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VERTEX AND EDGE TRANSITIVE, BUT NOT 1-TRANSITIVE, GRAPHS

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A (simple, undirected) graph G is *vertex transitive* if for any two vertices of G there is an automorphism of G that maps one to the other. Similarly, G is *edge transitive* if for any two edges $[a, b]$ and $[c, d]$ of G there is an automorphism f of G such that $\{c, d\}=\{f(a), f(b)\}$. A 1-*path* of G is an ordered pair (a, b) of (distinct) vertices a and b of G, such that a and b are joined by an edge. G is 1-*transitive* if for any two 1-paths (a, b) and (c, d) of G there is an automorphism f of G such that $c=f(a)$ and $d=f(b)$. A graph is *regular of valency d* if each of its vertices is incident with exactly d of its edges.

In [1, p. 59, 7.53] Tutte proved that if a connected graph, regular of odd valency, is both vertex and edge transitive, then it is 1-transitive. He states [1, p. 60] that it is not known if this theorem extends to connected graphs, regular of even valency. Our aim here is to prove the following

THEOREM. *For each integer $N \geq 2$, there exists a connected graph which is regular of valency $2N$, and which is vertex and edge transitive, but not 1-transitive.*

We start by constructing a wider class of graphs. Let N be any integer ≥ 2, and let m and n be any two integers ≥ 2 such that

$$2^m \equiv 1 \text{ (modulo } n). \tag{1}$$

With the ordered triple (N, m, n) we associate a graph $X(N, m, n)$ as follows: Its vertex set is identified with the Cartesian product

$$W = R(m) \times [R(n) \times \cdots ((N-1) \text{ copies}) \cdots \times R(n)],$$

where $R(m)$ and $R(n)$ denote the rings of integers modulo m and n, respectively; and its edges are formed by joining two vertices whenever they can be written as

$$(i, a_2, a_3, \ldots, a_N) \quad \text{and} \quad (i+1, b_2, b_3, \ldots, b_N) \tag{2}$$

($i \in R(m)$; $a_k, b_k \in R(n)$, $k=2, 3, \ldots, N$), where either $b_r=a_r$ for $r=2, 3, \ldots N$, or there is exactly one $s \in \{2, 3, \ldots, N\}$ for which $b_s \neq a_s$, in which case $b_s=a_s+2^i$ (the operations being interpreted in the appropriate ring). From the congruence (1) it follows that the edges are well-defined.

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From the construction it is clear that for any edge E of the graph, the two terminal vertices of E differ either in their first entries only, in which case E will be called of *type* 1, or in their first and also their kth entries, for exactly one $k \in \{2, 3, \ldots, N\}$, in which case E will be called of *type* k. We note that if $m \geq 3$, then the graph is regular of valency $2N$, each vertex being incident with exactly two edges of each of the N types. Although it is not needed (since eventually we need consider only connected components), we remark that the graph is connected.

PROPOSITION 1. *The graph* $X(N, m, n)$ *is vertex and edge transitive.*

Proof. Denoting, generically, a vertex and its image (with respect to a mapping from W to W) by $(i, a_2, a_3, \ldots, a_N)$ and $(i', b_2, b_3, \ldots, b_N)$, respectively, we define mappings S_k, R, and T_k $(k=2, 3, \ldots, N)$ from W to W as follows (with the operations being interpreted in the appropriate ring):

$$S_k\colon i' = i; \qquad b_k = a_k+1; \qquad b_j = a_j, \quad j \neq k, \quad j \in \{2, 3, \ldots, N\};$$

$$R\colon i' = i-1; \qquad b_j = 1+2^{-1}a_j \quad (j = 2, 3, \ldots, N)$$

(By (1) the element 2^{-1} is defined in $R(n)$);

$$T_k\colon i' = i; \qquad b_k = 2^i - \sum_{r=2}^{N} a_r; \qquad b_j = a_j, \quad j \neq k, j \in \{2, 3, \ldots, N\}.$$

It is easily verified that these mappings are automorphisms of the graph.

Under S_k $(k=2, 3, \ldots, N)$ the kth entry of a vertex increases by 1 (modulo n), while the other entries remain unchanged. It follows that under the successive application of suitable powers of the mappings S_k $(k=2, 3, \ldots, N)$, any given vertex may be transformed to a vertex whose entries, with the possible exception of the first, are equal to 2. Under a suitable power of R the latter vertex may be transformed to the one all of whose entries are equal to 2. We deduce that the graph is vertex transitive.

Let a denote the vertex all of whose entries are equal to 2, and let b denote the vertex whose first entry is equal to 3, and whose other entries are equal to 2. The respective mappings R, $S_k^{2N-4} \circ T_k$, and $S_k^{2N-4} \circ T_k \circ R$ (the compositions being read from right to left), with $k=2, 3, \ldots, N$, are found to transform the edge $[a, b]$ to each of the other edges incident with a. Since the graph is vertex transitive, this implies that it is also edge transitive.

For some choices of the triples (N, m, n) (we mention, for the sake of interest, the triples (2, 3, 7), (2, 6, 7), and (2, 4, 5)), we have checked that the corresponding graphs $X(N, m, n)$ are also 1-transitive. However, we proceed to show that the graphs which correspond to the triples of the form $(N, 6, 9)$, are not 1-transitive. For the cases $N \geq 3$ we do this by showing that the terminal vertices of an edge can be distinguished topologically by the patterns formed by the hexagons with which they are incident. For the case $N=2$ we also need hexagons that are not incident with a terminal vertex of the edge.

Henceforth we assume the values: N arbitrary (≥ 2), $m=6$, $n=9$.

LEMMA 1. *The graph* $X(N, 6, 9)$ *contains no p-gon, for* $p \leq 5$. *The hexagons in the graph are of the following types* (*with all choices being realizable*):

(H1) *There is a sense of traversal of the hexagon in which the first entries of the vertices encountered increase monotonically by* 1 (modulo 6). *Each pair of opposite edges are of the same type.*

(H2) *In traversing the hexagon, the first entries of the vertices alternate between two values* i *and* $i+1$ ($i \in R(6)$). *Each pair of opposite edges are of the same type, with different pairs being of different types.*

(H3) *The hexagon may be traversed in such a manner that the first entries of the vertices occur in the sequence*:

$$i, i+1, i+2, i+1, i, i+1,$$

for some $i \in R(6)$. *The edges alternate between two distinct types.*

Proof. By construction the graph contains no p-gon for $p=1$ or 2. From (2) it follows that adjacent vertices have first entries differing by 1 (in $R(m)$), and we deduce, firstly, since m ($=6$) is even, that the graph contains no p-gon with p odd (so that it is bipartite), and secondly, that for $p=4$ and 6, the arrangements of the first entries of the vertices (written as sequences, according to a traversal of the p-gon) can only be of the following types:

In the case $p=4$:

(Q1) $$i, i+1, i+2, i+1,$$

and

(Q2) $$i, i+1, i, i+1,$$

for $i \in R(6)$;

In the case $p=6$:

The sequences described under (H1), (H2) and (H3) above, as well as:

(H3*) $$i, i+1, i+2, i+1, i+2, i+1,$$

and

(H4) $$i, i+1, i+2, i+3, i+2, i+1,$$

for $i \in R(6)$.

We choose a representative p-gon C ($p=4$ or 6), and proceed to determine, for each of the above sequences, the possible arrangements of edge types along C. For any $k \in \{2, 3, \ldots, N\}$ and any edge E of C, let the quantity $c(k, E)$ be defined to be 1 or 0 according as to whether or not the edge E is of type k. We shall need to refer to the following simple facts:

(a) For any edge E of C it holds that $c(k, E) = 1$ for at most one $k \in \{2, 3, \ldots, N\}$ (the type of an edge being well-defined).

(b) If two consecutive edges of C have the property that the two vertices which belong to one, but not both, of the edges have equal first entries, then they cannot be of the same type. (For by (2) the edges would have to coincide.)

Finally, the edges of C will be denoted by $E_1, E_2, \ldots, E_p$, according to a traversal of C, with the traversal chosen such that the first entries of the vertices encountered occur, in order, according to the given sequence.

Let $k \in \{2, 3, \ldots, N\}$. If a and b are two consecutive vertices of C, joined by an edge E, and if the first entry of the vertex a is equal to $j \in R(6)$, then it follows from (2) that the kth entries a_k and b_k of a and b, respectively, are related by the first or second of the equations in $R(9)$:

$$b_k = a_k + c(k, E) \cdot 2^j$$

and

$$b_k = a_k - c(k, E) \cdot 2^{j-1},$$

according as to whether the first entry of b is equal to $j+1$ or $j-1$ (modulo 6). We call the term $c(k, E) \cdot 2^j$, or respectively, $-c(k, E) \cdot 2^{j-1}$, the *change* in the kth entry as we pass from the vertex a to the vertex b along the edge E. In a traversal of C, where we start and end at the same vertex, the changes in the kth entries of the successive vertices encountered, must sum to zero. Thus we associate with C a linear equation (in $R(9)$) in the p variables $c(k, E_i)$, $i = 1, 2, \ldots, p$, with coefficients independent of k. From its solution, and the conditions (a) and (b), we readily determine the possible combinations of edge types along C.

Since the case $p = 6$, sequence (H3), turns out to be of some further interest (see Remark below), we give the detail for it. The equation in $R(9)$ associated with C is seen to be the following:

$$\begin{aligned} c(k, E_1)2^i + c(k, E_2)2^{i+1} - c(k, E_3)2^{i+1} - c(k, E_4)2^i \\ + c(k, E_5)2^i - c(k, E_6)2^i = 0. \end{aligned} \tag{3}$$

This is equivalent to the congruence:

$$c(k, E_1) + c(k, E_5) - 2c(k, E_3) \equiv c(k, E_4) + c(k, E_6) - 2c(k, E_2) \tag{4}$$

(modulo 9). In view of (b) no two consecutive edges of C can be of the same type, and it follows in particular that there is at least one value k' of k such that $c(k', E_2) \neq c(k', E_3)$. Since the variables in (4) can assume only the values 0 and 1, this implies that the solution of the equation (4), for the value k' of k, is of the form:

$$c(k', E_2) = c(k', E_4) = c(k', E_6) = e$$

and

$$c(k', E_3) = c(k', E_5) = c(k', E_1) = 1 - e,$$

for $e=0$ or 1. (We note that in (3) this corresponds to the equality: $2^i+2^i=2^{i+1}$.) Thus there are three alternate edges of C that are of the same type $k' \in \{2, 3, \ldots, N\}$. By (b) none of the remaining three edges can be of type k', so that if they are not all of type 1, then one of them is of a type $k'' \neq k'$, $k'' \in \{2, 3, \ldots, N\}$. Setting $k=k''$ in (4), we find that, by (a) one side of (4) reduces to 0, while the presence of one nonzero term on the other side necessitates that each term on that side is nonzero. It follows that all three of the remaining edges are of the same type ($\neq k'$). This confirms the statement (H3) in the lemma.

A similar analysis applies to the other cases. In particular, it is found that the sequences (Q1) and (Q2) (for $p=4$), and (H3*) and (H4) (for $p=6$), cannot be realized in the graph, since the solutions to the corresponding equations in $R(9)$ are in contradiction with (b). As the case $p=4$ is thereby eliminated, it follows that the graph contains no quadrilateral. For the remaining sequences (given in (H1) and (H2)) for $p=6$, the edge type distributions are found as stated.

It is a matter of routine to verify that each instance of the solutions given in (H1), (H2) and (H3), can be realized in the graph. This concludes the proof of the lemma.

REMARK. Let ρ be any permutation on $R(6)$ which inverts the natural cyclic ordering on $R(6)$ (in which case it has the form $\rho: i \to c-i$, $i \in R(6)$, for some $c \in R$ (6)). Under application of ρ to the first entries of the vertices, we find that the procedure for the classification of the hexagons, as given in the above proof, is self-dual with respect to the types (H1), (H2), and (H4), while the roles of (H3) and (H3*) are interchanged in all respects but the following: the equality $2^i+2^i=2^{i+1}$ in $R(9)$, which determines the solutions for the case (H3), is not preserved when the mapping ρ is applied to the exponents (so that there are no corresponding solutions for the case (H3*)). This asymmetry, in relation to ρ, between the cases (H3) and (H3*) (expressed concretely by the fact that (H3) is represented in the graph, while (H3*)is not), is the circumstance which finally will enable us to establish the non-1-transitivity of the graph.

For any subset $\{E_1, E_2, \ldots, E_t\}$ of the set of edges of the graph $X(N, 6, 9)$, let $S(E_1, E_2, \ldots, E_t)$ denote the subgraph whose edges are the edges $E_1, E_2, \ldots, E_t$, and whose vertices are their terminal vertices. Call two edges *adjacent* if they are distinct and have a common terminal vertex. Two adjacent edges E and F of the graph will be called *opposed* if the three vertices of $S(E, F)$ have distinct first entries, and *properly opposed* if they are opposed and of different type. Using the scheme of hexagons as given in Lemma 1, we count possibilities, and readily verify:

LEMMA 2. *Let E and F be any two adjacent edges of the graph $X(N, 6, 9)$. If E and F are (are not) properly opposed, then the subgraph $S(E, F)$ is contained in exactly $N+1(N)$ distinct hexagons. In each case the hexagons are otherwise disjoint.*

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Corollary. *Adjacent and properly opposed edges remain so under automorphisms of the graph.*

We now prove:

Proposition 2. *The graphs $X(N, 6, 9)$ are not 1-transitive.*

Proof. We separate the cases $N \geq 3$ and $N=2$.

The case $N \geq 3$. Let $E=[a, b]$ be any given edge of the graph, the notation being chosen such that the first entry of a is one less (modulo 6) than the first entry of b. Since $N \geq 3$, there exist two distinct edges $D_1=[d_1, a]$ and $D_2=[d_2, a]$, which are a.p.o. (adjacent and properly opposed) to E, both having the vertex a in common with E. The edges E, D_1 and D_2 are of distinct types. Let the type of D_i be k_i $(i=1, 2)$. Let C_i be the edge a.p.o. to D_i, having the vertex d_i in common with D_i, and being of type k_{3-i} $(i=1, 2)$. If A is an automorphism of the graph which interchanges the vertices a and b, then by the above Corollary, A must map the subgraph $S(C_1, D_1, D_2, C_2)$ to a subgraph $S(C_1', D_1', D_2', C_2')$, where $D_1'=[b, d_1']$ and $D_2'=[b, d_2']$ are edges, a.p.o. to E, having the vertex b in common with E, while C_i' $(i=1, 2)$ is an edge, a.p.o. to D_i', having the vertex d_i' in common with D. However, the subgraph $S(C_1, D_1, D_2, C_2)$ is found to be contained in a hexagon (of type (H3)), while by considering the first entries of its vertices, we deduce from Lemma 1 that the subgraph $S(C_1', D_1', D_2', C_2')$ is not (see Remark).

The case $N=2$. In this case an edge can be of one of two types. It follows that for any given edge E, there is a unique polygon P in the graph such that P contains E as an edge, and such that each two adjacent edges in P are properly opposed. Let $E=[a, b]$, the notation being such that the first entry of a is one less (modulo 6) than the first entry of b. We define $E_1=[b, b_1]$ and $F_1=[a_1, a]$ to be edges, a.p.o. to E, having the vertices b and a, respectively, in common with E. Define, inductively, for $i=2, 3, 4, 5$, the edges $E_i=[b_{i-1}, b_i]$ and $F_i=[a_i, a_{i-1}]$ to be the edges, a.p.o. to E_{i-1} and F_{i-1}, respectively, with b_{i-1} and a_{i-1} being the respective common vertices. A direct verification shows that these edges are all distinct (in fact, in tracing out P, we find it to be an 18-gon). Since by Lemma 2 the subgraph $S(E_1, E_2)$ is contained in three otherwise disjoint hexagons, and each vertex is of valency 4, it follows that the subgraph $S(E_1, E_2, E_3)$ is contained in a unique hexagon. Since the first entries of the respective vertices b, b_1, b_2, and b_3, increase in steps of 1 (modulo 6), we have by Lemma 1 that this hexagon can only be of type (H1). Taking the types of edges in consideration, we see that in traversing this hexagon, starting at the vertex b_3, we can leave P, and traverse edges K_1 and K_2, both distinct from E_4 and E_5. Similarly, the subgraph $S(F_1, F_2, F_3)$ is contained in a unique hexagon, also of the type (H1), which can be traversed, starting at the vertex a_3, and then leaving P, to traverse edges L_1 and L_2, both distinct from F_4 and F_5. If A is an automorphism of the graph which interchanges a and b, then A must map P to itself, and must interchange the subgraphs $S(L_2, L_1, F_4, F_5)$ and

$S(K_2, K_1, E_4, E_5)$. However, the subgraph $S(L_2, L_1, F_4, F_5)$ is found to be contained in a hexagon (of type (H3)), while by considering the first entries of its vertices, we deduce from Lemma 1 that the subgraph $S(K_2, K_1, E_4, E_5)$ is not (see Remark).

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