g.c.d. $(z,w) = 1$, such that $\mu(f) = |f(z,w)|$. As was shown in [10], $\mu(f) = 1$ for any integral primitive indefinite Hermitian form if d belongs to the sequence in (1). If f is integral, the g.c.d. $(z,w) = 1$ for any solution of the equation $f(z,w) = 1$ in O_d. Hence, formula (12) is true for any f in the case under consideration, and Lemma 1 is equivalent to the following.

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THEOREM 1. *Let $d = 1, 2$, or p, prime $p \equiv 3 \pmod 4$. Let C be a circle or a straight line in the complex plane. The Fuchsian group $\Gamma = \mathrm{Stab}\big(C, \mathrm{PSL}(2, O_d)\big)$ is non-elementary if and only if $\rho(\Gamma) < \infty$.*

For a binary rational Hermitian form f, the equation $f(z, w) = 0$ has a solution in O_d if and only if $\Delta(f) = |\alpha|^2$ for some $\alpha \in \mathbf{Q}(\sqrt{-d})$ [10]. By (5), the circle C with equation $f(z, 1) = 0$ contains a point in $\mathbf{Q}(\sqrt{-d})$ if and only if $r(C)^2$ is the norm of some element of the field $\mathbf{Q}(\sqrt{-d})$.

THEOREM 2. *Let $d = 1, 2$, or p, prime $p \equiv 3 \pmod 4$. A maximal non-elementary Fuchsian subgroup Γ of $\mathrm{PSL}(2, O_d)$ is non-cocompact if and only if $\rho(\Gamma) = |\alpha|$ for some $\alpha \in \mathbf{Q}(\sqrt{-d})$.*

REMARK. In [12], for any d, $\rho(\Gamma)$ is defined to be equal to $1/\nu(f)$. With this definition, the assumption that d is as in (1) can be omitted in the statements of Theorem 1 and 2.

PROOF. Let f be a rational Hermitian form with matrix F. If f is anisotropic, the circle C with equation $f(z, 1) = 0$ contains no point in $\mathbf{Q}(\sqrt{-d})$. Hence, Γ contains no parabolic element and, therefore, is cocompact.

Let

$$f(z, w) = a|z|^2 + b\bar{z}w + \bar{b}z\bar{w} + c|w|^2 = 0, \tag{13}$$

for some $z, w \in O_d$, which can be written in the form

$$\bar{z}A + \bar{w}B = 0, \tag{14}$$

where

$$A = az + bw, \quad B = \bar{b}z + cw. \tag{15}$$

For any $n \in O_d$ satisfying the equation

$$\mathrm{Re}(nwA) = 0 \tag{16}$$

or, what is the same, the equation $\mathrm{Re}(nzB) = 0$, a tedious calculation shows that matrix

$$T = \begin{vmatrix} nzw + 1 & nw^2 \\ -nz^2 & -nzw + 1 \end{vmatrix},$$

for which z/w is a fixed point, satisfies the condition $TFT^* = F$. Thus, T belongs to the group Γ which is, therefore, non-cocompact.

Lemma 2 shows that the problem of classification of the conjugacy classes of maximal non-elementary Fuchsian subgroups of G in $\mathrm{PGL}(2, O_d)$ is equivalent to the problem of classification of $\mathrm{PGL}(2, O_d)$-equivalency classes of multiples of rational indefinite Hermitian forms. The last problem was partly solved in [10]. Denote the discriminant of the field $\mathbf{Q}(\sqrt{-d})$ by D. Theorem 1 of [10] implies the following.

THEOREM 3. *Let $d = 1, 2$, or p, prime $p \equiv 3 \pmod 4$. A maximal non-elementary Fuchsian subgroup of* $\mathrm{PSL}(2, O_d)$ *is conjugate in* $\mathrm{PGL}(2, O_d)$ *to one and only one of the groups* $\mathrm{PSU}(f_{\ell,m,c}O_d)$. *The binary rational indefinite Hermitian forms $f_{\ell,m,c}$ are defined by the following conditions:*

$$a = 1, \quad b = D^{-1/2}(m+\omega\ell), \quad \ell, m, c \in \mathbf{Z},$$
$$0 \leq m < d/2, \quad \ell = 0, \quad \text{if } d \equiv 3 \pmod 4,$$
$$m = 0, 1, \text{ or } 2, \quad \ell = 0 \text{ or } 1, \quad \text{if } d = 2,$$
$$m + i\ell = 0, 1, \text{ or } (1+i), \quad \text{if } d = 1.$$

COROLLARY. *Let d belong to the sequence in (1). Two maximal non-elementary Fuchsian subgroups Γ and Γ' of* $\mathrm{PSL}(2, O_d)$ *are conjugate in* $\mathrm{PGL}(2, O_d)$ *if and only if $\rho(\Gamma) = \rho(\Gamma')$.*

REMARKS. Let C be a rational circle in $\mathbf{C}$ with equation (4). Let S_f be the hemisphere in H^3 on C. S_f is a hyperbolic plane under the restriction of the hyperbolic metric in H^3. The Fuchsian group $\Gamma = \mathrm{Stab}\big(C, \mathrm{PSL}(2, O_d)\big)$ acts discontinuously on S_f. We shall show that the region in S_f satisfying the inequalities (cf. Swan [9])

$$|\mu z - \lambda|^2 + |\mu|^2 t^2 > 1 \tag{17}$$

for all $(\lambda\,\mu) = (1 \quad 0)T^t$, $T \in \Gamma$, is the Dirichlet polygon $D(e)$ for Γ with center $e = \big(-b/a, r(C)\big)$ (see [1], p. 226, for the definition of $D(e)$). Indeed, let $T \in \Gamma$ be fixed. Let λ/μ and e_T be the images of ∞ and e under transformation T in H^3. Since the isometric circle of T^{-1} in $\mathbf{C}$ is $|\mu z - \lambda| = 1$, hemisphere $S(\lambda/\mu)$, the boundary in (17), is orthogonal to S_f. λ/μ is the reflection of ∞ in $S(\lambda/\mu)$. Since the feet of the perpendiculars from ∞ and μ/λ to S_f are e and e_T respectively, e_T is the reflection of e in $S(\lambda/\mu)$. Thus, for x in S_f, the inequality (17) is reduced to $d(x, e_T) > d(x, e)$ where $d(x, y)$ is the distance between points x and y in the hyperbolic plane S_f.

Since the region $t > 1$ satisfies (17) for any pair $\mu, \lambda \in O_d$, the circle $t > 1$ in S_f belongs to $D(e)$. The area of this circle equals $2\pi\big(r(C) - 1\big)$. Thus, we have

$$\mathrm{Area}(S_f/\Gamma) > 2\pi(\rho(\Gamma) - 1).$$

Let $N(D)$ denote the number of sides of the polygon $D(e)$. Then

$$N(D) > \pi\rho(\Gamma),$$

since the radius of any hemisphere, the boundary in (17), is less than or equal to 1 and, as was mentioned above, it is orthogonal to S_f.

Finally, as follows from Theorem 3, for d in the sequence in (1),

$$\sum_r 1 \sim \frac{d}{2}x \quad (x \longrightarrow \infty)$$

where r runs through all the values of $\rho(\Gamma) < x$.

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